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# Loewner slit domains driven by the cubic root function

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# LÖWNER SLIT DOMAINS DRIVEN BY THE CUBIC ROOT FUNCTION

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Abstract. We consider the chordal Löwner differential equation with the model driving function $\sqrt[3]{t}$. Holomorphic and singular solutions are represented by their series. It is shown that a disposition of values of different singular solutions is monotonic, and solutions to the Löwner equation map slit domains onto the upper half-plane. The slit is a $C^1$-curve which is tangential to $\mathbb{R}$.

## 1. Introduction

The Löwner differential equation introduced by K. Löwner [8] served a source to study properties of univalent functions on the unit disk. Nowadays it is of growing interest in many areas, see, e.g., [9]. The half-plane version of the Löwner equation appeared later (see, e.g., [1]) and became popular a few years ago. Among other problems, the Löwner equation describes curves slitting the upper half-plane $\mathbb{H}$. Suppose that $\gamma(t)$, $t \geq 0$, is a simple curve in $\mathbb{H}$ emanating from the real axis $\mathbb{R}$. There exists a unique conformal mapping $w = f(z,t)$ that takes the domain $D(t) = \mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ in such a way that near infinity it has the expansion

$$f(z,t) = z + \frac{c(t)}{z} + O\left(\frac{1}{z^2}\right). \tag{1}$$

The coefficient $c(t)$ is continuously increasing with $t$. Therefore, it is possible to choose the parameterization of the curve so that $c(t) = 2t$, and this is assumed in the sequel. The conformal maps $f(z,t)$ are continuously extended onto the closure of $D(t)$ and satisfy the *chordal* Löwner ordinary differential equation

$$\frac{df(z,t)}{dt} = \frac{2}{f(z,t) - \lambda(t)}, \quad f(z,0) = z, \quad z \in \mathbb{H}. \tag{2}$$

The driving function $\lambda(t)$ is continuous and real-valued,

$$\lambda(t) = f(\gamma(t),t), \quad \gamma(t) = f^{-1}(\lambda(t),t). \tag{3}$$

Conversely, the Löwner equation (2) has a solution $f(z,t)$ for any given continuous real-valued driving function $\lambda(t)$.

It is known [10], [6] that if $\lambda(t)$ in (2) belongs to the Lipschitz class of order $1/2$ and $\|\lambda\|_{1/2} < 4$, then solutions $f(z,t)$ to (2) map $D(t)$ onto $\mathbb{H}$ where $\gamma(t)$ is a Jordan qausisymmetric curve which is non-tangential to $\mathbb{R}$. Conversely, if $f(z,t)$ is

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a mapping from a slit domain $D(t)$ with a Jordan quasisymmetric slit $\gamma(t)$ which is non-tangential to $\mathbb{R}$, then $\lambda \in \text{Lip}(1/2)$. Note that Kufarev [5] proposed a counterexample of the non-slit mapping for the *radial* Löwner equation in the disk. For the chordal Löwner equation, Kufarev's example corresponds to $\lambda(t) = 3\sqrt{2}\sqrt{1-t}$.

Point out important papers [4] and [7] describing solutions to the chordal Löwner equation (2). The authors of [4] studied partial cases of $\lambda(t)$ which admit integrating of (2) in quadratures. Lind, Marshall and Rohde [7] examined tangential and non-tangential self-intersections of $\gamma(t)$ in $\mathbb{H}$ for $\lambda(t) = 4\sqrt{1-t}$. They proved a form of stability of the non-tangential self-intersection of $\lambda(t) = k\sqrt{1-t}$ for $k > 4$ and many other interesting properties.

The results of Marshall and Rohde [10] imply that if $D(t)$ omits an arc $\gamma(t)$ tangential to $\mathbb{R}$, then the corresponding function $f(z,t)$ is generated by (2) with $\lambda(t) \in \text{Lip}\,\alpha$, $\alpha < 1/2$. Prokhorov and Vasil'ev [11] showed that a circular arc $\gamma(t)$ tangential to $\mathbb{R}$ equals $f^{-1}(\lambda(t),t)$ for $f(z,t)$ generated by $\lambda(t) \in \text{Lip}(1/3)$.

Notice that the original version of the chordal Löwner equation differs from (2) by the multiplier (-1) in its right-hand side. The relation between the two versions looks like changing time $t$ by $(-t)$. This is not a coincidence because, generally, putting $\lambda_1(-t) := \lambda(T-t)$ for a constant $T > t$ and solving Löwner's equation for both $f(z,T)$ with $\lambda(t)$ and $f(z,-T)$ with $\lambda_1(-t)$ we obtain the equality $f(z,-T) = f^{-1}(w,T)$.

The Löwner equation (2) continually generates singular points of $f(z,t)$ that are mapped by $f(z,t)$ onto the points $\lambda(t)$ in the $w$-plane when $f(z,t)$ are slit mappings. That is why the curve $\gamma(t)$ is called the trace of the Löwner evolution or the line of singularities, see [4]. Evidently, having an explicit solution $f(z,t)$ to (2) we obtain the line of singularities by (3). But this is a relatively rare case. However, it is possible to find singular solutions to (2) without formulas for $f(z,t)$. This is a way to draw a geometrical characteristic for solutions to the Löwner equation in the cases when $f(z,t)$ is not given explicitly.

The integrability cases of (2) are invariant under certain transformations of $\lambda(t)$ which mean that $\gamma(t)$ is transformed correspondingly. Thus $\lambda(t) \to \lambda(t) + b$, $b \in \mathbb{R}$, is compensated by $f(z,t) \to f(z-b,t) + b$, and $t \to \alpha^2 t$, $\alpha > 0$, is compensated by $\lambda(t) \to (1/\alpha)\lambda(\alpha^2 t)$ and the scaling $f(z,t) \to (1/\alpha) f(\alpha z, \alpha^2 t)$. Therefore, assume without loss of generality that $\lambda(0) = 0$ which is equivalent to $\gamma(0) = 0$.

This article is focused on the asymptotics for singular solutions $f(0,t)$, $t \to +0$, in the case $\lambda(t) = \sqrt[3]{t} \in \text{Lip}(1/3)$, that is

$$\frac{df(z,t)}{dt} = \frac{2}{f(z,t) - \sqrt[3]{t}}, \quad f(z,0) = z, \quad \text{Im}\, z \geq 0. \tag{4}$$

For real $z \neq 0$, solutions $f(z,t)$ to (4) locally exist, $f(z,t)$ increases with $t$ for $z > 0$, and $f(z,t)$ decreases with $t$ for $z < 0$. Singular solutions $f^s(0,t)$ to (4) are not unique. We will show that there are exactly two of them, the *upper* singular solution $f^+(0,t)$ determined as

$$f^+(0,t) := \inf_{z>0} f(z,t), \tag{5}$$

and the *lower* singular solution $f^-(0,t)$ determined as

$$f^-(0,t) := \sup_{z<0} f(z,t). \tag{6}$$

Both $f^-(0,t)$ and $f^+(0,t)$ are continuous for $t \geq 0$ small enough and satisfy (4) for $t > 0$. Besides, $f^-(0,t)$ is strictly decreasing, $f^+(0,t)$ is strictly increasing, and $f^-(0,t) < \sqrt[3]{t} < f^+(0,t)$ for $t > 0$.

The driving function $\lambda(t) = \sqrt[3]{t}$ is chosen as a typical function of the Lipschitz class Lip(1/3). We do not try to cover the most general case but hope that the model function serves a demonstration for a wider class.

In Section 2 we consider holomorphic solutions to (4) represented by power series and give asymptotics to the radius of convergence of the series.

In Section 3 the series expansions for singular solutions to (4) are derived. It is shown that a disposition of values of different singular solutions is monotonic.

The results of Section 3 are applied in Section 4 to show that solutions to the Löwner equation map slit domains onto the upper half-plane. The slit is a $C^1$-curve which is tangential to $\mathbb{R}$.

## 2. Representation of holomorphic solutions

Change variables $\sqrt[3]{t} = \tau$ in (4) and consider the ordinary differential equation (4) as an analytic differential equation having holomorphic solutions. Describe an asymptotic representation of holomorphic solutions in the following lemma.

**Lemma 1.** *For $\epsilon > 0$, let*

$$f(\epsilon,t) = \epsilon + \sum_{n=1}^{\infty} a_n(\epsilon) t^{n/3} \tag{7}$$

*be a solution of equation (4) holomorphic with respect to $\tau = \sqrt[3]{t}$. Then the series in (7) converges for*

$$|t| \leq \epsilon^3 + o(\epsilon^3), \quad \epsilon \to +0. \tag{8}$$

*Proof.* Denote $t = \tau^3$, and $g(\epsilon,\tau) := f(\epsilon,\tau^3)$. Transform equation (4) to a differential equation for $g(\epsilon,\tau)$,

$$\frac{dg}{d\tau} = \frac{6\tau^2}{g-\tau}, \tag{9}$$

with the initial data $g|_{\tau=0} = \epsilon$. Equation (9) can be regarded as an analytic differential equation $g' = w(g,\tau)$ in a neighborhood of $(g_0,\tau_0) = (\epsilon,0)$. The Cauchy problem for equation (9) has a holomorphic solution

$$g(\epsilon,\tau) = \epsilon + \sum_{n=1}^{\infty} a_n(\epsilon)\tau^n,$$

where the series converges for $|\tau| < R(\epsilon)$.

Estimate the convergence radius $R(\epsilon)$ according to the Cauchy theorem, see, e.g., [2, p.10]: if the right-hand side $w(g,\tau)$ in (9) is holomorphic on a product of the closed disks $|g-\epsilon| \leq \rho_1$ and $|\tau| \leq r_1$, and satisfies there $|w(g,\tau)| \leq M$, then the series $\sum_{n=1}^{\infty} a_n(\epsilon)\tau^n$ representing the solution $g(\epsilon,\tau)$ of equation (9) converges in the disk

$$|\tau| < R(\epsilon) = r_1\left(1 - \exp\left\{-\frac{\rho_1}{2Mr_1}\right\}\right).$$

The above Cauchy theorem is true for

$$\rho_1 + r_1 < \epsilon, \quad \text{and} \quad M = \frac{6r_1^2}{\epsilon - (\rho_1 + r_1)}.$$

This implies that for $\rho_1 + r_1 = \epsilon - \delta$, $\delta > 0$,

$$R(\epsilon) = r_1\left(1 - \exp\left\{-\frac{\epsilon - \delta - r_1}{12r_1^2}\delta\right\}\right).$$

So $R(\epsilon)$ depends on $\delta$ and $r_1$. Maximum of $R$ with respect to $\delta$ is obtained for $\delta = (\epsilon - r_1)/2$. Hence, this maximum is equal to

$$R_1(\epsilon) = r_1\left(1 - \exp\left\{-\frac{(\epsilon - r_1)^2}{48r_1^3}\right\}\right), \tag{10}$$

where $R_1(\epsilon)$ depends on $r_1$. Let us find a maximum of $R_1$ with respect to $r_1 \in (0,\epsilon)$. Notice that $R_1$ vanishes for $r_1 = 0$ and $r_1 = \epsilon$. Therefore the maximum of $R_1$ is attained for a certain root $r_1 = r_1(\epsilon) \in (0,\epsilon)$ of the derivative of $R_1$ with respect to $r_1$. To simplify the calculations we put $r_1(\epsilon) = \epsilon c(\epsilon)$, $0 < c(\epsilon) < 1$. Now the derivative of $R_1$ vanishes for $c(\epsilon)$ satisfying

$$1 - \exp\left\{-\frac{(1-c)^2}{48\epsilon c^3}\right\}\left(1 + \frac{(1-c)(3-c)}{48\epsilon c^3}\right) = 0. \tag{11}$$

Choose a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\epsilon_n = 0$, such that $c(\epsilon_n)$ converge to $c_0$ as $n \to \infty$. Suppose that $c_0 < 1$. Then

$$\exp\left\{-\frac{(1-c(\epsilon_n))^2}{48\epsilon_n c^3(\epsilon_n)}\right\}\left(1 + \frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}\right) < 1$$

for $n$ large enough. Therefore $c(\epsilon_n)$ is not a root of equation (11) for $\epsilon = \epsilon_n$ and $n$ large enough. This contradiction claims that $c_0 = 1$ for every sequence $\{\epsilon_n > 0\}_{n=1}^{\infty}$ tending to 0 with $\lim_{n\to\infty} c(\epsilon_n) = c_0$. So we proved that $c(\epsilon) \to 1$ as $\epsilon \to +0$.

Consequently, the maximum of $R_1$ with respect to $r_1$ is attained for $r_1(\epsilon) = \epsilon c(\epsilon) = \epsilon(1 + o(1))$ as $\epsilon \to +0$. Let $R_2 = R_2(\epsilon)$ denote the maximum of $R_1$ with respect to $r_1$. It follows from (10) that

$$R_2(\epsilon) = r_1(\epsilon)\left(1 - \exp\left\{-\frac{(\epsilon - r_1(\epsilon))^2}{48r_1^3(\epsilon)}\right\}\right) = \epsilon c(\epsilon)\left(1 - \exp\left\{-\frac{(1-c(\epsilon))^2}{48\epsilon c^3(\epsilon)}\right\}\right). \tag{12}$$

Examine how fast does $c(\epsilon)$ tends to 1 as $\epsilon \to +0$. Choose a sequence $\{\epsilon_n > 0\}_{n=1}^{\infty}$, $\lim_{n\to\infty}\epsilon_n = 0$, such that the sequence $(1 - c(\epsilon_n))^2/\epsilon_n$ converges to a non-negative number or to $\infty$. Denote

$$l := \lim_{n\to\infty}\frac{(1-c(\epsilon_n))^2}{\epsilon_n}, \quad 0 \le l \le \infty.$$

If $0 < l < \infty$, then $(1 - c(\epsilon_n))/\epsilon_n$ tends to $\infty$, and equation (11) with $\epsilon = \epsilon_n$ has no roots for $n$ large enough.

If $l = 0$, then, according to (11), $\lim_{n\to\infty}(1 - c(\epsilon_n))/\epsilon_n = 0$, and

$$\exp\left\{-\frac{(1-c(\epsilon_n))^2}{48\epsilon c^3(\epsilon)}\right\}\left(1+\frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}\right) =$$

$$\left(1-\frac{(1-c(\epsilon_n))^2}{48\epsilon_n c^3(\epsilon_n)}+o\left(\frac{(1-c(\epsilon_n))^2}{\epsilon_n}\right)\right)\left(1+\frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}+\right) =$$

$$1+\frac{1-c(\epsilon_n)}{24\epsilon_n}+o\left(\frac{1-c(\epsilon_n)}{\epsilon_n}\right).$$

This implies again that equation (11) with $\epsilon = \epsilon_n$ has no roots for $n$ large enough.

Thus the only possible case is $l = \infty$ for all sequences $\{\epsilon_n > 0\}_{n=1}^{\infty}$ converging to 0. It follows from (12) that

$$R_2(\epsilon) = \max_{0<r_1(\epsilon)<\epsilon} R_1(\epsilon) = \epsilon + o(\epsilon). \tag{13}$$

In other words, the series representing $g(\epsilon,\tau)$ converges for $|\tau| < \epsilon + o(\epsilon)$. Hence the series (7) converges for $|t| < \tau^3 = (\epsilon + o(\epsilon))^3$ which implies the statement of Lemma 1 with (8) and completes the proof.

□

To formulate the next lemma let us introduce certain denotations. For $\epsilon > 0$, set

$$a_1 = 0, \quad a_2 = 0, \quad a_k = \frac{6}{k\epsilon^{k-2}}, \quad k = 3,4,5, \tag{14}$$

and

$$a_n = \frac{1}{n\epsilon}\left[(n-1)a_{n-1} - \sum_{k=3}^{n-3}(n-k)a_{n-k}a_k\right], \quad n \geq 6. \tag{15}$$

**Lemma 2.** *For $\epsilon > 0$, let $f(\epsilon, t)$ given by (7) be a solution of equation (4). Then coefficients $a_n(\epsilon)$, $n \geq 1$, satisfy equations (14) and (15).*

*Proof.* The function $g(\epsilon,\tau) = f(\epsilon,\tau^3)$ solves equation (9) and satisfies

$$\frac{dg}{d\tau} = \sum_{n=1}^{\infty} n a_n(\epsilon)\tau^{n-1}.$$

From here and (9) we have

$$\sum_{n=1}^{\infty} n a_n(\epsilon)\tau^{n-1}\left[\epsilon - \tau + \sum_{n=1}^{\infty} a_n(\epsilon)\tau^n\right] = 6\tau^2. \tag{16}$$

Equate coefficients at the same powers in both sides of (16) and obtain equations (14) and (15). By Lemma 1 the series $\sum_{n=1}^{\infty} a_n(\epsilon)\tau^n$ converges if $|\tau| \leq \epsilon + o(\epsilon)$, $\epsilon \to +0$. The latter condition is equivalent to $\epsilon \geq |\tau| + o(|\tau|)$, $\tau \to 0$. This completes the proof. □

**Remark 1.** *Evidently, a similar conclusion with the same formulas (14) and (15) is true for* $\epsilon < 0$. *The convergence radius also is not less than* $|\epsilon| + o(|\epsilon|)$, $\epsilon \to -0$, *which gives that* $|\epsilon| \geq |\tau| + o(|\tau|)$, $\tau \to 0$.

## 3. Representation of singular solutions

The right-hand side $w(g, \tau)$ of the differential equation $g' = w(g, \tau)$ in (9) is rational. The Cauchy problem (9) has holomorphic solutions in a neighborhood of every point $(g_0, \tau_0)$ where $w(g, \tau)$ is holomorphic. If $(g_0, \tau_0)$ is a moving singular point of $w(g, \tau)$, then equation (9) with the initial data $g|_{\tau=\tau_0} = g_0$ has a singular solution expanded in powers of $(\tau - \tau_0)^{1/m}$, $m \in \mathbb{N}$, see, e.g., [2, p.33].

Let $g(z, \tau)$ be a holomorphic solution of (9) with the initial data $g|_{\tau=0} = z$, $z \neq 0$, in a neighborhood of $(z, 0)$. For $\tau$ close to 0, the function $g(z, \tau)$ is holomorphic with respect to $z$ outside a bounded set $K_\tau$ containing the origin.

For $\tau = 0$, the point $(g_0, \tau_0) = (0, 0)$ is a moving singular point of (9). Let

$$g^s(0, \tau) = \sum_{n=1}^{\infty} a_{n/m} \tau^{n/m} \tag{17}$$

be a singular solution to (9). Note that $g^s$ is not necessarily unique. It depends on the path along which $z$ approaches to 0, $z \notin K_\tau$. Substitute (17) into (9) and see that

$$\sum_{n=1}^{\infty} \frac{n a_{n/m} \tau^{n/m-1}}{m} \left( \sum_{n=1}^{\infty} a_{n/m} \tau^{n/m} - \tau \right) = 6\tau^2. \tag{18}$$

Equating coefficients at the same powers in both sides of (18) obtain that $m = 1$ and

$$a_1(a_1 - 1) = 0. \tag{19}$$

This equation gives two possible values $a_1 = 1$ and $a_1 = 0$ to two singular solutions $g^+(0, \tau)$ and $g^-(0, \tau)$. In both cases equation (18) implies recurrent formulas for coefficients $a_n^+$ and $a_n^-$ of $g^+(0, \tau)$ and $g^-(0, \tau)$ respectively,

$$a_1^+ = 1, \;\; a_2^+ = 6, \;\; a_n^+ = -\sum_{k=2}^{n-1} k a_k^+ a_{n+1-k}^+, \;\; n \geq 3, \tag{20}$$

$$a_1^- = 0, \;\; a_2^- = -3, \;\; a_n^- = \frac{1}{n} \sum_{k=2}^{n-1} k a_k^- a_{n+1-k}^-, \;\; n \geq 3, \tag{21}$$

Thus we have proved the following theorem.

**Theorem 1.** *To* $t = 0$ *there correspond the two singular solutions* $f^+(0, t)$ *and* $f^-(0, t)$ *to equation (4),*

$$f^+(0, t) = \sum_{n=1}^{\infty} a_n^+ t^{n/3}, \quad f^-(0, t) = \sum_{n=1}^{\infty} a_n^- t^{n/3},$$

*where the coefficients* $a_n^+$, $a_n^-$, $n \geq 1$, *satisfy equations (20) and (21).*

**Remark 2.** *Since* $f^+(0,t) = \sqrt[3]{t} + 6\sqrt[3]{t^2} + o(\sqrt[3]{t^2})$ *and* $f^-(0,t) = -3\sqrt[3]{t^2} + o(\sqrt[3]{t^2})$ *as* $t \to 0$, *the inequality*

$$f^-(0,t) < \sqrt[3]{t} < f^+(0,t)$$

*holds for all* $t > 0$ *small enough.*

**Remark 3.** *Expansions (17) for* $g^+(0,\tau)$ *and* $g^-(0,\tau)$ *with coefficients given by (20) and (21) show that* $\tau = 0$ *is a removable singular point for* $g(0,\tau)$.

Theorem 1 represents the singular solutions of the Löwner equation at the initial moment $t = 0$. Pose the problem to find all other singular solutions which appear at $t > 0$.

For $t_0 > 0$, there is a hull $K_{t_0} \subset \mathbb{H}$ such that $f(\cdot, t_0)$ maps $\mathbb{H} \setminus K_{t_0}$ onto $\mathbb{H}$. We refer to [7] for definitions and more details. The hull $K_{t_0}$ is driven by $\sqrt[3]{t}$. The function $f(\cdot, t_0)$ is extended continuously onto the set of prime ends on $\partial(\mathbb{H} \setminus K_{t_0})$ and maps this set onto $\mathbb{R}$. One of the prime ends is mapped on $\sqrt[3]{t_0}$. Let $z := \gamma(t_0)$ represent this prime end. As soon as $f(\gamma(t_0), t_0) = \sqrt[3]{t_0}$, the function $f(\gamma(t_0), t)$, $t \geq t_0$, is a singular solution to equation (4). Similarly to Theorem 1, $f(\gamma(t_0), t)$ is not necessarily unique. It depends on the path along which $z$ approaches to $\gamma(t_0)$, $z \neq K_t$.

According to [2, p.33] $f(\gamma(t_0), t)$ is expanded in series with powers $(t - t_0)^{n/m}$, $m \in \mathbb{N}$,

$$f(\gamma(t_0), t) = \sqrt[3]{t_0} + \sum_{n=1}^{\infty} b_{n/m}(t - t_0)^{n/m}. \tag{22}$$

Substitute (22) into (4) and see that

$$\sum_{n=1}^{\infty} \frac{nb_{n/m}(t - t_0)^{n/m-1}}{m} \times$$

$$\left( \sum_{n=1}^{\infty} b_{n/m}(t - t_0)^{n/m} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2 \cdot 5 \dots (3n-4)}{n!} \frac{(t - t_0)^n}{(3t_0)^n} \right) = 2. \tag{23}$$

Equating coefficients at the same powers in both sides of (23) obtain that $m = 2$ and

$$(b_{1/2})^2 = 4. \tag{24}$$

This equation gives two possible values $b_{1/2} = 2$ and $b_{1/2} = -2$ to two singular solutions $f^+(\gamma(t_0), t)$ and $f^-(\gamma(t_0), t)$. In both cases equation (23) implies recurrent formulas for coefficients $b^+_{n/2}$ and $b^-_{n/2}$ of $f^+(\gamma(t_0), t)$ and $f^-(\gamma(t_0, t)$ respectively,

$$b^+_{1/2} = 2, \quad b^+_{n/2} = \frac{1}{n+1} \left( c_{n/2} - \frac{1}{2} \sum_{k=2}^{n-1} k b^+_{k/2} (b^+_{(n+1-k)/2} - c_{(n+1-k)/2}) \right), \quad n \geq 2, \tag{25}$$

(26)

$$b^-_{1/2} = -2,\ \ b^-_{n/2} = \frac{1}{n+1}\left(c_{n/2} + \frac{1}{2}\sum_{k=2}^{n-1} k b^-_{k/2}(b^-_{(n+1-k)/2} - c_{(n+1-k)/2})\right),\ \ n \geq 2,$$

where

$$c_{(2k-1)/2} = 0,\ \ c_k = \frac{(-1)^{k-1}2\cdot 5\ldots(3k-4)}{3^k t_0^{k-1/3} k!},\ \ k = 1, 2, \ldots\ . \tag{27}$$

Thus we have proved the following theorem.

**Theorem 2.** *To $t = t_0 > 0$ there corespond the two singular solutions $f^+(\gamma(t_0), t)$ and $f^-(\gamma(t_0), t)$ to equation (4),*

$$f^+(\gamma(t_0), t) = \sqrt[3]{t_0} + \sum_{n=1}^{\infty} b^+_{n/2}(t-t_0)^{n/2},\quad f^-(\gamma(t_0), t) = \sqrt[3]{t_0} + \sum_{n=1}^{\infty} b^-_{n/2}(t-t_0)^{n/2},$$

*where the coefficients $b^+_{n/2}$ and $b^-_{n/2}$, $n \geq 1$, satisfy equations (25), (26) and (27).*

**Remark 4.** *Since*

$$f^+(\gamma(t_0), t) = \sqrt[3]{t_0} + 2\sqrt{t-t_0} + o(\sqrt{t-t_0}),\ f^-(\gamma(t_0), t) = \sqrt[3]{t_0} - 2\sqrt{t-t_0} + o(\sqrt{t-t_0}),$$

$$\sqrt[3]{t} = \sqrt[3]{t_0} + \frac{1}{3t_0}(t-t_0) + o(t-t_0),\quad t \to t_0 + 0,$$

*the inequality*

$$f^-(\gamma(t_0), t) < \sqrt[3]{t} < f^+(\gamma(t_0), t)$$

*holds for all $t > t_0$ close to $t_0$.*

Let us connect Remarks 2 and 4 by additional inequalities characterizing relations between the singular solutions.

**Theorem 3.** *For $t > 0$ small enough and $0 < t_0 < t$, the following inequalities*

$$f^-(0, t) < f^-(\gamma(t_0), t) < \sqrt[3]{t} < f^+(\gamma(t_0), t) < f^+(0, t)$$

*hold.*

*Proof.* To show that $f^+(\gamma(t_0), t) < f^+(0, t)$ let us subtract equations

$$\frac{df^+(0,t)}{dt} = \frac{2}{f^+(0,t) - \sqrt[3]{t}},\quad f^+(0,0) = 0,$$

$$\frac{df^+(\gamma(t_0),t)}{dt} = \frac{2}{f^+(\gamma(t_0),t) - \sqrt[3]{t}},\quad f^+(\gamma(t_0), t_0) = \sqrt[3]{t_0},$$

and obtain

$$\frac{d(f^+(0,t) - f^+(\gamma(t_0),t))}{dt} = \frac{2(f^+(\gamma(t_0),t) - f^+(0,t))}{(f^+(0,t) - \sqrt[3]{t})(f^+(\gamma(t_0),t) - \sqrt[3]{t})},$$

which we write as

$$\frac{d\log(f^+(0,t)-f^+(\gamma(t_0),t))}{dt}=\frac{-2}{(f^+(0,t)-\sqrt[3]{t})(f^+(\gamma(t_0),t)-\sqrt[3]{t})}.$$

Suppose that $T>t_0$ is the smallest number for which $f^+(0,T)=f^+(\gamma(t_0),T)$. This implies that

$$\int_{t_0}^{T}\frac{dt}{(f^+(0,t)-\sqrt[3]{t})(f^+(\gamma(t_0),t)-\sqrt[3]{t})}=\infty. \tag{28}$$

To evaluate the integral in (28) we should study the behavior of $f^+(\gamma(t_0),t)-\sqrt[3]{t}$ with the help of differential equation

(29)

$$\frac{d(f^+(\gamma(t_0),t)-\sqrt[3]{t})}{dt}=\frac{2}{f^+(\gamma(t_0),t)-\sqrt[3]{t}}-\frac{1}{3\sqrt[3]{t^2}}=\frac{\sqrt[3]{t}+6\sqrt[3]{t^2}-f^+(\gamma(t_0),t)}{3\sqrt[3]{t^2}(f^+(\gamma(t_0),t)-\sqrt[3]{t})}.$$

From (20) calculate that $a_3^+=-72$. According to Theorem 1

$$f^+(0,t)=\sqrt[3]{t}+6\sqrt[3]{t^2}-72t+o(t),\quad t\to+0.$$

There exists a number $T'>0$ such that for $0<t<T'$, $\sqrt[3]{t}+6\sqrt[3]{t^2}>f^+(0,t)$. Consequently, the right-hand side in (29) is positive for $0<t<T'$. Note that $T'$ does not depend on $t_0$. The condition "$t>0$ small enough" in Theorem 3 is understood from now as $0<t<T'$. We see from (29) that for such $t$, $f^+(\gamma(t_0),t)-\sqrt[3]{t}$ is increasing with $t$, $t_0<t<T<T'$. Therefore, the integral in the left-hand side of (28) is finite. The contradiction against (28) denies the existence of $T$ with the prescribed properties which proves the third and the fourth inequalities in Theorem 3.

The rest of inequalities of Theorem 3 are proved similarly and even easier. To show that $f^-(\gamma(t_0),t)>f^-(0,t)$ let us subtract equations

$$\frac{df^-(0,t)}{dt}=\frac{2}{f^-(0,t)-\sqrt[3]{t}},\quad f^-(0,0)=0,$$

$$\frac{df^-(\gamma(t_0),t)}{dt}=\frac{2}{f^-(\gamma(t_0),t)-\sqrt[3]{t}},\quad f^-(\gamma(t_0),t_0)=\sqrt[3]{t_0},$$

and obtain

$$\frac{d(f^-(0,t)-f^-(\gamma(t_0),t))}{dt}=\frac{2(f^-(\gamma(t_0),t)-f^-(0,t))}{(f^-(0,t)-\sqrt[3]{t})(f^-(\gamma(t_0),t)-\sqrt[3]{t})},$$

which we write as

$$\frac{d\log(f^-(\gamma(t_0),t)-f^-(0,t))}{dt}=\frac{-2}{(f^-(0,t)-\sqrt[3]{t})(f^-(\gamma(t_0),t)-\sqrt[3]{t})}.$$

Suppose that $T>t_0$ is the smallest number for which $f^-(\gamma(t_0),T)=f^-(0,T)$. This implies that

(30)
$$\int_{t_0}^{T} \frac{dt}{(f^-(0,t)-\sqrt[3]{t})(f^-(\gamma(t_0),t)-\sqrt[3]{t})} = \infty.$$

To evaluate the integral in (30) we should study the behavior of $f^-(\gamma(t_0),t)-\sqrt[3]{t}$ with the help of differential equation

(31)
$$\frac{d(f^-(\gamma(t_0),t)-\sqrt[3]{t})}{dt} = \frac{2}{f^-(\gamma(t_0),t)-\sqrt[3]{t}} - \frac{1}{3\sqrt[3]{t^2}} = \frac{\sqrt[3]{t}+6\sqrt[3]{t^2}-f^-(\gamma(t_0),t)}{3\sqrt[3]{t^2}(f^-(\gamma(t_0),t)-\sqrt[3]{t})}.$$

According to Theorem 1
$$f^-(0,t) = -3\sqrt[3]{t^2} + o(\sqrt[3]{t^2}), \quad t \to +0.$$
There exists a number $T'' > 0$ such that for $0 < t < T''$, $\sqrt[3]{t}+6\sqrt[3]{t^2} > f^-(0,t)$. Consequently, the right-hand side in (31) is positive for $0 < t < T''$. We see from (31) that for such $t$, $f^-(0,t)-\sqrt[3]{t}$ is decreasing with $t$, $t_0 < t < T < T''$. Therefore, the integral in the left-hand side of (30) is finite. The contradiction against (30) denies the existence of $T$ with the prescribed properties which completes the proof. □

The sense of Theorem 3 is in existence of an universal neighborhood of $t = 0$ where the declared inequalities are true while Remark 4 states existence of neighborhoods depending on $t_0$.

Compare equations (19) and (24) which give the main terms in asymptotics for singular solutions $f^{\pm}(0,t)$ and $f^{\pm}(\gamma(t_0),t)$. They reflect different kinds of singularities at $t = 0$ and $t = t_0 > 0$, the first corresponds to a circular arc tangential to $\mathbb{R}$ [11], the second corresponds to results in [3].

Add and complete the inequalities of Theorem 3 by the following statements demonstrating a monotonic disposition of values of different singular solutions.

**Theorem 4.** *For $t > 0$ small enough and $0 < t_1 < t_0 < t$, the following inequalities*
$$f^-(\gamma(t_1),t) < f^-(\gamma(t_0),t), \quad f^+(\gamma(t_0),t) < f^+(\gamma(t_1),t)$$
*hold.*

*Proof.* Similarly to Theorem 3 let us subtract equations
$$\frac{df^+(\gamma(t_1),t)}{dt} = \frac{2}{f^+(\gamma(t_1),t)-\sqrt[3]{t}}, \quad f^+(\gamma(t_1),t_1) = \sqrt[3]{t_1},$$
$$\frac{df^+(\gamma(t_0),t)}{dt} = \frac{2}{f^+(\gamma(t_0),t)-\sqrt[3]{t}}, \quad f^+(\gamma(t_0),t_0) = \sqrt[3]{t_0},$$
and obtain
$$\frac{d(f^+(\gamma(t_1),t)-f^+(\gamma(t_0),t))}{dt} = \frac{2(f^+(\gamma(t_0),t)-f^+(\gamma(t_1),t))}{(f^+(\gamma(t_1),t)-\sqrt[3]{t})(f^+(\gamma(t_0),t)-\sqrt[3]{t})},$$

which we write as

$$\frac{d\log(f^+(\gamma(t_1),t)-f^+(\gamma(t_0),t))}{dt}=\frac{-2}{(f^+(\gamma(t_1),t)-\sqrt[3]{t})(f^+(\gamma(t_0),t)-\sqrt[3]{t})}.$$

Suppose that $T>t_0$ is the smallest number for which $f^+(\gamma(t_1),T)=f^+(\gamma(t_0),T)$. This implies that

$$\int_{t_0}^{T}\frac{dt}{(f^+(\gamma(t_1),t)-\sqrt[3]{t})(f^+(\gamma(t_0),t)-\sqrt[3]{t})}=\infty. \tag{32}$$

To evaluate the integral in (32) apply to (29) and obtain that there exists a number $T'>0$ such that for $0<t<T'$, $f^+(\gamma(t_0),t)-\sqrt[3]{t}$ is increasing with $t$, $t_0<t<T<T'$. Therefore, the integral in the left-hand side of (32) is finite. The contradiction against (32) denies the existence of $T$ with the prescribed properties which proves the second inequality of Theorem 4.

To prove the first inequality of Theorem 4 let us subtract equations

$$\frac{df^-(\gamma(t_1),t)}{dt}=\frac{2}{f^-(\gamma(t_1),t)-\sqrt[3]{t}},\quad f^-(\gamma(t_1),t_1)=\sqrt[3]{t_1},$$

$$\frac{df^-(\gamma(t_0),t)}{dt}=\frac{2}{f^-(\gamma(t_0),t)-\sqrt[3]{t}},\quad f^-(\gamma(t_0),t_0)=\sqrt[3]{t_0},$$

and obtain after dividing by $f^-(\gamma(t_1),t)-f^-(\gamma(t_0),t)$

$$\frac{d\log(f^-(\gamma(t_0),t)-f^-(\gamma(t_1),t))}{dt}=\frac{-2}{(f^-(\gamma(t_1),t)-\sqrt[3]{t})(f^-(\gamma(t_0),t)-\sqrt[3]{t})}.$$

Suppose that $T>t_0$ is the smallest number for which $f^-(\gamma(t_0),T)=f^-(\gamma(t_1),T)$. This implies that

$$\int_{t_0}^{T}\frac{dt}{(f^-(\gamma(t_1),t)-\sqrt[3]{t})(f^-(\gamma(t_0),t)-\sqrt[3]{t})}=\infty. \tag{33}$$

To evaluate the integral in (33) we apply to (31) and obtain that $\sqrt[3]{t}+6\sqrt[3]{t^2}>\sqrt[3]{t}>f^-(0,t)$. Consequently, the right-hand side in (31) is positive and we see that $f^-(0,t)-\sqrt[3]{t}$ is decreasing with $t$, $t_0<t<T$. Therefore, the integral in the left-hand side of (33) is finite. The contradiction against (33) denies the existence of $T$ with the prescribed properties which completes the proof. □

## 4. Mapping from a slit domain

Show that $f(z,t)$ is a mapping from a slit domain $D(t)=\mathbb{H}\setminus\gamma(t)$.

**Theorem 5.** *Let $f(z,t)$ be a solution to the Löwner equation (4). Then $f(\cdot,t)$ maps $D(t)=\mathbb{H}\setminus\gamma(t)$ onto $\mathbb{H}$ for $t>0$ small enough where $\gamma(t)$ is a curve in $\mathbb{H}$ emanating from the origin.*

*Proof.* Theorems 3 and 4 describe the structure of the pre-image of $\mathbb{H}$ under $f(\cdot,t)$. All the singular solutions $f^{\pm}(0,t)$, $f^{\pm}(\gamma(t_0),t)$, $0<t_0<t<T'$, are real-valued and satisfy the inequalities of Theorems 3 and 4. So the segment $I=[f^-(0,t),f^+(0,t)]$ consists of points $f^-(\gamma(\tau),t)$, $0\le\tau<t$, to the left from the point $\sqrt[3]{\tau}$ inside $I$, and of points $f^+(\gamma(\tau),t)$, $0\le\tau<t$, to the right from $\sqrt[3]{\tau}$. All these points $f^{\pm}(\gamma(\tau),t)$ belong to the boundary $\mathbb{R}=\partial\mathbb{H}$. This means that all the points $\gamma(\tau)$, $0\le\tau<t$, belong to the boundary $\partial(\mathbb{H}\setminus K_t)$ of $\mathbb{H}\setminus K_t$. Moreover, every point $\gamma(\tau)$ except for the tip determines exactly two prime ends corresponding to $f^+(\gamma(\tau),t)$ and $f^-(\gamma(\tau),t)$. Evidently, $\gamma(\tau)$ is continuous on $[0,t]$. This proves that $\gamma(\tau)$ represents a curve $\gamma:=K_t$ with prime ends corresponding to points on different sides of $\gamma$ and complete the proof. □

**Remark 5.** *In the text of this article we treated $f^{\pm}(0,t)$ as singular solutions to (4) appearing at $t=0$, except for formulas (5) and (6) giving another definitions. Besides its own sense, Theorem 5 proved that both definitions of $f^{\pm}(0,t)$ are equivalent.*

Theorem 5 is of principal character. Indeed, the fact that $f(z,t)$ is a mapping from a slit domain for $t>0$ small enough is extended to arbitrary $t>0$ by standard means.

**Theorem 6.** *Let $f(z,t)$ be a solution to the Löwner equation (4). Then $f(\cdot,t)$ maps $D(t)=\mathbb{H}\setminus\gamma(t)$ onto $\mathbb{H}$ for all $t>0$ where $\gamma(t)$ is a $C^1$-curve, except probably for the point $\gamma(0)=0$.*

*Proof.* Theorem 5 states that $f(\cdot,t)$ maps $\mathbb{H}\setminus\gamma(t)$ onto $\mathbb{H}$ for a curve $\gamma(t)$ and $t>0$ small enough.

Fix $t_0>0$ from this neighborhood of $t=0$. Denote $g(w,t)=f^{-1}(w,t)$ an inverse of $f(z,t)$, and $h(w,t):=f(g(w,t_0),t)$, $t\ge t_0$. The arc $\gamma[t_0,t]:=K_t\setminus K_{t_0}$ is mapped by $f(z,t_0)$ onto a curve $\gamma_1(t)$ in $\mathbb{H}$ emanating from $\sqrt[3]{t_0}\in\mathbb{R}$. So the function $h(w,t)$ is well defined on $\mathbb{H}\setminus\gamma_1(t_0)$, $t\ge t_0$. Expand $h(w,t)$ near infinity,

$$h(w,t)=g(w,t_0)+\frac{2t}{g(w,t_0)}+O\left(\frac{1}{g^2(w,t_0)}\right)=w+\frac{2(t-t_0)}{w}+O\left(\frac{1}{w^2}\right).$$

Such expansion satisfies (1) after changing variables $t\to t-t_0$. The function $h(w,t)$ satisfies the differential equation

$$\frac{dh(w,t)}{dt}=\frac{2}{h(w,t)-\sqrt[3]{t}},\quad h(w,t_0)=w,\quad w\in\mathbb{H}.$$

This equation becomes the Löwner differential equation if $t_1:=t-t_0$, $h_1(w,t_1):=h(w,t_0+t_1)$,

$$\frac{dh_1(w,t_1)}{dt_1}=\frac{2}{h_1(w,t_1)-\sqrt[3]{t_1+t_0}},\quad h_1(w,0)=w,\quad w\in\mathbb{H}. \tag{34}$$

The driving function $\lambda(t_1)=\sqrt[3]{t_1+t_0}$ in (34) is analytic for $t_1\ge0$. It is known [1, p.59] that under this condition $h_1(w,t_1)$ maps $\mathbb{H}\setminus\gamma_1$ onto $\mathbb{H}$ where $\gamma_1$ is a $C^1$-curve in $\mathbb{H}$ emanating from $\lambda(0)=\sqrt[3]{t_0}$. The same does the function $h(w,t)$.

Go back to $f(z,t) = h(f(z,t_0),t)$ and see that $f(z,t)$ maps $\mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$, $\gamma(t) = \gamma[0,t_0] \cup \gamma[t_0,t]$, and $\gamma[t_0,t]$ is a $C^1$-curve. Tending $t_0$ to 0 we prove that $\gamma(t)$ is a $C^1$-curve, except probably for the point $\gamma(0) = 0$. This completes the proof. □

**Corollary 1.** *Under the conditions of Theorem 6, the curve $\gamma(t)$ is tangential to $\mathbb{R}$ at $t = 0$.*

*Proof.* It is known [10] that if a slit $\gamma(t)$ driven by a driving function $\lambda(t)$ in (4) is a quasiarc which meets $\mathbb{R}$ at a nonzero angle, then $\lambda(t)$ belongs to the Lipschitz class Lip(1/2). As soon as the conditions of Theorem 6 suppose that $\lambda(t) = \sqrt[3]{t} \in$ Lip(1/3) and has a nonvanishing norm , then the $C^1$-curve $\gamma(t)$ driven by $\sqrt[3]{t}$ must be tangential to $\mathbb{R}$. □

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