THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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Y. Ameur, H. Hedenmalm and N. Makarov

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RANDOM NORMAL MATRICES AND WARD IDENTITIES

YACIN AMEUR, HÅKAN HEDENMALM, AND NIKOLAI MAKAROV

Abstract. We consider the random normal matrix ensemble associated with a potential in the plane of sufficient growth near infinity. It is known that asymptotically as the order of the random matrix increases indefinitely, the eigenvalues approach a certain equilibrium density, given in terms of Frostman's solution to the minimum energy problem of weighted logarithmic potential theory. At a finer scale, we may consider fluctuations of eigenvalues about the equilibrium. In the present paper, we give the correction to the expectation of the fluctuations, and we show that the potential field of the corrected fluctuations converge on smooth test functions to a Gaussian free field with free boundary conditions on the droplet associated with the potential.

1. Introduction

1.1. **Synopsis.** Given a suitable real-valued "weight function" Q in the plane, it is understood how to associate a corresponding (weighted) random normal matrix ensemble (in short: RNM-ensemble). Under reasonable conditions on Q, the eigenvalues of matrices picked randomly from the ensemble will condensate on a certain compact subset $S = S_Q$ of the complex plane, as the order of the matrices tends to infinity. The set S is called the *droplet* of the ensemble. It is well-known that the droplet may be obtained in terms of weighted logarithmic potential theory and, that in its turn, the droplet determines the classical equilibrium distribution of the eigenvalues (Frostman's equilibrium measure).

In this paper, we prove a formula for the expectation of fluctuations about the equilibrium distribution, for linear statistics of the eigenvalues of random normal matrices. We also prove the convergence of the potential fields corresponding to corrected fluctuations to a Gaussian free field on S with free boundary conditions.

Our approach is based on the *Ward identities*, which are identities satisfied by the joint intensities of the point-process of eigenvalues, and which follow from the reparametrization invariance of the partition function of the ensemble. Ward identities are well-known in field theories. Analogous results in the random Hermitian matrix theory are known due to Johansson [14], in the case of a polynomial weight. Informally speaking, the Ward identities (also known as the "loop equation") work nicely in the Hermitian setting because the eigenvalues are then real and the entire real line is easily accessed from the complement in the complex plane. It is also important that the correlation kernel (also called the Christoffel-Darboux kernel) in the Hermitian ensembles obtains a very compact form in terms of orthogonal polynomials (cf., e.g., [19]). This does not happen for the RNM-ensembles. However, here we may instead rely on the Tian-Zelditch et al. local asymptotic expansion theory for weighted Bergman kernels (cf. [7]). In the RNM-ensembles case, the interior of the droplet cannot be reached from the exterior, and for this reason the standard arguments based on the "loop equation" do not carry over. To obtain the structure of the fluctuations up to the boundary of the droplet we combine the Ward identities which provide global information with the local interior information from the Bergman kernel expansion theory.

So, effectively we use the Ward identities to supply the needed information near the boundary of the droplet, which is assumed to be a smooth Jordan curve. This permits us to finish the argument.

1.2. **General notation.** By $\mathbb{D}(a, r)$ we mean the open Euclidean disk with center a and radius r. The special case $\mathbb{D}(0, 1)$ is simplified to $\mathbb{D}$. By $\mathrm{dist}_{\mathbb{C}}$ we mean the Euclidean distance in the complex plane $\mathbb{C}$. If A_n and B_n are expressions depending on a positive integer n, we write $A_n \lesssim B_n$ to indicate that $A_n \le CB_n$ for all n large enough where C is independent of n, and usually also independent of other relevant parameters. The notation $A_n \asymp B_n$ means that $A_n \lesssim B_n$ and $B_n \lesssim A_n$. If $z = x + \mathrm{i}y$ is the decomposition of a complex number into real and imaginary parts, we introduce the following notational conventions. We write $\partial = \partial_z := \frac{1}{2}(\partial/\partial x - \mathrm{i}\partial/\partial y)$ and $\bar\partial = \bar\partial_z := \frac{1}{2}(\partial/\partial x + \mathrm{i}\partial/\partial y)$ for the usual complex derivatives. We also write $\Delta := \Delta_z = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for the Laplacian, and introduce the notation $\mathbb{\Delta} = \mathbb{\Delta}_z := \frac{1}{4}\Delta_z$ for a quarter of the usual Laplacian, because it will appear many times naturally as a consequence of $\mathbb{\Delta}_z = \partial_z\bar\partial_z$. The ∇ operator is defined by $\nabla f := (\partial f/\partial x, \partial f/\partial y)$, so that ∇f becomes $\mathbb{C}^2$-valued when f is $\mathbb{C}$-valued and differentiable. It is easy to check that

$$|\nabla f|^2 = 2(|\partial f|^2 + |\bar\partial f|^2).$$

We let $\mathrm{d}A(z) = \mathrm{d}^2 z = \mathrm{d}x\mathrm{d}y$ denote the area measure in the complex plane $\mathbb{C}$. Given suitable functions f and g, such that $fg \in L^1(\mathbb{C})$, we write

$$\langle f, g\rangle_{\mathbb{C}} := \int_{\mathbb{C}} fg\,\mathrm{d}A. \tag{1.1}$$

With a slight abuse of notation we extend this notation to the setting of a continuous function f and a Borel measure μ (we need to require that $f \in L^1(\mathbb{C}, \mu)$):

$$\langle f, \mu\rangle_{\mathbb{C}} := \int_{\mathbb{C}} f\,\mathrm{d}\mu. \tag{1.2}$$

We will at times need to understand the bilinear form $\langle \cdot, \cdot\rangle_{\mathbb{C}}$ defined by (1.1) more liberally, that is, in the sense of distribution theory. In the expression $\langle f, g\rangle_{\mathbb{C}}$, we will then usually think of f as the test function and g as the distribution. Next, if Γ is a rectifiable curve in $\mathbb{C}$, we let $\mathrm{d}s = \mathrm{d}s_\Gamma$ denote arc length measure along Γ, and for suitable functions f, g we write

$$\langle f, g\rangle_\Gamma := \int_\Gamma fg\,\mathrm{d}s. \tag{1.3}$$

2. Random normal matrix ensembles

2.1. **The distribution of eigenvalues.** Let $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ be a suitable lower semi-continuous function subject to the growth condition

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 1. \tag{2.1}$$

We refer to Q as the *weight function* or the *potential*.

Let $\mathrm{NM}[n]$ be the set of all $n \times n$ normal matrices M, i.e., matrices with $MM^* = M^*M$. The *partition function* on $\mathrm{NM}[n]$ associated with Q is the function

$$\mathcal{Z}_n = \int_{\mathrm{NM}[n]} \mathrm{e}^{-2n\,\mathrm{trace}[Q(M)]}\,\mathrm{d}M_n,$$

where dM_n is the Riemannian volume form on NM[n] inherited from the space $\mathbb{C}^{n^2}$ of all $n \times n$ matrices, and where trace[Q] : NM[n] $\to \mathbb{R} \cup \{+\infty\}$ is the random variable

$$\mathrm{trace}[Q](M) := \sum_{\lambda_j \in \mathrm{spec}(M)} Q(\lambda_j),$$

i.e., it is the usual trace of the matrix $Q(M)$. We equip NM[n] with the probability measure

$$\mathrm{dProb}_{\mathrm{NM}[n]} := \frac{1}{\mathcal{Z}_n} \mathrm{e}^{-2n\,\mathrm{trace}[Q](M)}\, \mathrm{d}M_n,$$

and speak of the *random normal matrix ensemble* or "RNM-ensemble" associated with Q. The measure $\mathrm{Prob}_{\mathrm{NM}[n]}$ induces a probability measure Prob_n on the space $\mathbb{C}^n$ of eigenvalues, which is known as the *density of states* in external field Q; it is given by

$$\mathrm{dProb}_n(\lambda) := \frac{1}{Z_n} \mathrm{e}^{-H_n(\lambda)} \mathrm{d}A^{\otimes n}(\lambda), \quad \lambda = (\lambda_j)_1^n \in \mathbb{C}^n. \tag{2.2}$$

Here, we have put

$$H_n(\lambda) := \sum_{j,k:j\neq k} \log \frac{1}{|\lambda_j - \lambda_k|} + 2n \sum_{j=1}^{n} Q(\lambda_j), \tag{2.3}$$

while $\mathrm{d}A^{\otimes n}(\lambda) = \mathrm{d}A(\lambda_1) \cdots \mathrm{d}A(\lambda_n)$ denotes Lebesgue measure in $\mathbb{C}^n$ and Z_n is the normalization constant giving Prob_n unit mass. By a slight abuse of language, we will refer to Z_n as the partition function of the ensemble.

We notice that H_n is the energy (Hamiltonian) of a system of n identical point charges in the plane located at the points λ_j, under the influence of the external field $2nQ$. In this interpretation, Prob_n is the law of the Coulomb gas in the external magnetic field $2nQ$ (at inverse temperature $\beta = 2$). In particular, this explains the repelling nature of the eigenvalues of random normal matrices; they tend to be very spread out in the vicinity of the droplet, just like point charges would.

Let us consider the n-point configuration ("set" with possible repeated elements) $\{\lambda_j\}_1^n$ of eigenvalues of a normal matrix picked randomly with respect to $\mathrm{Prob}_{\mathrm{NM}[n]}$. In an obvious manner, the measure Prob_n induces a probability law on the n-point configuration space; this is the law of the *n-point process* $\Lambda_n = \{\lambda_j\}_1^n$ associated to Q.

It is well-known that the process Λ_n is *determinantal*. This means that there exists an Hermitian function K_n, called the *correlation kernel of the process* such that the density of states can be represented in the form

$$\mathrm{dProb}_n(\lambda) = \frac{1}{n!} \det\left[\mathrm{K}_n(\lambda_j, \lambda_k)\right]_{j,k=1}^n \mathrm{d}A_n(\lambda), \qquad \lambda \in \mathbb{C}^n.$$

Here, we have

$$\mathrm{K}_n(z, w) = \mathrm{k}_n(z, w)\, \mathrm{e}^{-n(Q(z)+Q(w))},$$

where k_n is the reproducing kernel of the space $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$, i.e., the space of all analytic polynomials of degree at most $n-1$ with norm induced from the usual L^2-space on $\mathbb{C}$ associated with the weight function e^{-2nQ}. Alternatively, we can regard K_n as the reproducing kernel for the subspace

$$L^2_{n,Q}(\mathbb{C}) := \left\{p\, \mathrm{e}^{-nQ} :\ p \text{ is an analytic polynomial of degree less than } n\right\} \subset L^2(\mathbb{C});$$

in particular, we have the frequently useful identities

$$f(z) = \int_{\mathbb{C}} f(w)\,\overline{\mathrm{K}_n(z,w)}\,\mathrm{d}A(w), \qquad f \in L^2_{n,Q}(\mathbb{C}),$$

and

$$\int_{\mathbb{C}} \mathrm{K}_n(z,z)\,\mathrm{d}A(z) = n.$$

We refer to [8], [20], [10], [11], [3], [15] for more details on point-processes and random matrices.

2.2. **The equilibrium measure and the droplet.** We are interested in the asymptotic distribution of eigenvalues as n, the order of the random matrix, increases indefinitely. Let u_n denote the one-point function of Prob_n:

$$u_n(\lambda) := \frac{1}{n}\mathrm{K}_n(\lambda,\lambda), \quad \lambda \in \mathbb{C}.$$

Given a suitable function f on $\mathbb{C}$, we associate the random variable $\mathrm{Tr}_n[f]$ on the probability space $(\mathbb{C}^n, \mathrm{Prob}_n)$ via

$$\mathrm{Tr}_n[f](\lambda) := \sum_{i=1}^{n} f(\lambda_i).$$

We reserve the notation $\mathbb{E}_n$ for the expectation with respect to Prob_n; then e.g.

$$\mathbb{E}_n\left(\mathrm{Tr}_n[f]\right) = n\int_{\mathbb{C}} f\,u_n\,\mathrm{d}A.$$

According to Johansson (see [11]), we have the weak-star convergence of the measures

$$\mathrm{d}\sigma_n(z) := u_n(z)\,\mathrm{d}A(z)$$

to some compactly supported probability measure $\sigma = \sigma_Q$ on $\mathbb{C}$. This probability measure σ is the Frostman equilibrium measure of the logarithmic potential theory with external field Q. We briefly recall the definition and some basic properties of this probability measure, cf. [17] and [11] for a more detailed exposition.

We write $S = S_Q := \operatorname{supp}\sigma_Q$ and assume that Q is C^2-smooth in some neighbourhood of S. Then S is compact and $\Delta Q \ge 0$ holds on S; moreover, $\sigma = \sigma_Q$ is absolutely continuous with density (we recall that $\Delta\!\!\!\!\Delta = \frac{1}{4}\Delta$)

$$u := \frac{1}{2\pi}\,1_S\Delta Q = \frac{2}{\pi}\,1_S\Delta\!\!\!\!\Delta Q. \tag{2.4}$$

We refer to the compact set S_Q as the *droplet* corresponding to the external field Q. If we let δ_w stand for the unit point mass at a point $w \in \mathbb{C}$, we may form the empirical measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$. Here, as before, the λ_j are the eigenvalues of a random normal matrix, so the empirical measure is a stochastic probability measure. As $n \to +\infty$, we have almost surely that the empirical measure converges to the Frostman equilibrium measure σ.

Our present goal is to describe the fluctuations of the density field $\mu_n = \sum_{j=1}^{n}\delta_{\lambda_j}$ around the equilibrium. More precisely, we will study the distribution (linear statistic)

$$f \mapsto \langle f, \mu_n\rangle_{\mathbb{C}} - n\langle f,\sigma\rangle_{\mathbb{C}} = \mathrm{Tr}_n[f] - n\langle f,\sigma\rangle_{\mathbb{C}}, \qquad f \in C_0^\infty(\mathbb{C}).$$

We will denote by ν_n the measure with density $n(u_n - u)$, i.e.,

$$\langle f, \nu_n\rangle_{\mathbb{C}} := \mathbb{E}_n\left(\mathrm{Tr}_n[f]\right) - n\langle f,\sigma\rangle_{\mathbb{C}} = n\,\langle f, \sigma_n - \sigma\rangle_{\mathbb{C}}, \qquad f \in C_0^\infty(\mathbb{C}).$$

2.3. **Assumptions on the potential.** To state the main results of the paper we make the following three assumptions:

(A1) (smoothness) Q is real-analytic (written $Q \in C^\omega$) in some neighborhood of the droplet $S = S_Q$;

(A2) (regularity) $\Delta Q \neq 0$ in S;

(A3) (topology) ∂S is a C^ω-smooth Jordan curve.

We will comment on the nature and consequences of these assumptions later. Let us agree to write

$$L = L_Q := \log \Delta Q.$$

This function is well-defined and C^ω-smooth in a neighborhood of the droplet S.

2.4. **The Neumann jump operator.** We will use the following general system of notation. If g is a continuous function defined in a neighborhood of S, then we write g^S for the continuous and bounded function in $\mathbb{C}$ with the following properties: in S, g^S equals g, while in the complement $\mathbb{C} \setminus S$, g^S is harmonic. It is clear that this determines g^S uniquely.

If g is smooth on S, then

$$\mathcal{N}_\Omega g := -\frac{\partial g|_S}{\partial \mathrm{n}}, \qquad \Omega := \mathrm{int}(S),$$

where n is the (exterior) unit normal of Ω. We define the normal derivative $\mathcal{N}_{\Omega^*} g$ for the complementary domain $\Omega^* := \mathbb{C} \setminus S$ analogously. If both normal derivatives exist, then we define the *Neumann jump*:

$$\mathcal{N} g \equiv \mathcal{N}_{\partial S} g := \mathcal{N}_\Omega g + \mathcal{N}_{\Omega^*} g.$$

By Green's formula we have the identity (of measures)

$$\Delta g^S \mathrm{d}A = 1_\Omega \Delta g \, \mathrm{d}A + \mathcal{N}[g^S] \, \mathrm{d}s, \tag{2.5}$$

where $\mathrm{d}s$ is the arc-length measure on ∂S. Here, Δg^S is understood as a distribution, which is why $\Delta g^S \mathrm{d}A$ need to be absolutely continuous with respect to area measure.

We now verify (2.5). Let ϕ be a test function. The left hand side in (2.5) applied to ϕ is

$$\langle \phi, \Delta g^S \rangle_{\mathbb{C}} = \int_C \phi \Delta g^S \mathrm{d}A = \int_C g^S \Delta \phi \, \mathrm{d}A = \int_S g \Delta \phi \, \mathrm{d}A + \int_{\mathbb{C} \setminus S} g^S \Delta \phi \, \mathrm{d}A,$$

while the right hand side applied to ϕ equals

$$\langle \phi, 1_\Omega \Delta g \rangle_{\mathbb{C}} + \langle \phi, \mathcal{N}[g^S] \rangle_{\partial S} = \int_S \phi \Delta g \, \mathrm{d}A + \int_{\partial S} \phi \mathcal{N}[g^S] \, \mathrm{d}s.$$

So, we need to check that

$$\int_S (g \Delta \phi - \phi \Delta g) \mathrm{d}A + \int_{\mathbb{C} \setminus S} (g^S \Delta \phi - \phi \Delta g^S) \mathrm{d}A = \int_{\partial S} \phi \mathcal{N}(g^S) \mathrm{d}s.$$

But this is an immediate consequence of Green's formula applied to the regions Ω and Ω^* separately, and (2.5) follows.

2.5. **Main results.** We shall prove the following results, which were announced in [3].

Theorem 2.1. *For all test functions $f \in C_0^\infty(\mathbb{C})$, the limit*

$$\langle f, \nu \rangle_{\mathbb{C}} := \lim_{n\to+\infty} \langle f, \nu_n \rangle_{\mathbb{C}}$$

exists, and

$$\langle f, \nu \rangle_{\mathbb{C}} = \frac{1}{8\pi}\left\{ \int_S (\Delta f + f\Delta L)\mathrm{d}A + \int_{\partial S} f\mathcal{N}(L^S)\mathrm{d}s \right\}.$$

Equivalently, we have the convergence as $n \to +\infty$

$$\mathrm{d}\nu_n \to \mathrm{d}\nu = \frac{1}{8\pi}\Delta\left(1_S + L^S\right)\mathrm{d}A,$$

in the sense of distribution theory.

Theorem 2.2. *Let $h \in C_0^\infty(\mathbb{C})$ be a real-valued test function. Then, as $n \to +\infty$, we have the convergence in distribution*

$$\mathrm{Tr}_n\, h - \mathbb{E}_n \,\mathrm{Tr}_n\, h \;\to\; N\left(0, \frac{1}{4\pi}\int_{\mathbb{C}} |\nabla h^S|^2 \mathrm{d}A\right).$$

The last formula is to be understood in the sense of convergence of the random variables to a normal law in distribution. As noted in [3], the result may be restated in terms of convergence of random fields to a Gaussian field with free boundary conditions.

2.6. **Derivation of Theorem 2.2.** By appealing to the variational approach employed by Johansson in [14], we now show that the Gaussian convergence in Theorem 2.2 follows from a generalized version of Theorem 2.1, which we now state.

We fix a real-valued test function $h \in C_0^\infty(\mathbb{C})$ and consider the perturbed potential

$$Q_n^h := Q - \frac{1}{n}h.$$

We denote by Prob_n^h the density of states associated with the perturbed potential Q_n^h (cf. (2.2)) given by

$$\mathrm{dProb}_n^h(\lambda) := \frac{1}{Z_n^h}\mathrm{e}^{-H_n^h(\lambda)}\mathrm{d}A^{\otimes n}(\lambda), \quad \lambda = (\lambda_j)_1^n \in \mathbb{C}^n, \tag{2.6}$$

where Z_n^h is the appropriate normalization constant ("partition function") and

$$H_n^h(\lambda) = \sum_{j,k:j\neq k} \log\frac{1}{|\lambda_j - \lambda_k|} + 2n\sum_{j=1}^n Q(\lambda_j) - 2\sum_{j=1}^n h(\lambda_j).$$

We let $\mathbb{E}_n^h$ denote expectation with respect to the perturbed law Prob_n^h. We also write u_n^h for the one-point function associated with the density of states Prob_n^h, and σ_n^h for the probability measure with density u_n^h (i.e., $\mathrm{d}\sigma_n^h = u_n^h \mathrm{d}A$). We let ν_n^h denote the measure $n(\sigma_n^h - \sigma)$, that is,

$$\langle f, \nu_n^h \rangle_{\mathbb{C}} := n\langle f, \sigma_n^h - \sigma \rangle_{\mathbb{C}} = \mathbb{E}_n^h\, \mathrm{Tr}_n[f] - n\langle f, \sigma\rangle_{\mathbb{C}}. \tag{2.7}$$

Theorem 2.3. *For all $f \in C_0^\infty(\mathbb{C})$ we have the convergence as $n \to +\infty$*

$$\langle f, \nu_n^h - \nu_n \rangle_{\mathbb{C}} \to \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S \mathrm{d}A.$$

Here, the dot stands for the inner product of vectors. We supply a proof of Theorem 2.3 in Section 5.

Lemma 2.4. *Theorem 2.2 is a consequence of Theorem 2.3.*

Proof. We write $X_n := \mathrm{Tr}_n[h] - \mathbb{E}_n \mathrm{Tr}_n[h]$ and let $a_n^h(\tau) := \mathbb{E}_n^{\tau h} X_n$. Here $\tau \geq 0$ is a parameter, and $\mathbb{E}_n^{\tau h}$ denotes expectation corresponding to the potential $Q - \frac{1}{n}\tau h$. We note that $a_n^h(0) = 0$ because $\mathbb{E}_n X_n = 0$. In view of Theorem 2.3, we have that as $n \to +\infty$,

$$a_n^h(\tau) \to \tau a \quad \text{where} \quad a = \frac{1}{2\pi} \int_{\mathbb{C}} |\nabla h^S|^2 \mathrm{d}A. \tag{2.8}$$

Here, τ is assumed to be fixed. Next, we put

$$F_n(\tau) := \log \mathbb{E}_n[\mathrm{e}^{\tau X_n}], \qquad 0 \leq \tau \leq 1,$$

and observe that $F_n(0) = 0$ and that F_n is convex. Then a calculation (see, e.g., [3], pp. 66–67) verifies that we have

$$F_n'(\tau) = \mathbb{E}_n^{\tau h} X_n = a_n^h(\tau), \tag{2.9}$$

so the convexity of F_n means that $\tau \mapsto a_n^h(\tau)$ is increasing. In particular, we may derive from this that the convergence (2.8) is uniform in $0 \leq \tau \leq 1$. By integrating (2.9), we then find that

$$\log \mathbb{E}_n[\mathrm{e}^{X_n}] = F_n(1) = F_n(1) - F_n(0) = \int_0^1 F_n'(\tau)\, \mathrm{d}\tau \ \to \ \frac{a}{2}, \qquad \text{as} \quad n \to +\infty. \tag{2.10}$$

We see from (2.10) that all the moments of X_n converge to the moments of the normal $N(0, a)$ distribution. It is well-known that this implies convergence in distribution, viz. Theorem 2.2 follows. □

2.7. **Comments.**

2.7.1. *Related Work.* The one-dimensional analog of the weighted RNM theory is the more well-known random Hermitian matrix theory, which was studied by Johansson in the important paper [14]. Indeed, Johansson obtained results not only for random Hermitian matrix ensembles, but for more general (one-dimensional) β-ensembles. The paper [14] was one of our main sources of inspiration for the present work. Earlier, in [3], we obtained the convergence in Theorems 2.1 and 2.2 for test functions supported in the interior of the droplet; see also [6]. In [3], we announced Theorems 2.1 and 2.2 and proved several consequences of them, e.g. the convergence of the Berezin measures, rooted at a point in the exterior to S, to harmonic measure. Rider and Virág [16] proved Theorems 2.1 and 2.2 in the special case $Q(z) = |z|^2$ (the *Ginibre ensemble*). The paper [9] contains results in this direction for β-Ginibre ensembles for some special values of β.

Our main technique, the method of Ward identities, is common practice in field theories. This method uses the reparametrization invariance of the partition function to obtain exact differential relations satisfied by the joint intensity functions of the ensemble. In particular, the method was applied on the physical level by Wiegmann, Zabrodin et al. to study RNM-ensembles as well as more general OCP-ensembles. See, e.g., the papers [21], [22], [23], [24]. A one-dimensional version of the Ward identity was also used by Johansson in [14].

Finally, we wish to mention that one of the topics in this paper, the behavior of fluctuations near the boundary, is analyzed from another perspective in the forthcoming paper [4].

2.7.2. *Assumptions on the potential.* Here, we comment on the assumptions (A1)–(A3) which are made on the potential Q.

The C^ω-smoothness assumption (A1) is natural for the study of fluctuation properties near the boundary of the droplet. (For test functions supported in the interior, one can do with less regularity.)

Using Sakai's theory [18], it can be shown that the conditions (A1) and (A2) imply that ∂S is a union of finitely many C^ω-smooth curves with a finite number of singularities of known types. It is not difficult to complete a proof using arguments from [12], Section 4. We rule out the singularities by the regularity assumption in (A3). What happens in the presence of singularities is probably an interesting topic, which we have not approached. Without singularities, the boundary of the droplet is a union of finitely many C^ω-smooth Jordan curves. The topological ingredient in assumption (A3) means that we only consider the case of a single boundary component. Our methods extend without difficulty to the case of a multiply connected droplet. The disconnected case requires further analysis, and is not considered in this paper.

2.7.3. *Droplets and potential theory.* Here, we state the properties of the droplet that will be needed for our analysis. Proofs for these properties can be found in [17], [11].

We will write $\check{Q}$ for the maximal subharmonic function $\le Q$ which grows like $\log|z| + \mathrm{O}(1)$ when $|z| \to +\infty$. We have that $\check{Q} = Q$ on S while $\check{Q}$ is $C^{1,1}$-smooth in $\mathbb{C}$ and

$$\check{Q}(z) = Q^S(z) + G(z, \infty), \qquad z \in \mathbb{C} \setminus S,$$

where G is the classical Green's function of $\mathbb{C} \setminus S$. In particular, if

$$U^\sigma(z) = \int_{\mathbb{C}} \log \frac{1}{|z-\zeta|} \mathrm{d}\sigma(\zeta)$$

denotes the logarithmic potential of the equilibrium measure, then

$$\check{Q} + U^\sigma \equiv c_Q, \tag{2.11}$$

where c_Q is a Robin-type constant.

The following proposition sums up the basic properties of the droplet and the function $\check{Q}$. We write $W^{2,\infty}$ for the usual Sobolev space of functions with bounded second order partial derivatives.

Proposition 2.5. *Suppose Q satisfies (A1)–(A3). Then ∂S is a C^ω-smooth Jordan curve, $\check{Q} \in W^{2,\infty}(\mathbb{C})$, and therefore*

$$\partial \check{Q} = [\partial Q]^S.$$

Furthermore, we have

$$Q(z) - \check{Q}(z) \asymp \delta_{\partial S}(z)^2, \qquad z \in \mathbb{C} \setminus S, \quad \text{as } \delta_{\partial S}(z) \to 0, \tag{2.12}$$

where $\delta_{\partial S}(z)$ denotes the distance from z to the droplet.

2.7.4. *Joint intensities.* We will occasionally use the intensity k-point function of the process Λ_n. This is the function defined by

$$R_n^{(k)}(z_1,\dots,z_k) := \lim_{\varepsilon\to 0}\frac{\mathrm{Prob}_n\left(\bigcap_{j=1}^k\{\Lambda_n\cap\mathbb{D}(z_j,\varepsilon)\neq\emptyset\}\right)}{(\pi\varepsilon^2)^k} = \det\left[\mathrm{K}_n(z_i,z_j)\right]_{i,j=1}^k.$$

In particular, $R_n^{(1)} = nu_n$.

2.7.5. *Organization of the paper.* We will derive the following statement which combines Theorems 2.1 and 2.3.

Master formula: Let ν_n^h be the measure defined in (2.7). Then

$$\lim_{n\to+\infty}\langle f,\nu_n^h\rangle_{\mathbb{C}} = \frac{1}{8\pi}\left\{\int_S(\Delta f + f\Delta L)\mathrm{d}A + \int_{\partial S} f\mathcal{N}(L^S)\,\mathrm{d}s\right\} + \frac{1}{2\pi}\int_{\mathbb{C}}\nabla f^S\cdot\nabla h^S\mathrm{d}A. \tag{2.13}$$

Our proof of this formula is based on the limit form of the Ward identities which we discuss in the next section. To justify this limit form we need to estimate certain error terms; this is done in Section 4. In the proof, we refer to some basic estimates of polynomial Bergman kernels, which we collect in the appendix. The proof of the main theorem is completed in Section 5.

3. Ward identities

3.1. **Exact identities.** For an appropriate function v on $\mathbb{C}$ we define a random variable $W_n^+[v]$ on the probability space $(\mathbb{C}^n, \mathrm{Prob}_n)$ by

$$W_n^+[v] := \frac{1}{2}\sum_{j,k:j\neq k}\frac{v(\lambda_j)-v(\lambda_k)}{\lambda_j-\lambda_k} - 2n\ \mathrm{Tr}_n[v\partial Q] + \mathrm{Tr}_n[\partial v].$$

The minimal requirement on v is that the above expression should be well-defined.

Proposition 3.1. *Let $v:\mathbb{C}\to\mathbb{C}$ be Lipschitz-continuous with compact support. Then*

$$\mathbb{E}_n W_n^+[v] = 0.$$

Proof. The proof is based on the observation that the value of the partition function

$$Z_n := \int_{\mathbb{C}^n}\mathrm{e}^{-H_n(z)}\mathrm{d}A^{\otimes n}(z)$$

remains unchanged under a change of variables. Here, H_n is the Hamiltonian given by (2.3). We will need to analyze the change of the volume element as well as the change of the Hamiltonian under the change of variables. To simplify the notation, we write

$$W_n^+[v] = \mathrm{I}_n[v] - \mathrm{II}_n[v] + \mathrm{III}_n[v]$$

where (a.e.)

$$\mathrm{I}_n[v](z) := \frac{1}{2}\sum_{j,k:j\neq k}\frac{v(z_j)-v(z_k)}{z_j-z_k},\quad \mathrm{II}_n[v]\,(z) = 2\sum_{j=1}^n\partial Q(z_j)\,v(z_j),\quad \mathrm{III}_n[v](z) = \sum_{j=1}^n\partial v(z_j).$$

We consider the change of variables $z_j = \phi(\zeta_j) := \zeta_j + \xi\, v(\zeta_j)$, for $1\le j\le n$. Here, $\xi\in\mathbb{C}$ is assumed to be close to 0. The corresponding area element is

$$\mathrm{d}A(z_j) = \left(|\partial\phi(\zeta_j)|^2 - |\bar\partial\phi(\zeta_j)|^2\right)\mathrm{d}A(\zeta_j) = \left\{1 + 2\,\mathrm{Re}[\xi\,\partial v(\zeta_j)] + \mathrm{O}(|\xi|^2)\right\}\mathrm{d}A(\zeta_j),$$

so that the corresponding volume element becomes

$$\mathrm{d}A^{\otimes n}(z) = \Big\{1 + 2\,\mathrm{Re}(\xi\,\mathrm{III}_n[v](\zeta)) + \mathrm{O}(|\xi|^2)\Big\}\,\mathrm{d}A^{\otimes n}(\zeta).$$

We turn to the Hamiltonian after the change of variables. We note that

$$\begin{aligned}\log|z_i - z_j|^2 &= \log|\zeta_i-\zeta_j|^2 + \log\left|1+\xi\frac{v(\zeta_i)-v(\zeta_j)}{\zeta_i-\zeta_j}\right|^2\\ &= \log|\zeta_i-\zeta_j|^2 + 2\,\mathrm{Re}\left(\xi\frac{v(\zeta_i)-v(\zeta_j)}{\zeta_i-\zeta_j}\right) + \mathrm{O}(|\xi|^2),\end{aligned}$$

so that

$$\sum_{j,k:j\neq k}\log\frac{1}{|z_j-z_k|} = \sum_{j,k:j\neq k}^{n}\log\frac{1}{|\zeta_j-\zeta_k|} - 2\,\mathrm{Re}[\xi\,\mathrm{I}_n(\zeta)| + \mathrm{O}(|\xi|^2), \tag{3.1}$$

as $|\xi|\to 0$. The external potential Q changes according to

$$Q(z_j) = Q(\zeta_j+\xi\,v(\zeta_j)) = Q(\zeta_j) + 2\,\mathrm{Re}\left(\xi\partial Q(\zeta_j)\,v(\zeta_j)\right) + \mathrm{O}(|\xi|^2),$$

so that

$$2n\sum_{j=1}^{n}Q(z_j) = 2n\sum_{j=1}^{n}Q(\zeta_j) + 2\,\mathrm{Re}[\xi\,\mathrm{II}_n(\zeta)] + \mathrm{O}(|\xi|^2). \tag{3.2}$$

Putting things together, we see that (3.1) and (3.2) imply that the Hamiltonian H_n given by (2.3) changes according to

$$H_n(z) = H_n(\zeta) + 2\,\mathrm{Re}\,(-\xi\,\mathrm{I}_n(\zeta) + \xi\,\mathrm{II}_n(\zeta)) + \mathrm{O}(|\xi|^2). \tag{3.3}$$

We find that after the change of variables, the partition function equals

$$Z_n = \int_{\mathbb{C}^n}\mathrm{e}^{-H_n(z)}\,\mathrm{d}A_n(z) = \int_{\mathbb{C}^n}\mathrm{e}^{-H_n(\zeta)-2\,\mathrm{Re}[-\xi\,\mathrm{I}_n(\zeta)+\xi\,\mathrm{II}_n(\zeta)]+\mathrm{O}(|\xi|^2)}\left(1+2\,\mathrm{Re}[\xi\,\mathrm{III}_n(\zeta)]+\mathrm{O}(|\xi|^2)\right)\mathrm{d}A^{\otimes n}(\zeta).$$

As the value of Z_n does not depend on the value of the small complex parameter ξ, a simple argument based on Taylor's formula gives that

$$\mathrm{Re}\left\{\xi\int_{\mathbb{C}^n}(\mathrm{III}_n(\zeta)+\mathrm{I}_n(\zeta)-\mathrm{II}_n(\zeta))\;\mathrm{e}^{-H_n(\zeta)}\,\mathrm{d}A^{\otimes n}(\zeta)\right\} = 0, \tag{3.4}$$

that is, $\mathrm{Re}(\xi\,\mathbb{E}_n W_n^+[v]) = 0$. Considering that ξ is an arbitrary complex number which is close enough to 0, the claimed assertion $\mathbb{E}_n(W_n^+[v]) = 0$ is immediate. □

By applying Proposition 3.1 to the perturbed potential $Q_n^h = Q - \frac{1}{n}h$, we obtain the identity

$$\mathbb{E}_n^h W_{n,h}^+[v] = 0, \tag{3.5}$$

where $\mathbb{E}_n^h$ is the expectation operation with respect to the weight Q_n^h, and

$$W_{n,h}^+[v] := W_n^+[v] + 2\,\mathrm{Tr}_n[v\partial h]. \tag{3.6}$$

If we write

$$V_n[v] = \frac{1}{2n}\sum_{i,j:i\neq j}\frac{v(\lambda_i)-v(\lambda_j)}{\lambda_i-\lambda_j},$$

we may reformulate (3.5) and (3.6) in the following fashion:

$$\mathbb{E}_n^h V_n[v] = 2\mathbb{E}_n^h \operatorname{Tr}_n[v\partial Q] - \langle \partial v + 2v\partial h, \sigma_n^h\rangle_{\mathbb{C}} = \langle 2nv\partial Q - 2v\partial h - \partial v, \sigma_n^h\rangle_{\mathbb{C}}. \tag{3.7}$$

Here, we recall that σ_n^h is the measure with density u_n^h.

3.2. **Some logarithmic potentials.** We recall from (2.11) that $\check{Q}$ may be written as

$$\check{Q} = c_Q - U^\sigma,$$

where c_Q is a Robin-type constant and U^σ is the usual logarithmic potential associated with σ. More generally, if μ is a finite Borel measure in $\mathbb{C}$, with finite moment

$$\int_{\mathbb{C}} |\zeta| \,\mathrm{d}|\mu|(\zeta) < +\infty,$$

its logarithmic potential U^μ is the function

$$U^\mu(z) = \int_{\mathbb{C}} \log \frac{1}{|z-\zeta|} \mathrm{d}\mu(\zeta), \qquad z \in \mathbb{C}.$$

We introduce the following function, associated with the measure σ_n^h:

$$Q^{\circledcirc}_{n,h} := c_Q - U^{\sigma_n^h}. \tag{3.8}$$

We write $Q^{\circledcirc}_n := Q^{\circledcirc}_{n,0}$ in case $h = 0$. Then, as $n \to +\infty$, $Q^{\circledcirc}_{n,h} \to \check{Q}$, uniformly in $\mathbb{C}$. Moreover, the way things are set up, we have

$$\Delta Q^{\circledcirc}_{n,h} = 2\pi\, u_n^h.$$

Using the estimates of the one-point function u_n^h developed in Lemma 4.1 and Theorem 4.2, it is not difficult to show that $\nabla Q^{\circledcirc}_{n,h} \to \nabla\check{Q}$, uniformly in $\mathbb{C}$.

3.3. **Cauchy kernels.** For each $z \in \mathbb{C}$, let κ_z denote the function

$$\kappa_z(\lambda) = \frac{1}{z-\lambda},$$

so that $z \mapsto \langle \kappa_z, \sigma\rangle_{\mathbb{C}}$ is the Cauchy transform of the the measure σ. By Proposition 2.5, we have

$$\langle \kappa_z, \sigma\rangle_{\mathbb{C}} = 2\partial\check{Q}(z).$$

We will also need the Cauchy integral $\langle \kappa_z, \sigma_n^h\rangle_{\mathbb{C}}$. We observe that

$$\langle \kappa_z, \sigma_n^h\rangle_{\mathbb{C}} = 2\partial Q^{\circledcirc}_{n,h}(z), \qquad z \in \mathbb{C}. \tag{3.9}$$

We now introduce the function

$$D_n^h(z) := \langle \kappa_z, \nu_n^h\rangle_{\mathbb{C}},$$

and write $D_n := D_n^0$ in case $h = 0$. In terms of the function $Q^{\circledcirc}_{n,h}$, we have

$$D_n^h = 2n\partial[Q^{\circledcirc}_{n,h} - \check{Q}], \quad \text{and} \quad \bar\partial D_n^h = n\pi(u_n^h - u), \tag{3.10}$$

and if f is a test function, then

$$\langle f, \nu_n^h\rangle_{\mathbb{C}} = \frac{1}{\pi}\int_{\mathbb{C}} f\bar\partial D_n^h \,\mathrm{d}A = -\frac{1}{\pi}\int_{\mathbb{C}} D_n^h \bar\partial f \,\mathrm{d}A. \tag{3.11}$$

Let K_n^h denote the correlation kernel with respect to the weight $Q_n^h = Q - \frac{1}{n}h$. In terms of D_n^h, we may rewrite the $V_n[v]$ term which appears in the Ward identity as follows.

Lemma 3.2. *We have that*

$$\mathbb{E}_n^h V_n[v] = 2n \int_{\mathbb{C}} v u_n^h \partial \check{Q}\, \mathrm{d}A + \int_{\mathbb{C}} v u_n^h D_n^h \mathrm{d}A - \frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} |\mathrm{K}_n^h(z,w)|^2 \mathrm{d}A^{\otimes 2}(z,w).$$

Proof. After all, we have

$$\mathbb{E}_n^h V_n[v] = \frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} R_{n,h}^{(2)}(z,w)\, \mathrm{d}A^{\otimes 2}(z,w),$$

where

$$R_{n,h}^{(2)}(z,w) := \mathrm{K}_n^h(z,z)\mathrm{K}_n^h(w,w) - |\mathrm{K}_n^h(z,w)|^2.$$

Next, we see that

$$\begin{aligned}\frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} \mathrm{K}_n^h(z,z)\mathrm{K}_n^h(w,w)\, \mathrm{d}A^{\otimes 2}(z,w) &= \frac{1}{n} \int_{\mathbb{C}^2} \frac{v(z)}{z - w} \mathrm{K}_n^h(z,z)\mathrm{K}_n^h(w,w)\, \mathrm{d}A^{\otimes 2}(z,w) \\ = n \int_{\mathbb{C}^2} \frac{v(z)}{z - w} u_n^h(z) u_n^h(w)\, \mathrm{d}A^{\otimes 2}(z,w) &= n \int_{\mathbb{C}} v(z) u_n^h(z) \langle \kappa_z, \sigma_n^h \rangle_{\mathbb{C}}\, \mathrm{d}A(z) \\ = 2n \int_{\mathbb{C}} v u_n^h \partial Q_{n,h}^{\otimes}\, \mathrm{d}A &= 2n \int_{\mathbb{C}} v u_n^h \partial \check{Q}\, \mathrm{d}A + \int_{\mathbb{C}} v u_n^h D_n^h\, \mathrm{d}A,\end{aligned}$$

where we first used symmetry, second the identity $\mathrm{K}_n^h(z,z) \equiv n u_n^h(z)$, third, the equality (3.9), and fourth, the relation (3.10). The proof is complete. □

3.4. **Limit form of the Ward identity.** The main formula (2.13) will be derived from Theorem 3.3 below. In this theorem we make the following assumptions on the vector field v:

(3.4-*i*) v is bounded in $\mathbb{C}$;

(3.4-*ii*) v is Lipschitz-continuous in $\mathbb{C}$;

(3.4-*iii*) v is uniformly C^2-smooth in $\mathbb{C} \setminus \partial S$.

(The last condition means that the restriction of v to S and the restriction to $(\mathbb{C} \setminus S) \cup \partial S$ are both C^2-smooth.)

Theorem 3.3. *If v satisfies (3.4-i)–(3.4-iii), then as $n \to +\infty$,*

$$\frac{2}{\pi} \int_S v D_n^h \Delta Q\, \mathrm{d}A + \frac{2}{\pi} \int_{\mathbb{C} \setminus S} v \partial(\check{Q} - Q)\, \bar{\partial} D_n^h\, \mathrm{d}A \to -\frac{1}{2} \langle \partial v, \sigma \rangle_{\mathbb{C}} - 2 \langle v \partial h, \sigma \rangle_{\mathbb{C}}.$$

We postpone the proof to remark that it will be convenient to integrate by parts in the second integral in Theorem 3.3. To control the boundary term we can use the next lemma.

Lemma 3.4. *For big n, we have the estimate*

$$|D_n^h(z)| = \mathrm{O}\left(\frac{n}{|z|^2}\right) \quad \text{as } |z| \to +\infty,$$

where the implied constant is independent of n.

Proof. As u_n^h and u are both probability densities, we see that

$$\frac{D_n^h(z)}{n} = \int_{\mathbb{C}} \frac{u_n^h(\lambda) - u(\lambda)}{z-\lambda} \mathrm{d}A(\lambda) = \int \Big(\frac{1}{z-\lambda} - \frac{1}{z}\Big)(u_n^h(\lambda) - u(\lambda))\mathrm{d}A(\lambda).$$

Next, since

$$\frac{1}{z-\lambda} - \frac{1}{z} = \frac{1}{z^2}\,\frac{\lambda}{1-\lambda/z},$$

we need to show that the integrals

$$\int_{\mathbb{C}} \frac{|u_n^h(\lambda) - u(\lambda)|}{|1-\lambda/z|}\,|\lambda|\,\mathrm{d}A(\lambda)$$

are uniformly bounded. We use that for some positive constant C_1,

$$|u_n^h(\lambda) - u(\lambda)| \le u(\lambda) + u_n^h(\lambda) \le \frac{C_1}{1+|\lambda|^4}, \qquad \lambda \in \mathbb{C}, \tag{3.12}$$

which may be justified by appealing to the the basic estimate (cf. Lemma 4.1 below)

$$u_n^h(\lambda) \le C_2\,\mathrm{e}^{-2n(Q(\lambda)-\check{Q}(\lambda))}, \qquad \lambda \in \mathbb{C},$$

together with the growth assumption (2.1). It is here that we need n to be big enough. Next, in view of (3.12),

$$\int_{\mathbb{C}} \frac{|u_n^h(\lambda) - u(\lambda)|}{|1-\lambda/z|}\,|\lambda|\,\mathrm{d}A(\lambda) \le C_1 \int_{\mathbb{C}} \frac{(1+|\lambda|)^{-4}}{|1-\lambda/z|}\,|\lambda|\,\mathrm{d}A(\lambda) \le 100\,C_1,$$

where the estimate of the integral can be achieved by splitting the plane into the disk $\mathbb{D}(z, \frac{1}{2}|z|)$ and its complement, and by suitably estimating the integrand in each region. □

Since $\partial Q = \partial\check{Q}$ on S, an integration by parts argument leads to the following reformulation of Theorem 3.3.

Corollary 3.5. ("Limit Ward identity") *Suppose that v meets the conditions* (*3.4-i*)–(*3.4-iii*). *Then as $n \to +\infty$ we have the convergence*

$$\frac{2}{\pi}\int_{\mathbb{C}} \Big(v\Delta Q + \bar{\partial} v\,\partial(Q-\check{Q})\Big) D_n^h\,\mathrm{d}A \to -\frac{1}{2}\langle \partial v, \sigma\rangle_{\mathbb{C}} - 2\langle v\partial h, \sigma\rangle_{\mathbb{C}}.$$

3.5. **Error terms and the proof of Theorem 3.3.** We recall that the Ward identity (3.7) states that

$$\mathbb{E}_n^h V_n[v] = \langle 2nv\partial Q - 2v\partial h - \partial v, \sigma_n^h\rangle_{\mathbb{C}},$$

while Lemma 3.2 supplies the formula

$$\mathbb{E}_n^h V_n[v] = \langle 2nv\partial\check{Q} + vD_n^h, \sigma_n^h\rangle_{\mathbb{C}} - \frac{1}{2n}\int_{\mathbb{C}^2} \frac{v(z)-v(w)}{z-w}|\mathrm{K}_n^h(z,w)|^2\mathrm{d}A^{\otimes 2}(z,w).$$

As we equate the two, and perform some rearrangement, we arrive at

$$\begin{aligned} \langle 2nv\partial(\check{Q}-Q), \sigma_n^h - \sigma\rangle_{\mathbb{C}} + \langle vD_n^h, \sigma\rangle_{\mathbb{C}} &= -\langle 2v\partial h + \tfrac{1}{2}\partial v, \sigma_n^h\rangle_{\mathbb{C}} - \langle vD_n^h, \sigma_n^h - \sigma\rangle_{\mathbb{C}} \\ &\quad + \frac{1}{2n}\int_{\mathbb{C}^2} \frac{v(z)-v(w)}{z-w}|\mathrm{K}_n^h(z,w)|^2\mathrm{d}A^{\otimes 2}(z,w) - \frac{1}{2}\langle \partial v, \sigma_n^h\rangle_{\mathbb{C}}. \end{aligned} \tag{3.13}$$

In the rearrangement, we used the facts that σ is supported on S and that $\partial(\check{Q}-Q)=0$ on S. Let us introduce the *first error term* by

$$\epsilon^1_{n,h}[v] := \frac{1}{n}\int_{\mathbb{C}^2}\frac{v(z)-v(w)}{z-w}\,|\mathrm{K}^h_n(z,w)|^2\mathrm{d}A^{\otimes 2}(z,w) - \langle \partial v, \sigma^h_n\rangle_{\mathbb{C}}, \tag{3.14}$$

and the *second error term* by

$$\epsilon^2_{n,h}[v] := \langle vD^h_n, \sigma^h_n - \sigma\rangle_{\mathbb{C}} = \int_{\mathbb{C}} vD^h_n(u^h_n - u)\mathrm{d}A = -\frac{1}{2\pi n}\int_{\mathbb{C}}[D^h_n]^2\,\bar\partial v\,\mathrm{d}A. \tag{3.15}$$

We insert these error terms into (3.13), to obtain

$$\langle 2nv\partial(\check{Q}-Q), \sigma^h_n - \sigma\rangle_{\mathbb{C}} + \langle vD^h_n, \sigma\rangle_{\mathbb{C}} = -\langle 2v\partial h + \tfrac{1}{2}\partial v, \sigma^h_n\rangle_{\mathbb{C}} - \epsilon^2_{n,h}[v] + \frac{1}{2}\epsilon^1_{n,h}[v].$$

We next rewrite this relation in integral form using (3.10) and (2.4):

$$\frac{2}{\pi}\int_{\mathbb{C}\setminus S} v\partial(\check{Q}-Q)\,\bar\partial D^h_n\mathrm{d}A + \frac{2}{\pi}\int_S vD^h_n\Delta Q\,\mathrm{d}A = -\langle 2v\partial h + \tfrac{1}{2}\partial v, \sigma^h_n\rangle_{\mathbb{C}} + \frac{1}{2}\epsilon^1_{n,h}[v] - \epsilon^2_{n,h}[v]. \tag{3.16}$$

As $n\to+\infty$, we have the convergence $\sigma^h_n\to\sigma$ in the weak-star sense of measures. For $h=0$, this is Johansson's theorem (see [11]), while in this more general setting, we may, e.g., deduce it from the one-point function estimates in Lemma 4.1 and Theorem 4.2 below. We see from (3.16) that once we have established that $\epsilon^j_{n,h}[v]\to 0$ as $n\to+\infty$ for $j=1,2$, the assertion of Theorem 3.3 is immediate.

In the next section we will show that for each v satisfying conditions (3.4-*i*)-(3.4-*iii*), the error terms $\epsilon^j_{n,h}[v]$ tend to zero as $n\to+\infty$, for $j=1,2$, which finishes the proof of Theorem 3.3.

4. Estimates of the error terms

4.1. **Estimates of the kernel** K^h_n**.** We will use two different estimates, one which gives control in the interior int(S) of the droplet S, and another which gives control in the exterior domain $\mathbb{C}\setminus S$.

4.1.1. *The exterior estimate.* We recall that $\mathrm{K}^h_n(z,w)$ is the correlation kernel of the n-point process associated with potential $Q^h_n = Q-\frac{1}{n}h$. We have the following global estimate, which is particularly useful in the exterior of the droplet.

Lemma 4.1. *There exists a positive constant C which only depends on Q,h such that*

$$\mathrm{K}^h_n(z,z)\le Cn\,\mathrm{e}^{-2n(Q-\check{Q})(z)},\qquad z\in\mathbb{C}.$$

This estimate has been recorded (see, e.g., [2], Section 3) for the kernels K_n, i.e., in the case $h=0$. Since obviously

$$\int_{\mathbb{C}}|p|^2\mathrm{e}^{-2nQ^h_n}\mathrm{d}A \asymp \int_{\mathbb{C}}|p|^2\mathrm{e}^{-2nQ}\mathrm{d}A,$$

the norms of the point evaluation functionals are equivalent in the spaces $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$ and $\mathrm{Pol}_n(\mathrm{e}^{-2nQ^h_n})$. In terms of reproducing kernels, this means that $\mathrm{k}_n(z,z)\asymp \mathrm{k}^h_n(z,z)$, and so $\mathrm{K}_n(z,z)\asymp \mathrm{K}^h_n(z,z)$ as well. It follows that the case $h\neq 0$ does not require any separate treatment.

In the following, we shall use the notation

$$\delta_{\partial S}(z) := \mathrm{dist}_{\mathbb{C}}(z,\partial S)$$

and

$$\delta_n := n^{-1/2}[\log n]^2.$$

In view of our assumptions on the droplet (see Proposition 2.5), and the growth control (2.1) together with the definition of $\check{Q}$, we have, for some small but positive real parameter ε,

$$Q(z) - \check{Q}(z) \;\geq\; \varepsilon \min\left\{ \log(2 + |z|), \delta_{\partial S}(z)^2 \right\}, \qquad z \in \mathbb{C} \setminus S. \tag{4.1}$$

For big n, it follows that for any $N > 0$ there exists a constant C_N such that

$$\mathrm{K}_n^h(z) \leq C_N n^{-N}(1 + |z|)^{-3}, \qquad \text{for } z \in \mathbb{C} \setminus S \text{ with } \delta_{\partial S}(z) \geq \delta_n. \tag{4.2}$$

4.1.2. *The interior estimate.* Let us recall that we assume that Q is real-analytic in some neighbourhood of S. This means that we can lift Q to a complex analytic function of two variables in some neighbourhood in $\mathbb{C}^2$ of the conjugate-diagonal

$$\{(z, \bar{z}) :\; z \in S\} \subset \mathbb{C}^2.$$

We will use the same letter Q for this extension, so that, e.g.,

$$Q(z) = Q(z, \bar{z}).$$

We have

$$Q(z, w) = \overline{Q(\bar{w}, \bar{z})}$$

and

$$\partial_1 Q(z, \bar{z}) = \partial Q(z), \quad \partial_1\partial_2 Q(z, \bar{z}) = \partial\bar{\partial} Q(z) = \Delta Q(z), \quad \partial_1^2 Q(z, \bar{z}) = \partial^2 Q(z), \quad \text{etc.}$$

Using this extension and some technical mathematical machinery, one can show that for z, w confined to the interior of the droplet S, the leading contribution to the perturbed correlation kernel K_n^h is of the form

$$\mathrm{K}^\sharp_{n,h}(z, w) \;=\; \frac{2n}{\pi}(\partial_1\partial_2 Q)(z, \bar{w})\, \mathrm{e}^{n[2Q(z,\bar{w}) - Q(z) - Q(w)]}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}[2(z-w)\partial h(w) + (z-w)^2\partial^2 h(w)] + |z-w|^2 \Delta h(w)}. \tag{4.3}$$

The diagonal restriction of this approximate correlation kernel is

$$\mathrm{K}^\sharp_{n,h}(w, w) \;=\; \frac{2n}{\pi}\Delta Q(w) = \frac{n}{2\pi}\Delta Q(w).$$

Theorem 4.2. *Suppose that $z, w \in S$, with $\delta_{\partial S}(z) > 2\delta_n$ and $|z - w| < \delta_n$. Then*

$$|\mathrm{K}_n^h(z, w) - \mathrm{K}^\sharp_{n,h}(z, w)| \;=\; \mathrm{O}(1),$$

where the implied constant in $\mathrm{O}(1)$ *depends on Q, h, but not on n.*

Similar types of expansions are discussed e.g. in [5], [1], [2]. In the appendix, we mention the modifications of the standard approach which are required to obtain Theorem 4.2.

We now turn to the proof that the error terms $\epsilon^1_{n,h}[v]$ and $\epsilon^2_{n,h}[v]$ (cf. (3.14)–(3.15)) are negligible. Our proof is based on the above-mentioned estimates of the correlation kernels K_n^h.

4.2. **The first error term.** We start with the observation that if $w \in S$ and $\delta_{\partial S}(w) > 2\delta_n$ then at short distances the so-called Berezin kernel rooted at w

$$Б^{\langle w\rangle}_{n,h}(z) = \frac{|\mathrm{K}^h_n(z,w)|^2}{\mathrm{K}^h_n(w,w)}$$

is close to the heat kernel

$$H^{\langle w\rangle}_n(z) = \frac{1}{\pi} an\, \mathrm{e}^{-an|z-w|^2}, \qquad a := 2\Delta Q(w).$$

Both kernels determine probability measures indexed by w. Most of the heat kernel measure is concentrated in the disc $\mathbb{D}(w,\delta_n)$,

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} H^{\langle w\rangle}_n(z)\,\mathrm{d}A(z) \;=\; \mathrm{O}(n^{-N}) \quad \text{as} \;\; n\to+\infty, \tag{4.4}$$

where N denotes an arbitrary (large) positive number.

Lemma 4.3. *Suppose that $z, w \in S$, with $\delta_{\partial S}(w) > 2\delta_n$ and $|z-w| < \delta_n$. Then*

$$\left|Б^{\langle w\rangle}_{n,h}(z) - H^{\langle w\rangle}_n(z)\right| = \mathrm{O}(n^2\delta_n^3) = \mathrm{O}(n^{1/2}[\log n]^6) \quad \text{as} \;\; n\to+\infty,$$

where the implied constant only depends on Q, h.

Proof. In view of Theorem 4.2, we have

$$Б^{\langle w\rangle}_{n}(z) = \frac{|\mathrm{K}^\sharp_{n,h}(z,w)|^2}{\mathrm{K}^\sharp_{n,h}(w,w)} + \mathrm{O}(1),$$

where $\mathrm{K}^\sharp_{n,h}$ is as in (4.3). Next, we fix w and apply Taylor's formula to get that

$$\begin{aligned}\mathrm{Re}[2Q(z,\bar w) - Q(z) - Q(w)] &= 2\,\mathrm{Re}\, Q(z,\bar w) - Q(z) - Q(w)\\ &= 2\,\mathrm{Re}\left\{Q(w) + (z-w)\partial Q(w) + \tfrac{1}{2}(z-w)^2\partial^2 Q(w)\right\} - \Big\{Q(w) + 2\,\mathrm{Re}[(z-w)\partial Q(w)]\\ + \mathrm{Re}[(z-w)^2\partial^2 Q(w)] &+ |z-w|^2\Delta Q(w)\Big\} - Q(w) + \mathrm{O}(|z-w|^3) = -|z-w|^2\Delta Q(w) + \mathrm{O}(|z-w|^3).\end{aligned}$$

Using the explicit formula (4.3), we find that with $\alpha = 2\Delta Q(w)$,

$$\begin{aligned}\frac{|\mathrm{K}^\sharp_{n,h}(z,w)|^2}{\mathrm{K}^\sharp_{n,h}(w,w)} &= \frac{n}{\pi}\,[\alpha + \mathrm{O}(|z-w|)]\,\mathrm{e}^{-\alpha n|z-w|^2 + \mathrm{O}(n|z-w|^3) + \mathrm{O}(|z-w|^2)}\\ &= H^{\langle w\rangle}_n(z) + \mathrm{O}(n|z-w|) + \mathrm{O}(n^2|z-w|^3) + \mathrm{O}(n|z-w|^2),\end{aligned}$$

which does it. □

Corollary 4.4. *If $w \in S$ and $\delta_{\partial S}(w) > 2\delta_n$, then*

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} Б^{\langle w\rangle}_{n,h}(z)\,\mathrm{d}A(z) \;=\; \mathrm{O}(n^2\delta_n^5) = \mathrm{O}(n^{-1/2}[\log n]^{10}).$$

Proof. We notice that

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} \text{Б}_{n,h}^{\langle w\rangle}\mathrm{d}A = 1 - \int_{\mathbb{D}(w,\delta_n)} \text{Б}_{n,h}^{\langle w\rangle}\mathrm{d}A = 1 - \int_{\mathbb{D}(w,\delta_n)} H_n^{\langle w\rangle}\mathrm{d}A + \int_{\mathbb{D}(w,\delta_n)} (H_n^{\langle w\rangle} - \text{Б}_{n,h}^{\langle w\rangle})\mathrm{d}A$$
$$= \int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} H_n^{\langle w\rangle}\mathrm{d}A + \int_{\mathbb{D}(w,\delta_n)} (H_n^{\langle w\rangle} - \text{Б}_{n,h}^{\langle w\rangle})\,\mathrm{d}A.$$

The assertion now follows from Lemma 4.3 and the decay (4.4) of the heat kernel. □

Proposition 4.5. *Suppose that v meets the conditions (3.4-i)–(3.4-iii). Then $\epsilon^1_{n,h}[v] = \mathrm{O}(n^2\delta_n^5) = \mathrm{O}(n^{-1/2}[\log n]^{10})$ as $n \to +\infty$.*

Proof. We consider the auxiliary function

$$F_n^h[v](w) := \int_{\mathbb{C}} \left\{ \frac{v(z) - v(w)}{z - w} - \partial v(w) \right\} \text{Б}_{n,h}^{\langle w\rangle}(z)\,\mathrm{d}A(z), \qquad w \in \mathbb{C}.$$

Then the error term defined by (3.14) may be expressed in the form

$$\epsilon^1_{n,h}[v] = \int_{\mathbb{C}} u_n^h(w) F_n^h[v](w)\,\mathrm{d}A(w). \tag{4.5}$$

As v is globally Lipschitz-continuous, $F_n^h[v]$ is uniformly bounded; indeed, we have the estimate

$$\|F_n^h[v]\|_{L^\infty(\mathbb{C})} \le 2\|\nabla v\|_{L^\infty(\mathbb{C})}. \tag{4.6}$$

Let $\mathcal{B}_n$ be the thin "tube" around ∂S given by

$$\mathcal{B}_n := \{z \in \mathbb{C} :\ \delta_{\partial S}(z) < 2\delta_n\}, \tag{4.7}$$

Since the area of $\mathcal{B}_n$ is $\asymp \delta_n$, it follows that

$$\int_{\mathcal{B}_n} u_n(w)|F_n(w)|\,\mathrm{d}A \le 2\|\nabla v\|_{L^\infty(\mathbb{C})} \int_{\mathcal{B}_n} u_n(w)\,\mathrm{d}A \le 2C\|\nabla v\|_{L^\infty(\mathbb{C})} \int_{\mathcal{B}_n} \mathrm{d}A = \mathrm{O}(\delta_n)$$

as $n \to +\infty$, where C is the constant of Lemma 4.1, which only depends on Q, h. We turn to the estimation of the same integrand over the complementary set $\mathbb{C} \setminus \mathcal{B}_n$. For $w \in \mathbb{C} \setminus \mathcal{B}_n$ and $z \in \mathbb{C}$ with $|z - w| < \delta_n$, then both z, w lie in the same component of $\mathbb{C} \setminus \partial S$ where the assumptions on v tell us that v is uniformly C^2-smooth. By Taylor's formula, then, we have

$$v(z) = v(w) + (z - w)\partial v(w) + (\bar z - \bar w)\bar\partial v(w) + \mathrm{O}(|z - w|^2),$$

where the implied constant is uniform. As a consequence, we get that

$$\frac{v(z) - v(w)}{z - w} - \partial v(w) = \bar\partial v(w)\,\frac{\bar z - \bar w}{z - w} + \mathrm{O}(|z - w|),$$

with a uniform implied constant. This leads to

$$\int_{\mathbb{D}(w,\delta_n)} \left\{ \frac{v(z) - v(w)}{z - w} - \partial v(w) \right\} H_n^{\langle w\rangle}(z)\,\mathrm{d}A(z) = \bar\partial v(w) \int_{\mathbb{D}(w,\delta_n)} \frac{\bar z - \bar w}{z - w} H_n^{\langle w\rangle}(z)\,\mathrm{d}A(z) + \mathrm{O}(\delta_n) = \mathrm{O}(\delta_n),$$

by the radial symmetry of the heat kernel. Next we use Lemma 4.3 and the global Lipschitz-continuity of v to see that

$$\left| \int_{\mathbb{D}(w,\delta_n)} \left\{ \frac{v(z) - v(w)}{z - w} - \partial v(w) \right\} \left[\text{Б}_{n,h}^{\langle w\rangle}(z) - H_n^{\langle w\rangle}(z) \right] \mathrm{d}A(z) \right| = \mathrm{O}(n^2\delta_n^5),$$

uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. Finally, we use the global Lipschitz-continuity of v together with Corollary 4.4 to see that

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} \left| \frac{v(z)-v(w)}{z-w} - \partial v(w) \right| \mathrm{B}^{\langle w\rangle}_{n,h}(z)\, \mathrm{d}A(z) = \mathrm{O}(n^2\delta_n^5) \quad \text{as } n \to +\infty,$$

uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. Putting the above ingredients together, we realize that we have obtained that

$$|F_n^h(w)| = \left| \int_{\mathbb{C}} \left\{ \frac{v(z)-v(w)}{z-w} - \partial v(w) \right\} \mathrm{B}^{\langle w\rangle}_{n,h}(z)\, \mathrm{d}A(z) \right| = \mathrm{O}(\delta_n + n^2\delta_n^5) = \mathrm{O}(n^2\delta_n^5) = \mathrm{O}(n^{-1/2}[\log n]^{10})$$

as $n \to +\infty$, uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. This entails that

$$\int_{\mathbb{C}\setminus\mathcal{B}_n} |F_n^h(w)|\, u_n^h(w)\, \mathrm{d}A(w) = \mathrm{O}(n^2\delta_n^5) = \mathrm{O}(n^{-1/2}[\log n]^{10}),$$

which combined with the previous estimate of the integral over $\mathcal{B}_n$ gives the assertion of the proposition. □

Remark 4.6. If in Proposition 4.5 we weaken the assumptions on v to just asking v to be bounded and globally Lipschitz-continuous, it is possible to slightly modify the argument of the proof so as to obtain that $\epsilon^1_{n,h}[v] = \mathrm{o}(1)$ as $n \to +\infty$.

4.3. **The second error term.** We shall prove the following proposition.

Proposition 4.7. *We have that for some small* $\beta > 0$,

$$\epsilon^2_{n,h}[v] = -\frac{1}{2\pi n} \int_{\mathbb{C}} [D_n^h]^2 \bar\partial v\, \mathrm{d}A \;= \mathrm{O}(n^{-\beta/2}(\|v\|_{L^\infty(\mathbb{C})} + \|\nabla v\|_{L^\infty(\mathbb{C})})) \;= \mathrm{o}(1), \qquad \textit{as} \quad n \to +\infty.$$

The proof will involve certain estimates of the function

$$D_n^h(z) = \langle \kappa_z, \nu_n^h \rangle_{\mathbb{C}} = n\, \langle \kappa_z, \sigma_n^h - \sigma \rangle_{\mathbb{C}} = n \int_{\mathbb{C}} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\, \mathrm{d}A(\zeta).$$

It is convenient to split the integral into two parts:

$$D_n^h(z) = D_{n,\mathrm{I}}^h(z) + D_{n,\mathrm{II}}^h(z),$$

where

$$D_{n,\mathrm{I}}^h(z) := n \int_{\mathcal{B}_n} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\, \mathrm{d}A(\zeta), \quad D_{n,\mathrm{II}}^h(z) := n \int_{\mathbb{C}\setminus\mathcal{B}_n} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\, \mathrm{d}A(\zeta);$$

here, $\mathcal{B}_n$ is the thin "tube" around ∂S given by (4.7). Since

$$n[u_n^h(\zeta) - u(\zeta)] = K_n^h(\zeta,\zeta) - \frac{2n}{\pi} 1_S(\zeta) \Delta Q(\zeta),$$

we get from Theorem 4.2 that

$$n\,|u_n^h(\zeta) - u(\zeta)| = \mathrm{O}(1), \qquad \zeta \in S \setminus \mathcal{B}_n, \tag{4.8}$$

uniformly in ζ, as $n \to +\infty$. The estimate (4.2) supplies fast decay of $n|u_n^h - u|$ in $\mathbb{C} \setminus (S \cup \mathcal{B}_n)$, and together with the above estimate (4.8), this leads to the conclusion that

$$\|D_{n,\mathrm{II}}^h\|_{L^\infty(\mathbb{C})} \le n \sup_{z\in\mathbb{C}} \int_{\mathbb{C}\setminus\mathcal{B}_n} \frac{|u_n^h(\zeta) - u(\zeta)|}{|z-\zeta|}\, \mathrm{d}A(\zeta) = \mathrm{O}(1), \quad \text{as } n \to +\infty. \tag{4.9}$$

We turn to the estimation of $D_{n,\mathrm{I}}^h$.

Lemma 4.8. *We have*

$$\|D^h_{n,\mathrm{I}}\|_{L^\infty(\mathbb{C})} = \mathrm{O}(n^{1/2}[\log n]^3) \quad \text{as } n \to +\infty.$$

Proof. As we shall see, this follows from the trivial bound $\|u^h_n - u\|_{L^\infty(\mathbb{C})} = \mathrm{O}(1)$ as $n \to +\infty$, which is a consequence of Lemma 4.1. We just need to estimate the integral

$$\int_{\mathcal{B}_n} \frac{\mathrm{d}A(\zeta)}{|z-\zeta|}.$$

Without loss of generality, we can take $z = 0$ and replace B_n by the rectangle $|x| < 1$, $|y| < \delta_n$ (with $z = x + \mathrm{i}y$). We have

$$\int_{\mathcal{B}_n} \frac{\mathrm{d}A(\zeta)}{|\zeta|} = \int_{-1}^{1} \mathrm{d}x \int_{-\delta_n}^{\delta_n} \frac{\mathrm{d}y}{\sqrt{x^2+y^2}} = \mathrm{I} + \mathrm{II},$$

where I is the integral where x is confined to the interval $-\delta_n < x < \delta_n$, and II is the remaining term. By passing to polar coordinates we see that

$$\mathrm{I} \asymp \int_0^{\delta_n} \frac{r\mathrm{d}r}{r} = \delta_n = n^{-1/2}[\log n]^2,$$

and

$$\mathrm{II} \asymp \delta_n \int_{\delta_n}^{1} \frac{\mathrm{d}x}{x} \asymp \delta_n \log\frac{1}{\delta_n} = \mathrm{O}(n^{-1/2}[\log n]^3) \quad \text{as } n \to +\infty.$$

The assertion of the lemma is immediate. □

Lemma 4.8 and (4.9) together give us the following estimate of the second error term:

$$\epsilon^2_{n,h}[v] \le C_3\,[\log n]^6 \|\nabla v\|_{L^\infty(\mathbb{C})}, \tag{4.10}$$

for some positive constant C_3. This comes rather close but is still weaker than what we want. Our strategy will be to use (4.10) and iterate the argument with the Ward identity. This will supply a better estimate in the *interior* of the droplet.

Lemma 4.9. *For big n, we have that for some positive constant C_4,*

$$|D^h_{n,\mathrm{I}}(z)| \le C_4\,\frac{[\log n]^6}{\delta_{\partial S}(z)^3}, \qquad z \in S.$$

Proof. Let ψ be a function of Lipschitz norm less than 1 supported inside the droplet S, i.e., $\|\nabla\psi\|_{L^\infty(\mathbb{C})} \le 1$. Then we have

$$|\epsilon^1_{n,h}[\psi]| \le 2, \qquad |\epsilon^2_{n,h}[\psi]| \le C_3[\log n]^6,$$

where the constants do not depend on ψ; the first estimate follows from a combination of (4.5) and (4.6), and the second one is just (4.10). By (3.16) applied to the function ψ, we have

$$\frac{2}{\pi}\int_S \psi D^h_n \Delta Q\,\mathrm{d}A = -\langle 2\psi\partial h + \tfrac{1}{2}\partial\psi, \sigma^h_n\rangle_{\mathbb{C}} + \frac{1}{2}\epsilon^1_{n,h}[\psi] - \epsilon^2_{n,h}[\psi],$$

and therefore,

$$\left|\frac{2}{\pi}\int_S \psi D^h_n \Delta Q\,\mathrm{d}A\right| \le 2\|\psi\|_{L^\infty(\mathbb{C})}\|\nabla h\|_{L^\infty(\mathbb{C})} + \frac{1}{2}\|\nabla\psi\|_{L^\infty(\mathbb{C})} + \frac{1}{2}|\epsilon^1_{n,h}[\psi]| + |\epsilon^2_{n,h}[\psi]| \le C_5[\log n]^6, \tag{4.11}$$

for a suitable positive constant C_5. The claimed estimate is trivial for $z \in S$ with $\delta_{\partial S}(z) \le n^{-1/3}$, as it is a consequence of the global estimate of Lemma 4.8. In the remaining case when $z \in S$ has $\delta_{\partial S}(z) \ge n^{-1/3}$, we consider the function

$$\psi(\zeta) = \max\left\{\frac{\frac{1}{2}\delta_{\partial S}(z) - |\zeta - z|}{\Delta Q(\zeta)},\ 0\right\}.$$

Then ψ has Lipschitz norm $\asymp 1$, and by the analyticity of $D^h_{n,\mathrm{I}}$ in $S \setminus \mathcal{B}_n$, we get the mean value identity

$$\int_S \psi(\zeta) D^h_{n,\mathrm{I}}(\zeta)\Delta Q(\zeta)\,\mathrm{d}A(\zeta) = 2\pi D^h_{n,\mathrm{I}}(z)\int_0^{\frac{1}{2}\delta_{\partial S}(z)} (\tfrac{1}{2}\delta_{\partial S}(z) - r)\, r\mathrm{d}r = \frac{\pi}{24}[\delta_{\partial S}(z)]^3 D^h_{n,\mathrm{I}}(z).$$

Combined with (4.11), this gives the claimed estimate. □

4.3.1. *Metric flow.* We need to introduce a family of curves $\Gamma[\varepsilon]$, where $\Gamma[0] = \Gamma = \partial S$, for $0 \le \varepsilon \ll 1$. Let $\mathrm{n}_\Gamma(\zeta)$ denote the *exterior* unit normal vector to Γ at the point $\zeta \in \Gamma$. The curve $\Gamma[\varepsilon]$ consists of the points

$$\zeta[\varepsilon] = \zeta[\varepsilon, \Gamma] = \zeta + \varepsilon \mathrm{n}_\Gamma(\zeta), \qquad \zeta \in \Gamma.$$

As Γ is a real-analytically smooth Jordan curve, so is $\Gamma[\varepsilon]$ for small ε. We now check that the normal vector is preserved under the flow of curves $\Gamma[\varepsilon]$:

$$\mathrm{n}_{\Gamma[\varepsilon]}(\zeta[\varepsilon]) = \mathrm{n}_\Gamma(\zeta), \qquad \zeta \in \Gamma. \tag{4.12}$$

To this end, let Γ be parametrized with positive orientation by $\zeta(t)$, $0 \le t \le 1$, where $\zeta'(t) \ne 0$ and $\zeta(0) = \zeta(1)$. Then $\mathrm{n}_\Gamma(\zeta(t)) = -\mathrm{i}\zeta'(t)/|\zeta'(t)|$, so that

$$\zeta[\varepsilon](t) = \zeta(t) + \varepsilon \mathrm{n}_\Gamma(\zeta(t)) = \zeta(t) - \mathrm{i}\varepsilon\frac{\zeta'(t)}{|\zeta'(t)|}$$

parametrizes $\Gamma[\varepsilon]$. A calculation gives that

$$(\zeta[\varepsilon])'(t) = \zeta'(t)\Big(1 + \varepsilon\,\mathrm{Im}[\bar\zeta'(t)\zeta''(t)]\Big),$$

which means that the two tangents point in the same direction. As a consequence, the two normals point in the same direction as well. The curves $\Gamma[\varepsilon]$ are obtained as the expansion front for a cloud of photons initially confined to the droplet S.

Finally, we need an estimate of $D^h_{n,\mathrm{I}}$ in the *exterior* of the droplet S. This will be done in the next subsection by reflecting the previous interior estimate in the curve $\Gamma := \partial S$. We then use the following lemma. Let us fix some sufficiently small positive number, e.g., $\beta = \frac{1}{10}$ will do, and define $\Gamma_n := \Gamma[n^{-\beta}]$, in the above notation. For big n, Γ_n is then a C^ω-smooth curve in $\mathbb{C} \setminus S$ which is very close to $\Gamma = \partial S$. The complement $\mathbb{C} \setminus \Gamma_n$ has two connectivity components; let Ω_n be component which is bounded, and $\Omega_n^\circledast$ the remaining component, which is unbounded.

Let $L^2(\Gamma_n)$ denote the usual L^2 space of functions on Γ_n with respect to arc-length measure.

Lemma 4.10. *We have that*

$$\|D^h_{n,\mathrm{I}}\|^2_{L^2(\Gamma_n)} = \mathrm{O}(n^{1-\frac{1}{2}\beta}) \quad \text{as } n \to +\infty.$$

Given this estimate, we can complete the proof of Proposition 4.7 as follows.

Proof of Proposition 4.7. By the correlation kernel decay in (4.2) and the uniform estimate of D_n^h supplied by (4.9) and Lemma 4.8, we have that

$$\begin{aligned}\epsilon^2_{n,h}[v] = \int_{\mathbb{C}} vD_n^h(u_n^h - u)\mathrm{d}A &= \int_{\Omega_n} vD_n^h(u_n^h - u)\mathrm{d}A + \mathrm{O}(n^{-100}\|v\|_{L^\infty(\mathbb{C})}) \\ &= \frac{1}{2\pi n}\int_{\Omega_n} v\bar\partial([D_n^h]^2)\mathrm{d}A + \mathrm{O}(n^{-100}\|v\|_{L^\infty(\mathbb{C})}) \\ &= -\frac{1}{2\pi n}\int_{\Omega_n}[D_n^h]^2\bar\partial v\,\mathrm{d}A + \frac{1}{4\pi n}\int_{\Gamma_n}[D_n^h(z)]^2 v(z)\,\mathrm{d}z + \mathrm{O}(n^{-100}\|v\|_{L^\infty(\mathbb{C})}),\end{aligned}$$

if we use the Cauchy-Green formula. As a consequence, we find that

$$|\epsilon^2_{n,h}[v]| \le \frac{1}{2\pi n}\|D_n^h\|^2_{L^2(\Omega_n)}\|\nabla v\|_{L^\infty(\mathbb{C})} + \frac{1}{4\pi n}\|D_n^h\|^2_{L^2(\Gamma_n)}\|v\|_{L^\infty(\mathbb{C})} + \mathrm{O}(n^{-100}\|v\|_{L^\infty(\mathbb{C})}).$$

The second term is taken care of by Lemma 4.10. To estimate the first term, we consider the set

$$\mathcal{A}_n = \{z \in \mathbb{C} :\ \delta_{\partial S}(z) < n^{-\beta}\}.$$

The area of $\mathcal{A}_n$ is $\asymp n^{-\beta}$, and in $S \setminus \mathcal{A}_n$ we have $|D^h_{n,\mathrm{I}}| = \mathrm{O}(n^{3\beta}[\log n]^6)$ (cf. Lemma 4.9). Inside $\mathcal{A}_n$, we apply the uniform bound of Lemma 4.8. Since we have that $\Omega_n \subset S \cup \mathcal{A}_n$, we find that

$$\begin{aligned}\|D_n^h\|^2_{L^2(\Omega_n)} = \int_{\Omega_n}|D_n^h|^2\mathrm{d}A &\le \int_{S\cup\mathcal{A}_n}|D_n^h|^2\mathrm{d}A \\ &= \int_{\mathcal{A}_n}|D_n^h|^2\mathrm{d}A + \int_{S\setminus A_n}|D_n^h|^2\mathrm{d}A = \mathrm{O}\Big(n^{1-\beta}[\log n]^2 + n^{6\beta}[\log n]^{12}\Big),\end{aligned}$$

whence

$$\|D_n^h\|^2_{L^2(\Omega_n)} = \mathrm{O}(n^{1-\frac{1}{2}\beta}).$$

This finishes the proof of the proposition. □

4.4. **The proof of Lemma 4.10.** We first establish the following fact, uniformly as $n \to +\infty$:

$$\Big|\mathrm{Im}\Big\{\mathrm{n}_\Gamma(\zeta)\,D^h_{n,\mathrm{I}}(\zeta[n^{-\beta}])\Big\}\Big| \ =\ \mathrm{O}(n^{\frac{1}{2}-\frac{1}{4}\beta}), \qquad \zeta \in \Gamma. \tag{4.13}$$

Proof. Without loss of generality, we may take $\zeta = 0$ and $\mathrm{n}_\Gamma(\zeta) = \mathrm{i}$. Then the tangent to Γ at 0 is horizontal, so Γ is the graph of a function $y = y(x)$ where $y(x) = \mathrm{O}(x^2)$ as $x \to 0$. We will show that

$$\Big|\mathrm{Re}\Big\{D^h_{n,\mathrm{I}}(\mathrm{i}n^{-\beta}) - D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})\Big\}\Big| \ = \mathrm{O}(n^{\frac{1}{2}-\frac{1}{4}\beta}). \tag{4.14}$$

This implies the desired estimate (4.13), because by Lemma 4.9, there exists a positive constant C_6 such that

$$|D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})| \le C_6 n^{3\beta}[\log n]^6 \le n^{\frac{1}{2}-\frac{1}{3}\beta},$$

where the right-hand side estimate is valid for big n, provided $\beta < \frac{3}{20}$. To obtain (4.14), we notice that

$$I := \mathrm{Re}\Big\{D^h_{n,\mathrm{I}}(\mathrm{i}n^{-\beta}) - D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})\Big\} = n\int_{\mathcal{B}_n}\mathrm{Re}\left\{\frac{1}{z+\mathrm{i}n^{-\beta}} - \frac{1}{z-\mathrm{i}n^{-\beta}}\right\}(u_n^h(z) - u(z))\,\mathrm{d}A(z).$$

We next subdivide the thin belt $\mathcal{B}_n$ into two parts:

$$\mathcal{B}_n^1 := \mathcal{B}_n \cap \{x+\mathrm{i}y :\ \max\{|x|,|y|\} \le n^{-\gamma}\}, \qquad \mathcal{B}_n^2 := \mathcal{B}_n \setminus \mathcal{B}_n^1,$$

where γ is a parameter with $0<\gamma<\beta<\frac{1}{2}$ (we have some freedom here). The part $\mathcal{B}_n^1$ is the local part, and $\mathcal{B}_n^2$ is the remainder. This allows us to split the integral I accordingly: $I=I^1+I^2$, where

$$I^1:=n\int_{\mathcal{B}_n^1}\mathrm{Re}\left\{\frac{1}{\bar z-\mathrm{i}n^{-\beta}}-\frac{1}{z-\mathrm{i}n^{-\beta}}\right\}(u_n^h(z)-u(z))\,\mathrm{d}A(z)$$

and

$$I^2:=n\int_{\mathcal{B}_n^2}\mathrm{Re}\left\{\frac{1}{z+\mathrm{i}n^{-\beta}}-\frac{1}{z-\mathrm{i}n^{-\beta}}\right\}(u_n^h(z)-u(z))\,\mathrm{d}A(z);$$

note that in the first formula, we use that for complex numbers ξ, we have $\mathrm{Re}\,\bar\xi=\mathrm{Re}\,\xi$. By Lemma 4.1, the function u_n^h-u is uniformly bounded, say $|u_n^h-u|\le C_7$, so that

$$|I^1|\le C_7\,n\int_{\mathcal{B}_n^1}\frac{2|\,\mathrm{Im}\,z|}{|(z-\mathrm{i}n^{-\beta})(z+\mathrm{i}n^{-\beta})|}\,\mathrm{d}A(z). \tag{4.15}$$

The analogous estimate involving I^2 reads:

$$|I^2|\le C_7\,n\int_{\mathcal{B}_n^2}\frac{2n^{-\beta}}{|(z-\mathrm{i}n^{-\beta})(z+\mathrm{i}n^{-\beta})|}\,\mathrm{d}A(z). \tag{4.16}$$

Since curve Γ is parametrized by $y=y(x)$ with $y(x)=\mathrm{O}(x^2)$ as $x\to 0$, we see that $|\,\mathrm{Im}\,z|=|y|=\mathrm{O}(n^{-2\gamma})$ on $\mathcal{B}_n^1$. Moreover, geometric considerations lead to

$$|(z-\mathrm{i}n^{-\beta})(z+\mathrm{i}n^{-\beta})|\gtrsim|\,\mathrm{Re}\,z|^2+n^{-2\beta}=x^2+n^{-2\beta},\qquad z=x+\mathrm{i}y\in\mathcal{B}_n^1,$$

and

$$|(z-\mathrm{i}n^{-\beta})(z+\mathrm{i}n^{-\beta})|\asymp|z|^2,\qquad z\in\mathcal{B}_n^2.$$

As we combine the above estimates with (4.15) and (4.16), and recall that $\mathcal{B}_n$ is a thin belt of width $\asymp\delta_n=n^{-1/2}[\log n]^2$ around $\Gamma=\partial S$, we realize that

$$|I^1|\lesssim n^{1-2\gamma}\delta_n\int_{-n^{-\gamma}}^{n^{\gamma}}\frac{\mathrm{d}t}{t^2+n^{-2\beta}}=n^{1+\beta-2\gamma}\delta_n\int_{-n^{\beta-\gamma}}^{n^{\beta-\gamma}}\frac{\mathrm{d}\tau}{1+\tau^2}\le\pi n^{\frac12+\beta-2\gamma}[\log n]^2 \tag{4.17}$$

and

$$|I^2|\lesssim n^{1-\beta}\delta_n\int_{n^{-\gamma}}^{1}\frac{\mathrm{d}t}{t^2}\asymp n^{1+\gamma-\beta}\delta_n=n^{\frac12+\gamma-\beta}[\log n]^2. \tag{4.18}$$

If we now pick $\gamma:=\frac{2}{3}\beta$, we obtain

$$I=I^2+I^2=\mathrm{O}\Big(n^{\frac12-\frac13\beta}[\log n]^2\Big)\quad\text{as }\ n\to+\infty,$$

which is even better than claimed. As a consequence, (4.13) follows. □

To finish the proof of Lemma 4.10, we let n_{Γ_n} be the exterior unit normal of Γ_n. By (4.12), n_{Γ_n} at the point $\zeta[n^{-\beta}]\in\Gamma_n$ for $\zeta\in\Gamma$ is the same as $\mathrm{n}_\Gamma(\zeta)$. So, from (4.13) we may derive that

$$|\,\mathrm{Im}[\mathrm{n}_{\Gamma_n}D^h_{n,\mathrm{I}}]|=\mathrm{O}(n^{\frac12-\frac14\beta}),\qquad\text{as }\ n\to+\infty, \tag{4.19}$$

uniformly on Γ_n. Next, let $\mathbb{D}^e:=\{z:\,|z|>1\}$ denote the exterior disk, and consider the conformal map

$$\phi_n:\Omega_n^{\circledast}\to\mathbb{D}^e,$$

which fixes the point at infinity ($\phi_n(\infty)=\infty$). We put

$$G_n^h:=\frac{\phi_n}{\phi_n'}\,D^h_{n,\mathrm{I}}.$$

Being the Cauchy transform of a density supported in $\mathcal{B}_n$, the function $D^h_{n,\mathrm{I}}(z)$ is holomorphic in $\mathbb{C}\setminus\mathcal{B}_n$, with decay rate $\mathrm{O}(|z|^{-1})$ as $|z|\to+\infty$. More precisely,

$$D^h_{n,\mathrm{I}}(z)=\frac{n}{z}\int_{\mathcal{B}_n}(u^h_n-u)\mathrm{d}A+\mathrm{O}(n|z|^{-2})\quad\text{as }\ |z|\to+\infty. \tag{4.20}$$

The quotient $\phi_n(z)/\phi_n'(z)$ grows like $z+\mathrm{O}(1)$ as $|z|\to+\infty$, so the function G^h_n gets to be bounded near infinity. Since $\Omega_n^{\circledast}\subset\mathbb{C}\setminus\mathcal{B}_n$ for big n, G^h_n is holomorphic and bounded in $\Omega^{\circledast}_n$ for big n. The conformal mappings ϕ_n are uniformly smooth in $\Omega^{\circledast}_n$, as a consequence of the smoothness assumptions on Γ which lead to the corresponding uniform smoothness of Γ_n. In particular, we have $|\phi_n'|\asymp 1$ in $\Omega^{\circledast}_n$ uniformly in n for big n. The Green function for the point at infinity in $\Omega^{\circledast}_n$ may be expressed as $\log|\phi_n|$, and at the boundary Γ_n its gradient points in the outward normal direction. This means that the outward unit normal n_{Γ_n} is

$$\mathrm{n}_{\Gamma_n}(z)=\frac{\bar\partial\log|\phi_n(z)|}{|(\bar\partial\log|\phi_n(z)|)|}=\frac{\bar\phi_n'(z)|\phi_n(z)|}{\bar\phi_n(z)|\phi_n'(z)|}=\frac{\phi_n(z)|\phi_n'(z)|}{\phi_n'(z)|\phi_n(z)|},\qquad z\in\Gamma_n.$$

It follows that

$$\mathrm{Im}[\mathrm{n}_{\Gamma_n}D^h_{n,\mathrm{I}}]=\frac{|\phi_n'|}{|\phi_n|}\,\mathrm{Im}[G^h_n]\quad\text{on }\ \Gamma_n,$$

so we see that (4.19) asserts that

$$\big\|\,\mathrm{Im}[G^h_n]\big\|_{L^\infty(\Gamma_n)}=\mathrm{O}(n^{\frac12-\frac14\beta}),\quad\text{as }\ n\to+\infty. \tag{4.21}$$

Moreover,

$$G^h_n(\infty)=\mathrm{O}(1),\quad\text{as }\ n\to+\infty, \tag{4.22}$$

as we easily see from the decay information (4.20) and from the identity

$$n\int_{\mathcal{B}_n}(u^h_n-u)\mathrm{d}A=\int_{S\setminus\mathcal{B}_n}(\mathrm{K}^\sharp_{n,h}-\mathrm{K}^h_n)\,\mathrm{d}A-\int_{\mathbb{C}\setminus(S\cup\mathcal{B}_n)}\mathrm{K}^h_n\mathrm{d}A,$$

if we recall the estimate (4.2) and Theorem 4.2. The harmonic conjugation operator is bounded in the setting of H^p spaces on the unit disk (or on the exterior disk if we like), and as we transfer this result to the context of $\Omega^{\circledast}_n$, we get from (4.21) that

$$\|G^h_n\|_{L^p(\Gamma_n)}\le C_p\|\,\mathrm{Im}\,G_n\|_{L^p(\Gamma_n)}+\mathrm{O}(1)=\mathrm{O}_p(n^{\frac12-\frac14\beta}),\quad\text{as }\ n\to+\infty,$$

for any fixed p, $1<p<+\infty$, where we use a subscript to indicate the dependence on p. The special case $p=2$ gives us the assertion of Lemma 4.10. □

5. Proof of the main formula

In this section we will use the limit form of the Ward identity (Corollary 3.5) to derive our main formula (2.13): for every test function f the limit

$$\langle f,\nu^h\rangle_{\mathbb{C}}:=\lim_{n\to+\infty}\langle f,\nu^h_n\rangle_{\mathbb{C}}$$

exists and equals

$$\langle f,\nu^h\rangle_{\mathbb{C}}=\frac{1}{8\pi}\left\{\int_S(\Delta f+f\Delta L)\mathrm{d}A+\int_{\partial S}f\mathcal{N}(L^S)\mathrm{d}s\right\}+\frac{1}{2\pi}\int_{\mathbb{C}}\nabla f^S\cdot\nabla h^S\mathrm{d}A. \tag{5.1}$$

5.1. **Decomposition of the test function.** The following statement uses our assumption that ∂S is a C^ω-smooth Jordan curve.

Lemma 5.1. *Let $f \in C^\infty(\mathbb{C})$ be bounded. Then f has the following representation:*

$$f = f_+ + f_- + f_0,$$

where

(i) *all three functions are C^∞-smooth and bounded in $\mathbb{C}$,*

(ii) $\bar\partial f_+ = 0$ *and* $\partial f_- = 0$ *in* $\mathbb{C} \setminus S$,

(iii) $f_0 = 0$ *on* $\Gamma = \partial S$.

Proof. Let us consider a conformal map

$$\phi : \mathbb{D}^e \to \mathbb{C} \setminus S,$$

which preserves the point at infinity; here, $\mathbb{D}^e = \{z \in \mathbb{C} : |z| > 1\}$ is the exterior disk. The smoothness assumptions on Γ imply that ϕ is extremely smooth (e.g., real-analytic on $\mathbb{T}$). The restriction of the function $F := f \circ \phi$ to $\mathbb{T}$ is in $C^\infty(\mathbb{T})$, and so it has a Fourier series representation

$$F(\zeta) = \sum_{j=-\infty}^{+\infty} a_j \zeta^j, \qquad \zeta \in \mathbb{T}.$$

The functions

$$F_+(z) = \sum_{j=0}^{+\infty} a_{-j} z^{-j}, \qquad F_-(z) = \sum_{j=1}^{+\infty} \frac{a_j}{\bar z^j},$$

are then well-defined in the closed exterior disk $\bar{\mathbb{D}}^e$, and C^∞-smooth up to the boundary. It is easy to extend F_+, F_- to C^∞-smooth functions on all of $\mathbb{C}$. Likewise, we may extend ϕ to C^∞-smooth diffeomorphism $\phi : \mathbb{C} \to \mathbb{C}$. We put

$$f_+ := F_+ \circ \phi^{-1}, \qquad f_- := F_- \circ \phi^{-1},$$

and realize that f_+, f_- are both C^∞-smooth and bounded, and that (ii) holds. Finally, we put

$$f_0 := f - f_+ - f_-.$$

It is automatic that f_0 is C^∞-smooth and bounded, and that f_0 vanishes on $\Gamma = \partial S$. □

Conclusion. It is enough to prove the main formula (5.1) only for functions of the form $f = f_+ + f_- + f_0$ as in the last lemma with an *additional* assumption that f_0 is supported inside any given neighborhood of the droplet S. Indeed, either side of the formula (5.1) will not change if we "kill" f_0 outside the neighborhood. The justification is immediate by Lemma 4.1 (exterior decay).

In what follows we will choose a neighborhood O of S such that the potential Q is real-analytic, strictly subharmonic in O, and

$$\partial Q \neq \partial \check{Q} \quad \text{in} \quad O \setminus S,$$

and will assume $\mathrm{supp}(f_0) \subset O$.

5.2. **The choice of the vector field in the Ward identity.** We will now compute the limit

$$\langle f, \nu^h \rangle_{\mathbb{C}} := \lim_{n\to+\infty} \langle f, \nu_n^h \rangle_{\mathbb{C}}$$

(and prove its existence) in the case where

$$f = f_+ + f_0.$$

To apply the limit Ward identity (see Corollary 3.5)

$$\frac{2}{\pi}\int_{\mathbb{C}} \left\{ v\Delta Q + \bar\partial v\, \partial(Q - \check{Q}) \right\} D_n^h \mathrm{d}A \;\to\; -\langle \tfrac{1}{2}\partial v + 2v\partial h, \sigma\rangle_{\mathbb{C}} \quad \text{as } n \to +\infty, \tag{5.2}$$

we set

$$v = v_+ + v_0,$$

where

$$v_0 = \frac{\bar\partial f_0}{\Delta Q}\, 1_S + \frac{f_0}{\partial(Q-\check{Q})}\, 1_{\mathbb{C}\setminus S}, \tag{5.3}$$

and

$$v_+ = \frac{\bar\partial f_+}{\Delta Q}\, 1_S. \tag{5.4}$$

Here, we need the additional assumption made on the support of f_0. We may combine the above to

$$v = \frac{\bar\partial f}{\Delta Q}\, 1_S + \frac{f_0}{\partial(Q-\check{Q})}\, 1_{\mathbb{C}\setminus S}.$$

We calculate that

$$v\Delta Q + \bar\partial v\, \partial(Q-\check{Q}) = \bar\partial f \quad \text{on } \mathbb{C}\setminus\partial S,$$

but to plug this information into (5.2), we need to that it is an identity in the sense of distribution theory on all of $\mathbb{C}$. This will be all right if, e.g., v is Lipschitz-continuous near ∂S. We would then also need to know that v satisfies the conditions (3.4-*i*)–(3.4-*iii*).

Lemma 5.2. *The vector field v defined above is bounded and globally Lipschitz-continuous. Moreover, the restrictions of v to S and to $S^{\circledast} := (\mathbb{C}\setminus S)\cup\partial S$ are both C^∞-smooth.*

Proof. The vector field v_+ is C^∞-smooth and supported on S, as $\bar\partial f_+$ is C^∞-smooth and supported on S, and $\Delta Q \neq 0$ on S. It remains to handle the vector field v_0. We need to check the following items:

(*i*) $v_0|_S$ and $v_0|_{S^{\circledast}}$ are both C^∞-smooth, and

(*ii*) v_0 is continuous across ∂S.

Proof of (*i*). It is clear from the defining formula that $v_0|_S$ is C^∞-smooth. As for $v_0|_{S^{\circledast}}$, we have $v_0 = f_0/g$ in $\mathbb{C}\setminus S$ where $g = \partial(Q-\check{Q})$. Since the statement is local, we consider a conformal map ψ that takes a neighbourhood of a boundary point in ∂S onto a neighbourhood of a point in $\mathbb{R}$ and takes (parts of) ∂S to $\mathbb{R}$. We fix the map so that (locally) $S^{\circledast}$ is mapped into the upper half plane $y \geq 0$. If we denote $F = f_0 \circ \psi$ and $G = g \circ \psi$, then $F = 0$ and $G = 0$ on $\mathbb{R}$. Moreover, locally,

G is the restriction to $y \ge 0$ of a real-analytic function, with non-vanishing partial derivative G'_y. Thus it is enough to check that

$$\frac{F(x+\mathrm{i}y)}{y} = \int_0^1 \frac{\partial F}{\partial y}(x+\mathrm{i}y\tau)\,\mathrm{d}\tau$$

has bounded derivatives of all orders. But this is pretty obvious, since we may differentiate under the integral sign.

Proof of (*ii*). Let $\mathrm{n} = \mathrm{n}_\Gamma(\zeta)$ be the exterior unit normal with respect to S. By Taylor's formula, we have

$$f_0(\zeta+\delta\mathrm{n}) = \delta\frac{\partial f_0}{\partial \mathrm{n}}(\zeta) + \mathrm{O}(\delta^2) = 2\delta\,\bar\partial f_0(\zeta)\,\overline{\mathrm{n}(\zeta)} + \mathrm{O}(\delta^2).$$

Similarly, if $g := \partial(Q-\check{Q})$ on $\mathbb{C}\setminus S$, then g extends to a real-analytically smooth function on $S^{\circledast}$. This real-analytic function has a unique extension to a neighborhood of $S^{\circledast}$, which we also denote by g. Since $g = 0$ on ∂S and $\bar\partial g = \Delta Q$ in $\mathbb{C}\setminus S$, Taylor's formula gives that

$$g(\zeta+\delta\mathrm{n}) = \delta\frac{\partial g}{\partial \mathrm{n}}(\zeta) + \mathrm{O}(\delta^2) = 2\delta\,\bar\partial g(\zeta)\,\overline{n(\zeta)} + \mathrm{O}(\delta^2).$$

It follows that

$$\frac{f_0(\zeta+\delta\mathrm{n})}{g(\zeta+\delta\mathrm{n})} = \frac{\bar\partial f_0(\zeta)}{\Delta Q(\zeta)} + \mathrm{O}(\delta),$$

and an inspection shows that the implied constant is locally uniform in $\zeta \in \Gamma = \partial S$. This shows that v_0 is continuous along Γ. □

We have now established that the vector field $v = v_0 + v_+$ meets the conditions (3.4-*i*)–(3.4-*iii*). This gives us the following result.

Corollary 5.3. *If* $f = f_0 + f_+$, *then*

$$\langle f, \nu^h\rangle_{\mathbb{C}} = \frac{1}{4}\langle \partial v, \sigma\rangle_{\mathbb{C}} + \langle v\partial h, \sigma\rangle_{\mathbb{C}}.$$

Proof. The conclusion is immediate from (5.2) and (3.11). □

5.3. **The conclusion of the proof.**

5.3.1. (*a*). We now turn to the general case

$$f = f_+ + f_0 + f_-.$$

In view of Corollary 5.3, we have

$$\langle f_+, \nu^h\rangle_{\mathbb{C}} = \frac{1}{4}\langle \partial v_+, \sigma\rangle_{\mathbb{C}} + \langle v_+\partial h, \sigma\rangle_{\mathbb{C}},$$

where v_+ is given by (5.4). By applying complex conjugation to this relation, while using the fact that the perturbation h is real-valued and the measures ν_n^h are all real-valued (and so the limit ν^h is real-valued, too), we get a similar expression for f_-:

$$\langle f_-, \nu^h\rangle_{\mathbb{C}} = \frac{1}{4}\langle \bar\partial v_-, \sigma\rangle_{\mathbb{C}} + \langle v_-\bar\partial h, \sigma\rangle_{\mathbb{C}},$$

where

$$v_- := \frac{\partial f_-}{\partial\bar\partial Q} \cdot 1_S. \tag{5.5}$$

Adding up the three contributions from f_+, f_-, f_0, we find that

$$\langle f, \nu^h\rangle_{\mathbb{C}} = \frac{1}{4}\langle \partial v_0, \sigma\rangle_{\mathbb{C}} + \langle v_0 \partial h, \sigma\rangle_{\mathbb{C}} + \frac{1}{4}\langle \partial v_+, \sigma\rangle_{\mathbb{C}} + \langle v_+ \partial h, \sigma\rangle_{\mathbb{C}} + \frac{1}{4}\langle \bar\partial v_-, \sigma\rangle_{\mathbb{C}} + \langle v_- \bar\partial h, \sigma\rangle_{\mathbb{C}}.$$

The expression we get when we put $h = 0$ is $\nu = \nu^0$, so that

$$\langle f, \nu\rangle_{\mathbb{C}} = \frac{1}{4}\langle \partial v_0 + \partial v_+ + \bar\partial v_-, \sigma\rangle_{\mathbb{C}} \tag{5.6}$$

and

$$\langle f, \nu^h - \nu\rangle_{\mathbb{C}} = \langle v_0 \partial h + v_+ \partial h + v_- \bar\partial h, \sigma\rangle_{\mathbb{C}}. \tag{5.7}$$

5.3.2. (*b*) *The computation of ν.* We recall that

$$\mathrm{d}\sigma = \frac{1}{2\pi} 1_S \, \Delta Q \, \mathrm{d}A = \frac{2}{\pi} 1_S \, \Delta Q \, \mathrm{d}A \quad \text{and} \quad L = \log \Delta Q.$$

Using (5.6), we compute

$$\begin{aligned}
\langle f, \nu\rangle_{\mathbb{C}} &= \frac{1}{2\pi}\int_S \left\{ \partial\left(\frac{\bar\partial f_0 + \bar\partial f_+}{\partial\bar\partial Q}\right) + \bar\partial\left(\frac{\partial f_-}{\partial\bar\partial Q}\right)\right\} \Delta Q \, \mathrm{d}A \\
&= \frac{1}{2\pi}\int_S \left\{ \Delta(f_0 + f_+ + f_-) - \bar\partial f_0 \, \partial \log \Delta Q - \bar\partial f_+ \partial \log \Delta Q - \partial f_- \bar\partial \log \Delta Q \right\} \mathrm{d}A \\
&= \frac{1}{2\pi}\int_S \left\{ \Delta f - \bar\partial f_0 \, \partial L - \bar\partial f_+ \, \partial L - \partial f_- \, \bar\partial L \right\} \mathrm{d}A.
\end{aligned}$$

At this point, we modify L outside some neighborhood of S to get a smooth function with compact support. We will still use the notation L for the modified function. The last expression clearly does not change as a result of this modification. We can now transform the part of the integral which involves L as follows using the Cauchy-Green formula:

$$\begin{aligned}
-\int_S \left\{ \bar\partial f_0 \, \partial L + \bar\partial f_+ \, \partial L + \partial f_- \, \bar\partial L \right\} \mathrm{d}A &= \int_S f_0 \, \Delta L \, \mathrm{d}A + \int_{\mathbb{C}} \left\{ \bar\partial f_+ \, \partial L + \partial f_- \, \bar\partial L \right\} \mathrm{d}A \\
&= \int_S f_0 \, \Delta L \, \mathrm{d}A + \int_{\mathbb{C}} (f_+ + f_-) \, \Delta L \, \mathrm{d}A = \int_S f \, \Delta L \, \mathrm{d}A + \int_{\mathbb{C}\setminus S} f^S \, \Delta L \, \mathrm{d}A.
\end{aligned}$$

In other words, we have that

$$\langle f, \nu\rangle_{\mathbb{C}} = \frac{1}{8\pi}\left\{ \int_S (\Delta f + f \Delta L) \, \mathrm{d}A + \int_{\mathbb{C}\setminus S} f^S \Delta L \, \mathrm{d}A \right\}.$$

We remark that the formula for $\langle f, \nu\rangle_{\mathbb{C}}$ was stated in this form in [3].

Finally, we express the last integral in terms of the Neumann jump. By Green's formula, we have

$$\begin{aligned}\int_{\mathbb{C}\setminus S} f^S \Delta L \, \mathrm{d}A = \int_{\mathbb{C}\setminus S} \left(f^S \Delta L - L\,\Delta f^S\right) \mathrm{d}A = \int_{\partial S} \left(f^S \frac{\partial L}{\partial \mathrm{n}^{\circledast}} - L^S \frac{\partial f^S}{\partial \mathrm{n}^{\circledast}}\right) \mathrm{d}s \\ = \int_{\partial S} \left(f^S \frac{\partial L}{\partial \mathrm{n}^{\circledast}} - f^S \frac{\partial L^S}{\partial \mathrm{n}^{\circledast}}\right) \mathrm{d}s = \int_{\partial S} f \mathcal{N}(L^S)\, \mathrm{d}s,\end{aligned}$$

where the last step just involved the definition of the Neumann jump. In an intermediate step we used that

$$\int_{\partial S} \left(f^S \frac{\partial L^S}{\partial \mathrm{n}^{\circledast}} - L^S \frac{\partial f^S}{\partial \mathrm{n}^{\circledast}}\right) \mathrm{d}s = \int_{\mathbb{C}\setminus S} (f^S \Delta L^S - L^S \Delta f^S) \mathrm{d}A = 0,$$

which comes from Green's formula. Here, $\mathrm{n}^{\circledast}$ is the unit normal vector which point into S. In conclusion, we arrive at

$$\langle f, \nu \rangle_{\mathbb{C}} = \frac{1}{8\pi} \left\{ \int_S (\Delta f + f \Delta L)\, \mathrm{d}A + \int_{\partial S} f \mathcal{N}(L^S) \mathrm{d}s \right\}. \tag{5.8}$$

5.3.3. (*c*) *The computation of* $\nu^h - \nu$. Using the identity (5.7), we can deduce that

$$\langle f, \nu^h - \nu \rangle_{\mathbb{C}} = \frac{2}{\pi} \int_S \left\{ \bar\partial f_+ \partial h + \partial f_- \bar\partial h + \bar\partial f_0 \partial h \right\} \mathrm{d}A = \frac{1}{2\pi} \int \nabla f^S \cdot \nabla h^S \, \mathrm{d}A. \tag{5.9}$$

This is because of the following calculations:

$$\int_S \bar\partial f_+ \partial h \, \mathrm{d}A = \int_{\mathbb{C}} \bar\partial f_+ \partial h \, \mathrm{d}A = -\frac{1}{4} \int_{\mathbb{C}} f_+ \Delta h \, \mathrm{d}A = \frac{1}{4} \int_{\mathbb{C}} \nabla f_+ \cdot \nabla h \, \mathrm{d}A,$$

and analogously

$$\int_S \partial f_- \bar\partial h \, \mathrm{d}A = \frac{1}{4} \int_{\mathbb{C}} \nabla f_- \cdot \nabla h \, \mathrm{d}A;$$

moreover, on the other hand, we have

$$\int_S \bar\partial f_0 \partial h \, \mathrm{d}A = -\frac{1}{4} \int_S f_0 \Delta h \, \mathrm{d}A = \frac{1}{4} \int_S \nabla f_0 \cdot \nabla h \, \mathrm{d}A.$$

The above three identities lead to

$$\langle f, \nu^h - \nu \rangle_{\mathbb{C}} = \frac{1}{2\pi} \left\{ \int_S \nabla f \cdot \nabla h \, \mathrm{d}A + \int_{\mathbb{C}\setminus S} \nabla f^S \cdot \nabla h \, \mathrm{d}A \right\},$$

and if we use that

$$\int_{\mathbb{C}\setminus S} \nabla f^S \cdot \nabla h \, \mathrm{d}A = \int_{\mathbb{C}\setminus S} \nabla f^S \cdot \nabla h^S \, \mathrm{d}A,$$

which is a consequence of the fact that harmonic functions minimize the Dirichlet norm, we arrive at (5.9) right away. By combining (5.8) and (5.9), we see that

$$\langle f, \nu^h \rangle_{\mathbb{C}} = \frac{1}{8\pi} \left\{ \int_S (\Delta f + f \Delta L)\, \mathrm{d}A + \int_{\partial S} f \mathcal{N}(L^S)\, \mathrm{d}s \right\} + \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S \, \mathrm{d}A,$$

and the main formula (5.1) has been completely established. □

Appendix 1: The proof of Theorem 4.2

5.4. **Polynomial Bergman spaces.** We consider a potential Q, which is C^∞-smooth on some neighborhood of the droplet $S = S_Q$, with $\Delta Q > 0$ on S, and subject to the usual growth condition (2.1). We denote by $\mathrm{k}_n = \mathrm{k}_{n,Q}$ the reproducing kernel for the space $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$, which is given explicitly in the form

$$\mathrm{k}_n(z,w) = \sum_{j=0}^{n-1} p_j(z)\bar{p}_j(w),$$

where the polynomial p_j is of degree j, is orthogonal to all the polynomials of lower degree than j, and is normalized to have unit length in $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$. The corresponding correlation kernel is the function

$$\mathrm{K}_n(z,w) = \mathrm{k}_n(z,w)\,\mathrm{e}^{-n[Q(z)+Q(w)]}.$$

The following result was obtained in [1] based on the approach developed in [7], assuming that Q was real-analytically smooth near S [the statement in [1] involves an asymptotic expansion, and here, we are considering only the main term of that expansion]. We recall the definition (4.3) of the approximate kernel $K_n^\sharp$ (with $h = 0$):

$$\mathrm{K}_n^\sharp(z,w) \;=\; \frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\,\mathrm{e}^{n[2Q(z,\bar{w})-Q(z)-Q(w)]}. \tag{5.10}$$

Theorem 5.4. *Suppose that $z, w \in S$, with $\delta_{\partial S}(z) > 2\delta_n$ and $|z-w| < \delta_n$. Then*

$$|\mathrm{K}_n(z,w) - \mathrm{K}_n^\sharp(z,w)| \;=\; \mathrm{O}(1),$$

where the implied constant in O(1) *depends on Q, but not on n.*

We make three observations which will be important to us.

(*a*) Firstly, we see from the proof in [1] (cf. [2], [7]) that the assertion remains valid if the assumption $\delta_{\partial S}(z) > 2\delta_n$ is relaxed to $\delta_{\partial S}(z) > \frac{3}{2}\delta_n$.

(*b*) Secondly, we note that the smoothness assumption on Q is excessive for the result to hold. The real-analyticity assumption is just a matter of convenience, as the it allows us to define the *lift* of Q (also denoted by Q) via locally convergent power series. Here, the lift of Q is the function $Q(z,w)$ which is holomorphic in (z,w) near the conjugate diagonal $z = \bar{w}$, with the defining property

$$Q(z,\bar{z}) = Q(z).$$

It is explained in, e.g., [7] how to define the lift if Q is only C^∞-smooth, in terms of almost holomorphic extensions. So, with very little effort, Theorem 5.4 holds if Q is only C^∞-smooth in a neighborhood of S. What is more essential is that $\Delta Q > 0$ is assumed on S.

(*c*) Thirdly, the way the implied constant in Theorem 5.4 depends on Q is important. Here, what we need is that it can be assumed locally uniform in Q if the weight Q gets perturbed slightly. Again, this is easy to check, as only the behavior of the first few partial derivatives of Q near the given point $z \in S$ play a role for the estimate.

5.5. **The perturbed polynomial correlation kernel.** We now consider the perturbed potential $Q_n^h := Q - \frac{1}{n}h$, where h is a compactly supported C^∞-smooth test function, and n is big. In view of the above comments (a)–(c), we may apply Theorem 5.4 to this perturbed potential. Let $\check{Q}_n^h$ denote the maximal subharmonic function $\le Q_n^h$ which grows as $\log|z| + \mathrm{O}(1)$ when $|z| \to +\infty$. If we consider the coincidence set

$$S_{n,h}^* := \{z \in \mathbb{C} : \check{Q}_n^h(z) = Q_n^h(z)\},$$

the corresponding support $S_{n,h}$ of the Frostman measure is obtained by removing all point from $S_{n,h}^*$ which are Q_n^h-shallow, that is, have a neighborhood such that $1_{S_{n,h}^*}\Delta Q$ vanishes a.e. there (cf. [11]). From the uniform version of Theorem 5.4 outlined above ((a)–(c)), we get that if K_n^h denotes the polynomial correlation kernel associated with the perturbed potential Q_n^h, we find that

$$|\mathrm{K}_n^h(z,w) - \mathrm{K}_{n,h}^\#(z,w)| = \mathrm{O}(1),$$

provided that $z, w \in S_{n,h}$ with $\delta_{\partial S_{n,h}}(z) > \frac{3}{2}\delta_n$ and $|z - w| < \delta_n$. Here, the approximate correlation kernel $\mathrm{K}_{n,h}^\#$ is as in (4.3). It remains for us to convince ourselves that if $\delta_{\partial S}(z) > 2\delta_n$ holds, then $\delta_{\partial S_{n,h}}(z) > \frac{3}{2}\delta_n$ holds as well, for big n. That is to say, we need to know that the boundary $\partial S_{n,h}$ cannot move too far into the interior of the droplet S, for big n. In this setting, we have Proposition 3.1 of [12], which asserts that the maximal distance that the boundary $\partial S_{n,h}$ can move in the direction interior to the droplet S is $\mathrm{O}(n^{-1/2})$. Since $\mathrm{O}(n^{-1/2}) = \mathrm{o}(\delta_n)$ as $n \to +\infty$, we then have

$$\delta_{\partial S}(z) > 2\delta_n \implies \delta_{\partial S_{n,h}}(z) > \frac{3}{2}\delta_n$$

for big n, and the assertion Theorem 4.2 follows.

5.6. **The perturbed obstacle problem.** Strictly speaking, the formulation of Proposition 3.1 in [12] is not adapted to our present setting, so we should say some words about why the statement applies here. Let $\tilde{Q}_n^h := \check{Q} - \frac{1}{n}h_n$, where h_n is defined to equal h in $S \setminus C_n$, and to equal $h^S + An^{-1/2}$ in $\mathbb{C} \setminus S$, for a suitable positive constant A. Here, C_n is the interior band

$$C_n := \{z \in S : \delta_{\partial S}(z) < n^{-1/2}\}$$

around ∂S. Now, if A is big enough, we can use (4.1) to see that

$$\tilde{Q}_n^h = \check{Q} - \frac{1}{n}h_n = \check{Q} - \frac{1}{n}h^S - \frac{A}{n^{3/2}} \le Q - \frac{1}{n}h = Q_n^h \quad \text{on } \mathbb{C} \setminus S.$$

On the other hand,

$$\tilde{Q}_n^h = \check{Q} - \frac{1}{n}h_n = \check{Q} - \frac{1}{n}h = Q_n^h \quad \text{on } S \setminus C_n.$$

We would like to have the following properties globally:

$$\tilde{Q}_n^h \le Q_n^h \quad \text{and} \quad \Delta\tilde{Q}_n^h \ge 0, \tag{5.11}$$

for then we could assert that $\check{Q}_n^h = Q_n^h$ holds on $S \setminus C_n$, so that $S \setminus C_n \subset S_{n,h}^*$, and in a second step, that $S \setminus C_n \subset S_{n,h}$. To accomplish (5.11), we need to require that $h_n \ge h$ on C_n and that $\Delta h_n \le n\Delta Q$ on S. Let $h^{(S)}$ be a C^∞-smooth real-valued global extension of the restriction of h^S to $\mathbb{C} \setminus S$. Moreover, let $\chi_n \in C^\infty(\mathbb{C})$ be a "cut-off" function with $0 \le \chi_n \le 1$ globally, while $\chi_n = 1$ on $S \setminus C_n$ and $\chi_n = 0$ on $\mathbb{C} \setminus S$. We can find such a χ_n with $|\nabla\chi_n| \lesssim n^{1/2}$ and $|\nabla^2\chi_n| \lesssim n$. Here, $\nabla^2 f$ is the 2×2 matrix of all the second order partial derivatives (the Hessian). We put

$$h_n := \chi_n h + (1 - \chi_n)(h^{(S)} + An^{-1/2}) = \chi_n(h - h^{(S)}) + h^{(S)} + An^{-1/2}(1 - \chi_n),$$

and observe that $h_n \geq h$ holds if

$$h - h^{(S)} \leq An^{-1/2} \quad \text{on } C_n,$$

which we easily fulfill but jacking up the constant A, since $h - h^{(S)} = \mathrm{O}(\delta_{\partial S}) = \mathrm{O}(n^{-1/2})$ on C_n. As for the Laplacian control, we have

$$\Delta h_n = (h - h^S)\Delta\chi_n + \nabla\chi_n \cdot \nabla(h - h^S) + \chi_n\Delta(h - h^S) + \Delta h^{(S)} - An^{-1/2}\Delta\chi_n,$$

which leads to

$$\Delta h_n = \mathrm{O}(n^{1/2}) \quad \text{as } n \to +\infty,$$

uniformly on C_n. So, for big n, we are assured that $\Delta h_n \leq n\Delta Q$ holds on S.

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AMEUR: DEPARTMENT OF MATHEMATICS, LULEÅ UNIVERSITY OF TECHNOLOGY, 971 87 LULEÅ, SWEDEN
E-mail address: yacin.ameur@gmail.com

HEDENMALM: DEPARTMENT OF MATHEMATICS, THE ROYAL INSTITUTE OF TECHNOLOGY, S – 100 44 STOCKHOLM, SWEDEN
E-mail address: haakanh@math.kth.se

MAKAROV: MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA
E-mail address: makarov@caltech.edu