THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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M. Grochowski

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THE STRUCTURE OF REACHABLE SETS AND GEOMETRIC OPTIMALITY OF SINGULAR TRAJECTORIES FOR CERTAIN AFFINE CONTROL SYSTEMS IN $\mathbb{R}^3$.

MAREK GROCHOWSKI

ABSTRACT. In this paper we investigate the structure of reachable sets from the origin for a class of analytic control affine system characterized, among other things, by possessing two singular trajectories initiating at the origin. The aim of the paper is to establish the connection between the minimal number of analytic functions needed for describing reachable sets and the number of geometrically optimal singular trajectories. The paper is written in a language of the sub-Lorentzian geometry. Also the sub-Lorentzian geometry methods are used to prove theorems.

1. Introduction

1.1. **Preliminaries.** This paper[1][2] is a continuation of the research started in [9], [10], and devoted to the study of reachable sets for noncontact sub-Lorentzian structures on $\mathbb{R}^3$, as well as for affine control systems induced by them. Similarly as in [9], [10] our objective is to investigate the interrelation of the structure of reachable sets from a given point q_0 for the mentioned systems and geometric optimality of singular trajectories starting at q_0 or - speaking in the sub-Lorentzian language - geometric optimality of timelike abnormal curves starting at q_0 (a trajectory of a control system starting from a point q_0 is said to be geometrically optimal if it is contained in the boundary of the reachable set from q_0 - cf. [1]). The paper is arranged in such a way, that in first four sections we develop the theory in the sub-Lorentzian setting, and section 5 contains applications of the obtained results to control affine systems.

For all facts and notions from the sub-Lorentzian geometry the reader is referred to the previous papers by the author (see [7] and its reference section; see also [11]). Here we recall only those basic facts that are needed for stating the results. Let M be a smooth manifold and let H be a smooth distribution on M of constant rank. For a point $q \in M$ and an integer $k \in \mathbb{N}$ we define H_q^k to be the linear subspace in H_q generated by all vectors of the form $[X_1, [X_2, ..., [X_{i-1}, X_i]...]](q)$, where $X_1, ..., X_i$ are smooth (local) sections of H defined in a a neighbourhood of q, and $i \leq k$. We say that H is *a bracket generating distribution* if for every $q \in M$ there exists an $i = i(q)$ such that $H_q^i = T_qM$. Now, by *a sub-Lorentzian structure* on a manifold M we mean a pair (H, g). The triple (M, H, g) is called *a sub-Lorentzian manifold.* Take a point $q \in M$; a vector $v \in H_q$ is said to be *timelike* if $g(v, v) < 0$, *nonspacelike*

Key words and phrases. sub-Lorentzian manifolds, geodesics, reachable sets, geometric optimality, vector distributions, a Martinet surface.

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if $g(v,v) \leq 0$, and *null* if $g(v,v)=0$ but $v \neq 0$. *A time orientation* of (M,H,g) is, by definition, a continuous timelike vector field on M. Suppose (M,H,g) to be time-oriented by X and let $v \in H_q$ be a nonspacelike vector. We say that v is *future directed* if $g(v,X(q))<0$. A curve $\gamma:[a,b] \longrightarrow M$ is called *horizontal* if it is absolutely continuous, $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[a,b]$, and $\dot{\gamma}$ is square integrable with respect to some Riemannian metric on M. From now on all curves are supposed horizontal. We will also use the following abbreviations: *t.* for 'timelike', *nspc.* for 'nonspacelike', and *f.d.* for 'future directed'. We say that a curve $\gamma:[a,b] \longrightarrow M$ is t.f.d. (resp. nspc.f.d., null f.d.) if so is $\dot{\gamma}(t)$ a.e. on $[a,b]$. Fix a point $q_0 \in M$ and its neighbourhood U. *The (future) timelike (resp. nonspacelike, null) reachable set from q_0 relative to U* is defined to be the set of all points in U that can be reached from q_0 by a t.f.d. (resp. nspc.f.d., null f.d.) curve entirely contained in U. They are denoted respectively by $I^+(q_0,U)$, $J^+(q_0,U)$, $N^+(q_0,U)$. In the general case all we can say about reachable sets is that $intI^+(q_0,U) \neq \varnothing$, and that the three reachable sets have the same interiors and closures with respect to U. In order to be able to say something more, we need to make certain assumption on U. To this end let us notice that if U is sufficiently small, then our sub-Lorentzian metric can be extended to a Lorentzian metric, say $\tilde{g}$, on U. So U is said to be *a normal neighborhood* of q_0 if it is a convex normal neighbourhood of q_0 with respect to $\tilde{g}$, and U is contained in some other convex normal neighborhood of q_0 with respect to $\tilde{g}$ (see [9] for a constructive definition of normal neighbourhoods). Now, if U is a normal neighbourhood of q_0, we know that $J^+(q_0,U)$ is closed with respect to U and moreover $cl_U\left(intI^+(q_0,U)\right) = cl_U\left(intN^+(q_0,U)\right) = J^+(q_0,U)$, where cl_U stands for the closure with respect to U. Note that unlike the Lorentzian case, the boundary $\tilde{\partial} J^+(q_0,U)$ (here and below $\tilde{\partial}$ means the boundary with respect to U) may contain timelike curves starting from q_0. It can be proved [7] that such curves are *abnormal curves* for the underlaying distribution H (see [12] for a definition); they are also Goh curves (cf. [1]) but we do not need this latter fact in this paper. Let $X_0,...,X_k$ be an orthonormal frame for (H,g) defined on an open set U. We define the so-called *geodesic (sub-Lorentzian) Hamiltonian* $\mathcal{H}:T^*U \longrightarrow \mathbb{R}$, by formula $\mathcal{H}(q,p) = -\frac{1}{2}\left\langle p, X_0(q)\right\rangle + \frac{1}{2}\sum_{i=1}^{k}\left\langle p, X_i(q)\right\rangle$ (it is possible to define $\mathcal{H}$ in a global and invariant way - see [7]). Now a curve $\gamma:[a,b] \longrightarrow U$ is said to be *a Hamiltonian geodesic* if it can be represented as $\gamma(t) = \pi \circ \Gamma(t)$ where $\pi: T^*M \longrightarrow M$ is the canonical projections and $\dot{\Gamma} = \overrightarrow{\mathcal{H}}$, $\overrightarrow{\mathcal{H}}$ being the Hamiltonian vector field corresponding to the function $\mathcal{H}$. Hamiltonian geodesics do not change their causal character; moreover null f.d. Hamiltonian geodesics are [7] locally geometrically optimal. Finally, let U be an open subset in (M,H,g), and suppose that $\varphi: U \longrightarrow \mathbb{R}$ is a smooth function. *The horizontal gradient* of the function φ is defined to be the vector field $\nabla_H\varphi$ such that $d_q\varphi(v) = g(\nabla_H\varphi(q), v)$ for every $v \in H_q$, $q \in U$. One easily makes sure that if $\nabla_H\varphi$ is null f.d. on U, and $\gamma:[a,b] \longrightarrow U$ is t.f.d. (nspc.f.d.) then the function $[a,b] \ni t \longrightarrow \varphi(((t))$ is decreasing (nonincreasing).

1.2. **Statement of the results.** In papers [6], [8] we studied contact sub-Lorentzian structures on $\mathbb{R}^3$. On the other hand, in [9], [10] (generalized) Martinet sub-Lorentzian structures of Hamiltonian type of order k were studied, i.e. structures that, among other conditions imposed on them, are not contact on a hypersurface. As a next step it is reasonable to consider structures with the simplest non-smooth Martinet surface S, i.e. where S is a union of transversely intersecting smooth hypersurfaces. In order to formulate necessary assumption, let us introduce a notion of a *hyperbolic angle* on a sub-Lorentzian manifold (M, H, g). Let $v_1, v_2 \in H_q$, $q \in M$, be t.f.d. vectors. The hyperbolic angle between v_1 and v_2 is the number $\sphericalangle(v_1, v_2) \geq 0$ defined by

$$\cosh \sphericalangle(v_1, v_2) = -\frac{g(v_1, v_2)}{\|v_1\| \, \|v_2\|},$$

where $\|v\| = |g(v,v)|^{1/2}$; by the reverse Schwarz inequality $-\frac{g(v_1,v_2)}{\|v_1\|\|v_2\|} \geq 1$, so the definition makes sense. If $L_1 = Span\{v_1\}$, $L_2 = Span\{v_2\}$ are 1-dimensional timelike linear subspaces in H_q with v_1, v_2 being chosen to be t.f.d., then we put

$$\sphericalangle(L_1, L_2) = \sphericalangle(v_1, v_2).$$

Now we come to the precise definition of type of sub-Lorentzian structures we are going to consider in this paper. So let H be a bracket generating distribution of constant rank equal to 2, defined in a neighbourhood U of $0 \in \mathbb{R}^3$. We say that H satisfies the condition $(M_{2,2})$ if the following conditions are satisfied:
(i) there exist smooth hypersurfaces S_1, S_2 such that the intersection $\Gamma = S_1 \cap S_2$ is smooth of dimension 1 and contains the origin; moreover for each $q \in \Gamma$, $\dim(T_qS_i \cap H_q) = 1$, $i = 1, 2$;
(ii) H defines a contact structure on $U \setminus (S_1 \cup S_2)$;
(iii) $H_q^2 = H_q$, and $H_q^3 = T_qR^3$ whenever $q \in (S_1 \cup S_2) \setminus \Gamma$;
(iv) $H_q^4 = T_qR^3$ whenever $q \in \Gamma$.
The set $S = S_1 \cup S_2$ will be called *the Martinet surface* for H. Note that S is foliated by abnormal curves for the distribution H. Next we chose a Lorentzian metric g on H in the way similar as in [9], [10]:
(v) the field of directions $S_i \ni q \longrightarrow T_qS_i \cap H_q$ is timelike, $i = 1, 2$;
(vi) the function $S_1 \cap S_2 \ni q \longrightarrow \sphericalangle(T_qS_1 \cap H_q, T_qS_2 \cap H_q)$ is constant;
(vii) the abnormal curves foliating S are, up to a change of parameter, t.f.d. Hamiltonian geodesics.
As we shall see, this latter assumption is used only in the process of constructing normal forms.

We will say that a sub-Lorentzian structure (H, g) is of type $M_{2,2}$ if it satisfies conditions (i),...,(vii). The sub-Lorentzian structure (H, g) is analytic if all objects entering its definition (e.g. the Martinet surface) are analytic.

Theorem 1.1. *Let (H, g) be a time-oriented analytic sub-Lorentzian structure of type $M_{2,2}$ defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, provided that U is sufficiently small, there exist analytic coordinates x, y, z on U in which (H, g) has an orthonormal frame in the normal form*

$$\begin{array}{l} X = \dfrac{\partial}{\partial x} + y\varphi(y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}) + \dfrac{1}{2}y\,(y - c_1x)\,(y - c_2x)\,(1 + \psi)\dfrac{\partial}{\partial z} \\ Y = \dfrac{\partial}{\partial y} - x\varphi(y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}) - \dfrac{1}{2}x\,(y - c_1x)\,(y - c_2x)\,(1 + \psi)\dfrac{\partial}{\partial z} \end{array}, \tag{1.1}$$

where X is a time orientation, c_1, c_2 are constants such that $-1 < c_2 < c_1 < 1$, $S = \{y = c_1x\} \cup \{y = c_2x\}$ is the Martinet surface for H, and φ, ψ are analytic functions on U, $\psi(0, 0, z) = 0$.

Using theorem 1.1 we investigate the structure of reachable sets. Let $W(c_1, c_2) = c_1c_2 + 2c_1 - 2c_2 - 1$. Then we can prove

Theorem 1.2. *Suppose that (H, g) is a sub-Lorentzian structure defined on a suitably small normal neighbourhood U of the origin by an orthonormal frame X, Y in the normal form (1.1) with X being a time orientation. Then, if $W(c_1, c_2) > 0$, there exist two analytic functions η_1, $\eta_2 : U \longrightarrow \mathbb{R}$ such that the reachable sets from the origin for (H, g) are of the form*

$$J^+(0,U) = N^+(0,U) = A_1 \cup A_2,$$

$$I^+(0,U) = \operatorname{int}(A_1 \cup A_2),$$

where
$A_1 = \{(x, y, z) \in U : \ \eta_1(x, y, z) \le 0\} \cap \{x \ge 0,\ z \ge 0\}$,
$A_2 = \{(x, y, z) \in U : \ \eta_2(x, y, z) \le 0\} \cap \{x \ge 0,\ z \le 0\}$.
In particular the three reachable sets are semi-analytic.

Note that in cases covered by theorem 1.2 there are no timelike curves in the boundary.

Theorem 1.3. *Suppose that (H, g) is a sub-Lorentzian structure defined on a suitably small normal neighbourhood U of the origin by an orthonormal frame X, Y in the normal form (1.1) with X being a time orientation. Then, if $W(c_1, c_2) < 0$, there exist six analytic functions η_1, η_2, $\xi_{ij} : U \longrightarrow \mathbb{R}$, $i, j = 1, 2$, and two 2-dimensional semi-analytic sets Σ_1, Σ_2 with the property that $U \cap \{x \ge 0\} \cap \{z \ge 0\} \cap \{c_2 x \le y \le x\} \setminus \Sigma_1$ has two connected components Σ_1^+, Σ_1^-, and $U \cap \{x \ge 0\} \cap \{z \le 0\} \cap \{-x \le y \le c_1 x\} \setminus \Sigma_2$ has two connected components Σ_2^+, Σ_2^-, such that*

$$J^+(0,U) = A_1 \cup ... \cup A_6,$$

$$I^+(0,U) = \operatorname{int}(A_1 \cup ... \cup A_6) \cup A_7 \cup A_8,$$

$$N^+(0,U) = \operatorname{int}(A_1 \cup ... \cup A_4) \cup \left(\{\eta_1 = 0\} \cap \Sigma_1^+\right) \cup \left(\{\eta_2 = 0\} \cap \Sigma_2^-\right)$$

where
$A_1 = \{\eta_1 \le 0\} \cap \Sigma_1^+$,
$A_2 = \{\xi_{11} \le 0\} \cap \Sigma_1^-$,
$A_3 = \{\xi_{12} \le 0\} \cap \{-x \le y \le c_2 x\} \cap \{z \ge 0\} \cap U$,
$A_4 = \{\eta_2 \le 0\} \cap \Sigma_2^-$,
$A_5 = \{\xi_{21} \le 0\} \cap \Sigma_2^+$,
$A_6 = \{\xi_{22} \le 0\} \cap \{c_1 x \le y \le x\} \cap \{z \le 0\} \cap U$,
$A_7 = \{y = c_1 x,\ x \ge 0,\ z = 0\} \cap U$,
$A_8 = \{y = c_2 x,\ x \ge 0,\ z = 0\} \cap U$.
In particular, the three reachable sets are semi-analytic

Note that in cases covered by theorem 1.3 there are two timelike curves on the boundary $\tilde{\partial} J^+(0, U)$. It is also seen that in such cases neither $I^+(0, U)$ is open nor $N^+(0, U)$ is closed.

Similarly as it was done in some previous papers by the author, all above results can be applied to control affine systems. More details will be given in section 5, where we investigate reachable sets for control systems of the form $\dot{q} = X + uY$, where $u \in [a, b]$, and X, Y are given by (1.1). We obtain the following

Corollary 1.1. *Suppose that $W \ne 0$. If the system $\dot{q} = X + uY$, $u \in [a, b]$, has k geometrically optimal singular trajectories initiating at q_0 (in our cases $k = 0, 1, 2$) then one needs $2 + 2k$ analytic functions for describing reachable sets from q_0.*

The above corollary also holds for all cases treated in [10]. Thus the presence on the boundary of a singular trajectory initiating at q_0 increases (in most cases) by two the number of analytic functions needed for describing reachable set from q_0, and it would be interesting to know if this observation can be extended to a more general class of (not necessarily affine) control systems.

1.3. **Organizations of the paper.** Section 2 is devoted to computing reachable sets for the so-called flat structures - they correspond to normal forms (1.1) with φ and ψ set to be equal to zero. In section 3 we compute normal forms. More precisely we prove theorem 3.1 which gives normal forms in a more general situation than that treated in the present paper and which can be a starting point for further studies. Theorem 1.1 is then a corollary of theorem 3.1. In section 4 we generalize global result from section 2 to local results in a general (i.e. not flat) situation in cases where $W(c_1, c_2) \neq 0$ (the polynomial W is defined at the beginning of section 2). In section 5 we apply the results obtained for sub-Lorentzian structures to control affine systems.

2. Reachable sets in the flat case.

In this section we study reachable sets from the origin for the sub-Lorentzian structure $(\hat{H}, \hat{g})$ defined by an orthonormal basis

$$\hat{X} = \frac{\partial}{\partial x} + \frac{1}{2} y(y - c_1 x)(y - c_2 x)\frac{\partial}{\partial z}$$
$$\hat{Y} = \frac{\partial}{\partial y} - \frac{1}{2} x(y - c_1 x)(y - c_2 x)\frac{\partial}{\partial z} \tag{2.1}$$

with a time orientation $\hat{X}$ where we assume that $-1 < c_2 < c_1 < 1$. Let $S_i = \{(x, y, z) : \ y = c_i x\}$. We see that the Martinet surface S in our case is equal to $S_1 \cup S_2$. The structure (or a metric) $(\hat{H}, \hat{g})$ will be called *flat*. This is because (2.1) is a particular case of (1.1) where φ and ψ has been set to zero. Hence any structure as in (1.1) can be regarded as a perturbation of a flat structure - see section 4 for some applications of this observation.

As in the previous papers by the author, the key role in the process of constructing functions describing reachable sets is played by the signs of the z-coordinates of the fields

$$\hat{X} - \hat{Y} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{1}{2}(y + x)(y - c_1 x)(y - c_2 x)\frac{\partial}{\partial z} \tag{2.2}$$

$$\hat{X} + \hat{Y} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{2}(y - x)(y - c_1 x)(y - c_2 x)\frac{\partial}{\partial z} \tag{2.3}$$

Let

$$\Gamma_1 = \{(x, x, z) : \ x, z \in \mathbb{R}\}$$

and

$$\Gamma_2 = \{(x, -x, z) : \ x, z \in \mathbb{R}\}.$$

Similarly as in [8], [9], [10] we construct two functions $\hat{\eta}_1$, $\hat{\eta}_2$. $\hat{\eta}_1$ is the solution to the Cauchy problem

$$(\hat{X} - \hat{Y})(\eta) = 0, \quad \eta|_{\Gamma_1}(x, x, z) = z,$$

and $\hat{\eta}_2$ is the solution to the Cauchy problem

$$(\hat{X} + \hat{Y})(\eta) = 0, \quad \eta|_{\Gamma_2}(x, -x, z) = -z.$$

After calculations we obtain

$$\hat{\eta}_1(x,y,z) = z - \frac{x^2-y^2}{48}\left(\left(7c_1c_2 - 2c_1 - 2c_2 + 1\right)x^2 + 4\left(c_1c_2 - 2c_1 - 2c_2 + 1\right)xy + \left(c_1c_2 - 2c_2 - 2c_1 + 7\right)y^2\right)$$

and

$$\hat{\eta}_2(x,y,z) = -z - \frac{x^2-y^2}{48}\left(\left(7c_1c_2 + 2c_1 + 2c_2 + 1\right)x^2 - 4\left(c_1c_2 + 2c_1 + 2c_2 + 1\right)xy + \left(c_1c_2 + 2c_2 + 2c_1 + 7\right)y^2\right).$$

As in the previous papers we need to know the horizontal gradient $\nabla_{\hat{H}}\hat{\eta}_i$ of the function $\hat{\eta}_i$ with respect to $(\hat{H},\hat{g})$, $i=1,2$. Clearly, $\nabla_{\hat{H}}\hat{\eta}_1 = -\hat{X}(\hat{\eta}_1)(\hat{X}-\hat{Y})$ where

(2.4)
$$\hat{X}(\hat{\eta}_1) = -\frac{1}{12}(x-y)\left(\left(7c_1c_2 - 2c_1 - 2c_2 + 1\right)x^2 + 4\left(c_1c_2 - 2c_1 - 2c_2 + 1\right)xy + \left(c_1c_2 - 2c_1 - 2c_2 + 7\right)y^2\right),$$

and $\nabla_{\hat{H}}\hat{\eta}_2 = -\hat{X}(\hat{\eta}_2)(\hat{X}+\hat{Y})$ where

(2.5)
$$\hat{X}(\hat{\eta}_2) = -\frac{1}{12}(x+y)\left(\left(7c_1c_2 + 2c_1 + 2c_2 + 1\right)x^2 - 4\left(c_1c_2 + 2c_1 + 2c_2 + 1\right)xy + \left(c_1c_2 + 2c_1 + 2c_2 + 7\right)y^2\right).$$

Let us define a polynomial:

$$W = c_1c_2 + 2c_1 - 2c_2 - 1. \tag{2.6}$$

As we are about to see the sign of W is determinative for the structure of the reachable set for (2.1). Indeed, it is easy to check that

$$\hat{X}(\hat{\eta}_1) = -\frac{1}{12}(x-y)\left(c_1c_2 - 2c_1 - 2c_2 + 7\right)\left(y - E_1x\right)\left(y - E_2x\right) \tag{2.7}$$

with

(2.8)
$$\begin{aligned} E_1 &= \tfrac{1}{c_1c_2-2c_1-2c_2+7}\left(-2c_1c_2 + 4c_1 + 4c_2 - 2 + \sqrt{-3\left(c_1c_2 - 2c_1 + 2c_2 - 1\right)W}\right) \\ E_2 &= \tfrac{1}{c_1c_2-2c_1-2c_2+7}\left(-2c_1c_2 + 4c_1 + 4c_2 - 2 - \sqrt{-3\left(c_1c_2 - 2c_1 + 2c_2 - 1\right)W}\right) \end{aligned}$$

and

$$\hat{X}(\hat{\eta}_2) = -\frac{1}{12}(x+y)\left(c_1c_2 + 2c_1 + 2c_2 + 7\right)\left(y - E_3x\right)\left(y - E_4x\right) \tag{2.9}$$

with

(2.10)
$$\begin{aligned} E_3 &= \tfrac{1}{c_1c_2+2c_1+2c_2+7}\left(2c_1c_2 + 4c_1 + 4c_2 + 2 + \sqrt{-3\left(c_1c_2 - 2c_1 + 2c_2 - 1\right)W}\right) \\ E_4 &= \tfrac{1}{c_1c_2+2c_1+2c_2+7}\left(2c_1c_2 + 4c_1 + 4c_2 + 2 - \sqrt{-3\left(c_1c_2 - 2c_1 + 2c_2 - 1\right)W}\right) \end{aligned}.$$

Let us notice here that $c_1c_2 - 2c_1 + 2c_2 - 1 = (c_1c_2 - 1) - 2(c_1 - c_2) < 0$ for all c_1, c_2 such that $-1 < c_2 < c_1 < 1$.

2.1. **The case $W < 0$.** This case is the simplest because as it can be seen from (2.7) and (2.9), $\nabla_{\hat{H}}\hat{\eta}_i$, $i=1,2$, is null f.d. in the whole sector $\{-x < y < x\}$. Thus using similar arguments as e.g. in [8] or [9] we have

Proposition 2.1. *If $W < 0$ then*

$$\hat{J}^+(0) = \hat{N}^+(0) = \hat{A}_1 \cup \hat{A}_2$$

and

$$\hat{I}^+(0) = \operatorname{int}(\hat{A}_1 \cup \hat{A}_2),$$

where
$\hat{A}_1 = \{(x,y,z) \in \mathbb{R}^3 : \ \hat{\eta}_1(x,y,z) \le 0\} \cap \{x \ge 0, \ z \ge 0\}$,
$\hat{A}_2 = \{(x,y,z) \in \mathbb{R}^3 : \ \hat{\eta}_2(x,y,z) \le 0\} \cap \{x \ge 0, \ z \le 0\}$.

Let us remark that
$\hat{\eta}_1(x,c_1x,0) = \frac{1}{48}x^4\,(c_1-1)^2\,(c_1+1)^2\,(c_1c_2-2c_1+2c_2-1) < 0$,
$\hat{\eta}_1(x,c_2x,0) = \frac{1}{48}x^4\,(c_2-1)^2\,(c_2+1)^2\,W < 0$,
$\hat{\eta}_2(x,c_1x,0) = \frac{1}{48}x^4\,(c_1-1)^2\,(c_1+1)^2\,W < 0$, and
$\hat{\eta}_2(x,c_2x,0) = \frac{1}{48}x^4\,(c_2-1)^2\,(c_2+1)^2\,(c_1c_2-2c_1+2c_2-1) < 0$.
This means that there are no geometrically optimal timelike abnormal curves starting from 0, and what follows only two functions suffice to describe the reachable sets.

2.2. **The case $W > 0$.** Now, E_i, $i = 1,...,4$, are real so this case is more complicated. First of all we must examine constants E_1,..., E_4. Obviously $E_2 < E_1$ and $E_4 < E_3$. Moreover we have two lemmas.

Lemma 2.1. *The following inequalities hold true: $-1 < E_2 < E_1 < 1$, $E_i < c_i$, $i = 1,2$.*

Proof. The proof relies on straightforward computations. For instance $E_1 < 1$ is equivalent to

$$\sqrt{-3\,(c_1c_2-2c_1+2c_2-1)\,(c_1c_2+2c_1-2c_2-1)} < 3c_1c_2-6c_1-6c_2+9, \tag{2.11}$$

where $3c_1c_2-6c_1-6c_2+9 = 3\,(W+4-4c_1)$ is positive. Squaring both sides of (2.11) we see that (2.11) is equivalent to

$$12\,(c_2-1)\,(c_1-1)\,(c_1c_2-2c_1-2c_2+7) > 0$$

which is of course true. In the similar way we prove the remaining inequalities. ∎

In the similar manner we prove

Lemma 2.2. *The following inequalities hold true: $-1 < E_4 < E_3 < 1$, $c_2 < E_4$, $c_1 < E_3$.*

Proof. We prove for instance that $c_2 < E_4$. By (2.10) this is equivalent to

$$-\,(c_2+2)\,(2c_2+c_1c_2-1-2c_1) > \sqrt{-3\,(c_1c_2-2c_1+2c_2-1)\,(c_1c_2+2c_1-2c_2-1)}$$

which, in turn, is equivalent to

$$\begin{aligned}(c_2+2)^2\,(2c_2+c_1c_2-1-2c_1)^2 + 3\,(c_1c_2-2c_1+2c_2-1)\,(c_1c_2+2c_1-2c_2-1)\\ = (c_2-1)\,(c_2+1)\,(c_1c_2+2c_1+2c_2+7)\,(2c_2+c_1c_2-1-2c_1) > 0.\end{aligned}$$

∎

Using (2.7), (2.9) we conclude that $\nabla_{\hat{H}}\hat{\eta}_1$ is null f.d. on $\{E_1x < y < x\}$ while $\nabla_{\hat{H}}\hat{\eta}_2$ is null f.d. on $\{-x < y < E_4x\}$. Hence we need more functions to describe the reachable sets from the origin.

First we will compute $\hat{J}^+(0) \cap \{z \ge 0\}$. It is natural to consider the following Cauchy problems:

$$(\hat{X}+\hat{Y})(\eta) = 0, \ \eta(x,c_2x,z) = z$$

with the solution equal to

$$\hat{\xi}_{11}(x,y,z)=z-\frac{1}{12}\frac{(x-y)(y-c_2x)^2}{(1-c_2)^2}\left(\left(-2c_1c_2+3c_1-c_2\right)x+\left(-c_1+3c_2-2\right)y\right)$$

and

$$(\hat{X}-\hat{Y})(\eta)=0,\eta(x,c_2x,z)=z$$

with the solution equal to

$$\hat{\xi}_{12}(x,y,z)=z-\frac{1}{12}\frac{(x+y)(c_2x-y)^2}{(c_2+1)^2}\left(\left(2c_1c_2+3c_1-c_2\right)x+\left(c_1-3c_2-2\right)y\right).$$

Now we examine horizontal gradients $\nabla_{\hat{H}}\hat{\xi}_{1i}$, $i=1,2$. So $\nabla_{\hat{H}}\hat{\xi}_{11}=-\hat{X}(\hat{\xi}_{11})(\hat{X}+\hat{Y})$ with

$$\hat{X}(\hat{\xi}_{11})=\frac{(y-c_2x)^2}{3(c_2-1)^2}\left(\left(2c_1c_2-3c_1+c_2\right)x+\left(c_1-3c_2+2\right)y\right) \tag{2.12}$$

and $\nabla_{\hat{H}}\hat{\xi}_{12}=-\hat{X}(\hat{\xi}_{12})(\hat{X}-\hat{Y})$ with

$$\hat{X}(\hat{\xi}_{12})=-\frac{(y-c_2x)^2}{3(c_2+1)^2}\left(\left(2c_1c_2+3c_1-c_2\right)x+\left(c_1-3c_2-2\right)y\right). \tag{2.13}$$

Using (2.13) it is easy to see that $\nabla_{\hat{H}}\hat{\xi}_{12}$ is null f.d. in $\{-x<y<c_2x\}$. Indeed, if $c_1-3c_2-2\geq 0$ then we have $\left(2c_1c_2+3c_1-c_2\right)x+\left(c_1-3c_2-2\right)y\geq 2\left(c_2+1\right)\left(c_1+1\right)x>0$. If, on the other hand, $c_1-3c_2-2<0$ then $\left(2c_1c_2+3c_1-c_2\right)x+\left(c_1-3c_2-2\right)y>\left(2c_1c_2+3c_1-c_2\right)x+c_2\left(c_1-3c_2-2\right)x=3\left(c_2+1\right)\left(c_1-c_2\right)x>0$. Also, since $c_1-3c_2+2=c_1-c_2+2\left(1-c_2\right)>0$, we see that $\nabla_{\hat{H}}\hat{\xi}_{11}$ is null f.d. for $y<-\frac{2c_1c_2-3c_1+c_2}{c_1-3c_2+2}x$.

Lemma 2.3. $-\frac{2c_1c_2-3c_1+c_2}{c_1-3c_2+2}>c_1$.

Proof. Since $c_1-3c_2+2=c_1-c_2+2\left(1-c_2\right)>0$, the hypothesis is equivalent to $-\left(2c_1c_2-3c_1+c_2\right)-c_1\left(c_1-3c_2+2\right)=\left(1-c_1\right)\left(c_1-c_2\right)>0$ which is of course true. ∎

Now, by lemmas 2.1 and 2.3 it is clear that $E_1<-\frac{2c_1c_2+c_2-3c_1}{c_1+2-3c_2}$. We will compute the intersection $\{\hat{\eta}_1=0\}\cap\left\{\hat{\xi}_{11}=0\right\}\cap\left\{E_1x<y<-\frac{2c_1c_2+c_2-3c_1}{c_1+2-3c_2}x\right\}$. Evidently

$$\hat{\xi}_{11}\left(x,c_2x,z\right)-\hat{\eta}_1\left(x,c_2x,z\right)=-\frac{1}{48}x^4\left(c_2-1\right)^2\left(c_2+1\right)^2W<0. \tag{2.14}$$

On the other hand

$$\hat{\xi}_{11}\left(x,c_1x,z\right)-\hat{\eta}_1\left(x,c_1x,z\right)=-\frac{1}{48}\frac{\left(c_1-1\right)^2}{\left(c_2-1\right)^2}x^4\left(4\left(c_1-c_2\right)^3+\left(c_2-1\right)^2\left(c_1+1\right)^2\left(c_1c_2-2c_1+2c_2-1\right)\right). \tag{2.15}$$

We need the following

Lemma 2.4. *Let $f(x,y)=4(x-y)^3+(y-1)^2(x+1)^2(xy-2x+2y-1)$ be a function considered on the set $D=\{(x,y):\ -1<y<x<1\}$. Then $f<0$ on D.*

Proof. We look for stationary points of f. Any such point (x,y) must satisfy the equality

$$\frac{\partial f}{\partial x}(x,y)+\frac{\partial f}{\partial y}(x,y)=(x+1)\,(y-1)\,(x+y)\,(3xy+5y-5x-3)=0.$$

Clearly $3xy+5y-5x-3=3(xy-1)-5(x-y)<0$ in D, thus either $x=-1$or $y=1$, or $y=-x$. If $y=-x$ in D, we must have $x>0$, but then $\frac{\partial f}{\partial x}(x,-x)=-3x^5-20x^4-46x^3-23x-4<0$ while $\frac{\partial f}{\partial y}(x,-x)=3x^5+20x^4+46x^3+23x+4>0$. It follows that all stationary points of f are contained in ∂D. By direct calculation we make sure that $f_{|\partial D}\leq 0$ which implies $f<0$ in D. ∎

Lemma 2.4 and (2.15) give

$$\hat{\xi}_{11}\,(x,c_1x,z)-\hat{\eta}_1\,(x,c_1x,z)>0. \tag{2.16}$$

Moreover, $(X+Y)(\hat{\xi}_{11}-\hat{\eta}_1)=-(X+Y)(\hat{\eta}_1)<0$ on $\{c_2x<y<E_1x\}$, and $(X+Y)(\hat{\xi}_{11}-\hat{\eta}_1)>0$ on $\{E_1x<y<x\}$. Now let us sum up what we already know. By (2.14) the expression $\hat{\xi}_1-\hat{\eta}_1$ (which is a homogeneous polynomial in x, y) is negative on $y=c_2x$ and decreases along the trajectories of $X+Y$ in $\{c_2x<y<E_1x\}$. Then $\hat{\xi}_{11}-\hat{\eta}_1$ starts to increase and for $y=c_1x$ it attains a positive value by (2.16). It follows that $\{\hat{\eta}_1=0\}\cap\left\{\hat{\xi}_{11}=0\right\}\cap\left\{E_1x<y<-\frac{2c_1c_2+c_2-3c_1}{c_1+2-3c_2}x\right\}$ is of the form $\{y=S_1x\}$ with $E_1<S_1<c_1$. In this way we arrive at

$$\hat{J}^+(0)\cap\{z\geq 0\}=\hat{A}_1\cup\hat{A}_2\cup\hat{A}_3$$

where

$$\hat{A}_1=\{\hat{\eta}_1\leq 0\}\cap\{S_1x\leq y\leq x\}\cap\{z\geq 0\}\,, \tag{2.17}$$

$$\hat{A}_2=\left\{\hat{\xi}_{11}\leq 0\right\}\cap\{c_2x\leq y\leq S_1x\}\cap\{z\geq 0\}\,, \tag{2.18}$$

$$\hat{A}_3=\left\{\hat{\xi}_{12}\leq 0\right\}\cap\{-x\leq y\leq c_2x\}\cap\{z\geq 0\}\,. \tag{2.19}$$

Now we examine $\hat{J}^+(0)\cap\{z\leq 0\}$. Let us consider two Cauchy problems:

$$(\hat{X}-\hat{Y})(\eta)=0,\ \eta(x,c_1x,z)=-z$$

with the solution equal to

$$\hat{\xi}_{21}(x,y,z)=-z-\frac{1}{12}\frac{(x+y)\,(c_1x-y)^2}{(1+c_1)^2}\left((c_1-2c_1c_2-3c_2)\,x+(3c_1-c_2+2)\,y\right),$$

and

$$(\hat{X}+\hat{Y})(\eta)=0,\ \eta(x,c_1x,z)=-z$$

with the solution equal to

$$\hat{\xi}_{22}(x,y,z)=-z+\frac{1}{12}\frac{(x-y)\,(y-c_1x)^2}{(1-c_1)^2}\left((-2c_1c_2-c_1+3c_2)\,x+(3c_1-c_2-2)\,y\right).$$

As above we need to know the regions where horizontal gradients $\nabla_{\hat{H}}\hat{\xi}_{2i}$, $i=1,2$, are suitably directed. After calculations

$$\hat{X}(\hat{\xi}_{21})=\frac{(c_1x-y)^2}{3\,(c_1+1)^2}\left((2c_1c_2-c_1+3c_2)\,x+(-3c_1+c_2-2)\,y\right) \tag{2.20}$$

and

$$\hat{X}(\hat{\xi}_{22}) = \frac{(y-c_1x)^2}{3(1-c_1)^2}\left((-2c_1c_2-c_1+3c_2)x+(3c_1-c_2-2)y\right). \tag{2.21}$$

(2.21) immediately yields that $\nabla_{\hat{H}}\hat{\xi}_{22}$ is null f.d. in $\{c_1x<y<x\}$. Indeed, $(-2c_1c_2-c_1+3c_2)x+(3c_1-c_2-2)y<-2(c_2-1)(c_1-1)x<0$ in this sector whenever $3c_1-c_2-2\geq 0$. On th other hand, if $3c_1-c_2-2\geq 0$ then $(-2c_1c_2-c_1+3c_2)x+(3c_1-c_2-2)y<(-2c_1c_2-c_1+3c_2)x+c_1(3c_1-c_2-2)x=3(c_1-1)(c_1-c_2)x<0$. Also by (2.20) we know that $\nabla_H\hat{\xi}_{21}$ is null f.d. for $y>-\frac{2c_1c_2-c_1+3c_2}{-3c_1+c_2-2}x$. Indeed, it is enough to notice that $-3c_1+c_2-2=-(c_1-c_2)-2(c_1+1)<0$.

Lemma 2.5. $-\frac{2c_1c_2-c_1+3c_2}{-3c_1+c_2-2}<c_2$.

Proof. Since $-3c_1+c_2-2=-(c_1-c_2)-2(c_1+1)<0$, the hypothesis is equivalent to $-(2c_1c_2-c_1+3c_2)-c_2(-3c_1+c_2-2)=(c_2+1)(c_1-c_2)>0$. ∎

Now it follows that $-\frac{2c_1c_2-c_1+3c_2}{-3c_1+c_2-2}<E_4$, and we will compute the intersection $\{\hat{\eta}_2=0\}\cap\left\{\hat{\xi}_{21}=0\right\}\cap\left\{-\frac{2c_1c_2-c_1+3c_2}{-3c_1+c_2-2}x<y<E_4x\right\}$. We proceed similarly as above. So first of all

$$\hat{\xi}_{21}(x,c_1x,z)-\hat{\eta}_2(x,c_1x,z)=-\frac{1}{48}x^4(c_1-1)^2(c_1+1)^2W<0. \tag{2.22}$$

Next

$$\hat{\xi}_{21}(x,c_2x,z)-\hat{\eta}_2(x,c_2x,z)=\frac{(c_2+1)^2}{48(c_1+1)^2}x^4\left(4(-c_1+c_2)^3-(c_1+1)^2(c_2-1)^2(c_1c_2+2c_2-2c_1-1)\right). \tag{2.23}$$

Lemma 2.6. *The function* $f(x,y)=4(-x+y)^3-(x+1)^2(y-1)^2(xy-2x+2y-1)$ *is positive on* $D=\{(x,y):\ -1<y<x<1\}$.

Proof. This is proved analogously as lemma 2.4. ∎

Lemma 2.6 and (2.23) give

$$\hat{\xi}_{21}(x,c_2x,z)-\hat{\eta}_2(x,c_2x,z)>0. \tag{2.24}$$

Now (2.22) and (2.24) imply that similarly as above the set $\{\hat{\eta}_2=0\}\cap\left\{\hat{\xi}_{21}=0\right\}\cap\left\{-\frac{2c_1c_2-c_1+3c_2}{-3c_1+c_2-2}x<y<E_4x\right\}$ is of the form $\{y=S_2x\}$ with $c_2<S_2<E_4$. We deduce that

$$\hat{J}^+(0)\cap\{z\leq 0\}=\hat{A}_4\cup\hat{A}_5\cup\hat{A}_6$$

where

$$\hat{A}_4=\{\hat{\eta}_2\leq 0\}\cap\{-x\leq y\leq S_2x\}\cap\{z\leq 0\}, \tag{2.25}$$

$$\hat{A}_5=\left\{\hat{\xi}_{21}\leq 0\right\}\cap\{S_2x\leq y\leq c_1x\}\cap\{z\leq 0\}, \tag{2.26}$$

$$\hat{A}_6=\left\{\hat{\xi}_{22}\leq 0\right\}\cap\{c_1x\leq y\leq x\}\cap\{z\leq 0\}. \tag{2.27}$$

We conclude this section with the following

Proposition 2.2. *If $W > 0$ then*

$$\hat{J}^+(0) = \hat{N}^+(0) = \hat{A}_1 \cup ... \cup \hat{A}_6,$$
$$\hat{I}^+(0) = \operatorname{int}(\hat{A}_1 \cup ... \cup \hat{A}_6) \cup A_7 \cup A_8$$

and

$$N^+(0) = \operatorname{int}(\hat{A}_1 \cup ... \cup \hat{A}_6) \cup (\{\hat{\eta}_1 = 0\} \cap \{S_1 x \le y \le x\} \cap \{z \ge 0\}) \cup \\ \cup (\{\hat{\eta}_2 = 0\} \cap \{-x \le y \le S_2 x\} \cap \{z \le 0\})$$

where $\hat{A}_1$,..., $\hat{A}_6$ are given by (2.17), (2.18), (2.19) and (2.25), (2.26), (2.27), respectively, and $A_7 = \{y = c_1 x,\ z = 0,\ x \ge 0\}$, $A_8 = \{y = c_2 x,\ z = 0,\ x \ge 0\}$.

In this way we see that there are two geometrically optimal timelike (abnormal) curves and we need six analytic functions to describe reachable sets.

2.3. **The case $W = 0$.** Using (2.8), (2.10), (2.7), (2.9) and the condition $W = 0$ we see that in the case under consideration

$$E_1 = E_2 = -\frac{2c_1 - 1}{c_1 - 2} = c_2,$$
$$E_3 = E_4 = \frac{2c_2 + 1}{c_2 + 2} = c_1,$$
$$\hat{\eta}_1(x, y, z) = z + \frac{x^2 - y^2}{12}\,(c_1 - 2)\,(y - c_2 x)^2\,,$$
$$\hat{\eta}_2(x, y, z) = -z - \frac{x^2 - y^2}{12}\,(c_2 + 2)\,(y - c_1 x)^2$$

and what follows

$$\hat{X}(\hat{\eta}_1) = \frac{1}{3}\,(x - y)\,(c_1 - 2)\left(y + \frac{2c_1 - 1}{c_1 - 2}x\right)^2 = \frac{1}{3}\,(x - y)\,(c_1 - 2)\,(y - c_2 x)^2\,,$$

$$\hat{X}(\hat{\eta}_2) = -\frac{1}{3}\,(x + y)\,(c_2 + 2)\left(y - \frac{2c_2 + 1}{c_2 + 2}x\right)^2 = -\frac{1}{3}\,(x + y)\,(c_2 + 2)\,(y - c_1 x)^2\,.$$

Thus $\nabla_{\hat{H}}\hat{\eta}_1$ is null f.d. on $\{-x < y < x\} \cap \{y \ne c_2 x\}$, and $\nabla_{\hat{H}}\hat{\eta}_2$ is null f.d. on $\{-x < y < x\} \cap \{y \ne c_1 x\}$. We can also see that $\hat{\eta}_1(x, c_2 x, 0) = \hat{\eta}_2(x, c_1 x, 0) = 0$. Moreover $\nabla_{\hat{H}}\hat{\xi}_{11}$ is null f.d. for $y < -\frac{2c_1c_2 - 3c_1 + c_2}{c_1 - 3c_2 + 2}x$, and as above we make sure that (2.16) holds together with

$$\hat{\xi}_{11}\,(x, c_2 x, z) - \hat{\eta}_1\,(x, c_2 x, z) = 0. \tag{2.28}$$

Similar reasoning as in subsection 2.2 shows that $\hat{\xi}_{11} - \hat{\eta}_1$ is nondecreasing along trajectories of $X + Y$ starting at $\{y = c_2 x\}$. It follows that $\hat{\xi}_{11} \ge \hat{\eta}_1$, and in turn

$$\left\{\hat{\xi}_{11} \le 0\right\} \cap \{c_2 x < y < x\} \subset \{\hat{\eta}_1 \le 0\} \cap \{c_2 x < y < x\}\,.$$

Moreover

$$\hat{\xi}_{12}(x, y, z) - \hat{\eta}_1(x, y, z) = -\frac{1}{48}\,(x + y)^4\,(c_2 - 1)^2\,\frac{W}{(c_2 + 1)^2} = 0$$

which is, in fact, clear without calculations, since both functions satisfy the same differential equation with the same boundary conditions on the hypersurface $\{y = c_2 x\}$. Similar reasoning shows that

$$\left\{\hat{\xi}_{21} \le 0\right\} \cap \{-x < y < c_1 x\} \subset \{\hat{\eta}_2 \le 0\} \cap \{-x < y < c_1 x\}$$

and

$$\hat{\xi}_{22}(x,y,z) - \hat{\eta}_2(x,y,z) = -\frac{1}{48}(x-y)^4 (c_1+1)^2 \frac{W}{(c_1-1)^2} = 0.$$

We sum up this subsection with the following

Proposition 2.3. *If $W = 0$ then*

$$\hat{J}^+(0) = \hat{N}^+(0) = \hat{A}_1 \cup \hat{A}_2$$

and

$$\hat{I}^+(0) = \operatorname{int}(\hat{A}_1 \cup \hat{A}_2),$$

where
$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ \hat{\eta}_1(x,y,z) \leq 0\right\} \cap \{x \geq 0, \ z \geq 0\}$,
$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \ \hat{\eta}_2(x,y,z) \leq 0\right\} \cap \{x \geq 0, \ z \leq 0\}$.

As we see this case is very exceptional as compared to the previous cases with $W \neq 0$. Namely, in spite of the fact that there are two geometrically optimal timelike curves, only two analytic functions suffice for describing reachable sets.

3. Normal forms.

In this section we consider more general sub-Lorentzian structures (H,g) than those dealt with in theorem 1.1. At first we describe the underlying distribution H. So let H be a rank 2 distribution defined on a neighbourhood U of the origin in $\mathbb{R}^3$. H will be said to satisfy the condition $(M_{l_1,\ldots,l_k})$ if it possesses the following properties:
(i) there exist smooth hypersurfaces $S_1,\ldots,S_k$ in U, such that the intersection $\Gamma = \bigcap_{i=1}^k S_i$ contains the origin and is smooth 1-dimensional; moreover for each $q \in \Gamma$ and every $i,j = 1,\ldots,k$, $i \neq j$, $\dim(T_qS_i \cap T_qS_j) = 1, \dim(T_qS_i \cap H_q) = 1$;
(ii) H defines a contact structure on $U \backslash \bigcup_{i=1}^k S_i$;
(iii) There exist positive integers $l_1,\ldots,l_k \geq 2$ such that for any fixed $i = 1,\ldots,k$, $H_q^l \subset H_q$, $l \leq l_i$, $H^{l_i+1} = T_qR^3$ on the set $S_i \backslash \Gamma$.
(iv) $H_q^l \subset H_q$, $l \leq l_1 + \ldots + l_k - k + 1$, $H_q^{l_1+\ldots+l_k-k+2} = T_qR^3$ for every $q \in \Gamma$.
Now we choose a Lorentzian metric g. We make two assumptions:
(v) for each $i = 1,\ldots,k$, the field of directions $S_i \ni q \longrightarrow T_qS_i \cap H_q$ is timelike.;
(vi) for every $i,j = 1,\ldots,k$, $i \neq j$, the function $S_i \cap S_j \ni q \longrightarrow \sphericalangle(T_qS_i \cap H_q, T_qS_j \cap H_q)$ is constant;
(vii) the abnormal curves foliating the surfaces $S_1,\ldots,S_k$ are, up to a change of parameter, t.f.d. Hamiltonian geodesics.

We will say that the structure (H,g) is of type $M_{l_1,\ldots,l_k}$ on U if (i),...,(vii) hold on U. Our aim is to prove the following

Theorem 3.1. *Let (H,g) be a time-oriented analytic sub-Lorentzian structure of type $M_{l_1,\ldots,l_k}$ defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, provided that U is sufficiently small, there exist analytic coordinates x,y,z on U in which (H,g) has an orthonormal frame in the normal form*

$$\begin{aligned} X &= \frac{\partial}{\partial x} + y\varphi\Big(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big) + \frac{1}{2}y\,(y-c_1x)^{l_1-1}\ldots(y-c_kx)^{l_k-1}(1+\psi)\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} - x\varphi\Big(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big) - \frac{1}{2}x\,(y-c_1x)^{l_1-1}\ldots(y-c_kx)^{l_k-1}(1+\psi)\frac{\partial}{\partial z} \end{aligned},$$

where X is a time orientation, $c_1, ..., c_k$ are constants such that $-1 < c_k < ... < c_1 < 1$, $S_i = \{y = c_i x\}$, $i = 1, ..., k$, and finally φ, ψ are analytic functions on U with $\psi(0,0,z) = 0$.

Clearly, it is seen that $S = \bigcup_{i=1}^{k} S_i$ is the Martinet surface for (H, g) in the described coordinates.

We start from the following

Lemma 3.1. *Suppose that (H, g) is analytic and satisfies the condition $M_{2,...,2}$ on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, provided that U is sufficiently small, there are analytic coordinates x, y, z defined on U in which (H, g) admits an orthonormal frame in the following form*

$$
\begin{aligned}
X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - y\,(y - c_1 x)\dots(y - c_k x)\,A\frac{\partial}{\partial z} \\
Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + x\,(y - c_1 x)\dots(y - c_k x)\,A\frac{\partial}{\partial z}
\end{aligned}
\tag{3.1}
$$

with X being a time orientation. Here A, B are analytic functions and $S_i = \{y = c_i x\}$, $i = 1, ..., k$.

Proof. By [6] we know that there are coordinates in which (H, g) has an orthonormal frame in the form

$$
\begin{aligned}
X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - yA\frac{\partial}{\partial z} \\
Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + xA\frac{\partial}{\partial z}
\end{aligned}
.
$$

Now, the z-coordinate of $[X, Y]$ is equal to

$$[X,Y]\,(z) = \left(1 - y^2 B\right)\left(A + x\frac{\partial A}{\partial x}\right) - x^2 y\frac{\partial A}{\partial y}B + xy^2\frac{\partial A}{\partial x}B + \left(1 + x^2 B\right)\left(A + y\frac{\partial A}{\partial y}\right).$$

By our assumptions $[X,Y]_{|y=c_i x}$ is horizontal, $i = 1, ..., k$. Take $i = 1$. Then there exist analytic functions $f(x,z)$, $g(x,z)$ such that $[X,Y]_{|y=c_1 x} = f(x,z)X_{|y=c_1 x} + g(x,z)Y_{|y=c_1 x}$. This leads us to the equality of the z-coordinates

$$2A + \left(\frac{\partial A}{\partial x} + c_1\frac{\partial A}{\partial y}\right)x + \left(1 - c^2\right)ABx^2 = \left(g(x,z) - c_1 f(x,z)\right)xA \tag{3.2}$$

where A and its derivatives should be evaluated at $(x, c_1 x, z)$. Suppose that $A(x, c_1 x, z)$ does not vanish identically. Then we can find a z such that $A(x, c_1 x, z) = a_m(z)x^m + o(x^m)$ as $x \longrightarrow 0$ with $a_m(z) \neq 0$ and $m > 0$. Since $x\left(\frac{\partial A}{\partial x} + c_1\frac{\partial A}{\partial y}\right)_{|y=c_1 x} = x\,(X + c_1 Y)\,(A)_{|y=c_1 x} = m a_m(z)x^m + o(x^m)$, (3.2) gives

$$(2 + m)\,a_m(z) = o(1),$$

so we arrive at $a_m(z) = 0$ which is a contradiction. In this way A may be replaced by the expression $(y - c_1 x)A$ for some other analytic function A.

Repeating the argument for $i = 2, ..., k$ we are lead to (3.1). ∎

Now suppose that our structure (H, g), which is given by (3.1) on a neighbourhood U of the origin, satisfies the condition $M_{l_1,...,l_k}$, $l_i \geq 2$, $i = 1, ..., k$.

Fix an index i. If we set A_1 to be the function defined by $y\,(y-c_i x)\,A_1 = y\,(y-c_1 x)\,...\,(y-c_k x)\,A$, the frame (3.1) takes the form

$$(3.3)\qquad \begin{aligned} X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - y\,(y-c_i x)\,A_1\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + x\,(y-c_i x)\,A_1\frac{\partial}{\partial z} \end{aligned}.$$

Applying the following change of coordinates

$$\begin{cases} \tilde{x} = x\cosh\varphi - y\sinh\varphi \\ \tilde{y} = -x\sinh\varphi + y\cosh\varphi \\ \tilde{z} = z \end{cases},$$

with $\tanh\varphi = c_i$ (i.e. $y - c_i x = \tilde{y}/\cosh\varphi$), and rewriting (3.3) we are led to

$$(3.4)\qquad \begin{aligned} X &= (\cosh\varphi)\,\tilde{X} - (\sinh\varphi)\,\tilde{Y} \\ Y &= -\,(\sinh\varphi)\,\tilde{X} + (\cosh\varphi)\,\tilde{Y} \end{aligned},$$

where

$$(3.5)\qquad \begin{aligned} \tilde{X} &= \frac{\partial}{\partial\tilde{x}} - \tilde{y}B(\tilde{y}\frac{\partial}{\partial\tilde{x}} + \tilde{x}\frac{\partial}{\partial\tilde{y}}) - \tilde{y}^2\tilde{A}\frac{\partial}{\partial\tilde{z}} \\ \tilde{Y} &= \frac{\partial}{\partial\tilde{y}} + \tilde{x}B(\tilde{y}\frac{\partial}{\partial\tilde{x}} + \tilde{x}\frac{\partial}{\partial\tilde{y}}) + \tilde{x}\tilde{y}\tilde{A}\frac{\partial}{\partial\tilde{z}} \end{aligned},$$

and $\tilde{A} = \frac{A_1}{\cosh\varphi}$. Obviously (cf. [14]) $\tilde{X}$, $\tilde{Y}$ is again an orthonormal frame for (H,g) with a time orientation $\tilde{X}$, and we can apply to it the same method as in [10], proposition 3.1. As a result, (3.5) can be written as

$$\begin{aligned} \tilde{X} &= \frac{\partial}{\partial\tilde{x}} - \tilde{y}B(\tilde{y}\frac{\partial}{\partial\tilde{x}} + \tilde{x}\frac{\partial}{\partial\tilde{y}}) - \tilde{y}^{l_i}\hat{A}\frac{\partial}{\partial\tilde{z}} \\ \tilde{Y} &= \frac{\partial}{\partial\tilde{y}} + \tilde{x}B(\tilde{y}\frac{\partial}{\partial\tilde{x}} + \tilde{x}\frac{\partial}{\partial\tilde{y}}) + \tilde{x}\tilde{y}^{l_i-1}\hat{A}\frac{\partial}{\partial\tilde{z}} \end{aligned},$$

with $\tilde{A} = \tilde{y}^{l_i-2}\hat{A}$. Passing again to (3.4) we obtain

$$\begin{aligned} X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - y\,(y-c_i x)^{l_i-1}\,(\cosh\varphi)^{l_i-1}\,\hat{A}\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + x\,(y-c_i x)^{l_i-1}\,(\cosh\varphi)^{l_i-1}\,\hat{A}\frac{\partial}{\partial z} \end{aligned},$$

i.e. to say

$$\begin{aligned} X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - y\,(y-c_1 x)\,...\,(y-c_i x)^{l_i-1}\,...\,(y-c_k x)\,A\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + x\,(y-c_1 x)\,...\,(y-c_i x)^{l_i-1}\,...\,(y-c_k x)\,A\frac{\partial}{\partial z} \end{aligned}$$

for a suitably chosen analytic function A. Repeating the same argument for every $i = 1, ..., k$, we are led to

$$(3.6)\qquad \begin{aligned} X &= \frac{\partial}{\partial x} - yB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + y\,(y-c_1 x)^{l_1-1}\,...\,(y-c_k x)^{l_k-1}\,A\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} + xB(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - x\,(y-c_1 x)^{l_1-1}\,...\,(y-c_k x)^{l_k-1}\,A\frac{\partial}{\partial z} \end{aligned}$$

for yet another analytic A about which we know that does not contain terms of the form $(y-c_ix)^l$. (3.6) is, in fact, all we can get without assuming (v).

Now we take (v) into account. To obtain a condition for A we need $l_1+...+l_k-k+1$ differentiations of $X(z)$ or $Y(z)$, i.e. we have to consider sections of $H^{l_1+...+l_k-k+2}$. So let W be a (local) section of $H^{l_1+...+l_k-k+2}$ defined near zero. Then, looking at (3.6), we see that $W(z)=CA+O(r)$, where by (v) $C\neq 0$ for suitable chosen W. Setting $x=y=0$ we arrive at $A(0,0,z)\neq 0$. Last stage is to renormalize the z-axis, so as to have $A(0,0,z)=\frac{1}{2}$. This can be done similarly as e.g. in [10]. To end the proof we write $\varphi=-B$, $\psi=2A-1$.

4. Reachable sets in the general case.

In this section, by (H,g) we denote a fixed time-oriented sub-Lorentzian metric of type $M_{2,2}$, defined on a normal neighbourhood U of the origin in $\mathbb{R}^3$. Throughout this section we assume that U is as small as we need. We may suppose that (H,g) is already transformed to the normal form. Let X,Y be an orthonormal frame for (H,g) given on U by (1.1). In cases $W>0$, $W<0$ we will use the same method to compute local reachable sets as in [8], [9], [10]. The mentioned method, however, does not work when $W=0$, and it seems impossible to arbitrate in advance what will be the structure of the reachable set in this case.

Let $X=\hat{X}+X_1$, $Y=\hat{Y}+Y_2$, $\hat{X},\hat{Y}$ are as in (2.1) and

$$\begin{aligned} Y_1 &= y\varphi(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})+\frac{1}{2}y\,(y-c_1x)\,(y-c_2x)\,\psi\frac{\partial}{\partial z} \\ Y_2 &= -x\varphi(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})-\frac{1}{2}x\,(y-c_1x)\,(y-c_2x)\,\psi\frac{\partial}{\partial z} \end{aligned}. \tag{4.1}$$

4.1. **The case $W<0$.** Similarly as in section 2, consider two Cauchy problems:

$$(X-Y)(\eta)=0,\quad \eta(x,x,z)=z$$

with the solution denoted by η_1, and

$$(X+Y)(\eta)=0,\quad \eta(x,-x,z)=-z$$

with the solution denoted by η_2. We write $\eta_1=\hat{\eta}_1+R_1$, $\eta_2=\hat{\eta}_2+R_2$. It is seen that R_1 and R_2 satisfy respectively

$$(X-Y)(R_1)=-(X_1-Y_1)(\hat{\eta}_1),\quad R_1(x,x,z)=0$$

and

$$(X+Y)(R_2)=-(X_1+Y_1)(\hat{\eta}_2),\quad R_2(x,-x,z)=0.$$

Clearly, $(X_1-Y_1)(\hat{\eta}_1)=O(r^4)$, $r=\sqrt{x^2+y^2+z^2}$. Since η_1-z is divisible by x^2-y^2 ($y=-x$ is the trajectory of $X-Y$ starting at $(0,0,0)$), we deduce that $R_1=O(r^5)$. Similarly, $R_2=O(r^5)$ which, in view of $\hat{\eta}_i=\pm z+O(r^4)$, means that η_i may be regarded as a perturbation of $\hat{\eta}_i$, $i=1,2$. Exactly e.g. as in subsection 4.1 of [10] we prove that $X(\eta_1)$ is divisible by $x-y$. It follows that $\nabla_H\eta_1=-X(\eta_1)(X-Y)$ where, by using (2.4), we have

$$X(\eta_1)=-\frac{1}{12}\,(x-y)\,((7c_1c_2-2c_1-2c_2+1)\,x^2+4\,(c_1c_2-2c_1-2c_2+1)\,xy+(c_1c_2-2c_1-2c_2+7)\,y^2+O(r^3)). \tag{4.2}$$

It follows that $X(\eta_1) < 0$ on $U \cap \{y < x\}$ and $\nabla_H \eta_1$ is null f.d. on $U \cap \{y < x\}$. Similarly
(4.3)

$$X(\eta_2) = -\frac{1}{12}(x+y)\left((7c_1c_2 + 2c_1 + 2c_2 + 1)\,x^2 - 4\,(c_1c_2 + 2c_1 + 2c_2 + 1)\,xy + (c_1c_2 + 2c_1 + 2c_2 + 7)\,y^2 + O(r^3)\right)$$

from which $X(\eta_2) < 0$ on $U \cap \{-x < y\}$ and hence $\nabla_H \eta_2$ is null f.d. on $U \cap \{-x < y\}$. We finish the proof of theorem 1.2 as in the subsection 2.1.

4.2. **The case** $W > 0$**.** Here $X(\eta_1)$ and $X(\eta_2)$ are again given by (4.2) and (4.3), respectively. This time however, $X(\eta_1) < 0$ on $\{(E_1 + \varepsilon)x < y < x\} \cap U$ and $X(\eta_2) < 0$ on $\{-x < y < (E_2 - \varepsilon)x\} \cap U$, where $\varepsilon > 0$ and will be supposed to be sufficiently small. Next we define functions ξ_{11}, ξ_{12} as solutions to the following Cauchy problems

$$(X + Y)(\eta) = 0,\ \eta(x, c_2x, z) = z$$

and

$$(X - Y)(\eta) = 0, \eta(x, c_2x, z) = z,$$

respectively. As above we write $\xi_{11} = \hat{\xi}_{11} + R_{11}$, $\xi_{12} = \hat{\xi}_{12} + R_{12}$, where e.g. R_{11} satisfies $(X + Y)(R_{11}) = -(X_1 + Y_1)(\hat{\xi}_{11})$, $R_{11}(x, c_2x, z) = 0$. It follows that $R_{11} = O(r^5)$, and similarly $R_{12} = O(r^5)$. So again we may think of ξ_{1i} as being perturbations of $\hat{\xi}_{1i}$, $i = 1, 2$. Now, since $X + c_2Y_{|y=c_2x} = \frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}$, $(X + c_2Y)(\xi_{11})_{|y=c_2x} = 0$ by definition of ξ_{11}. But also $(X + Y)(\xi_{11}) = 0$, from which $X(\xi_{11})_{|y=c_2x} = 0$, and therefore $X(\xi_{11})$ is divisible by $y - c_2x$. We prove analogously that also $X(\xi_{12})$ is divisible by $y - c_2x$. However it is not enough for our purposes and we need the following

Lemma 4.1. *$X(\xi_{11})$ and $X(\xi_{12})$ are divisible by $(y - c_2x)^2$.*

Proof. We prove the first statement. We already know that $X(\xi_{11}) = (y - c_2x)g$ for an analytic function g. Since $[X, X + Y] = 0$ on $\{y = c_2x\}$, it follows that $(X + Y)(X(\xi_{11})) = X(X + Y)(\xi_{11}) = 0$ on $\{y = c_2x\}$.

$$(X + Y)(X(\xi_{11})) = (1 - c_2 - (y - x)(c_2y + x)\varphi)\, g + O\left((y - c_2x)\right).$$

By setting $y = c_2x$ we arrive at $(1 - c_2)\left[1 + (c_2 + 1)x^2\varphi\right] g_{|y=c_2x} = 0$ and the proof is over since g must be divisible by $y - c_2x$ (recall that U is as small as we need).

The proof of the second statement is analogous. We notice that $[X, X - Y] = 0$ on $\{y = c_2x\}$, so $(X - Y)(X(\xi_{11})) = X(X - Y)(\xi_{11}) = 0$ on $\{y = c_2x\}$, and continue in the same manner. ∎

Making use of (2.12), (2.13) and the above lemma, $\nabla_H \xi_{11} = -X(\xi_{11})(X + Y)$ with

$$X(\xi_{11}) = \frac{(y - c_2x)^2}{3\,(c_2 - 1)^2}\left((2c_1c_2 - 3c_1 + c_2)\,x + (c_1 - 3c_2 + 2)\,y + O(r^2)\right)$$

and $\nabla_H \xi_{12} = -X(\xi_{12})(X - Y)$ with

$$X(\xi_{12}) = -\frac{(y - c_2x)^2}{3\,(c_2 + 1)^2}\left((2c_1c_2 + 3c_1 - c_2)\,x + (c_1 - 3c_2 - 2)\,y + O(r^2)\right).$$

This of course implies that $\nabla_H \xi_{11}$ is null f.d. on $\left\{c_2x < y < \left(-\frac{2c_1c_2 - 3c_1 + c_2}{c_1 - 3c_2 + 2} - \varepsilon\right)x\right\} \cap U$, and $\nabla_H \xi_{12}$ is null f.d. on $\{-x < y < c_2x\}$.

Using just presented considerations and remembering subsection 2.2 we may suppose that $\xi_{11} < \eta_1$ on $\{c_2x < y < (E_1+\varepsilon)x\} \cap U$, while $\xi_{11} > \eta_1$ on $\{(c_1-\varepsilon)x < y < x\} \cap U$ or, which is more convenient to us, that $\xi_{11} < \eta_1$ on $\{c_2x < y < (S_1-\varepsilon)x\} \cap U$, while $\xi_{11} > \eta_1$ on $\{(S_1+\varepsilon)x < y < x\} \cap U$.

Let us define a set Z_1 by

$$Z_1 = \{\eta_1 = 0\} \cap \{\xi_{11} = 0\} \cap \{(S_1-\varepsilon)x < y < (S_1+\varepsilon)\,x\} \cap U;$$

clearly Z_1 is a semi-analytic set (cf. [13]). As in [9] and [10] we make sure that $\dim Z_1 = 1$ and that Z_1 is made up of a single analytic curve entering the origin. Further, let us define a semi-analytic set by $\Sigma_1 = \rho^{-1}(\pi(Z_1) \cap U$ where $\rho : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the projection $(x, y, z) \longrightarrow (x, y)$, and let Σ_1^+, Σ_1^- be the two components of $U \cap \{x \geq 0\} \cap \{z \geq 0\} \cap \{c_2x \leq y \leq x\} \setminus \Sigma_1$, containing half-lines $y = x$, $x \geq 0$ and $y = c_2x$, $x \geq 0$, respectively. All what we have just said leads us to

$$J^+(0, U) \cap \{z \geq 0\} = A_1 \cup A_2 \cup A_3$$

where

$$A_1 = \{\eta_1 \leq 0\} \cap \Sigma_1^+,$$

$$A_2 = \{\xi_{11} \leq 0\} \cap \Sigma_1^-,$$

$$A_3 = \{\xi_{12} \leq 0\} \cap \{-x \leq y \leq c_2x\} \cap \{z \geq 0\} \cap U$$

as it is announced in theorem 1.3.

Quite similar considerations can be carried out to describe the set $J^+(0, U) \cap \{z \leq 0\}$. We only note that this time we define a 1-dimensional semi-analytic set

$$Z_2 = \{\eta_2 = 0\} \cap \{\xi_{21} = 0\} \cap \{(S_2-\varepsilon)x < y < (S_2+\varepsilon)\,x\} \cap U.$$

Then we set $\Sigma_2 = \rho^{-1}(\pi(Z_2) \cap U$, and define Σ_2^+, Σ_2^- to be the two connected components of $U \cap \{x \geq 0\} \cap \{z \leq 0\} \cap \{-x \leq y \leq c_1x\} \setminus \Sigma_2$, containing half-lines $y = c_1x$, $x \geq 0$ and $y = -x$, $x \geq 0$, respectively. Finally we obtain

$$J^+(0, U) \cap \{z \leq 0\} = A_4 \cup A_5 \cup A_6$$

where

$$A_4 = \{\eta_2 \leq 0\} \cap \Sigma_2^-,$$

$$A_5 = \{\xi_{21} \leq 0\} \cap \Sigma_2^+,$$

$$A_6 = \{\xi_{22} \leq 0\} \cap \{c_1x \leq y \leq x\} \cap \{z \leq 0\} \cap U.$$

This terminates the proof of theorem 1.3.

4.3. **The case $W = 0$.** Here, as it was mentioned earlier we are not able to say what the structure of reachable sets is. This is, for instance, because the relation $\hat{\eta}_1(x, c_2x, 0) = 0$ may no longer be true after perturbation. Therefore we cannot predict the sign of the expression $\eta_1(x, c_2x, 0)$ even in a small neighbourhood of the origin without computing higher order terms in the expression for η_i. We will not do it in this paper.

5. Applications to control affine systems.

Let X, Y be two linearly independent smooth vector fields defined on an open set U in $\mathbb{R}^3$ (or any 3-manifold), and consider the following affine control system with a scalar input

$$\dot{q} = X + uY, \quad u \in [a,b]. \tag{5.1}$$

Fix $q_0 \in U$ and denote by $\mathcal{A}_{[a,b]}(q_0,U)$ (resp. $\mathcal{A}_{(a,b)}(q_0,U)$, $\mathcal{A}_{\{a,b\}}(q_0,U)$) the set of endpoints of all trajectories of (5.1) contained in U that initiate at q_0 and are generated by measurable controls $u:[0,T] \longrightarrow [a,b]$ (resp. $u:[0,T] \longrightarrow (a,b)$, $\{u:[0,T] \longrightarrow \{a,b\}\}$) where $T = T(u)$ depends on a control. It was proved in [10] that the reachable sets $\mathcal{A}_{[a,b]}(q_0,U)$, $\mathcal{A}_{(a,b)}(q_0,U)$, $\mathcal{A}_{\{a,b\}}(q_0,U)$ are respectively equal to the future nonspacelike, timelike, and null reachable sets for the time-oriented sub-Lorentzian structure $(H^{a,b}, g^{a,b})$ defined by declaring the frame

$$Z^{a,b} = X + \tfrac{1}{2}(b+a)Y$$
$$W^{a,b} = \tfrac{1}{2}(b-a)Y$$

to be orthonormal with a time orientation $Z^{a,b}$.

Let us recall a notion of *singular trajectories* (cf. [4]) for affine control systems. So, a trajectory $\gamma:[0,T] \longrightarrow U$ is called *a singular trajectory* for (5.1) if it is generated by a control $u(t)$ with values in the open interval (a,b) and is an abnormal curve (see [12] for a definition) for the distribution $H = Span\{X,Y\}$. In terms of the sub-Lorentzian metric $(H^{a,b}, g^{a,b})$, singular trajectories are exactly timelike abnormal curves.

The aim of this section is to investigate reachable sets from the origin for the system as in (5.1) where X and Y are supposed to be given by (1.1). To save space, we will not give exact formulas for functions describing reachable sets. We will restrict ourselves only in examining the structure of reachable sets and it dependence on geometric optimality of singular trajectories.

The first evident observation is that

$$\mathcal{A}_{[a,b]}(q_0,U) \subset \{ax \leq y \leq bx\} \cap U$$

where U is supposed to be sufficiently small normal neighbourhood of the origin. In order to investigate e.g. the reachable set $\mathcal{A}_{[a,b]}(q_0,U)$, i.e. the future nonspacelike reachable set $J^+(0,U)$ for $(H^{a,b}, g^{a,b})$, it is enough (see [10], lemma 1.1) to consider the reachable set for the following affine control system

$$\dot{q} = Z^{a,b} + uW^{a,b}, \quad u \in [-1,1]. \tag{5.2}$$

5.1. **The case $c_1, c_2 \notin (a,b)$.** In this case there are no singular trajectories starting from the origin for (5.2). As in [8], [10] or as above, we investigate z-coordinates of $Z^{a,b} \pm W^{a,b}$. So

$$\left(Z^{a,b} + W^{a,b}\right)(z) = \frac{1}{2}\left(y - c_1 x\right)\left(y - c_2 x\right)\left(y - bx\right)\left(1 + \psi\right) \tag{5.3}$$

and

$$\left(Z^{a,b} - W^{a,b}\right)(z) = \frac{1}{2}\left(y - c_1 x\right)\left(y - c_2 x\right)\left(y - ax\right)\left(1 + \psi\right) \tag{5.4}$$

have opposite signs in $\{ax \leq y \leq bx\} \cap U$, therefore $\mathcal{A}_{[a,b]}(q_0,U)$ is described by two analytic functions.

5.2. **The case $c_2 \leq a < c_1 < b$ or $a < c_2 < b \leq c_1$.** In this case the system (5.2) has one singular trajectory initiating at the origin. Again we examine the signs of z-coordinates of $Z^{a,b} \pm W^{a,b}$. Thus, in the first case (5.3) on $\{y = ax\}$ and (5.4) on $\{y = bx\}$ are both positive, while in the second case they are both negative. Also (5.3) and (5.4) are negative on $\{y = c_1 x\}$ in the first case, while (5.3) and (5.4) are positive on $\{y = c_2 x\}$ in the second case. Therefore we arrive at the similar situation as in [10], and it can be proved that the same reasoning leads to the conclusion that the minimal number of analytic functions needed for describing $\mathcal{A}_{[a,b]}(q_0, U)$ is four.

5.3. **The case $a < c_2 < c_1 < b$.** In this case the system (5.2) has two singular trajectories initiating at the origin. In order to simplify the situation we make the following change of coordinates in (5.2):

$$\left\{ \begin{array}{l} \tilde{x} = x \\ \tilde{y} = -\frac{b+a}{b-a} x + \frac{2}{b-a} y \\ \tilde{z} = z \end{array} \right. . \tag{5.5}$$

The resulting system is

$$\dot{q} = \tilde{Z}^{a,b} + u\tilde{W}^{a,b} \tag{5.6}$$

where

$$\begin{array}{l} \tilde{Z}^{a,b} = \frac{\partial}{\partial \tilde{x}} + \varphi A^{a,b} + \frac{1}{2}\left(\frac{b-a}{2}\right)^3 \tilde{y}\,(\tilde{y} - \tilde{c}_1\tilde{x})\,(\tilde{y} - \tilde{c}_2\tilde{x})\,(1+\psi)\,\frac{\partial}{\partial \tilde{z}} \\ \tilde{W}^{a,b} = \frac{\partial}{\partial \tilde{y}} - \varphi B^{a,b} - \frac{1}{2}\left(\frac{b-a}{2}\right)^3 \tilde{x}\,(\tilde{y} - \tilde{c}_1\tilde{x})\,(\tilde{y} - \tilde{c}_2\tilde{x})\,(1+\psi)\,\frac{\partial}{\partial \tilde{z}} \end{array} \tag{5.7}$$

with

$$\begin{array}{l} A^{a,b} = \frac{1}{2}\tilde{y}\left(\frac{1}{2}\left(b^2-a^2\right)\tilde{x} + \frac{1}{2}(b-a)^2\tilde{y}\right)\frac{\partial}{\partial \tilde{x}} + \frac{1}{2}\tilde{y}\left(\frac{1}{2}\left(4-(a+b)^2\right)\tilde{x} - \frac{1}{2}\left(b^2-a^2\right)\tilde{y}\right)\frac{\partial}{\partial \tilde{y}} \\ B^{a,b} = \frac{1}{2}\tilde{x}\left(\frac{1}{2}\left(b^2-a^2\right)\tilde{x} + \frac{1}{2}(b-a)^2\tilde{y}\right)\frac{\partial}{\partial \tilde{x}} + \frac{1}{2}\tilde{x}\left(\frac{1}{2}\left(4-(a+b)^2\right)\tilde{x} - \frac{1}{2}\left(b^2-a^2\right)\tilde{y}\right)\frac{\partial}{\partial \tilde{y}} \end{array},$$

and

$$\tilde{c}_1 = \frac{2c_1 - b - a}{b-a}, \quad \tilde{c}_2 = \frac{2c_2 - b - a}{b-a}.$$

Note that by rescaling the $\tilde{z}$-axis we can always get rid of the factor $\left(\frac{b-a}{2}\right)^3$, so we no longer take care of it. Since the change of coordinates (5.5) is bi-analytic, transforms straight lines onto straight lines, and preserves geometric optimality of trajectories, the reachable set for (5.6) is described by the same number of analytic functions as $\mathcal{A}_{[a,b]}(q_0, U)$ and has the same number of geometrically optimal singular trajectories. Now we repeat the above arguments. Reachable sets for (5.6) with $\varphi = \psi = 0$ in (5.7) are computed according to section 2. Reachable sets for (5.6) with arbitrary φ and ψ in (5.7) are computed according to section 4 (here we should remark that although (5.7) does not coincide with (1.1), arguments still work since, as one easily checks, functions describing reachable sets for arbitrary φ and ψ are again perturbations of functions describing reachable sets for $\varphi = \psi = 0$). To sum up, $\mathcal{A}_{[a,b]}(q_0, U)$ is described by six analytic function when $W(\tilde{c}_1, \tilde{c}_2) < 0$ and by two analytic functions if $W(\tilde{c}_1, \tilde{c}_2) > 0$. We also know that in case $W(\tilde{c}_1, \tilde{c}_2) = 0$ and $\varphi = \psi = 0$ two analytic functions suffice.

In this way the proof of corollary 1.1 is over. To illustrate the presented results let us consider two examples.

Example 5.1. *Suppose that we are interested in the structure of the reachable set from the origin for the following affine control system* $\dot{q} = X + uY$, $u \in [-1, 4]$, *where* $X = \frac{\partial}{\partial x} + \frac{1}{2}y(y-2x)(y-3x)\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} - \frac{1}{2}x(y-2x)(y-3x)\frac{\partial}{\partial z}$. *This system is equivalent to the sub-Lorentzian structure* $(H^{-1,4}, g^{-1,4})$ *defined by an orthonormal basis* $\tilde{X}$, $\tilde{Y}$ *with* $\tilde{X}$ *being a time orientation, where*

$$\begin{array}{l} \tilde{X} = \frac{\partial}{\partial x} + \frac{3}{2}\frac{\partial}{\partial y} + \frac{1}{2}\left(y - \frac{3}{2}x\right)(y-2x)(y-3x)\frac{\partial}{\partial z} \\ \tilde{Y} = \frac{5}{2}\frac{\partial}{\partial y} - \frac{5}{4}x(y-2x)(y-3x)\frac{\partial}{\partial z} \end{array} .$$

According to (5.5) we change of coordinates as follows: $\tilde{x} = x$, $\tilde{y} = -\frac{3}{5}\tilde{x} + \frac{2}{5}\tilde{y}$, $\tilde{z} = z$, *and as a result we have*

$$\begin{array}{l} \tilde{X} = \frac{\partial}{\partial \tilde{x}} + \frac{1}{2}\left(\frac{5}{2}\right)^3 \tilde{y}\left(\tilde{y} - \frac{3}{5}\tilde{x}\right)\left(\tilde{y} - \frac{1}{5}\tilde{x}\right)\frac{\partial}{\partial \tilde{z}} \\ \tilde{Y} = \frac{\partial}{\partial \tilde{y}} - \frac{1}{2}\left(\frac{5}{2}\right)^3 \tilde{x}\left(\tilde{y} - \frac{3}{5}\tilde{x}\right)\left(\tilde{y} - \frac{1}{5}\tilde{x}\right)\frac{\partial}{\partial \tilde{z}} \end{array} . \tag{5.8}$$

We end up with $\tilde{c}_1 = \frac{3}{5}$, $\tilde{c}_2 = \frac{1}{5}$. *Since* $W(\frac{3}{5}, \frac{1}{5}) < 0$ *we know that there are no geometrically optimal singular trajectories, hence the reachable set from the origin for (5.8) can be described by two analytic functions.*

Example 5.2. *Consider again the system* $\dot{q} = X + uY$, $u \in [-1, 4]$, *where this time* $X = \frac{\partial}{\partial x} + \frac{1}{2}y(y-x)(y-3x)\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} - \frac{1}{2}x(y-x)(y-3x)\frac{\partial}{\partial z}$. *According to the above procedure we pass the the sub-Lorentzian structure induced by an orthonormal frame*

$$\begin{array}{l} \tilde{X} = \frac{\partial}{\partial x} + \frac{3}{2}\frac{\partial}{\partial y} + \frac{1}{2}(y-x)(y-3x)\left(y - \frac{3}{2}x\right)\frac{\partial}{\partial z} \\ \tilde{Y} = \frac{5}{2}\frac{\partial}{\partial y} - \frac{5}{4}x(y-x)(y-3x)\frac{\partial}{\partial z} \end{array} .$$

Making again the same change of coordinates we are led to

$$\begin{array}{l} \tilde{X} = \frac{\partial}{\partial \tilde{x}} + \frac{1}{2}\left(\frac{5}{2}\right)^3 \tilde{y}\left(\tilde{y} - \frac{3}{5}\tilde{x}\right)\left(\tilde{y} + \frac{1}{5}\tilde{x}\right)\frac{\partial}{\partial \tilde{z}} \\ \tilde{Y} = \frac{\partial}{\partial \tilde{y}} - \frac{1}{2}\left(\frac{5}{2}\right)^3 \tilde{x}\left(\tilde{y} - \frac{3}{5}\tilde{x}\right)\left(\tilde{y} + \frac{1}{5}\tilde{x}\right)\frac{\partial}{\partial \tilde{z}} \end{array} ,$$

so $\tilde{c}_1 = \frac{3}{5}$, $\tilde{c}_2 = -\frac{1}{5}$. *But this time* $W(\frac{3}{5}, -\frac{1}{5}) > 0$, *therefore we conclude that there are two geometrically optimal singular trajectories, and that we need six analytic functions to describe reachable sets.*

Faculty of Mathematics and Science, Cardinal Stefan Wyszyński University, ul. Dewajtis 5, 01-815 Waszawa, Poland

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland

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E-mail address: `m.grochowski@uksw.edu.pl`