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**An elementary proof of the vanishing of the
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algebra with values in the adjoint module**

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AN ELEMENTARY PROOF OF THE VANISHING OF THE SECOND COHOMOLOGY OF THE WITT AND VIRASORO ALGEBRA WITH VALUES IN THE ADJOINT MODULE

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ABSTRACT. By elementary and direct calculations the vanishing of the (algebraic) second Lie algebra cohomology of the Witt and the Virasoro algebra with values in the adjoint module is shown. This yields infinitesimal and formal rigidity of these algebras. The first (and up to now only) proof of this important result was given 1989 by Fialowski in an unpublished note. It is based on cumbersome calculations. Compared to the original proof the presented one is quite elegant and considerably simpler.

1. INTRODUCTION

Everywhere where symmetries of a system play a role, Lie groups and their infinitesimal versions, the Lie algebras, appear. By the use of the symmetry the system is better tractable, or even “solvable”. Symmetries with continuous but nevertheless finitely many degrees of freedom are related to finite-dimensional Lie groups and Lie algebras. Systems with an infinite number of independent symmetries appear in the theory of partial differential equations, conformal field theory, fluid dynamics, and at many other places in- and outside of mathematics. Consequently, the appearing Lie groups and Lie algebras are infinite dimensional. The simplest nontrivial infinite dimensional Lie algebras are the Witt algebra and its central extension the Virasoro algebra. The Witt algebra is related to the Lie algebra of the group of diffeomorphisms of the unit circle. The central extension comes into play as one is typically forced to consider projective actions if one wants to quantize a classical system or wants to regularize a field theory. There is a huge amount of literature about the application of these algebras. Here it is not the place to give even a modest overview about these applications. I will only make a reference to its appearance in conformal field theory [1], [15].

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The Witt algebra \mathcal{W} is the graded Lie algebra generated as vector space by the elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}.$$

The Virasoro algebra \mathcal{V} is a nontrivial one-dimensional central extension of \mathcal{W} . In fact it is the universal central extension. Detailed definitions and descriptions are given in Section 2. The goal of this article is to give an elementary proof that the second Lie algebra cohomology of both algebras with values in the adjoint modules will vanish. The result will be stated in Theorem 3.1 below. As explained in Section 3 these cohomology spaces are related to deformations of the algebras. In particular, if they vanish the algebras will be infinitesimal and formally rigid. Note that this does not mean that the algebras are analytically or geometrically rigid. Indeed, together with Alice Fialowski we showed in [5] that there exist natural, geometrically defined, nontrivial families of Lie algebras given by Krichever-Novikov type algebras [11], [13] associated to elliptic curves. These families appeared already in [14]. In these families the special fiber is the Witt (resp. Virasoro) algebra but all other fibers are non-isomorphic to it. Similar results for the current, resp. affine Lie algebras can be found in [6], [7].

The result on the vanishing of the second Lie algebra cohomology of the Witt algebra should clearly be attributed to Fialowski. There exists an unpublished manuscript [2] by her, dating from 1989, where she does explicit calculations. These calculations were quite cumbersome not really appealing for journal publication. Later, in the above-mentioned joint paper with her [5] we presented a sketch of a proof avoiding such kind of calculations. It is based on the calculations of the cohomology of the Lie algebra $Vect(S^1)$ of vector fields on S^1 with values in the adjoint module. Based on results of Tsujishita [16], Reshetnikov [12], and Goncharova [10] we showed that $H^*(Vect(S^1), Vect(S^1)) = \{0\}$. We argued that by density arguments the vanishing of the cohomology of the Witt algebra will follow. In a recent attempt to make the sketch of the density argument more precise we could only find a detailed proof by assuming that most of the cocycle values for the Witt algebra vanish. Hence, there was an incentive to return to a direct and elementary proof. Indeed, I found a very elementary, computational, but nevertheless reasonable short and elegant proof which is much simpler than Fialowski's original calculations [2]. As the Witt and Virasoro algebra are of fundamental importance inside of mathematics and in the applications, and up to now there is no complete published proof, the presented proof is for sure worthwhile to publish. The proof avoids the heavy machinery of Tsujishita, Reshetnikov, and Goncharova.¹ Of course, I do not exclude that there might be a way to make the density argument finally work without explicit calculations, but it is not necessary at all.

In Section 2 we give the definition of both the Witt and Virasoro algebra and make some remarks on their graded structure. The graded structure will play an important role in the article.

In Section 3 after recalling the definition of general Lie algebra cohomology the 2-cohomology of a Lie algebra with values in the adjoint module is considered in more detail. Some facts about its relation to deformations of the algebra are given to allow to judge the

¹As Fuks writes in his book [8, p.119] "Theorem 2.3.1 [Goncharova's theorem] still remains one of the most difficult theorems in the cohomology theory of infinite-dimensional Lie algebras."

importance of this cohomology. In particular, it is explained why the vanishing of it yields infinitesimal and formal rigidity (saying that all infinitesimal and formal deformations are trivial). The main result about the vanishing of the second cohomology and the rigidity for the Witt and Virasoro algebra is formulated in Theorem 3.1. The cohomology considered in this article is algebraic cohomology without any restriction on the cocycles.

In Section 4 we use the graded structure of the algebras to decompose their cohomology into graded pieces. It is quite easy then to show that for degree $d \neq 0$, the degree d parts will vanish. Hence, everything is reduced to degree zero. This is due to the fact that the grading is induced by the action of a special element. Here it is the element e_0 .

The vanishing of the degree zero part is more involved. It will be presented for the Witt algebra in Section 5. It is the computational core of the article, but the computations are elementary.

In Section 6 we extend this to the Virasoro algebra \mathcal{V} . We show that the vanishing of the cohomology for \mathcal{W} implies the same for its central extension \mathcal{V} .

As the core of the proof is elementary, I want to keep this spirit throughout the article. Hence, it will be rather self-contained and elementary. We will not use things like long exact cohomology sequences, etc. But see Remark 6.5 about the interpretation of the extension of our calculations to the Virasoro case in terms of such sequences.

2. THE ALGEBRAS

The *Witt algebra* \mathcal{W} is the Lie algebra generated as vector space by the elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}. \quad (2.1)$$

It can be realized as complexification of the Lie algebra of polynomial vector fields $Vect_{pol}(S^1)$ on the circle S^1 , which is a subalgebra of $Vect(S^1)$, the Lie algebra of all C^∞ vector fields on the circle. In this realization $e_n = \exp(in\varphi) \frac{d}{d\varphi}$. The Lie product is the usual bracket of vector fields.

An alternative realization is given as the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are holomorphic outside $\{0\}$ and $\{\infty\}$. In this realization $e_n = z^{n+1} \frac{d}{dz}$. A very important fact is that the Witt algebra is a \mathbb{Z} -graded Lie algebra. We define the degree by setting $\deg(e_n) := n$, then the Lie product between elements of degree n and of degree m is of degree $n + m$ (if nonzero). The homogeneous spaces \mathcal{W}_n of degree n are one-dimensional with basis e_n . Crucial for the following is the additional fact that the eigenspace decomposition of the element e_0 , acting via the adjoint action on \mathcal{W} coincides with the decomposition into homogeneous subspaces. This follows from

$$[e_0, e_n] = n e_n = \deg(e_n) e_n. \quad (2.2)$$

Another property, which will play a role, is that $[\mathcal{W}, \mathcal{W}] = \mathcal{W}$. That means that \mathcal{W} is a perfect Lie algebra. In fact,

$$e_n = \frac{1}{n}[e_0, e_n], \quad n \neq 0, \quad e_0 = \frac{1}{2}[e_{-1}, e_1]. \quad (2.3)$$

The *Virasoro algebra* \mathcal{V} is a one-dimensional central extension of \mathcal{W} . As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its

basis elements are \hat{e}_n , $n \in \mathbb{Z}$ and t with the Lie product

$$[\hat{e}_n, \hat{e}_m] = (m - n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - m^3)\delta_n^{-m} t, \quad [\hat{e}_n, t] = [t, t] = 0, \quad (2.4)$$

for all $n, m \in \mathbb{Z}$. Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero. If we set $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ then \mathcal{V} becomes a graded algebra. The algebra \mathcal{W} will only be a subspace and not a subalgebra of \mathcal{V} . But it will be a quotient. We project the central element t to 0 and \hat{x} to x and denote the projection by ν (note that ν is a Lie homomorphism) and get the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{V} \xrightarrow{\nu} \mathcal{W} \longrightarrow 0. \quad (2.5)$$

This sequence does not split, i.e. there does not exist a Lie homomorphism $\alpha : \mathcal{W} \rightarrow \mathcal{V}$ with $\nu \circ \alpha = id_{\mathcal{W}}$. The map $x \mapsto \hat{x}$ is only a linear splitting map. The non-existence of a Lie map means that the central extension is nontrivial.

In some abuse of notation we identify the element $\hat{x} \in \mathcal{V}$ with $x \in \mathcal{W}$ and after identification we have $\mathcal{V}_n = \mathcal{W}_n$ for $n \neq 0$ and $\mathcal{V}_0 = \langle e_0, t \rangle_{\mathbb{C}}$. Note that the relation (2.2) inducing the eigenspace decomposition for the grading element $\hat{e}_0 = e_0$ remains true. This sloppiness in notation has also its disadvantages. If we calculate the bracket, let us say $[e_i, e_{-1}]$, we have to be careful whether we calculate it in \mathcal{W} or in \mathcal{V} as it will differ by a multiple of the central element.

The expression $\frac{1}{12}(n^3 - m^3)\delta_n^{-m}$ is called a defining cocycle for the central extension. This form is given in a standard normalisation – others are possible. A more general prescription would be

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \alpha(x, y) t, \quad [V, t], \quad \alpha(x, y) \in \mathbb{C}. \quad (2.6)$$

The α is an alternating bilinear form on \mathcal{W} . The condition that the central extension should fulfill the Jacobi identity imposes that α fulfills

$$\alpha([x, y], z) + \alpha([y, z], x) + \alpha([z, x], y) = 0, \quad (2.7)$$

called the cocycle condition. If we change the linear splitting map then the cocycle α in (2.6) will change to α' . Both will differ by a coboundary, i.e. there exists a linear form $\phi : \mathcal{W} \rightarrow \mathbb{C}$ such that

$$\alpha(x, y) - \alpha'(x, y) = (\delta_1 \phi)(x, y) = \phi([x, y]). \quad (2.8)$$

In this case the two central extensions are called equivalent. The equivalence classes of central extensions are in 1:1 correspondence to the cohomology classes modulo coboundaries. In the language of the next section it is the space $H^2(\mathcal{W}, \mathbb{C})$ of cohomology classes of 2-cocycles with values in the trivial module \mathbb{C} . It is well-known that $\dim H^2(\mathcal{W}, \mathbb{C}) = 1$. In fact, as a side effect we will prove this in Section 6. There is always the trivial (split) central extension given by the trivial class [0]. Additionally, in our case, we have up to equivalence of the extension and the rescaling of the central element a unique nontrivial central extension. It is given by the Virasoro algebra (2.4). From $[\mathcal{W}, \mathcal{W}] = \mathcal{W}$ one can show that it is the universal central extension, meaning that all other one-dimensional central extensions will be quotients of it.

3. COHOMOLOGY AND DEFORMATIONS

Let us recall for completeness and further reference the definition of the *Lie algebra cohomology* of a Lie algebra W with values in a Lie module M over W . We denote the Lie module structure by $W \times M \rightarrow M$, $(x, m) \mapsto x \cdot m$. A k -cochain is an alternating k -multilinear map $W \times W \times \cdots \times W \rightarrow M$ (k copies of W). The vector space of k -cochains is denoted by $C^k(W; W)$. As we are dealing with infinite dimensional Lie algebras and modules, we have to be very careful whether we impose additional conditions, or even allow somewhere completions. Here we will deal exclusively with algebraic cohomology. Hence, no additional conditions.

Next we have the family of coboundary operators

$$\delta_k : C^k(W; W) \rightarrow C^{k+1}(W; W), \quad k \in \mathbb{N}, \quad \text{with} \quad \delta_{k+1} \circ \delta_k = 0. \quad (3.1)$$

Here we will only consider the second cohomology

$$\begin{aligned} \delta_2 \psi(x, y, z) &:= \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) \\ &\quad - x \cdot \psi(y, z) + y \cdot \psi(x, z) - z \cdot \psi(x, y), \\ (\delta_1 \phi)(x, y) &:= \phi([x, y]) - x \cdot \phi(y) + y \cdot \phi(x). \end{aligned} \quad (3.2)$$

A k -cochain ψ is called a *k-cocycle* if it lies in the kernel of the k -coboundary operator δ_k . It is called a *k-coboundary* if it lies in the image of the $(k-1)$ -coboundary operator.

By $\delta_k \circ \delta_{k-1} = 0$ the vector space quotient of cocycles modulo coboundaries is well-defined. It is called the vector space of *k-Lie algebra cohomology* of W with values in the module M . It is denoted by $H^k(W; M)$. Two cocycles which are in the same cohomology class are called *cohomologous*.

The trivial module is \mathbb{C} with the Lie action $x \cdot m = 0$, for all $x \in W$ and $m \in \mathbb{C}$. The second cohomology with values in the trivial module classifies equivalence classes of central extensions of W . Like in the Witt case, after choosing a representing cocycle α in the cocycle class $[\alpha] \in H^2(W; \mathbb{C})$ the Lie structure of the central extension \hat{W} will be given by (2.6). It is well-known that for the Witt algebra we have $\dim H^2(W; \mathbb{C}) = 1$ and that the class of the cocycle defining \mathcal{V} gives a basis.

The second cohomology of W with values in the adjoint module, $H^2(W; W)$, i.e. with module structure $x \cdot y := [x, y]$, is the cohomology to be studied here. As we will need the formula for explicit calculations later let me specialize the 2-cocycle condition (3.2)

$$\begin{aligned} \delta_2 \psi(x, y, z) &:= 0 = \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) \\ &\quad - [x, \psi(y, z)] + [y, \psi(x, z)] - [z, \psi(x, y)]. \end{aligned} \quad (3.3)$$

A 2-cocycle will be a coboundary if it lies in the image of the (1-)coboundary operator, i.e. there exists a linear map $\phi : W \rightarrow W$ such that

$$\psi(x, y) = (\delta_1 \phi)(x, y) := \phi([x, y]) - [\phi(x), y] - [x, \phi(y)]. \quad (3.4)$$

The second cohomology $H^2(W, W)$ is related to the deformations of the Lie algebra W . The Lie algebra W with its bracket $[\cdot, \cdot]$ can also be written with an anti-symmetric bilinear map

$$\mu_0 : W \times W \rightarrow W, \quad \mu_0(x, y) = [x, y],$$

fulfilling certain additional conditions corresponding to the Jacobi identity. On the same vector space W is modeled on, we consider a family of Lie structures

$$\mu_s = \mu_0 + s \cdot \psi_1 + s^2 \cdot \psi_2 + \cdots, \quad (3.5)$$

with bilinear maps $\psi_i : W \times W \rightarrow W$ such that $W_s := (W, \mu_s)$ is a Lie algebra and W_0 is the Lie algebra we started with. The family $\{W_s\}$ is a *deformation* of W_0 . For the deformation “parameter” we have different possibilities.

(1) The parameter s might be a variable which allows to plug in numbers $\alpha \in \mathbb{C}$. In this case W_α is a Lie algebra for every α for which the expression (3.5) is defined. The family can be considered as deformation over the affine line $\mathbb{C}[s]$ or over the convergent power series $\mathbb{C}\{\{s\}\}$. The deformation is called a *geometric* or an *analytic deformation* respectively.

(2) We consider s as a formal variable and we allow infinitely many terms in (3.5). It might be the case that μ_s does not exist if we plug in for s any other value different from 0. In this way we obtain deformations over the ring of formal power series $\mathbb{C}[[s]]$. The corresponding deformation is a *formal deformation*.

(3) The parameter s is considered as an infinitesimal variable, i.e. we take $s^2 = 0$. We obtain *infinitesimal deformations* defined over the quotient $\mathbb{C}[X]/(X^2) = \mathbb{C}[[X]]/(X^2)$.

Even more general situations for the parameter space can be considered. See [5],[6],[7] for a general mathematical treatment.

There is always the trivially deformed family given by $\mu_s = \mu_0$ for all values of s . Two families μ_s and μ'_s deforming the same μ_0 are *equivalent* if there exists a linear automorphism (with the same vagueness about the meaning of s)

$$\phi_s = id + s \cdot \alpha_1 + s^2 \cdot \alpha_2 + \cdots \quad (3.6)$$

with $\alpha_i : W \rightarrow W$ linear maps such that

$$\mu'_s(x, y) = \phi_s^{-1}(\mu_s(\phi_s(x), \phi_s(y))). \quad (3.7)$$

A Lie algebra (W, μ_0) is called *rigid* if every deformation μ_s of μ_0 is locally equivalent to the trivial family. Intuitively, this says that W cannot be deformed. This definition again depends crucially on the nature of the deformation parameter.

The deformation problem is related to $H^2(W; W)$ in the following way. First note that the ψ_i are alternating bilinear map from $W \times W \rightarrow W$, i.e. 2-cochains. If we write down the Jacobi identity for μ_s given by (3.5) then it can be immediately verified that the first non-vanishing ψ_i has to be a 2-cocycle, i.e. $\delta_2 \psi = 0$. Furthermore, if μ_t and μ'_t are equivalent then the corresponding ψ_i and ψ'_i are cohomologous, i.e. their difference is a coboundary.

The following results are well-known:

- (1) $H^2(W, W)$ classifies infinitesimal deformations of W up to equivalence. This is due to Gerstenhaber [9].
- (2) If $\dim H^2(W, W) < \infty$ then there exists a versal formal family for the formal deformations of W whose base is formally embedded into $H^2(W, W)$. This is due to Fialowski [3], and Fialowski and Fuks [4].

Hence, if $H^2(W, W) = 0$ then W is infinitesimally and formally rigid.

If the Lie algebra under consideration is finite dimensional then formal rigidity also implies analytic rigidity. This means that locally all families are trivial. As our examples with the Krichever-Novikov algebras show [5],[6] this is not true anymore in the infinite dimensional case.

After having recalled the general definitions, I formulate the main result of this article.

Theorem 3.1. *Both the 2nd cohomology of the Witt algebra W and of the Virasoro algebra \mathcal{V} with values in the adjoint module vanishes, i.e.*

$$H^2(\mathcal{W}; \mathcal{W}) = \{0\}, \quad H^2(\mathcal{V}; \mathcal{V}) = \{0\}. \quad (3.8)$$

In particular, both algebras are formally and infinitesimally rigid.

I refer to Section 1, the Introduction, for the history of this theorem. Here I only like to repeat that the first proof (at least in the Witt case) was given by Alice Fialowski by very cumbersome unpublished calculations [2]. The proof which I present in the following is elementary, but much more accessible and elegant as the original proof.

4. THE DEGREE DECOMPOSITION OF THE COHOMOLOGY

Recall that we consider algebraic cohomology, i.e. our 2-cochains $\psi \in C^2(W; W)$ are arbitrary alternating bilinear maps in the usual sense, i.e. for all $v, w \in W$ the cochain $\psi(v, w)$ will be a finite linear combination (depending on v and w) of basis elements in W .

The grading $W = \bigoplus_{n \in \mathbb{Z}} W_n$ will play an important role in the following. We call a k -cochain ψ homogeneous of degree d if there exists a $d \in \mathbb{Z}$ such that for all $i_1, i_2, \dots, i_k \in \mathbb{Z}$ and homogeneous elements $x_{i_l} \in W$, of $\deg(x_{i_l}) = i_l$, for $l = 1, \dots, k$ we have that

$$\psi(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in W_n, \quad \text{with} \quad n = \sum_{l=1}^k i_l + d. \quad (4.1)$$

The corresponding subspace of degree d homogeneous k -cochains is denoted by $C_{(d)}^k(W; W)$. Every k -cochain can be written as a formal infinite sum $\psi = \sum_{d \in \mathbb{Z}} \psi_{(d)}$. Note that evaluated for a fixed k -tuple of elements only a finite number of the summands will produce values different from zero.

Proposition 4.1. *The coboundary operators δ_k are operators of degree zero, i.e. applied to a k -cocycle of degree d they will produce a $(k+1)$ -cocycle also of degree d .*

Proof. We will need only $k = 1$ and $k = 2$ (but it will be true in general). We start with a such a cochain of degree d . If we evaluate the coboundary operators (3.3) and (3.4) for homogeneous elements x, y, z and use that our Lie algebra structures is graded then on the right hand side all terms are either zero or of degree $\deg(x) + \deg(y) + \deg(z) + d$, resp. of degree $\deg(x) + \deg(y) + d$. \square

In the following we will concentrate on $k = 2$ or $k = 1$. If $\psi = \sum_d \psi_{(d)}$ is a 2-cocycle then $\delta_2 \psi = \sum_d \delta_2 \psi_{(d)} = 0$. By Proposition 4.1 $\delta_2 \psi_{(d)}$ is either zero or of degree d . As we sum over different degrees and the terms cannot cancel if different from zero we obtain that ψ is cocycle if and only if all degree d components $\psi_{(d)}$ will be individually 2-cocycles.

Moreover, if $\psi_{(d)}$ is 2-coboundary, i.e. $\psi_{(d)} = \delta_1\phi$ with a 1-cochain ϕ , then we can find another 1-cochain ϕ' of degree d such that $\psi_{(d)} = \delta_1\phi'$.

We summarize as follows. Every cohomology class $\alpha \in H^2(W; W)$ can be decomposed as formal sum

$$\alpha = \sum_{d \in \mathbb{Z}} \alpha_{(d)}, \quad \alpha_{(d)} \in H_{(d)}^2(W; W), \quad (4.2)$$

where the latter spaces consists of classes of cocycles of degree d modulo coboundaries of degree d .

Next we will show for our algebras that the cohomology spaces of degree $d \neq 0$ will vanish. Hence, for the rest of this section let d be different from zero. The degree zero case needs a more involved treatment and will be done in the next sections. We start with a cocycle of degree $d \neq 0$ and make first a cohomological change $\psi' = \psi - \delta_1\phi$ with

$$\phi : W \rightarrow W, \quad x \mapsto \phi(x) = \frac{\psi(x, e_0)}{d}. \quad (4.3)$$

Recall e_0 is the element of either \mathcal{W} or \mathcal{V} which gives the degree decomposition. This implies (note that by definition $\phi(e_0) = 0$)

$$\begin{aligned} \psi'(x, e_0) &= \psi(x, e_0) - (\delta_1\phi)(x, e_0) = \psi(x, e_0) - \phi([x, e_0]) + [\phi(x), e_0] \\ &= d\phi(x) + \deg(x)\phi(x) - (\deg(x) + d)\phi(x) = 0. \end{aligned} \quad (4.4)$$

We evaluate (3.3) for the cocycle ψ' on the triple (x, y, e_0) and leave out the cocycle values which vanish due to (4.4):

$$\begin{aligned} 0 &= \psi'([y, e_0], x) + \psi'([e_0, x], y) - [e_0, \psi'(x, y)] \\ &= (\deg(y) + \deg(x) - (\deg(x) + \deg(y) + d))\psi'(x, y) = -d\psi'(x, y). \end{aligned} \quad (4.5)$$

As $d \neq 0$ we obtain $\psi'(x, y) = 0$ for all $x, y \in W$. We conclude

Proposition 4.2.

- (a) $H_{(d)}^2(\mathcal{W}; \mathcal{W}) = H_{(d)}^2(\mathcal{V}; \mathcal{V}) = \{0\}$, for $d \neq 0$.
- (b) $H^2(\mathcal{W}; \mathcal{W}) = H_{(0)}^2(\mathcal{W}; \mathcal{W})$, $H^2(\mathcal{V}; \mathcal{V}) = H_{(0)}^2(\mathcal{V}; \mathcal{V})$.

Remark 4.3. In fact the arguments also work for every \mathbb{Z} -graded Lie algebra W for which there exists an element e_0 such that the homogeneous spaces W_n are just the eigenspaces of e_0 under the adjoint action to the eigenvalue n . Such Lie algebras are called internally graded. See also Theorem 1.5.2 in [8].

5. DEGREE ZERO FOR THE WITT ALGEBRA

It remains the degree zero part. In this section we consider only the Witt algebra. Recall that the homogeneous subspaces of degree n are one-dimensional and generated by e_n . Hence, a degree zero cocycle can be written as $\psi(e_i, e_j) = \psi_{i,j}e_{i+j}$ and if it is a coboundary then it can be given as a coboundary of a linear form of degree zero:

$\phi(e_i) = \phi_i e_i$. The systems of $\psi_{i,j}$ and ϕ_i for $i, j \in \mathbb{Z}$ fix ψ and ϕ completely. If we evaluate (3.3) for the triple (e_i, e_j, e_k) we get for the coefficients

$$\begin{aligned} 0 = & (j-i)\psi_{i+j,k} - (k-i)\psi_{i+k,j} + (k-j)\psi_{j+k,i} \\ & - (j+k-i)\psi_{j,k} + (i+k-j)\psi_{i,k} - (i+j-k)\psi_{i,j}. \end{aligned} \quad (5.1)$$

For the coboundary we obtain

$$(\delta\phi)_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i). \quad (5.2)$$

Hence, ψ is a coboundary if and only if there exists a system of $\phi_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that

$$\psi_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i), \quad \forall i, j \in \mathbb{Z}. \quad (5.3)$$

A degree zero 1-cochain ϕ will be a 1-cocycle (i.e. $\delta_1\phi = 0$) if and only if $\phi_{i+j} - \phi_j - \phi_i = 0$. This has the solution $\phi_i = i\phi_1, \forall i \in \mathbb{Z}$. Hence, given a ϕ we can always find a ϕ' with $(\phi')_1 = 0$ and $\delta_1\phi = \delta_1\phi'$. In the following we will always choose such a ϕ' for our 2-coboundaries.

Step 1: We make a cohomological change

We start with a 2-cocycle ψ given by the system of $\psi_{i,j}$ and will modify it by adding a coboundary $\delta_1\phi$ with suitable ϕ to obtain $\psi' = \psi - \delta_1\phi$. We will determine ϕ inspired by the intended relation (5.3)

$$\psi_{i,1} = (1-i)(\phi_{i+1} - \phi_1 - \phi_i) = (1-i)(\phi_{i+1} - \phi_i). \quad (5.4)$$

Note that we could put $\phi_1 = 0$ by our normalization.

(a) Starting from $\phi_0 := -\psi_{0,1}$ we set in descending order for $i \leq -1$

$$\phi_i := \phi_{i+1} - \frac{1}{1-i}\psi_{i,1}. \quad (5.5)$$

(b) ϕ_2 cannot be fixed by (5.4), instead we use (5.3)

$$\psi_{-1,2} = 3(-\phi_2 - \phi_{-1}), \quad \text{yielding} \quad \phi_2 := -\phi_{-1} - \frac{1}{3}\psi_{-1,2}. \quad (5.6)$$

Then we have $\psi'_{-1,2} = 0$.

(c) We use again (5.4) to calculate recursively in ascending order ϕ_i , $i \geq 3$ by

$$\phi_{i+1} := \phi_i + \frac{1}{1-i}\psi_{i,1}. \quad (5.7)$$

For the cohomologous cocycle ψ' we obtain by construction

$$\psi'_{i,1} = 0, \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \psi'_{-1,2} = \psi'_{2,-1} = 0. \quad (5.8)$$

Step2: Show that ψ' is identical zero.

To avoid cumbersome notation we will denote the cohomologous cocycle by ψ .

Lemma 5.1. *Let ψ be a 2-cocycle of degree zero such that $\psi_{i,1} = 0, \forall i \in \mathbb{Z}$ and $\psi_{-1,2} = 0$, then ψ will be identical zero.*

Before we proof the lemma we use it to show

Proof. (Witt part of Theorem 3.1.) By Proposition 4.2 it is enough to consider degree zero cohomology. By the cohomological change done in Step 1 every degree zero cocycle ψ is cohomologous to a cocycle fulfilling the conditions of Lemma 5.1. But such a cocycle vanishes by the lemma. Hence, the original cocycle ψ is cohomological trivial. \square

Proof. (Lemma 5.1.) First, we note two special cases of (5.1) which will be useful in the following. For the index triple $(i, -1, k)$ we obtain

$$\begin{aligned} 0 = & -(i+1)\psi_{i-1,k} - (k-i)\psi_{i+k,-1} + (k+1)\psi_{k-1,i} \\ & - (-1+k-i)\psi_{-1,k} + (i+k+1)\psi_{i,k} - (i-1-k)\psi_{i,-1}. \end{aligned} \quad (5.9)$$

For the triple $(i, 1, k)$, and ignoring terms of the type $\psi_{i,1}$ which are zero by assumption, we obtain

$$0 = (1-i)\psi_{i+1,k} + (k-1)\psi_{k+1,i} + (i+k-1)\psi_{i,k}. \quad (5.10)$$

We will consider $\psi_{i,m}$ for certain values of $|m| \leq 2$ and finally make ascending and descending induction on m . We will call the coefficient $\psi_{i,m}$ coefficients of level m (and of level i by antisymmetry). By assumption the cocycle values of level 1 are all zero.

$$\boxed{m = 0}$$

By the antisymmetry we have $\psi_{1,i} = -\psi_{i,1} = 0$. In (5.10) we consider $k = 0$ this gives

$$0 = (1-i)(\psi_{i+1,0} - \psi_{i,0}). \quad (5.11)$$

Starting from $\psi_{1,0} = 0$ this implies for $i \leq 0$ that $\psi_{i,0} = 0$ and for $i \geq 3$ that $\psi_{i,0} = \psi_{2,0}$. Next we consider (5.9) for $k = 2, i = 0$ and obtain

$$0 = -\psi_{-1,2} - 2\psi_{2,-1} + 3\psi_{1,0} - \psi_{-1,2} + 3\psi_{0,2} + 3\psi_{0,-1}.$$

The $\psi_{-1,2}$ terms cancel and we know already $\psi_{1,0} = \psi_{-1,0} = 0$, hence $\psi_{2,0} = 0$. This implies

$$\psi_{i,0} = 0 \quad \forall i \in \mathbb{Z}. \quad (5.12)$$

$$\boxed{m = -1}$$

In (5.10) we set $k = -1$ and obtain (with $\psi_{0,i} = 0$)

$$-(i-1)\psi_{i+1,-1} + (i-2)\psi_{i,-1} = 0. \quad (5.13)$$

Hence,

$$\psi_{i,-1} = \frac{i-1}{i-2}\psi_{i+1,-1}, \quad \text{for } i \neq 2, \quad \psi_{i+1,-1} = \frac{i-2}{i-1}\psi_{i,-1}, \quad \text{for } i \neq 1. \quad (5.14)$$

The first formula implies starting from $\psi_{1,-1} = -\psi_{-1,1} = 0$ that $\psi_{i,-1} = 0$, for all $i \leq 1$. The second formula for $i = 2$ implies $\psi_{3,-1} = 0$ and hence $\psi_{i,-1} = 0$ for $i \geq 3$. But by assumption $\psi_{2,-1} = \psi_{-1,2} = 0$. Hence,

$$\psi_{i,-1} = 0 \quad \forall i \in \mathbb{Z}. \quad (5.15)$$

$$\boxed{m = -2}$$

We plug the value $k = -2$ into (5.10) and get for the terms not yet identified as zero

$$(1-i)\psi_{i+1,-2} + (i-3)\psi_{i,-2} = 0. \quad (5.16)$$

This yields

$$\psi_{i+1,-2} = \frac{i-3}{i-1} \psi_{i,-2}, \quad \text{for } i \neq 1, \quad \psi_{i,-2} = \frac{i-1}{i-3} \psi_{i+1,-2}, \quad \text{for } i \neq 3. \quad (5.17)$$

From the first formula we get $\psi_{3,-2} = -\psi_{2,-2}$, $\psi_{4,-2} = 0 \cdot \psi_{3,-2}$, and hence $\psi_{i,-2} = 0$, for all $i \geq 4$.

From the second formula we get starting from $\psi_{1,-2} = 0$ that $\psi_{i,-2} = 0$ for $i \leq 1$.

Altogether, $\psi_{i,-2} = 0$ for $i \neq 2, 3$. The value of $\psi_{2,-2} = -\psi_{3,-2}$ stays undetermined for the moment.

$m = 2$

We start from (5.9) for $k = 2$ and recall that terms of levels 0, 1, -1 are zero. This gives

$$-(i+1)\psi_{i-1,2} + (i+3)\psi_{i,2} = 0. \quad (5.18)$$

Hence,

$$\psi_{i,2} = \frac{i+1}{i+3} \psi_{i-1,2}, \quad \text{for } i \neq -3, \quad \psi_{i-1,2} = \frac{i+3}{i+1} \psi_{i,2}, \quad \text{for } i \neq -1. \quad (5.19)$$

From the first formula we start from $\psi_{-1,2} = 0$ and get $\psi_{i,2} = 0, \forall i \geq -1$. From the second we get $\psi_{-3,2} = -\psi_{-2,2}$, then $\psi_{-4,2} = 0$ and then altogether $\psi_{i,2} = 0$ for all $i \neq -2, -3$. The value $\psi_{-3,2} = -\psi_{-2,2}$ stays undetermined for the moment.

To find it we consider the index triple $(2, -2, 4)$ in (5.1) and obtain after leaving out terms which are obviously zero

$$0 = -2\psi_{6,-2} - 8\psi_{4,2} + 4\psi_{2,-2}. \quad (5.20)$$

From the level $m = 2$ discussion we get $\psi_{4,2} = 0$, from $m = -2$ we get $\psi_{6,-2} = 0$. This shows that $\psi_{2,-2} = \psi_{3,-2} = \psi_{-3,2} = 0$ and we can conclude

$$\psi_{i,-2} = \psi_{i,2} = 0, \quad \forall i \in \mathbb{Z}. \quad (5.21)$$

$m < -2$

We make induction assuming it is true for $m = 2, 1, 0, -1, -2$. We start from (5.9) for k and put the $k-1$ level element on the l.h.s. By this

$$(k+1)\psi_{i,k-1} = \text{terms of level } k \text{ and } -1.$$

By induction the terms on the r.h.s. are zero. Note that in this region $k < -1$, hence $k+1 \neq 0$, and $\psi_{i,k-1} = 0$, too.

$m > 2$

Again we make induction. Starting from (5.10) we get

$$(1-k)\psi_{i,k+1} = (1-i)\psi_{i+1,k} + (i+k-1)\psi_{i,k}. \quad (5.22)$$

As $k \geq 2$ the value of $1-k \neq 0$ and we get by induction trivially the statement for $k+1$.

Altogether we obtain $\psi_{i,k} = 0, \forall i, k \in \mathbb{Z}$. \square

The presented calculation is completely different and much simpler as in [2]

6. EXTENSION TO THE VIRASORO ALGEBRA

Here we show in an elementary way that also for the Virasoro algebra $H^2(\mathcal{V}, \mathcal{V}) = 0$. Hence \mathcal{V} is also infinitesimally and formally rigid. This shows the Virasoro part of Theorem 3.1. It is our intention not to use any higher techniques, but see Remark 6.5 at the end of this section.

First recall that by Proposition 4.2 it is enough to consider degree zero cocycles. We start with a degree zero 2-cocycle $\psi : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ of the Virasoro algebra. If we apply the Lie homomorphism ν we get the bilinear map $\nu \circ \psi$ which we restrict to $\mathcal{W} \times \mathcal{W}$

$$\psi' = \nu \circ \psi| : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}. \quad (6.1)$$

Unfortunately, in general ψ' will not be a 2-cocycle for the Witt algebra. We have to be careful as the Lie product for \mathcal{V} differs from that of \mathcal{W} by multiples of the central element. But we are allowed to make cohomologous changes.

Proposition 6.1. *Given a cocycle $\psi \in C^2(\mathcal{V}, \mathcal{V})$ there exists a cohomologous one $\tilde{\psi} \in C^2(\mathcal{V}, \mathcal{V})$ such that the bilinear map $\psi' = \nu \circ \tilde{\psi} \in C^2(\mathcal{W}, \mathcal{W})$.*

Proof. Let $x, y, z \in \mathcal{W}$. We have $[x, y]_{\mathcal{V}} = [x, y]_{\mathcal{W}} + \alpha(x, y) \cdot t$. We consider (3.3) with the bracket $[\cdot, \cdot]_{\mathcal{V}}$ and rewrite it in terms of $[\cdot, \cdot]_{\mathcal{W}}$ (but drop the index) and α . For the second group of terms we use that ν is a Lie homomorphism and they will not see the central elements. Only in the first group they will play a role. We get

$$\begin{aligned} 0 &= \psi'([x, y], z) + \psi'([y, z], x) + \psi'([z, x], y) \\ &\quad + \alpha([x, y], z) \nu \circ \psi(t, z) + \alpha([y, z], x) \nu \circ \psi(t, x) + \alpha([z, x], y) \nu \circ \psi(t, y) \\ &\quad - [x, \psi'(y, z)] + [y, \psi'(x, z)] - [z, \psi'(x, y)]. \end{aligned} \quad (6.2)$$

We will make a cohomological change for the cocycle ψ in \mathcal{V} such that $\psi(t, z)$ will have only central terms, i.e. $\nu \circ \psi(t, z)$ will vanish. Restricting it to $\mathcal{W} \times \mathcal{W}$ and projecting it by ν will define a 2-cocycle for \mathcal{W} .

As ψ is a degree zero cocycle, we have $\psi(e_1, t) = a e_1$. We set $\phi(t) := a e_0$, and $\phi(e_n) := 0$ for all $n \in \mathbb{Z}$. Let $\tilde{\psi} := \psi - \delta_1 \phi$. We calculate

$$\begin{aligned} \tilde{\psi}(e_1, t) &= \psi(e_1, t) - (\delta_1 \phi)(e_1, t) = a e_1 - \phi([e_1, t]) + [e_1, \phi(t)] + [\phi(e_1), t] \\ &= a e_1 - 0 - a e_1 + 0 = 0. \end{aligned} \quad (6.3)$$

The coefficients a_n (and b , but it will not play any role) are given by

$$\tilde{\psi}(e_n, t) = a_n e_n, \quad n \neq 0, \quad \tilde{\psi}(e_0, t) = a_0 e_n + b \cdot t. \quad (6.4)$$

Note that $a_1 = 0$. If we evaluate (3.3) for the triple (e_n, e_m, t) and leave out terms which are zero due to the fact, that t is central we get

$$0 = \tilde{\psi}([e_n, e_m]_{\mathcal{V}}, t) - [e_n, \tilde{\psi}(e_m, t)]_{\mathcal{V}} + [e_m, \tilde{\psi}(t, e_n)]_{\mathcal{V}}. \quad (6.5)$$

This implies (by evaluating the coefficients at the element e_{m+n})

$$(m - n)(a_{n+m} - a_m - a_n) = 0. \quad (6.6)$$

If we plug in $m = 1$ and use $a_1 = 0$ we get $a_n = 0$ for all $n \leq 1$ and $a_n = a_2$ for all $n \geq 2$. Now plugging in $m = 2, n = -2$ we obtain $a_2 = -a_{-2} = 0$. Hence altogether $a_n = 0$ for all n , and $\nu \circ \tilde{\psi}(w, t) = 0, \forall w \in \mathcal{V}$. \square

Hence, after a cohomologous change we may assume that our restricted and projected ψ' will be a 2-cocycle for the algebra \mathcal{W} with values in the adjoint module \mathcal{W} . In the last section we showed that it is cohomologically trivial, i.e. there exists a $\phi' : \mathcal{W} \rightarrow \mathcal{W}$ such that $\delta_1^{\mathcal{W}}\phi' = \psi' = \nu \circ \psi$. We denote for a moment the corresponding coboundary operators of the two Lie algebras by $\delta^{\mathcal{W}}$ and $\delta^{\mathcal{V}}$. By setting $\phi(t) := 0$ we extend ϕ' to a linear map $\phi : \mathcal{V} \rightarrow \mathcal{V}$. In particular, $\nu \circ \phi = \phi'$ if restricted to \mathcal{W} .

The 2-cocycle $\hat{\psi} = \psi - \delta_1^{\mathcal{V}}\phi$ will be a cohomologous cocycle for \mathcal{V} . If we apply ν then

$$\nu\hat{\psi} = \nu\psi - \nu\delta_1^{\mathcal{V}}\phi = \psi' - \delta_1^{\mathcal{W}}\phi' = 0. \quad (6.7)$$

Hence, $\hat{\psi}$ takes values in the kernel of ν , i.e.

$$\hat{\psi} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C} \cdot t, \quad (6.8)$$

with t the central element.

This implies that it is enough to show that every class $\psi : \mathcal{V} \rightarrow \mathcal{V}$ with values in the central ideal $\mathbb{C} \cdot t$ will be a coboundary to show the vanishing of $H^2(\mathcal{V}, \mathcal{V})$. In the following let ψ be of this kind.

As $\psi(x, y) = \psi_{x,y}t$ will be central the 2-cocycle condition (3.3) will reduce to

$$(\delta_2\psi)(x, y, z) = \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0. \quad (6.9)$$

The coboundary condition for $\phi : \mathcal{V} \rightarrow \mathbb{C} \cdot t$ reduces to

$$(\delta_1\phi)(x, y) = \phi([x, y]). \quad (6.10)$$

We can reformulate this as that the component function $\psi_{x,y} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ is a Lie algebra 2-cocycle for V with values in the trivial module, see Remark 6.4 below.

Lemma 6.2.

$$\psi(x, t) = 0, \quad \forall x \in \mathcal{V}. \quad (6.11)$$

Proof. We evaluate (6.9) for (e_i, e_j, t) . As t is central $[e_j, t] = [t, e_i] = 0$, hence only the first term in (6.9) will survive and we get $(j - i)\psi(e_{i+j}, t) = 0$. Choosing $j = 0$ we obtain $\psi(e_i, t) = 0$ for $i \neq 0$, choosing $i = -1, j = +1$ we obtain also $\psi(e_0, t) = 0$. As $\psi(t, t) = 0$ is automatic and $\{e_n, n \in \mathbb{Z}, t\}$ is a basis of \mathcal{V} this shows the result. \square

Next let $\phi : \mathcal{V} \rightarrow \mathbb{C} \cdot t$ be a linear map. We write for the component functions $\phi(e_i) = \phi_i t$, $\phi(t) = c \cdot t$. If we evaluate the 1-coboundary operator (6.10) for pairs of basis elements we get

$$\delta_1\phi(e_i, e_j) = \phi([e_i, e_j]) = \left((j - i)\phi_{i+j} - \frac{1}{12}(i^3 - i)\delta_i^{-j}c \right) t, \quad i, j \in \mathbb{Z}, \quad (6.12)$$

and by Lemma 6.2 we get $\delta_1\phi(e_i, t) = 0$, for all $i \in \mathbb{Z}$.

With respect to (6.12) we choose ϕ such that the cohomologous cocycle $\psi' = \psi - \delta_1\phi$ fulfills

$$\psi'(e_i, e_0) = 0, \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \psi'(e_1, e_{-1}) = \psi'(e_2, e_{-2}) = 0. \quad (6.13)$$

This is obtained by putting ²

$$\phi_i := -\frac{1}{i}\psi_{e_i, e_0}, \quad i \in \mathbb{Z}, i \neq 0, \quad \phi_0 := -(1/2)\psi_{e_1, e_{-1}}, \quad c := (-2\psi_{e_2, e_{-2}} - 8\phi_0). \quad (6.14)$$

Lemma 6.3. *The cocycle ψ' is identically zero.*

Proof. First we consider the 2-cocycle condition (6.9) for the triple (e_i, e_j, e_0) and obtain using $\psi'(e_i, t) = 0$ that

$$0 = 0 + (-j)\psi'(e_j, e_i) + i\psi'(e_i, e_j) = (i+j)\psi'(e_i, e_j). \quad (6.15)$$

This yields that $\psi'(e_i, e_j) = 0$ for $j \neq -i$. Next we show by induction that $\psi'(e_i, e_{-i}) = 0$ for all $i \geq 0$ (and by antisymmetry also for $i \leq 0$). Note that this is true for $i = 0, 1$ and 2 , see (6.13). Consider (6.9) for the triple $(e_n, e_{-(n-1)}, e_{-1})$, $n > 2$. After evaluation of the Lie products we get

$$0 = (-2n+1)\psi'(e_1, e_{-1}) + (n-2)\psi'(e_{-n}, e_n) + (-n+1)\psi'(e_{n-1}, e_{-(n-1)}). \quad (6.16)$$

The first and last term vanishes by induction. As $n > 2$ this implies $\psi'(e_n, e_{-n}) = 0$. As we have always $\psi'(x, t) = 0$ this shows the lemma. \square

Finally we obtain that the ψ we started with was a coboundary. This implies indeed $H^2(\mathcal{V}; \mathcal{V}) = \{0\}$.

Remark 6.4. As already mentioned above the components $\psi_{x,y}$ are 2-cocycles of \mathcal{V} with values in the trivial module, and coboundaries are exactly coboundaries in this cohomology. What we showed with our small calculation in Lemma 6.3 is that $H^2(\mathcal{V}, \mathbb{C}) = \{0\}$. Note that our calculation was rather direct and we did not a priori use the reduction to the degree zero part. The vanishing of this cohomology means that all central extensions of the Virasoro algebra are trivial, i.e. they split. Of course, from a more general point of view one knows this already as the Virasoro algebra is the universal central extension of \mathcal{W} . This follows from the fact that \mathcal{W} is a perfect Lie algebra, i.e. $[\mathcal{W}, \mathcal{W}] = \mathcal{W}$ and that $H(\mathcal{W}, \mathbb{C})$ is one-dimensional generated by the cocycle class of the defining cocycle for \mathcal{V} . Moreover, the above calculations can be extended to show the well-know fact that $\dim H^2(\mathcal{W}, \mathbb{C}) = 1$ and is generated by the class of the cocycle defining \mathcal{V} , see (2.4). As it was our intention to stay self-contained we preferred to do the short calculation here, instead making reference to general results.

Remark 6.5. The presented proof in this section might also be interpreted in terms of long exact cohomology sequences in Lie algebra cohomology. The exact sequence (2.5) of Lie algebras can also be considered as exact sequence of Lie modules over \mathcal{V} . For such sequences we have a long exact sequence in cohomology. If we consider only the level two we get

$$\dots \longrightarrow H^2(\mathcal{V}; \mathbb{C}) \longrightarrow H^2(\mathcal{V}; \mathcal{V}) \xrightarrow{\nu_*} H^2(\mathcal{V}; \mathcal{W}) \longrightarrow \dots \quad (6.17)$$

Proposition 6.1 shows that $\nu_* : H^2(\mathcal{V}; \mathcal{W}) \cong H^2(\mathcal{W}; \mathcal{W})$. That we have such a part of the long exact sequence was deduced above too. The calculations of Lemma 6.2 and Lemma 6.3 showed $H^2(\mathcal{V}; \mathbb{C}) = \{0\}$. From Section 5 we know $H^2(\mathcal{W}; \mathcal{W}) = \{0\}$. Hence, the sequence (6.17) implies $H^2(\mathcal{V}; \mathcal{V})$.

²In view of Remark 6.4 we will consider for this part cocycles of arbitrary degree.

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