THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Identities for square of vector fields in the infinitesimal representation of the symplectic group into the Siegel disk.

h. Airault

REPORT No. 9, 2011/2012, fall

ISSN 1103-467X
ISRN IML-R- -9-11/12- -SE+fall

Identities for square of vector fields in the infinitesimal representation
of the symplectic group of order 2
into the Siegel disk of 2×2 complex symmetric matrices

Helene Airault

Insset,Université de Picardie Jules Verne, 48, rue Raspail 02100 Saint-Quentin (Aisne)
LAMFA, UMR 6140, , Cnrs, 33, rue Saint-Leu, 80039, Amiens, Cedex, France

Abstract

We discuss the notion of Ornstein-Uhlenbeck operator on a complex manifold endowed with a Kählerian metric. We give the example of the Siegel disk.We consider the infinitesimal holomorphic representation of $Sp(2 \times 2)$, the symplectic group of order 2, into the Siegel disk $\mathcal{D}_2$ of symmetric complex 2×2 matrices. Let $\rho(v) = L(v) + \beta(v)\, I$, the first order differential operator on $\mathcal{D}_2$ associated to the element v in the Lie algebra $\mathcal{G}$ of $Sp(2 \times 2)$. We denote $L(v)$ a vector field, $\beta(v)$ a function on $\mathcal{D}_2$ and $\beta(v)I$ is the operator of multiplication by $\beta(v)$. For a suitable basis (e_k) in the Lie algebra $\mathcal{G}$, we show the existence of constants (a_k) such that the operator $\sum_k a_k \rho(e_k)^2$ is equal to the multiplication by a constant. Varying the coeficients in the modular factor of the representation, we obtain Ornstein-Uhlenbeck type operators on $\mathcal{D}_2$ of the form $\sum_k a_k \rho(e_k)\overline{L(e_k)}$ where $\overline{L(e_k)}$ is the complex conjugate of $L(e_k)$. In particular the Kählerian Laplacian on $\mathcal{D}_2$ is expressed as $\sum_k a_k L(e_k)\overline{L(e_k)}$. This extends some of the identities obtained for the Poincaré disk in [2], [3].

Contents

Acknowledgements. The author thanks the organizers of the session Complex Analysis and Integrable Systems 1 September-15 December 2011, and the participants of the seminar at Institut Mittag-Leffler *where this work has ben presented* Support by the Institut Mittag-Leffler (Djursholm, Sweden) during the session is gratefully acknowledged.

1 Introduction

Consider the holomorphic representation T_g of the symplectic group $G = Sp(2n)$ of order n into the Siegel disk $\mathcal{D}_n$ of $n \times n$ complex symmetric matrices $\mathcal{Z}$ such that $I - \mathcal{Z}\overline{\mathcal{Z}} > 0$.

This representation is defined as follows. Let $g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \in Sp(2n)$ where A, B are $n \times n$ complex matrices and where we assume that ${}^t g J g = J$, we denote J the $2n \times 2n$ matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $n \times n$ identity matrix. The action of G on $\mathcal{D}_n$ is given by

$$\mathcal{W} = k_g(\mathcal{Z}) = (A\mathcal{Z} + B)(\overline{B}\mathcal{Z} + \overline{A})^{-1} \tag{1.1}$$

We define the holomorphic representation of $Sp(2n)$ into $\mathcal{D}_n$ with the classical formula

$$(T_g\Phi)(\mathcal{Z}) = det(\overline{B}\mathcal{Z} + \overline{A})^{\gamma}\Phi(k_g(\mathcal{Z})) \tag{1.2}$$

where Φ is a holomorphic function on $\mathcal{D}_n$, γ is a constant. We call $det(\overline{B}\mathcal{Z} + \overline{A})^{\gamma}$ the modular factor. Let $\mathcal{D}$ be a complex domain and let G be a complex group operating on $\mathcal{D}$ by a holomorphic map $\mathcal{Z} \to k_g(\mathcal{Z})$, if $\mathcal{Z} \in \mathcal{D}$, then $k_g(\mathcal{Z}) \in \mathcal{D}$. Following [3],[8], for a holomorphic function Φ on $\mathcal{D}$, we define

$$(T_g\Phi)(\mathcal{Z}) = h_g(\mathcal{Z})\,\Phi(k_g(\mathcal{Z})) \tag{1.3}$$

where $h_g(\mathcal{Z})$ is a holomorphic function of $\mathcal{Z}$ with values in the set of complex numbers. We assume that $k_e(\mathcal{Z}) = \mathcal{Z}$ and that $h_e(\mathcal{Z}) = 1$ when e is the neutral element of G. The condition

$$T_{g_1g_2}\Phi(\mathcal{Z}) = T_{g_1}(T_{g_2}\Phi)(\mathcal{Z}) \tag{1.4}$$

must be satisfied. This implies that

$$h_{g_1g_2}(\mathcal{Z}) = h_{g_1}(\mathcal{Z})h_{g_2}k_{g_1}(\mathcal{Z}) \quad and \quad k_{g_1g_2}(\mathcal{Z}) = (k_{g_2}\, o\, k_{g_1})(\mathcal{Z}) \tag{1.5}$$

A particular solution of (1.5) is given by

$$h_g(\mathcal{Z}) = (\det(k'_g(\mathcal{Z})))^{\alpha} \tag{1.6}$$

where $k'_g(\mathcal{Z})$ is the complex holomorphic Jacobian matrix $(\frac{\partial k_m}{\partial z_j})$ of the map $\mathcal{Z} \to k_g(\mathcal{Z})$. For the representation (1.2), $h_g(\mathcal{Z})$ is obtained with (1.6). However for the representation of the 3-dimensional Heisenberg group given in [2], the factor $h_g(\mathcal{Z})$ is not like in (1.6). It would be interesting to know the relation between the factor $h_g(\mathcal{Z})$ and the action $k_g(\mathcal{Z})$ in the case of a holomorphic representation of $Diff_+(S)$ into the space of functions univalent in the unit disk. Such representations have been suggested by Kirillov, Neretin and Yureev, see [6], [7]. In the present work, from the Kählerian potential on $\mathcal{D}_n$, we recall how to obtain the Laplace-Beltrami operator. We define the complex Ornstein-Uhlenbeck operator (complex O-U operator) on $\mathcal{D}_n$. We calculate the infinitesimal representation of $Sp(2n)$ when $n = 2$, generalizing the calculation done for the Poincaré disk in [4]. We show that for $n = 2$, the Laplace-Beltrami operator and the complex O-U operator can be calculated as a linear combination of square of vector fields coming from the infinitesimal representation. This had been proved in [1],[2] for $n = 1$. Our main Theorems are given in Sections 8 and 9. They extend (3.20)-(3.21)- (3.23) obtained for the Poincaré disk.

2 Laplacian and O-U operators on a Kähler domain, their expressions in terms of the infinitesimal representation

In this section, we explain our motivation. A real measure μ on $\mathcal{D}$ is invariant for the operator D if

$$\int_{\mathcal{D}} (D\Psi)(\mathcal{Z}, \overline{\mathcal{Z}}) d\mu = 0 \tag{2.1}$$

for all differentiable Ψ such that the integral in (2.1) is well defined. The representation (1.3) is unitary in the space of square integrable holomorphic functions $L^2_{Hol}(\mu)$ if

$$\int |(T_g\Phi)(\mathcal{Z})|^2 \, d\mu(\mathcal{Z}) \; = \; \int |\Phi(\mathcal{Z})|^2 \, d\mu(\mathcal{Z}) \tag{2.2}$$

We say that μ is unitarizing for T_g. In the following, we relate invariant measures and unitarizing measures.

2.1 Kähler geometry and unitary holomorphic representations

The representation (1.2) is unitary in the space of square integrable holomorphic functions $L^2_{Hol}(\mu)$ when

$$\mu = \mu^\gamma = \exp(-\gamma \log(\det(I - \mathcal{Z}\overline{\mathcal{Z}}))dv \tag{2.3}$$

and dv is the volume measure on the Siegel disk $\mathcal{D}_n$. The manifold $\mathcal{D}_n$ is a Kähler manifold with Kähler potential

$$U(\mathcal{Z}, \overline{\mathcal{Z}}) = \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) \tag{2.4}$$

More generally, assume that $\mathcal{D}$ is a Kähler manifold. We write an element of $\mathcal{D}$ as

$$\mathcal{Z} = (z_1, z_2, \cdots, z_p, \cdots) \tag{2.5}$$

We assume the existence of a globally defined Kähler potential,

$$U = \log K(\mathcal{Z}, \overline{\mathcal{Z}}) \tag{2.6}$$

this means
1) $K(\mathcal{Z}, \overline{\mathcal{Z}})$ is a positive **real** valued function.
2) The metric on $\mathcal{D}$ is given by

$$ds^2 = -\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}}) dz_j d\overline{z_k} \tag{2.7}$$

We put

$$\omega = i \sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}}) dz_j \wedge d\overline{z_k} \tag{2.8}$$

If $\mathcal{D}$ is of complex dimension p, then

$$dv \;=\; (\omega)^{\wedge p} \tag{2.9}$$

defines the volume element on $\mathcal{D}$. We denote by Δ the Riemannian Laplace operator on $\mathcal{D}$ with the Kählerian metric (2.7). We define the matrix

$$P = (p_{jk}) \quad with \quad p_{jk} = \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}}) \tag{2.10}$$

Since K is a real function, we have

$${}^t\overline{P} \;=\; P \tag{2.11}$$

and we put

$$M = (m_{jk}) = \;constant\, \overline{P}^{-1} \tag{2.12}$$

Let

$$\Delta = \sum_{jk} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \tag{2.13}$$

then

1) Δ is a real operator, up to a multiplicative constant, it is the Laplace-Beltrami operator associated to the Kählerian metric.

2) Δ is an invariant operator with respect to the volume measure dv,

$$\int \Delta\Psi \, dv = 0 \tag{2.14}$$

3) Let

$$\Delta^c = \Delta - c \sum_{j,k} m_{jk} [\frac{\partial}{\partial z_j} \log K\,] \,\frac{\partial}{\partial \overline{z_k}} \tag{2.15}$$

where c is a constant. Then Δ^c has $\mu^c = \;constant\; \exp(-c\log K)\, dv$ for invariant measure ($\int \Delta^c \,\Psi \, d\mu^c = 0$). Following [8], we define a Berezinian measure as a probability measure μ^c of the form

$$\mu^c = \;constant\; \exp(-c\log K)\, dv \tag{2.16}$$

where K is the Kähler potential, dv is the volume on $\mathcal{D}$ and where the constant is a normalizing constant in order to have a probability measure.

Definition 2.1 *Let*

$$\Delta^c = \Delta - cV \quad with \quad V = \sum_{j,k} m_{jk} [\frac{\partial}{\partial z_j} \log K\,] \,\frac{\partial}{\partial \overline{z_k}} \tag{2.17}$$

then Δ^c is called complex Ornstein-Uhlenbeck (O-U operator) and

$$\Delta^c + \overline{\Delta^c} \tag{2.18}$$

is called real O-U operator. If μ^c defined in (2.16) is a unitarizing measure for the representation T_g, then the O-U operator is said to be associated to T_g..

The Berezinian measure μ^γ in (2.3) is associated to the representation (1.2). It .is an invariant measure for the complex Ornstein-Uhlenbeck operator on $\mathcal{D}_n$.

The aim of this note is to show that it is possible to express the Laplacian and the O-U operator in terms of the infinitesimal representation. We do the calculations on $\mathcal{D}_2$. In that case, we establish the formulas giving such expressions of Δ and Δ^c

The interest of such expressions comes from the infinite dimensional setting: The difficulty in infinite dimension is to define the volume measure dv and the unitarizing measure μ^c of the representation. Thus it is convenient to first define Δ and Δ^c and then obtain dv and μ^c as invariant measures associated respectively to the elliptic operators Δ and Δ^c.

2.2 Infinitesimal representation and associated differential operators

The infinitesimal representation is defined as follows: To $v \in \mathcal{G}$, the Lie algebra of G, we associate the differential operator

$$\rho(v)\Phi(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} T_{g_\epsilon}\Phi(\mathcal{Z}) \tag{2.19}$$

where

$$v = \frac{d}{d\epsilon}_{|\epsilon=0} g_\epsilon \quad with \quad g_0 = e \quad and \quad g_\epsilon \in G, \quad \epsilon \text{ is a real parameter} \tag{2.20}$$

The Lie algebra $\mathcal{G}$ is a real vector space. Let $[v_1, v_2]$ be the Lie bracket on $\mathcal{G}$, then

$$\rho([v_1, v_2]) = \rho(v_1)\rho(v_2) - \rho(v_2)\rho(v_1) \tag{2.21}$$

For a holomorphic function Φ, we have

$$\begin{aligned}\rho(v)\Phi(\mathcal{Z}) = & \frac{d}{d\epsilon}_{|\epsilon=0} k_{g_\epsilon}(\mathcal{Z})\,\Phi'(\mathcal{Z}) + [\frac{d}{d\epsilon}_{|\epsilon=0} h_{g_\epsilon}(\mathcal{Z})] \quad \Phi(\mathcal{Z}) \\ = & \ \alpha(v)(\mathcal{Z})\Phi'(\mathcal{Z}) + \beta(v)\,\Phi(\mathcal{Z}) \end{aligned} \tag{2.22}$$

where

$$\alpha(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} k_{g_\epsilon}(\mathcal{Z}) \quad , \qquad \beta(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} h_{g_\epsilon}(\mathcal{Z}) \tag{2.23}$$

and Φ' is the complex derivative of Φ. In other words, if $\mathcal{Z} = (z_1, z_2, \cdots)$ and $\mathcal{W} = k_g(\mathcal{Z})$ is given by

$$k_g(\mathcal{Z}) = ((k_g)_1(\mathcal{Z}),\, (k_g)_2(\mathcal{Z}), \cdots) \tag{2.24}$$

then

$$(\rho(v)\Phi)(\mathcal{Z}) = \sum_j \alpha_j(v)(\mathcal{Z})\frac{\partial}{\partial z_j}\,\Phi + \beta(v)\,\Phi(\mathcal{Z}) \tag{2.25}$$

with

$$\alpha_j(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0}(k_{g_\epsilon})_j(\mathcal{Z}), \quad j = 1, 2, \cdots \tag{2.26}$$

We denote $L(v)$ the vector field

$$(L(v)\Phi)(\mathcal{Z}) = \sum_j \alpha_j(v)(\mathcal{Z})\frac{\partial}{\partial z_j}\Phi = \frac{d}{d\epsilon}_{|\epsilon=0}\Phi(k_{g_\epsilon}(\mathcal{Z})) \quad then \quad \rho(v) = L(v) + \beta(v)I \tag{2.27}$$

In the case of $\mathcal{D}_2$, one can find a basis of the Lie algebra $\mathcal{G}$ of $Sp(2\times 2)$ and constants a_k such that

$$\sum_k a_k\rho(e_k)L(e_k) = 0 \tag{2.28}$$

Let K be the Kähler potential on $\mathcal{D}_2$ and

$$\Delta = \sum_{j,k} m_{jk}\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \tag{2.29}$$

be the associated Riemannian Laplacian on $\mathcal{D}_2$ for the Kähler metric, then we show that Δ can be written as

$$\Delta = \sum_k a_k L(e_k)\overline{L(e_k)} \tag{2.30}$$

The operator Δ^c defined by

$$\Delta^c = \Delta - c\sum_{j,k} m_{jk}[\frac{\partial}{\partial z_j}\log K]\,\frac{\partial}{\partial \overline{z}_k} \tag{2.31}$$

has for invariant measure μ^c and T_g is unitary in $L^2_{Hol}(\mu^c)$, the set of square integrable holomorphic functions. Moreover from (9.4)-(**??**),

$$\Delta^c = \sum_k a_k\rho(e_k)\overline{L(e_k)} = \Delta - \,cV \tag{2.32}$$

2.3 Adjoint for the first order differential operators related to the representation and unitarity

Definition 2.2 *A first order differential operator L on $\mathcal{D}$ is said to be holomorphic if it is of the form*

$$L = \sum_j a_j(\mathcal{Z})\frac{\partial}{\partial z_j} \tag{2.33}$$

where a_j are holomorphic functions. We say also that L is a holomorphic vector field.

Lemma 2.3 *Assume that T_g is unitary as in (2.2) where μ is a positive real measure, then for any holomorphic functions Φ and Ψ defined on $\mathcal{D}$, we have*

$$\int (\rho(v)\Phi)(\mathcal{Z})\overline{\Psi(\mathcal{Z})}d\mu(\mathcal{Z}) + \int \Phi(\mathcal{Z})\overline{(\rho(v)\Psi)(\mathcal{Z})}d\mu(\mathcal{Z}) = 0 \tag{2.34}$$

Proof. Let g_ϵ in (2.2), we take the derivative with respect to ϵ. It gives the result since $L(v)$ is a holomorphic vector field and we have

$$L(v)(\Phi\overline{\Psi}) = \overline{\Psi}L(v)(\Phi) \tag{2.35}$$

Notation 2.4 *Let V be a vector field, we denote $div_\mu V$ the function such that for any differentiable function F null outside a compact set in $\mathcal{D}$ and vanishing out of the support of μ, we have*

$$\int (div_\mu V)(\mathcal{Z})F(\mathcal{Z})d\mu(\mathcal{Z}) = \int (VF)(\mathcal{Z})d\mu(\mathcal{Z}) \tag{2.36}$$

Then (2.34), can be expressed as $div_\mu(L(v) + \overline{L(v)}) = -(\beta(v) + \overline{\beta(v)})$.

2.4 Second order differential operators related to the representation

In this section, we raise several questions in a more general setting. Let $\rho(v)$ and $L(v)$ as in (2.22)- (2.27). Let $(e_1, e_2, \cdots)$ be a basis of the Lie algebra $\mathcal{G}$ of G and real constants A_{jk}. We consider second order differential operators of the form

$$\sum_{j,k} A_{jk}\rho(e_j)\overline{L(e_k)} \tag{2.37}$$

Remark that the second order derivatives in (2.37) are all of the form $\frac{\partial^2}{\partial z_j \partial \overline{z_k}}$ There are no terms like $\frac{\partial^2}{\partial z_j \partial z_k}$ or $\frac{\partial^2}{\partial \overline{z_j} \partial \overline{z_k}}$ We ask the following questions

1) Does there exist constants A_{jk} and an appropriate basis $(e_1, e_2, \cdots)$ such that the Laplacian Δ is given by $\Delta = \sum_{j,k} A_{jk}L(e_j)\overline{L(e_k)}$ and such that the operator

$$\sum_{j,k} A_{jk}\rho(e_j)L(e_k) \tag{2.38}$$

is the multiplication by a constant.

2) Assume that μ_0 is a real measure such that

$$\int |(T_g\Phi)(\mathcal{Z})|^2 \, d\mu_0(\mathcal{Z}) \;=\; \int |\Phi(\mathcal{Z})|^2 \, d\mu_0(\mathcal{Z}) \quad for \quad T_g\Phi(\mathcal{Z}) = \Phi(k_g(\mathcal{Z})) \tag{2.39}$$

Is the measure μ_0 an invariant measure for the Laplacian Δ?

3) Consider the measure μ of the representation T_g such that (2.2) holds. With the same constants A_{jk}. and basis $(e_1, e_2, \cdots)$ as in (2.38), does the operator (2.37) has the measure μ for invariant measure?

4) If the answer to 3) is positive, how is the first order part in (2.37) expressed in terms of the derivatives of the Kähler potential on $\mathcal{D}$.

5) Assume μ is the measure of the unitary representation, that means: μ is a real positive measure and we have (2.2). Does there exist a second order differential operator like (2.37) and admitting μ as invariant measure?

3 The Poincaré disk

Let $G = Sp(2)$ be the group of matrices

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \quad whare \quad |a|^2 - |b|^2 = 1 \tag{3.1}$$

Taking the differential of $a\overline{a} - b\overline{b} = 1$ at the identity, it is immediate that the Lie algebra $\mathcal{G}$ of G is the set of matrices

$$v = \begin{pmatrix} i\alpha & \beta \\ \overline{\beta} & -i\alpha \end{pmatrix} \quad where \quad \alpha \text{ is real} \tag{3.2}$$

It is a real vector space of dimension 3. We take for basis of $\mathcal{G}$,

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.3}$$

Let $g_t^j = \exp(te_j)$,

$$g_1 = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \quad g_2 = \begin{pmatrix} ch(t/2) & sh(t/2) \\ sh(t/2) & ch(t/2) \end{pmatrix}, \quad g_3 = \begin{pmatrix} ch(t/2) & i\,sh(t/2) \\ -i\,sh(t/2) & ch(t/2) \end{pmatrix} \tag{3.4}$$

The Poincaré disk is the unit disk $\mathcal{D} = \{z\overline{z} < 1\}$ with the metric

$$ds^2 = -\frac{\partial^2}{\partial z \partial \overline{z}} \log(1 - z\overline{z}) = \frac{dz d\overline{z}}{(1 - z\overline{z})^2} \tag{3.5}$$

$$ds^2 = -\frac{\partial^2}{\partial z \partial \overline{z}} U(z, \overline{z}) \quad \text{with the Kähler potential} \quad U(z, \overline{z}) = \log(1 - z\overline{z}) \tag{3.6}$$

The group G acts on the unit disk,

$$if \quad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{we put } u = k_g(z) = \frac{az + b}{\overline{b}z + \overline{a}} \tag{3.7}$$

Then

$$k_{g_1 g_2}(z) = k_{g_1}(k_{g_2}(z)) \quad and \quad k_g'(z) = \frac{1}{(\overline{b}z + \overline{a})^2} \tag{3.8}$$

We have the two fundamental identities

$$\frac{du \wedge d\overline{u}}{(1 - u\overline{u})^2} = \frac{dz \wedge d\overline{z}}{(1 - z\overline{z})^2} \quad and \quad 1 - u\overline{u} = \frac{1 - z\overline{z}}{(\overline{b}z + \overline{a})(b\overline{z} + a)} \tag{3.9}$$

The second relation in (3.9) can be written as

$$U(k_g(z)) = U(z) \ + 2\Re \log(\bar{b}z + \bar{a}) \tag{3.10}$$

The holomorphic representation is given by

$$[T_g\Phi)(\mathcal{Z}) = (k_g'(\mathcal{Z}))^{\alpha}\Phi(k_g(\mathcal{Z})) \tag{3.11}$$

From (3.9), we deduce that the operator T_g is unitary in $L^2_{Hol}(\mu)$ with

$$d\mu = (1-z\overline{z})^{2\alpha}\frac{dz\,d\overline{z}}{(1-z\overline{z})^2} \tag{3.12}$$

By direct calculation, we prove

Theorem 3.1 *The measure μ is an invariant measure for*

$$\Delta^{OU} = (1-z\overline{z})^2[\frac{\partial^2}{\partial z\partial\overline{z}} + \alpha\frac{\partial}{\partial z}\log(1-z\overline{z})\frac{\partial}{\partial\overline{z}} + \alpha\frac{\partial}{\partial\overline{z}}\log(1-z\overline{z})\frac{\partial}{\partial z}] \tag{3.13}$$

We have

$$\Delta^{OU} = \Delta + \alpha V \tag{3.14}$$

where Δ is the Laplacian on the unit disk,

$$\Delta = (1-z\overline{z})^2\frac{\partial^2}{\partial z\partial\overline{z}} \tag{3.15}$$

Moreover the vector field

$$W = (1-z\overline{z})(\overline{z}\frac{\partial}{\partial\overline{z}} - z\frac{\partial}{\partial z}) \tag{3.16}$$

is a free-divergence vector field, this means $div_{\mu}(W) = 0$ or equivalently $\int (W\psi)d\mu = 0$.

.

Our problem is whether it is possible to find Δ^{OU} in terms of the infinitesimal representation. As in (2.19), we define

$$\rho(e_j)\Phi(z) = \frac{d}{d\epsilon}_{|\epsilon=0} T^{\gamma}_{g^j_\epsilon}\Phi(z) \ = \ \alpha_j(z)\,\frac{d}{dz}f(z) \ + \ \beta_j(z)\,f(z) \tag{3.17}$$

We put

$$L_j\Phi(z) = \ \alpha_j(z)\,\frac{d}{dz}\Phi(z) \tag{3.18}$$

then

$$\begin{aligned}
\rho(e_1)\Phi(z) &= \ -i[z\Phi'(z) \ + \ \gamma\,\Phi(z)] \\
\rho(e_2)\Phi(z) &= \ -\frac{1}{2}(1-z^2)\Phi'(z) \ + \ \gamma\,z\,\Phi(z) \\
\rho(e_3)\Phi(z) &= \ -\frac{i}{2}(1+z^2)\Phi'(z) \ - \ i\,\gamma\,z\,\Phi(z)
\end{aligned} \tag{3.19}$$

Theorem 3.2 *Let*

$$A = \rho(e_2)\overline{L_2} + \rho(e_3)\overline{L_3} - \rho(e_1)\overline{L_1} \tag{3.20}$$

We have

$$A = \frac{1}{2}(1 - z\overline{z})^2[\frac{\partial^2}{\partial z\partial\overline{z}} - \frac{2\alpha\,\overline{z}}{(1 - z\overline{z})}\frac{\partial}{\partial\overline{z}}]$$

or equivalently

$$A = \frac{1}{2}(1 - z\overline{z})^2[\frac{\partial^2}{\partial z\partial\overline{z}} - 2\alpha\frac{\partial}{\partial z}\log(1 - z\overline{z})\frac{\partial}{\partial\overline{z}}] \tag{3.21}$$

The measure μ^γ of the representation is an invariant measure for A. We have

$$\Delta^{OU} = A + \overline{A} \tag{3.22}$$

where Δ^{OU} is given by (3.13). Moreover,

$$[\rho(e_2)^2 + \rho(e_3)^2 - \rho(e_1)^2]\Phi = (\alpha^2 - \alpha)\,\Phi \tag{3.23}$$

Proof.

$$\rho(e_1)^2\Phi = -z^2\Phi''(z) - (1 + 2\gamma)z\Phi'(z) - \gamma^2\Phi(z)$$

$$\rho(e_2)^2\Phi = \frac{1}{4}(1 - z^2)^2\Phi''(z) - \frac{1 + 2\gamma}{2}z(1 - z^2)\Phi'(z) - \frac{\gamma}{2}(1 - (1 + 2\gamma)z^2)\Phi(z)$$

$$\rho(e_3)^2\Phi = -\frac{1}{4}(1 + z^2)^2\Phi''(z) - \frac{1 + 2\gamma}{2}z(1 + z^2)\Phi'(z) - \frac{\gamma}{2}(1 + (1 + 2\gamma)z^2)\Phi(z)$$

4 Metric and Laplacian on $\mathcal{D}_n$. Action of $Sp(2n)$.

Before specifying the calculation to $\mathcal{D}_2$, we recall for completeness some known formulae relative to the holomorphic representation of $Sp(2n)$ on $\mathcal{D}_n$.

4.1 Metric and Laplacian on $\mathcal{D}_n$,

The manifold $\mathcal{D}_n$ has complex dimension $n(n+1)/2$. We denote the coefficients of the matrix $\mathcal{Z}$.

$$\mathcal{Z} = (z_1, z_2, \cdots, z_{n(n+1)/2})$$

As in (2.4), let

$$K(\mathcal{Z}, \overline{\mathcal{Z}}) = \det(I - \mathcal{Z}\overline{\mathcal{Z}}) \quad \textit{and} \quad U(\mathcal{Z}, \overline{\mathcal{Z}}) = \log K(\mathcal{Z}, \overline{\mathcal{Z}}) \tag{4.1}$$

Since $\mathcal{Z}$ is symmetric, the transposed matrix $(I - \mathcal{Z}\overline{\mathcal{Z}})^t$ is equal to $I - \overline{\mathcal{Z}}\mathcal{Z}$ and

$$K(\mathcal{Z}, \overline{\mathcal{Z}}) = \overline{K(\mathcal{Z}, \overline{\mathcal{Z}})} \tag{4.2}$$

This shows that $K(\mathcal{Z}, \overline{\mathcal{Z}})$ is a **real valued function**. On the domain $\mathcal{D}_n$, we consider the Kählerian metric

$$ds^2 = -\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log \det(I - \mathcal{Z}\overline{\mathcal{Z}})\, dz_j d\overline{z}_k = \sum_{j,k} p_{jk} dz_j d\overline{z_k} \tag{4.3}$$

Let

$$P = (p_{jk}) \quad with \quad p_{jk} = \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log \det(I - \mathcal{Z}\overline{\mathcal{Z}}) \tag{4.4}$$

We have ${}^t\overline{P} = P$ and $\det P = -\, constant \times (\det(I - \mathcal{Z}\overline{\mathcal{Z}})^{-(n+1)}$. We have

$$\omega = i \sum_{j,k} p_{jk} dz_j \wedge d\overline{z_k} \;=\; i\,\overline{\partial}\partial\, U \tag{4.5}$$

then $d\omega = 0$, $\partial\omega = 0$, $\overline{\partial}\omega = 0$. We put

$$M = -4\overline{P}^{-1}, \quad M = (m_{jk}), \quad \Delta = \sum_{j,k} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \tag{4.6}$$

then

$$\sum_j \frac{\partial}{\partial z_j}(m_{jk} \det P) = 0 \tag{4.7}$$

and Δ is the Laplacian on $\mathcal{D}$ associated to the metric (4.3). The volume element is given by

$$dv = \frac{dz_1 \wedge\, d\overline{z_1} \wedge \cdots}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}}|^{n+1}} \tag{4.8}$$

Integration by parts shows that the volume is an invariant measure for Δ ($\int \Delta\psi\, dv = 0$).

4.2 The group $Sp(2n)$

Let J be the $2n \times 2n$ matix, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is the unit matrix of order n. The complex group $Sp(2n)$ is the set of $2n \times 2n$ complex matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which satisfy

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \overline{g} \tag{4.9}$$

and

$${}^t g\, J\, g \;=\; J \tag{4.10}$$

${}^t g$ is the transposed matrix of g. The condition (4.9) is equivalent to the fact that g is of the form $g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$. The condition (4.10) is equivalent to

$${}^t A\overline{B} \;=\; {}^t\overline{B}\, A \qquad and \qquad {}^t\overline{A}A -{}^t B\overline{B} = I \tag{4.11}$$

Remark 4.1 *If (A, B) is solution of (4.11) then (A, iB) is also solution of (4.11).*

4.3 The group $Sp(2n)$ of order n acts on $\mathcal{D}_n$

The group $G = Sp(2n)$ acts on the domain $\mathcal{D}_n$ with (1.1). Let

$$\mathcal{Z} = (z_1, z_2, \cdots) \to \mathcal{W} = (k_g(\mathcal{Z})_1, k_g(\mathcal{Z})_2, \cdots) = (w_1, w_2, \cdots) \tag{4.12}$$

The Jacobian matrix of the map $\mathcal{Z} \to k_g(\mathcal{Z})$ has determinant equal to

$$\det[(\frac{\partial}{\partial z_j} k_g(\mathcal{Z})_p)_{1 \le j,p \le \frac{n(n+1)}{2}}] = (\det(\,(\overline{B}\mathcal{Z} + \overline{A})^{-1})\,)^{n+1} \tag{4.13}$$

If $\mathcal{Z}$ is symmetric, the conditions on A and B imply that $k_g(\mathcal{Z}) = \mathcal{W}$ is also symmetric and we have, $I - k_g(\mathcal{Z})\overline{k_g(\mathcal{Z})} = ({}^t\overline{A} + \mathcal{Z}\,{}^t\overline{B})^{-1}\,(I - \mathcal{Z}\overline{\mathcal{Z}})\,(B\overline{\mathcal{Z}} + A)^{-1}$ thus

$$\det(I - k_g(\mathcal{Z})\overline{k_g(\mathcal{Z})}) = \det(I - \mathcal{Z}\overline{\mathcal{Z}}) \times \frac{1}{|\det(\overline{B}\mathcal{Z} + \overline{A})|^2} \tag{4.14}$$

which is the same as (3.9) for arbitrary n, see also the “cocycle “ identity in [6], p. 744. Since the Kählerian metric is

$$ds^2 = -\partial\overline{\partial}\log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) \tag{4.15}$$

it results from (4.14) that the metric ds^2 is invariant under the action of the group G. This extends the first identity in (3.9). In fact, since $\mathcal{Z}$ is **symmetric**, differentiating $(\overline{B}\mathcal{Z} + \overline{A})^{-1}(\overline{B}\mathcal{Z} + \overline{A})$, we deduce

$$d(\overline{B}\mathcal{Z} + \overline{A}) = -(\overline{B}\mathcal{Z} + \overline{A})^{-1}\overline{B}\,[d\mathcal{Z}]\,(\overline{B}\mathcal{Z} + \overline{A})^{-1} \tag{4.16}$$

and

$$d\mathcal{W} = (\mathcal{Z}\,{}^t\overline{B} + {}^t\overline{A})^{-1}[d\mathcal{Z}\,]\,(\overline{B}\mathcal{Z} + \overline{A})^{-1} \tag{4.17}$$

Let $\mathcal{W} = k_g(\mathcal{Z})$. We deduce from (4.14) that

$$\frac{dw_1 \wedge dw_2 \wedge \cdots \wedge dw_{n(n+1)/2}}{[\det(I - \mathcal{W}\overline{\mathcal{W}})]^{(n+1)/2}} = \frac{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n(n+1)/2}}{[\det(I - \mathcal{Z}\overline{\mathcal{Z}})]^{(n+1)/2}} \tag{4.18}$$

This generalizes (3.9). The representation (1.2) is unitary in $L^2_{Hol}(\mu^\gamma)$ with

$$d\mu^\gamma = \frac{dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_{n(n+1)/2} \wedge d\overline{z_{n(n+1)/2}}}{[\det(I - \mathcal{Z}\overline{\mathcal{Z}})]^{n+1+\gamma}} = \frac{dv}{[\det(I - \mathcal{Z}\overline{\mathcal{Z}})]^\gamma} \tag{4.19}$$

where dv is the volume measure on $\mathcal{D}_n$. Compare with (3.12). As in (3.13)-(3.16), we verify by simple calculation that

$$d\mu^\gamma = \exp[-\gamma\log\det(I - \mathcal{Z}\overline{\mathcal{Z}})]\,dv = \exp[-\gamma\log K(\mathcal{Z}, \overline{\mathcal{Z}})]\,dv \tag{4.20}$$

and that μ^γ is an invariant measure ($\int \Delta^\gamma\Psi\,d\mu^\gamma = 0$) for

$$\Delta^\gamma = \sum_{jk} m_{jk}\frac{\partial^2}{\partial z_j \partial\overline{z_k}} - \gamma\sum_{j,k} m_{jk}\frac{\partial}{\partial z_j}\log\det(I - \mathcal{Z}\overline{\mathcal{Z}})\frac{\partial}{\partial\overline{z_k}} \tag{4.21}$$

5 Metric and Laplacian on $\mathcal{D}_2$. Action of $Sp(2\times 2)$

For $n=2$, we calculate metric, laplacian and representation of section 4. Let

$$\mathcal{Z}=\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \quad \textit{and} \quad I-\mathcal{Z}\overline{\mathcal{Z}}=\begin{pmatrix} u & -w \\ -\overline{w} & v \end{pmatrix} \tag{5.1}$$

with

$$u=1-z_1\overline{z_1}-z_2\overline{z_2}, \quad v=1-z_3\overline{z_3}-z_2\overline{z_2}, \quad w=z_1\overline{z_2}+z_2\overline{z_3} \tag{5.2}$$

The condition $I-\mathcal{Z}\overline{\mathcal{Z}}>0$ means that the eigenvalues of the hermitian matrix $I-\mathcal{Z}\overline{\mathcal{Z}}$ are strictly positive. For u, v, w, it implies that $u>0$, $v>0$, $uv-w\overline{w}>0$: Thus $\mathcal{D}_2$ is a bounded domain in C^3.

In the case of the Poincaré disk, the Laplacian, see (3.15), is expressed as

$$\Delta=u_0^2\frac{\partial^2}{\partial z\partial\overline{z}} \quad \textit{where} \quad u_0=1-z\overline{z}$$

Similarly, in subsection 5.1, we obtain the coefficients of $\frac{\partial^2}{\partial z_j\partial\overline{z}_k}$ in the $\mathcal{D}_2$-Laplacian as functions of the coefficients of the matrix $I-\mathcal{Z}\overline{\mathcal{Z}}$, in that case in terms of u, v, w defined in (5.2). In subsection 5.2, we calculate the Jacobian of the map $\mathcal{Z}\to k_g(\mathcal{Z})$ and we determine by direct calculation the measue μ^γ which makes T_g a unitary operator.

5.1 Metric and Laplacian on $\mathcal{D}_2$

Let $K(\mathcal{Z},\overline{\mathcal{Z}})=\det(I-\mathcal{Z}\overline{\mathcal{Z}})$. Consider the matrix

$$P=(p_{ik})=(\frac{\partial^2}{\partial z_i\overline{\partial z_k}}\log K(\mathcal{Z},\overline{\mathcal{Z}}))=(\frac{\partial^2}{\partial z_i\overline{\partial z_k}}\log(uv-w\overline{w})) \tag{5.3}$$

Lemma 5.1 *We have* $\overline{P}={}^tP$,

$$P=\frac{1}{(uv-w\overline{w})^2}\begin{pmatrix} -v^2 & -2v\overline{w} & -\overline{w}^2 \\ -2vw & -2(uv+w\overline{w}) & -2u\overline{w} \\ -w^2 & -2uw & -u^2 \end{pmatrix} \tag{5.4}$$

and

$$\det P=-\frac{2}{(uv-w\overline{w})^3}=-2\,[\det(I-\mathcal{Z}\overline{\mathcal{Z}})]^{-3} \tag{5.5}$$

Proof

$$\frac{\partial}{\partial z_1}(uv-w\overline{w})=-\overline{z_1}v-\overline{z_2}\overline{w}=-\overline{z_1}(1-z_3\overline{z_3})-z_3\overline{z_2}^2$$

$$\frac{\partial^2}{\partial z_1\partial\overline{z_1}}(uv-w\overline{w})=z_3\overline{z_3}-1, \qquad \frac{\partial^2}{\partial z_1\partial\overline{z_2}}(uv-w\overline{w})=-2\overline{z_2}z_3$$

$$\frac{\partial^2}{\partial z_1\partial\overline{z_3}}(uv-w\overline{w})=\overline{z_1}z_3$$

$$\frac{\partial}{\partial z_2}(uv - w\overline{w}) = -\overline{z_2}(u+v) - \overline{z_1}w - \overline{z_3 w} = -2\overline{z_2}(1 - z_2\overline{z_2}) - 2z_2\overline{z_1 z_3}$$

$$\frac{\partial^2}{\partial z_2 \partial \overline{z_2}}(uv - w\overline{w}) = 4z_2\overline{z_2} - 2 \qquad \frac{\partial^2}{\partial z_2 \partial \overline{z_3}}(uv - w\overline{w}) = -2z_2\overline{z_1}$$

$$\frac{\partial}{\partial z_3}(uv - w\overline{w}) = -u\overline{z_3} - w\overline{z_2} = -\overline{z_3}(1 - z_1\overline{z_1}) - z_1\overline{z_2}^2, \qquad \frac{\partial^2}{\partial z_3 \partial \overline{z_3}}(uv - w\overline{w}) = -(1 - z_1\overline{z_1})$$

We have

$$\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log \phi = \frac{1}{\phi^2}[\phi \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} - (\frac{\partial \phi}{\partial z_i})(\frac{\partial \phi}{\partial \overline{z_j}})]$$

For $\phi = uv - w\overline{w}$, we put

$$a_{ij} = \phi \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} - (\frac{\partial \phi}{\partial z_i})(\frac{\partial \phi}{\partial \overline{z_j}})$$

We calculate

$$a_{11} = -v^2, \quad a_{12} = -2v\overline{w}, \quad a_{13} = -\overline{w}^2, \quad a_{22} = -2(uv + w\overline{w})$$

We deduce $\det P$ with Sarrus rule

$$\det \begin{pmatrix} -v^2 & -2v\overline{w} & -\overline{w}^2 \\ -2vw & -2(uv + w\overline{w}) & -2u\overline{w} \\ -w^2 & -2uw & -u^2 \end{pmatrix} = -2\,(uv - w\overline{w})^3$$

Lemma 5.2 *Let* $M = -4\overline{P}^{-1} = (m_{jk})$ *then*

$$M = (m_{ij}) = \begin{pmatrix} 4u^2 & -4uw & 4w^2 \\ -4u\overline{w} & 2(uv + w\overline{w}) & -4vw \\ 4\overline{w}^2 & -4v\overline{w} & 4v^2 \end{pmatrix} \tag{5.6}$$

Theorem 5.3 *(Laplacian on $\mathcal{D}_2$).*
Let

$$\Delta = \sum_{j,k} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \tag{5.7}$$

We have

$$\Delta = 4u^2 \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + 2(uv + w\overline{w}) \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + 4v^2 \frac{\partial^2}{\partial z_3 \partial \overline{z_3}} - 4u(w \frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + \overline{w} \frac{\partial^2}{\partial z_2 \partial \overline{z_1}})$$
$$-4v(\overline{w} \frac{\partial^2}{\partial z_3 \partial \overline{z_2}} + w \frac{\partial^2}{\partial z_2 \partial \overline{z_3}}) + 4(w^2 \frac{\partial^2}{\partial z_1 \partial \overline{z_3}} + \overline{w}^2 \frac{\partial^2}{\partial z_3 \partial \overline{z_1}}) \tag{5.8}$$

Consider the metric

$$H = -\sum_{j,k} \frac{\partial}{\partial z_j \partial \overline{z_k}} [\log \det(I - \mathcal{Z}\overline{\mathcal{Z}})] dz_j d\overline{z_k} \tag{5.9}$$

Then H defines a Kählerian metric on $\mathcal{D}_2$ and Δ is the associated Laplacian.

5.2 Jacobian of the action of $Sp(2 \times 2)$ on $\mathcal{D}_2$

. Consider the complex matrices of order 2,

$$\mathcal{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \quad \textit{and} \quad \mathcal{W} = k_g(\mathcal{Z}) = (A\mathcal{Z} + B)(\overline{B}\mathcal{Z} + \overline{A})^{-1}$$

$$\mathcal{W} = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \tag{5.10}$$

In order to calculate the jacobian of the map $\mathcal{Z} \to k_g(\mathcal{Z})$, we write

$$k_g : (z_1, z_2, z_3) \to (w_1, w_2, w_3)$$

and we express $dw_1 \wedge dw_2 \wedge dw_3$ in terms of $dz_1 \wedge dz_2 \wedge dz_3$.

Lemma 5.4

$$dw_1 \wedge dw_2 \wedge dw_3 = (\det[(\overline{B}\mathcal{Z} + \overline{A})^{-1}])^3 \, dz_1 \wedge dz_2 \wedge dz_3 \tag{5.11}$$

Proof. Denote

$$N = (\overline{B}\mathcal{Z} + \overline{A})^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

where C_1, C_2 are the columns of the matrix N. According to (4.17),

$$\begin{pmatrix} dw_1 & dw_2 \\ dw_2 & dw_3 \end{pmatrix} = {}^tN \begin{pmatrix} dz_1 & dz_2 \\ dz_2 & dz_3 \end{pmatrix} N$$

Making the product of the matrices, it gives

$$dw_1 = \alpha_1^2 dz_1 + 2\alpha_1\alpha_3 dz_2 + \alpha_3^2 dz_3 = {}^tC_1 \, d\mathcal{Z} \, C_1$$

$$dw_2 = \alpha_1\alpha_2 dz_1 + (\alpha_1\alpha_4 + \alpha_2\alpha_3)dz_2 + \alpha_3\alpha_4 dz_3 = {}^tC_1 \, d\mathcal{Z} \, C_2$$

$$dw_3 = \alpha_2^2 dz_1 + 2\alpha_2\alpha_4 dz_2 + \alpha_4^2 dz_3 = {}^tC_2 \, d\mathcal{Z} \, C_2$$

We find

$$dw_1 \wedge dw_3 = (\alpha_1\alpha_4 - \alpha_2\alpha_3)[2\alpha_1\alpha_2 dz_1 \wedge dz_2 + (\alpha_1\alpha_4 + \alpha_2\alpha_3)dz_1 \wedge dz_3 + 2\alpha_3\alpha_4 dz_2 \wedge dz_3]$$

and

$$dw_1 \wedge dw_2 \wedge dw_3 = (\alpha_1\alpha_4 - \alpha_2\alpha_3)^3 \, dz_1 \wedge dz_2 \wedge dz_3$$

Lemma 5.5 *We have*

$$\frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}}))|^3} = \frac{dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \wedge dw_3 \wedge d\overline{w_3}}{|\det(I - \mathcal{W}\overline{\mathcal{W}})|^3} \tag{5.12}$$

Proof.

$$dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \wedge dw_3 \wedge d\overline{w_3} = \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det[\overline{B}\mathcal{Z} + \overline{A}]|^6}$$

We calculate

$$\frac{dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \wedge dw_3 \wedge d\overline{w_3}}{|\det(I - \mathcal{W}\overline{\mathcal{W}})|^p}$$

It is invariant if $p = 3$.

Theorem 5.6 *The representation (1.2) is unitary for μ^γ where*

$$d\mu^\gamma(z) = \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^3 \, |\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^\gamma} \tag{5.13}$$

Proof. We put $w = k_g(z)$, we have

$$I = \int T_g f(z)\overline{T_g f(z)} d\mu(z) = \int |\det(\overline{B}\mathcal{Z} + \overline{A})|^{2\gamma} f(w)\overline{f(w)} d\mu(z)$$

By (6.12),

$$I = \int \frac{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^\gamma}{|\det(I - \mathcal{W}\overline{\mathcal{W}})|^\gamma} f(w)\overline{f(w)} d\mu(z) = \int f(w)\overline{f(w)} d\mu(w)$$

Remark 5.7 *The calculation of the jacobian for n arbitrary.is more difficult. We may use [5], p. 53, where the expression for the volume element of $\mathcal{D}$ is given explicitly.*

6 The Lie algebra $\mathcal{SP}_{2n}$, $n = 2$

We need the classical lemma,

Lemma 6.1 *The Lie algebra $\mathcal{SP}_{2n}$ of $Sp(2n)$ has real dimension equal to $2n^2 + n$.*

Proof. We take the differentials of (4.9) and (4.10) at $g = Id$ and we denote dg the differential of g, this gives

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} dg \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = d\overline{g} \tag{6.1}$$

and

$$d\,({}^t g)\, J + J\, dg = 0 \tag{6.2}$$

From (6.1) and (6.2), we deduce that dg is of the form

$$e = \begin{pmatrix} a_1 & a_2 \\ \overline{a_2} & \overline{a_1} \end{pmatrix} \tag{6.3}$$

where the $n \times n$ matrices a_1, a_2 satisfy

$$a_1 + {}^t\overline{a_1} = 0 \quad \textit{and} \quad a_2 \text{ is complex symmetric} \tag{6.4}$$

The set of matrices $a_2 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ has **real** dimension

$$2\frac{n(n+1)}{2} = n(n+1)$$

and the set of matrices $a_1 = \begin{pmatrix} i\alpha & \beta \\ -\overline{\beta} & i\delta \end{pmatrix}$ where α and δ are real matrices has **real** dimension

$$2\frac{n(n-1)}{2} + n = n^2$$

This proves the lemma. •

For our purpse, we consider the following basis of the Lie algebra $\mathcal{SP}_{2n}$, $n = 2$.

Lemma 6.2 *For $n = 2$, we have for basis of the Lie algebra*

$$dg = \begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix}$$

where a_2 is one of the matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$$

$$e_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{6.5}$$

and

$$dg = \begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix}$$

where a_1 is one of the matrices

$$e_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_9 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{10} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \tag{6.6}$$

For

$$e = \begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} 0 & u_2 \\ \overline{u_2} & 0 \end{pmatrix} \text{ then } [e,u] = \begin{pmatrix} a_2\overline{u_2} - u_2\overline{a_2} & 0 \\ 0 & \overline{a_2}u_2 - \overline{u_2}a_2 \end{pmatrix}$$

To calculate the infinitesimal representation, we need to exponentiate the matiices in (6.5)-(6.6). We see that the Lie algebra $\mathcal{SP}_{2n}$, $n = 2$ is a direct sum of seven subspaces. These are

$$E_1 = \{\tau_1 e_1 + \tau_2 e_2\}, \quad E_2 = \{\tau_1 e_3 + \tau_2 e_4\}, \quad E_3 = \{\tau e_5\}, \quad E_4 = \{\tau e_6\}, \tag{6.7}$$

$$F_5 = \{\alpha e_9 + \delta e_{10}\}, \quad F_6 = \{\tau e_7\}, \quad F_7 = \{\tau e_8\} \tag{6.8}$$

where all parameters $\tau_1, \cdots$ are real.

6.1 Exponential of the matrices of the basis (6.5)-(6.6)

In order to calculate the infinitesimal representation as in (2.19)-(2.22), we exponentiate the matices as follows. We consider the case $n = 2$, but we keep in mind the number of parameters in the subspaces (E_j), (F_j), for arbitrary n. The calculation is elementary and is summarized in (6.9)-...-(6.15)

(E_1) (n parameters):

$$a_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \quad with \quad \tau_1, \quad \tau_2 \; real$$

then

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau_1) & 0 & sh(\tau_1) & 0 \\ 0 & ch(\tau_2) & 0 & sh(\tau_2) \\ sh(\tau_1) & 0 & ch(\tau_1) & 0 \\ 0 & sh(\tau_2) & 0 & ch(\tau_2) \end{pmatrix} \tag{6.9}$$

(E_2) (n parameters):

$$a_2 = \begin{pmatrix} i\tau_1 & 0 \\ 0 & i\tau_2 \end{pmatrix} \quad with \quad \tau_1, \quad \tau_2 \; real$$

then

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau_1) & 0 & i\,sh(\tau_1) & 0 \\ 0 & ch(\tau_2) & 0 & i\,sh(\tau_2) \\ i\,sh(\tau_1) & 0 & ch(\tau_1) & 0 \\ 0 & i\,sh(\tau_2) & 0 & ch(\tau_2) \end{pmatrix} \tag{6.10}$$

(E_3) ($\frac{n(n-1)}{2}$ parameters):

$$a_2 = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \quad with \quad \tau \; real$$

then

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau) & 0 & 0 & sh(\tau) \\ 0 & ch(\tau) & sh(\tau) & 0 \\ 0 & sh(\tau) & ch(\tau) & 0 \\ sh(\tau) & 0 & 0 & ch(\tau) \end{pmatrix} \tag{6.11}$$

(E_4) ($\frac{n(n-1)}{2}$ parameters)

$$a_2 = \begin{pmatrix} 0 & i\,\tau \\ i\,\tau & 0 \end{pmatrix} \quad with \quad \tau \; real$$

then

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau) & 0 & 0 & i\,sh(\tau) \\ 0 & ch(\tau) & i\,sh(\tau) & 0 \\ 0 & -i\,sh(\tau) & ch(\tau) & 0 \\ -i\,sh(\tau) & 0 & 0 & ch(\tau) \end{pmatrix} \tag{6.12}$$

Now consider $a_1 = \begin{pmatrix} i\alpha & \beta \\ -\overline{\beta} & i\delta \end{pmatrix}$ where α and δ are real.

(E_5) (n parameters):

$$a_1 = \begin{pmatrix} i\alpha & 0 \\ 0 & i\delta \end{pmatrix} \quad \textit{with} \quad \alpha,\, \delta \quad \textit{real}$$

then

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 & 0 & 0 \\ 0 & e^{i\delta} & 0 & 0 \\ 0 & 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 0 & e^{-i\delta} \end{pmatrix} \tag{6.13}$$

(E_6) ($\frac{n(n-1)}{2}$ parameters):

$$a_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix} \quad \textit{with} \quad \tau \quad \textit{real}$$

then

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{pmatrix} \tag{6.14}$$

(E_7) ($\frac{n(n-1)}{2}$ parameters):

$$a_1 = \begin{pmatrix} 0 & i\tau \\ i\tau & 0 \end{pmatrix} \quad \textit{with} \quad \tau \quad \textit{real}$$

then

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} \cos(\tau) & i\sin(\tau) & 0 & 0 \\ i\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -i\sin(\tau) \\ 0 & 0 & -i\sin(\tau) & \cos(\tau) \end{pmatrix} \tag{6.15}$$

7 The infinitesimal representation, $n = 2$

As in section 2, from (1.2), we have

$$\rho(v)\Phi(\mathcal{Z}) = \sum_{j=1}^{n(n+1)/2} \frac{d}{d\epsilon}_{|\epsilon=0} (k_{g_\epsilon}(\mathcal{Z}))_j \frac{\partial}{\partial z_j}\Phi(\mathcal{Z}) \,+\, \gamma \frac{d}{d\epsilon}_{|\epsilon=0} \det(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon})\, \Phi(\mathcal{Z}) \tag{7.1}$$

with

$$k_{g_\epsilon}(\mathcal{Z}) = (A_\epsilon \mathcal{Z} + B_\epsilon)(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon})^{-1} \quad \textit{and} \quad \frac{d}{d\epsilon}_{|\epsilon=0} g_\epsilon = v \tag{7.2}$$

As in (2.27), define $L(v)$ corresponding to the infinitesimal representation $\rho(v)$ when $\gamma = 0$,

$$L(v)\Phi(\mathcal{Z}) = \sum_{j=1}^{n(n+1)/2} \frac{d}{d\epsilon}_{|\epsilon=0} (k_{g_\epsilon}(\mathcal{Z}))_j \frac{\partial}{\partial z_j}\Phi(\mathcal{Z}) \quad and \quad \rho(v) = L(v) + \gamma\beta(v)I$$

In the following, we calculate $\rho(v)$ for each vector v of the basis given in section 6. We consider separately each case $(E_1), \cdots, (E_7)$ as given in section 6. We consider α_1, α_2 as in (6.3))-(6.4)). We denote $\rho(\tau_1)$, $\rho(\tau_2)$ corresponding to $\alpha_1 = 0$, $\alpha_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. Then

$$\widetilde{\rho}(\tau_1), \quad \widetilde{\rho}(\tau_2) \quad \text{correspond to } \alpha_1 = 0. \quad \alpha_2 = \begin{pmatrix} i\,\tau_1 & 0 \\ 0 & i\,\tau_2 \end{pmatrix}$$

$$\rho(\tau) \quad \text{corresponds to } \alpha_1 = 0. \quad \alpha_2 = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \quad \tau \text{ is real}$$

$$\widetilde{\rho}(\tau) \quad \text{corresponds to } \alpha_1 = 0. \quad \alpha_2 = \begin{pmatrix} i\,\tau & 0 \\ 0 & i\,\tau \end{pmatrix} \quad \tau \text{ is real}$$

and

$$\rho(\alpha), \quad \rho(\delta) \quad \text{correspond to } \alpha_2 = 0. \quad \alpha_1 = \begin{pmatrix} i\alpha & 0 \\ 0 & i\delta \end{pmatrix}$$

$$\rho_m(\tau) \quad \text{corresponds to } \alpha_2 = 0. \quad \alpha_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix} \quad \tau \text{ is real}$$

$$\widetilde{\rho}_m(\tau) \quad \text{corresponds to } \alpha_2 = 0. \quad \alpha_1 = \begin{pmatrix} 0 & i\,\tau \\ i\,\tau & 0 \end{pmatrix} \quad \tau \text{ is real}$$

We deduce

Theorem 7.1 *We identify* $\mathcal{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$ *with* $\mathcal{Z} = (z_1, z_2, z_3) \in C^3$. *The infinitesimal representation is given by*

(1)
$$\rho(\tau_1)f = (1 - z_1^2)\frac{\partial f}{\partial z_1} - z_1 z_2 \frac{\partial f}{\partial z_2} - z_2^2 \frac{\partial f}{\partial z_3} + \gamma z_1\, f$$

$$\rho(\tau_2)f = -z_2^2 \frac{\partial f}{\partial z_1} - z_3 z_2 \frac{\partial f}{\partial z_2} + (1 - z_3^2)\frac{\partial f}{\partial z_3} + \gamma z_3\, f$$

(2)
$$\widetilde{\rho}(\tau_1)f = i(1 + z_1^2)\frac{\partial f}{\partial z_1} + i z_1 z_2 \frac{\partial f}{\partial z_2} + i z_2^2 \frac{\partial f}{\partial z_3} - i\,\gamma z_1\, f$$

$$\widetilde{\rho}(\tau_2)f = i z_2^2 \frac{\partial f}{\partial z_1} + i\, z_3 z_2 \frac{\partial f}{\partial z_2} + i(1 + z_3^2)\frac{\partial f}{\partial z_3} - i\,\gamma z_3\, f$$

(3)
$$\rho(\tau)f = -2 z_1 z_2 \frac{\partial f}{\partial z_1} + (1 - (z_1 z_3 + z_2^2))\frac{\partial f}{\partial z_2} - 2 z_3 z_2 \frac{\partial f}{\partial z_3} + 2\,\gamma z_2\, f$$

$$\text{(4)} \qquad \widetilde{\rho}(\tau)f = 2i\, z_1 z_2 \frac{\partial f}{\partial z_1} \; + \, i\,(1 + (z_1 z_3 + z_2^2))\frac{\partial f}{\partial z_2} \; + \, 2i\, z_3 z_2 \frac{\partial f}{\partial z_3} \; - \, 2\, i\gamma z_2\, f$$

$$\text{(5)} \quad \rho(\alpha)f = 2iz_1 \frac{\partial f}{\partial z_1} \; + iz_2 \frac{\partial f}{\partial z_2} \; - \, i\gamma\, f \quad , \qquad \rho(\delta)f = iz_2 \frac{\partial f}{\partial z_2} \; + 2iz_3 \frac{\partial f}{\partial z_3} \; - \, i\gamma\, f$$

$$\text{(6)} \qquad \rho_m(\tau)f = 2z_2 \frac{\partial f}{\partial z_1} \; + (z_3 - z_1)\frac{\partial f}{\partial z_2} - 2z_2 \frac{\partial f}{\partial z_3}$$

$$\text{(7)} \qquad \widetilde{\rho}_m(\tau)f = 2iz_2 \frac{\partial f}{\partial z_1} \; + i(z_3 + z_1)\frac{\partial f}{\partial z_2} + 2iz_2 \frac{\partial f}{\partial z_3}$$

8 Identities for square of vector fields in the infinitesimal representation

In Theorem 7.1, the operators are of the form $\rho_j = L_j + \gamma\, \phi_j\, I$ where L_j is a first order operator with holomorphic coefficients and $\phi_j(z_1, z_2, z_3)$ is a holomorphic function.

Theorem 8.1 *Taking $\gamma = 0$, we have*

$$2(\, \rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 + \rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 - (\rho(\alpha)^2 + \rho(\delta)^2) + \rho(\tau)^2 + \widetilde{\rho}(\tau)^2 - (\rho_m(\tau)^2 + \widetilde{\rho}_m(\tau)^2) = 0 \tag{8.1}$$

In other words, there exist constants A_j such that

$$\mathcal{C} = \sum_j A_j L_j^2 = 0 \tag{8.2}$$

Proof.

$$\rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 = \, -4\, z_1^2 \frac{\partial^2}{\partial z_1^2} - \, 4\, z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} \; - \, 4\, z_2^2 \frac{\partial^2}{\partial z_1 \partial z_3} \; - \, 4z_1 \frac{\partial}{\partial z_1} \; - \, 2z_2 \frac{\partial}{\partial z_2}$$

$$\rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 = \, -4\, z_3^2 \frac{\partial^2}{\partial z_3^2} - \, 4\, z_3 z_2 \frac{\partial^2}{\partial z_3 \partial z_2} \; - \, 4\, z_2^2 \frac{\partial^2}{\partial z_1 \partial z_3} \; - \, 4z_3 \frac{\partial}{\partial z_3} \; - \, 2z_2 \frac{\partial}{\partial z_2}$$

$$\rho(\alpha)^2 + \rho(\delta)^2 =$$

$$-2z_2^2 \frac{\partial^2}{\partial z_2^2} - 4\, z_1^2 \frac{\partial^2}{\partial z_1^2} - 4z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} - 4z_3^2 \frac{\partial^2}{\partial z_3^2} - 4z_3 z_2 \frac{\partial^2}{\partial z_3 \partial z_2} - (4z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + 4z_3 \frac{\partial}{\partial z_3})$$

thus

$$D_1 \;=\; \rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 \;+\; \rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 \;-\; (\rho(\alpha)^2 + \rho(\delta)^2) =$$

$$2z_2^2\frac{\partial^2}{\partial z_2^2} - 8z_2^2\frac{\partial^2}{\partial z_3 \partial z_1} \;-\; 4z_1\frac{\partial}{\partial z_1} - 4z_2\frac{\partial}{\partial z_2} - 4z_3\frac{\partial}{\partial z_3} \;+\; (4z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + 4z_3\frac{\partial}{\partial z_3}) \quad (8.3)$$

On the other hand, with (E_3)-(E_4), we have

$$D_2 \;=\; \rho(\tau)^2 + \widetilde{\rho}(\tau)^2 =$$

$$-4(z_1z_3 + z_2^2)\frac{\partial^2}{\partial z_2^2} \;-\; 4z_1z_2\frac{\partial^2}{\partial z_1 \partial z_2} \;-\; 4z_3z_2\frac{\partial^2}{\partial z_3 \partial z_2} \;-\; 4z_1\frac{\partial}{\partial z_1} - 4z_2\frac{\partial}{\partial z_2} - 4z_3\frac{\partial}{\partial z_3} \quad (8.4)$$

and from (E_6)-(E_7),

$$D_3 \;=\; \rho_m(\tau)^2 + \widetilde{\rho}_m(\tau)^2 \;=$$

$$-4z_1z_3\frac{\partial^2}{\partial z_2^2} \;-\; 4z_1z_2\frac{\partial^2}{\partial z_1 \partial z_2} \;-\; 4z_3z_2\frac{\partial^2}{\partial z_3 \partial z_2} - 16z_2^2\frac{\partial^2}{\partial z_3 \partial z_1} - 8z_2\frac{\partial}{\partial z_2} - 4z_1\frac{\partial}{\partial z_1} \;-\; 4z_3\frac{\partial}{\partial z_3} \quad (8.5)$$

Thus, if $\gamma = 0$,

$$\mathcal{C} = 2D_1 + D_2 - D_3 = 0 \quad (8.6)$$

This proves the Theorem. •

Theorem 8.2 *Assume $\gamma \neq 0$, then with the same constants as in (8.2)), we have*

$$\sum_j A_j \phi_j L_j = 0 \quad (8.7)$$

Proof. From (E_1) and (E_2),

$$T_1 = \phi(\tau_1)L(\tau_1) + \phi(\tau_2)L(\tau_2) + \widetilde{\phi(\tau_1)}\widetilde{L(\tau_1)} + \widetilde{\phi(\tau_2)}\widetilde{L(\tau_2)} = 2\gamma(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3}) \quad (8.8)$$

From $(E_3)$and (E_4),

$$T_2 = \phi(\tau)L(\tau) + \widetilde{\phi(\tau)}\widetilde{L(\tau)} = 4\gamma\, z_2\frac{\partial}{\partial z_2} \quad (8.9)$$

From (E_5)

$$T_4 = \phi(\alpha)L(\alpha) + \phi(\delta)L(\delta) = 2\gamma[z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}] \quad (8.10)$$

Adding, we find $2T_1 - 2T_4 + T_2 = 0$.•

Corollary 8.3

$$\sum_j A_j(\rho_j + \overline{\rho_j})\overline{L_j} = \sum_j A_j L_j \overline{L_j} \;+\; \gamma \sum_j A_j \phi_j \overline{L_j} \quad (8.11)$$

9 Laplacian and O-U operator on $\mathcal{D}_2$ in terms of the infinitesimal representation of $Sp(2n)$, $n = 2$

With the same constants as in (8.1)-(8.2), we calculate $\sum_j A_j L_j \overline{L_j}$ and prove that it is the Laplacian on $\mathcal{D}_2$. Then we calculate the first order operator $U = \sum_j A_j \phi_j \overline{L_j}$, the "drift".

9.1 The Laplacian

Theorem 9.1 *Assume that $\gamma = 0$: Consider the vector fields as in (8.1), then*

$$2(\,\rho(\tau_1)\overline{\rho(\tau_1)} + \widetilde{\rho}(\tau_1)\overline{\widetilde{\rho}(\tau_1)} \,+\, \rho(\tau_2)\overline{\rho(\tau_2)} + \widetilde{\rho}(\tau_2)\overline{\widetilde{\rho}(\tau_2)} \,-\, (\rho(\alpha)\overline{\rho(\alpha)} + \rho(\delta)\overline{\rho(\delta)}) +$$

$$\rho(\tau)\overline{\rho(\tau)} + \widetilde{\rho}(\tau)\overline{\widetilde{\rho}(\tau)} - (\rho_m(\tau)\overline{\rho_m(\tau)} + \widetilde{\rho}_m(\tau)\overline{\widetilde{\rho}_m(\tau)}) \tag{9.1}$$

is the Laplace-Beltrami Δ on $\mathcal{D}_2$.

Proof. Taking $\gamma = 0$, we have

$$\mathcal{D}_1 = \rho(\tau_1)\overline{\rho(\tau_1)} \,+\, \widetilde{\rho}(\tau_1)\overline{\widetilde{\rho}(\tau_1)} \,+\, \rho(\tau_2)\overline{\rho(\tau_2)} \,+\, \widetilde{\rho}(\tau_2)\overline{\widetilde{\rho}(\tau_2)} \,-\, (\rho(\alpha)\overline{\rho(\alpha)} \,+\, \rho(\delta)\overline{\rho(\delta)}) =$$

$$2[\,(1 - z_1\overline{z}_1)^2 \,+\, z_2^2\overline{z_2}^2]\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} \,+\, 2[\,(1 - z_3\overline{z}_3)^2 \,+\, z_2^2\overline{z_2}^2]\frac{\partial^2}{\partial z_3 \partial \overline{z_3}} +$$

$$2z_2\overline{z_2}[\,-1 + z_1\overline{z}_1 + z_3\overline{z_3}]\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} +$$

$$2[z_1\overline{z_2}(z_1\overline{z}_1 - 1) + z_2^2\overline{z_2}\overline{z_3}]\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} \,+ 2[z_3\overline{z_2}(z_3\overline{z}_3 - 1) + z_2^2\overline{z_2}\overline{z_1}]\frac{\partial^2}{\partial z_3 \partial \overline{z_2}} +$$

$$2[z_2\overline{z_1}(z_1\overline{z}_1 - 1) + z_2 z_3\overline{z_2}^2]\frac{\partial^2}{\partial z_2 \partial \overline{z_1}} \,+ 2[z_2\overline{z_3}(z_3\overline{z}_3 - 1) + z_2 z_1\overline{z_2}^2]\frac{\partial^2}{\partial z_2 \partial \overline{z_3}} +$$

$$2[z_1^2\overline{z_2}^2 + z_2^2\overline{z_3}^2]\frac{\partial^2}{\partial z_1 \partial \overline{z_3}} \,+\, 2[z_3^2\overline{z_2}^2 + z_2^2\overline{z_1}^2]\frac{\partial^2}{\partial z_3 \partial \overline{z_1}}$$

Then

$$\mathcal{D}_2 = \rho(\tau)\overline{\rho(\tau)} \,+\, \widetilde{\rho}(\tau)\overline{\widetilde{\rho}(\tau)} \,=$$

$$8\,z_1\overline{z_1}z_2\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} \,+\, 2(1 + (z_1 z_3 + z_2^2)(\overline{z_1}\,\overline{z_3} \,+\, \overline{z_2}^2))\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} \,+\, 8\,z_3\overline{z_3}z_2\overline{z_2}\frac{\partial^2}{\partial z_3 \partial \overline{z_3}} +$$

$$4z_1 z_2(\overline{z_1}\,\overline{z_3} \,+\, \overline{z_2}^2))\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} \,+\, 4\overline{z_1}\,\overline{z_2}(z_1 z_3 \,+\, z_2^2))\frac{\partial^2}{\partial \overline{z_1} \partial z_2} +$$

$$8\,z_1\overline{z_3}z_2\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_3}} \,+\, 8\,z_3\overline{z_1}z_2\overline{z_2}\frac{\partial^2}{\partial z_3 \partial \overline{z_1}} +$$

$$4\,\overline{z_2}\overline{z_3}(z_1 z_3 + z_2^2)\frac{\partial^2}{\partial z_2 \partial \overline{z_3}} \,+\, 4\,z_2 z_3(\overline{z_1}\,`\overline{z_3} + \,\overline{z_2}^2)\frac{\partial^2}{\partial z_3 \partial \overline{z_2}}$$

and

$$\mathcal{D}_3 = \rho_m(\tau)\overline{\rho_m(\tau)} + \widetilde{\rho}_m(\tau)\overline{\widetilde{\rho}_m(\tau)} =$$
$$8z_2\overline{z_2}\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 8z_2\overline{z_2}\frac{\partial^2}{\partial z_3\partial\overline{z_3}} + 2(z_3\overline{z_3}+z_1\overline{z_1})\frac{\partial^2}{\partial z_2\partial\overline{z_2}} +$$
$$4z_2\overline{z_3}\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + 4z_3\overline{z_2}\frac{\partial^2}{\partial z_2\partial\overline{z_1}} + 4z_1\overline{z_2}\frac{\partial^2}{\partial z_2\partial\overline{z_3}} + 4z_2\overline{z_1}\frac{\partial^2}{\partial z_3\partial\overline{z_2}}$$

Adding all, we find

$$\Delta = 2\mathcal{D}_1 + \mathcal{D}_2 - \mathcal{D}_3 =$$
$$4(1-z_1\overline{z_1}-z_2\overline{z_2})^2\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 4(1-z_3\overline{z_3}-z_2\overline{z_2})^2\frac{\partial^2}{\partial z_3\partial\overline{z_3}}$$
$$+[(4z_2\overline{z_2}-2)(-1+z_1\overline{z_1}+z_3\overline{z_3}) + 2(z_1z_3+z_2^2)(\overline{z_1z_3}+\overline{z_2}^2)]\frac{\partial^2}{\partial z_2\partial\overline{z_2}}$$
$$+4(z_1\overline{z_1}+z_2\overline{z_2}-1)[(z_1\overline{z_2}+z_2\overline{z_3})\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + (z_2\overline{z_1}+z_3\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_1}}]$$
$$+4(z_3\overline{z_3}+z_2\overline{z_2}-1)[(z_3\overline{z_2}+z_2\overline{z_1})\frac{\partial^2}{\partial z_3\partial\overline{z_2}} + (z_2\overline{z_3}+z_1\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_3}}]$$
$$+4(z_1\overline{z_2}+z_2\overline{z_3})^2\frac{\partial^2}{\partial z_1\partial\overline{z_3}} + 4(z_3\overline{z_2}+z_2\overline{z_1})^2\frac{\partial^2}{\partial z_3\partial\overline{z_1}} \tag{9.2}$$

As in (5.1))-(5.2)), consider the matrices $\mathcal{Z}$ and $I-\mathcal{Z}\overline{\mathcal{Z}}$ with $u = 1-z_1\overline{z_1}-z_2\overline{z_2}$, $v = 1-z_3\overline{z_3}-z_2\overline{z_2}$ and $w = z_1\overline{z_2}+z_2\overline{z_3}$. We express the coefficients of the operator (9.2) with u, v, w. We see that Δ in (9.2) is equal to

$$\Delta = 4u^2\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 2(uv+w\overline{w})\frac{\partial^2}{\partial z_2\partial\overline{z_2}} + 4v^2\frac{\partial^2}{\partial z_3\partial\overline{z_3}} - 4u(w\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + \overline{w}\frac{\partial^2}{\partial z_2\partial\overline{z_1}})$$
$$-4v(\overline{w}\frac{\partial^2}{\partial z_3\partial\overline{z_2}} + w\frac{\partial^2}{\partial z_2\partial\overline{z_3}}) + 4(w^2\frac{\partial^2}{\partial z_1\partial\overline{z_3}} + \overline{w}^2\frac{\partial^2}{\partial z_3\partial\overline{z_1}}) \tag{9.3}$$

and this is the Laplacian in (5.8).

9.2 Case $\gamma \neq 0$. The O-U operator on $\mathcal{D}_2$

Theorem 9.2 *Assume that $\gamma \neq 0$, we write as in section 7, $\rho = L + \gamma\phi I$. Then*

$$2(\rho(\tau_1)\overline{L(\tau_1)} + \widetilde{\rho}(\tau_1)\overline{\widetilde{L}(\tau_1)} + \rho(\tau_2)\overline{L(\tau_2)} + \widetilde{\rho}(\tau_2)\overline{\widetilde{L}(\tau_2)} - (\rho(\alpha)\overline{L(\alpha)} + \rho(\delta)\overline{L(\delta)}) +$$
$$\rho(\tau)\overline{L(\tau)} + \widetilde{\rho}(\tau)\overline{\widetilde{L}(\tau)} - (\rho_m(\tau)\overline{L_m(\tau)} + \widetilde{\rho}_m(\tau)\overline{\widetilde{L}_m(\tau)}) \tag{9.4}$$

is equal to

$$\Delta - \gamma V \quad where \quad V = \sum_{j,k} m_{jk}\frac{\partial}{\partial z_j}\log\det(I-\mathcal{Z}\overline{\mathcal{Z}})\frac{\partial}{\partial\overline{z_k}} \tag{9.5}$$

The O-U operator $\Delta - \gamma V$ has $\mu^\gamma = \exp(-\gamma\log\det(I-\mathcal{Z}\overline{\mathcal{Z}}))dv$ for invariant measure, dv is the volume measure on $\mathcal{D}_2$: Mreover μ^γ is the measure which makes the representation unitary.

The proof will result from the following two lemmas

Lemma 9.3 *We have*

$$\overline{z_2}u - \overline{z_3 w} = \overline{z_2}v - \overline{z_1}\, w \tag{9.6}$$

As in (5.6)-(5.7), let

$$M = (m_{ij}) = \begin{pmatrix} 4u^2 & -4uw & 4w^2 \\ -4u\overline{w} & 2(uv + w\overline{w}) & -4vw \\ 4\overline{w}^2 & -4v\overline{w} & 4v^2 \end{pmatrix} \quad \text{and} \quad \Delta = \sum_{ij} m_{ij} \frac{\partial^2}{\partial z_i \partial \overline{z_j}}$$

Let

$$d\mu_\gamma = \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{(uv - w\overline{w})^{\gamma+3}}$$

We have

$$\int \Delta f \, d\mu^\gamma = 4\gamma \int U\, f d\mu^\gamma \tag{9.7}$$

with

$$U = (\overline{w z_2} - \overline{z_1} u)\frac{\partial}{\partial \overline{z_1}} + (w\overline{z_2} - \overline{z_3} v)\frac{\partial}{\partial \overline{z_3}} + (\overline{w z_3} - \overline{z_2} u)\frac{\partial}{\partial \overline{z_2}} \tag{9.8}$$

In particular

$$U = \sum_{j,k} m_{jk} \frac{\partial}{\partial z_j} \log\det(i - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial}{\partial \overline{z_k}} \tag{9.9}$$

Proof. Itegrating by parts (9.7) (in $\frac{\partial}{\partial z}$), this gives $\int \Delta f \, d\mu^\gamma = I_1 + I_2 + I_3$. with

$$I_1 = -\int [\frac{\partial}{\partial z_1}(\frac{4u^2}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_2}(-\frac{4u\overline{w}}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_3}(\frac{4\overline{w}^2}{(uv - w\overline{w})^{\gamma+3}})] \frac{\partial}{\partial \overline{z_1}} f$$

$$I_2 = -\int [\frac{\partial}{\partial z_1}(-\frac{4uw}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_2}(2\frac{uv + w\overline{w}}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_3}(-\frac{4v\overline{w}}{(uv - w\overline{w})^{\gamma+3}})] \frac{\partial}{\partial \overline{z_2}} f$$

$$I_3 = -\int [\frac{\partial}{\partial z_1}(\frac{4w^2}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_2}(-\frac{4vw}{(uv - w\overline{w})^{\gamma+3}}) + \frac{\partial}{\partial z_3}(\frac{4v^2}{(uv - w\overline{w})^{\gamma+3}})] \frac{\partial}{\partial \overline{z_3}} f$$

To calculate I_1, let

$$A_1 = \frac{\partial}{\partial z_1}(4u^2) - \frac{\partial}{\partial z_2}(4u\overline{w}) + \frac{\partial}{\partial z_3}(4\overline{w}^2) = 12(\overline{w z_2} - u\overline{z_1})$$

and

$$\begin{aligned} B_1 &= -(\gamma+3)[4u^2 \frac{\partial}{\partial z_1} - 4u\overline{w}\frac{\partial}{\partial z_2} + 4\overline{w}^2 \frac{\partial}{\partial z_3}](uv - w\overline{w}) \\ &= -4(\gamma+3)(uv - w\overline{w})(\overline{w z_2} - \overline{z_1} u) \end{aligned} \tag{9.10}$$

This gives the first term in (9.8). In the same way, to calculate I_2, let

$$A_2 = \frac{\partial}{\partial z_2}(2(uv + w\overline{w})) - \frac{\partial}{\partial z_1}(4uw) - \frac{\partial}{\partial z_3}(4v\overline{w}) = -6(u+v)\overline{z_2} + 6(w\overline{z_1} + \overline{w z_3})$$

$$B_2 = -(\gamma+3)[2(uv+w\overline{w})\frac{\partial}{\partial z_2} - 4uw\frac{\partial}{\partial z_1} - 4v\overline{w}\frac{\partial}{\partial z_3}](uv-w\overline{w})$$
$$= 2(\gamma+3)(uv-w\overline{w})(\overline{z_2}(u+v) - \overline{z_3}\overline{w} - \overline{z_1}w) \quad (9.11)$$

We deduce

$$A_2 + \frac{B_2}{(uv-w\overline{w})} = 2\gamma[(u+v)\overline{z_2} - w\overline{z_1} - \overline{w}\overline{z_3}]$$

From the important identity (9.6), we obtain the second term in (9.8). For I_3, we put

$$A_3 = \frac{\partial}{\partial z_1}(4w^2) + \frac{\partial}{\partial z_2}(-4vw) + \frac{\partial}{\partial z_3}(4v^2) = 12(w\overline{z_2} - v\overline{z_3})$$

and

$$B_3 = -(\gamma+3)[4w^2\frac{\partial}{\partial z_1} - 4vw\frac{\partial}{\partial z_2} + 4v^2\frac{\partial}{\partial z_3}](uv-w\overline{w})$$
$$= -4(\gamma+3)(uv-w\overline{w})(-v\overline{z_3} + w\overline{z_2}) \quad (9.12)$$
$$A_3 + \frac{B_3}{(uv-w\overline{w})} = 4\gamma(v\overline{z_3} - w\overline{z_2})$$

This gives (9.8). From (9.10)-(9.11)-(9.12)), we deduce that U is given by (9.9).

In the next lemma, we relate U to the infinitesimal representation. With the constants A_j are as in (8.1)-(8.2)-(9.1)-(9.4), we put

$$\Delta^\gamma = \sum_j A_j(\rho_j + \overline{\rho_j})\,\overline{L_j} = \sum_j A_j L_j \overline{L_j} + \gamma \sum_j A_j \phi_j\,\overline{L_j} \quad (9.13)$$

We have proved that $\sum_j A_j L_j \overline{L_j}$ is the Laplacian on $\mathcal{D}_2$.

Lemma 9.4 *For the first order part in (9.13)*

$$\gamma \sum_j A_j(\phi_j + \overline{\phi_j})\,\overline{L_j}$$

is equal to $-4\gamma U$*,*

$$\sum_j A_j\phi_j\overline{L_j} = 4(\overline{z_1}u - \overline{w}\overline{z_2})\frac{\partial}{\partial \overline{z_1}} + 4(\overline{z_3}v - w\overline{z_2})\frac{\partial}{\partial \overline{z_3}} + 4(\overline{z_2}u - \overline{w}\overline{z_3})\frac{\partial}{\partial \overline{z_2}} \quad (9.14)$$

Proof. By (8.7), we have $\sum_j A_j\overline{\phi_j}\,\overline{L_j} = 0$. On the other hand,

$$\sum_j A_j\phi_j\overline{L_j} = 2U_1 - 2U_4 + U_2 \quad (9.15)$$

with

$$U_1 = \phi(\tau_1)\overline{H(\tau_1)} + \widetilde{\phi}(\tau_1)\overline{\widetilde{H(\tau_1)}} + \phi(\tau_2)\overline{H(\tau_2)} + \widetilde{\phi}(\tau_2)\overline{\widetilde{H(\tau_2)}} =$$

$$-2z_1\overline{z_1}[\overline{z_1}\frac{\partial}{\partial\overline{z_1}} + \overline{z_2}\frac{\partial}{\partial\overline{z_2}}] - 2z_3\overline{z_3}[\overline{z_2}\frac{\partial}{\partial\overline{z_2}} + \overline{z_3}\frac{\partial}{\partial\overline{z_3}}] - 2z_1\overline{z_2}^2\frac{\partial}{\partial\overline{z_3}} - 2z_3\overline{z_2}^2\frac{\partial}{\partial\overline{z_1}}$$

$$U_4 = \phi(\alpha)\overline{H(\alpha)} + \phi(\delta)\overline{H(\delta)} = -2[\overline{z_1}\frac{\partial}{\partial\overline{z_1}} + \overline{z_2}\frac{\partial}{\partial\overline{z_2}} + \overline{z_3}\frac{\partial}{\partial\overline{z_3}}]$$

$$U_2 = \phi(\tau)\overline{H(\tau)} + \widetilde{\phi}(\tau)\overline{\widetilde{H(\tau)}} = -8z_2\overline{z_2}[\overline{z_1}\frac{\partial}{\partial\overline{z_1}} + \overline{z_3}\frac{\partial}{\partial\overline{z_3}}] - 4z_2(\overline{z_1 z_3} + \overline{z_2}^2)\frac{\partial}{\partial\overline{z_2}}$$

From the previous identities, we deduce (9.14).

References

[1] H. Airault, H. Ouerdiane, Ornstein-Uhlenbeck operators and unitarizing measures in the Poincaré disk, *Bull. Sci. Math.,* **133** (2009), 671-692.

[2] H. Airault, H. Ouerdiane, Invariant measure for some differential operators and unitarizing measure for the representation of a Lie group. Examples in finite dimension, *preprint, Banach Center Publications* (2011).

[3] F. A. Berezin, Quantization in complex symmetric spaces, *Izv. Akad. Nauk SSSR* **39** (1975) 341-379.

[4] A. B. Cruzeiro and P. Malliavin, Stochstic calculus of variations on complex line bundle and construction of unitarizing measures for the Poincaré disk *J. Funct. Anal.* **256** (2009) n2, 385-408.

[5] L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains,* Translation of Mathematical Monographsn Vol. 6, AMS Providence, 1963.

[6] A. A. Kirillov, Geometric approach to discrete series of unirreps for Vir, *J. Math. Pures Appl.* **77** (1998) n2, 735-746.

[7] Yu. A. Neretin, Representations of Virasoro and Affine Lie algebras, *Encyclopaedia of . Math. Sciences, Non-commutative harmonic analysis* **22** (1994) 157-225.

[8] P. Malliavin, Invariant or quasi-invariant probability measures for infinite dimensional groups II- Unitarizing measures or Berezinian measures, *Jpn. J. Math.* **3** (2008), n1, 19-47.