THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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J. Lewis

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p Harmonic Measure in Simply Connected Domains Revisited

John L. Lewis*†
Department of Mathematics, University of Kentucky
Lexington, KY 40506-0027, USA

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Abstract

Let Ω be a bounded simply connected domain in the complex plane, $\mathbb{C}$. Let N be a neighborhood of $\partial\Omega$, let p be fixed, $1 < p < \infty$, and let $\hat{u}$ be a positive weak solution to the p Laplace equation in $\Omega \cap N$. Assume that $\hat{u}$ has zero boundary values on $\partial\Omega$ in the Sobolev sense and extend $\hat{u}$ to $N \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $N \setminus \Omega$. Then there exists a positive finite Borel measure $\hat{\mu}$ on $\mathbb{C}$ with support contained in $\partial\Omega$ and such that

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \, dA = - \int \phi \, d\hat{\mu}$$

whenever $\phi \in C_0^\infty(N)$. In this paper we continue our studies in [BL05], [L06], [LNP11], [LNV] by establishing endpoint type results for the Hausdorff dimension of this measure in simply connected domains. Our results are similar to the well known result of Makarov [M85] concerning harmonic measure in simply connected domains.

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*email: john@ms.uky.edu

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1 Introduction

Let $\mathbb{C}$ denote the complex plane and let dA be Lebesgue measure on $\mathbb{C}$. If $O \subset \mathbb{C}$ is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the Banach space of equivalence classes of functions $\hat{u}$ with distributional gradient $\nabla \hat{u} = (\hat{u}_x, \hat{u}_y)$, and norm

$$\|\hat{u}\|_{1,q} = \|\hat{u}\|_q + \|\nabla \hat{u}\|_q < \infty$$

where $\|\cdot\|_q$ denotes the usual Lebesgue q norm in O. Denote infinitely differentiable functions with compact support in O by $C_0^\infty(O)$ and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Throughout this paper $\Omega \subset \mathbb{C}$ is a bounded simply connected domain. Let N be a neighborhood of $\partial\Omega$, p fixed, $1 < p < \infty$, and suppose that $\hat{u}$ is a positive weak solution to the p Laplace equation in $\Omega \cap N$ That is, $\hat{u} \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \, dA = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Equivalently we say that $\hat{u}$ is p harmonic in $\Omega \cap N$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in $\Omega \cap N$, then $\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$, in the classical sense, where $\nabla\cdot$ denotes divergence. We assume that $\hat{u}$ has zero boundary values on $\partial\Omega$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(N)$, then $\hat{u}\,\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend $\hat{u}$ to $N \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $N \setminus \Omega$. Then $\hat{u} \in W^{1,p}(N)$ and it follows from (1.1), as in [HKM93], that there exists a positive finite Borel measure $\hat{\mu}$ on $\mathbb{C}$ with support contained in $\partial\Omega$ and the property that

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \, dA = -\int \phi \, d\hat{\mu} \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. We note that if $\partial\Omega$ is smooth enough, then $d\hat{\mu} = |\nabla \hat{u}|^{p-1} \, ds$. Also if $p = 2$ and if $\hat{u}$ is the Green function for Ω with pole at $x \in \Omega$ then $\hat{\mu}$ coincides with harmonic measure at x. In this paper for fixed $p, 1 < p < \infty, p \neq 2$, we continue our study of the Hausdorff dimension of $\hat{\mu}$ (denoted H-dim $\hat{\mu}$) defined by

$$\text{H-dim } \hat{\mu} = \inf\{\alpha : \text{ there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^\alpha(E) = 0 \text{ and } \hat{\mu}(E) = \hat{\mu}(\partial\Omega)\}, \tag{1.3}$$

where $H^\alpha(E)$, for $\alpha \in \mathbf{R}_+$, is the α-dimensional Hausdorff measure of E defined below. In order to state our results we shall need some more notation: Denote points in the complex plan by $z = x_1 + ix_2$ and put $B(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ whenever $z \in \mathbb{C}$ and $r > 0$. Let $d(E, F)$ denote the distance between the sets $E, F \subset \mathbb{C}$. If $\lambda > 0$ is a positive function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$ define H^λ Hausdorff measure on $\mathbb{C}$ as follows: For fixed $0 < \delta < r_0$ and $E \subseteq \mathbb{R}^2$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, \ldots$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Then

$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

In case $\lambda(r) = r^\alpha$ we write H^α for H^λ.

In [LNP11] we proved the following theorem which generalized earlier results in [BL05], [L06].

Theorem A. *Given $p, 1 < p < \infty, p \neq 2$, let $\hat{u}, \hat{\mu}$ be as in (1.1), (1.2), and suppose Ω is simply connected. Put*

$$\tilde{\lambda}(r) = r \exp[A\sqrt{\log 1/r \, \log\log 1/r}], 0 < r < 10^{-6}.$$

Then the following is true.

(α) *If $p > 2$, there exists $A = A(p) \leq -1$ such that $\hat{\mu}$ is concentrated on a set of σ finite $H^{\tilde{\lambda}}$ measure.*

(β) *If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that $\hat{\mu}$ is absolutely continuous with respect to H^{λ}.*

We note that Theorem A easily implies

$$\text{H-dim } \hat{\mu} \leq 1 \text{ for } p > 2 \text{ while H-dim } \hat{\mu} \geq 1 \text{ when } 1 < p < 2. \tag{1.4}$$

In this paper we improve the results in [LNP11] by proving:

Theorem 1. *Given $p, 1 < p < \infty, p \neq 2$, let $\hat{u}, \hat{\mu}$ be as in (1.1), (1.2), and suppose Ω is simply connected. Put*

$$\lambda(r) = \lambda(r, A) = r \exp[A\sqrt{\log 1/r \, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

Then the following is true.

(a) *If $p > 2$, then $\hat{\mu}$ is concentrated on a set of σ finite H^1 measure.*

(b) *If $1 < p < 2$, then $\hat{\mu}$ is absolutely continuous with respect to H^{λ} provided $A = A(p) \geq 1$ is large enough. Moreover $A(p)$ is bounded on (3/2, 2).*

Remark: Makarov in [M85] (see also [M90],[P92], and [GM05]), essentially proved Theorem 1 for harmonic measure with respect to a point in Ω (the $p = 2$ case) . Moreover, in this case it suffices in (b) to take $A = 6\sqrt{(\sqrt{24} - 3)/5}$, see [HK07]. We note that Makarov also showed that (b) is sharp for harmonic measure in the sense that there exist simply connected domains Ω for which harmonic measure is mutually singular with respect to H^{λ} provided A in the definition of λ is small enough. We do not know if an analogous sharpness result holds when $1 < p < 2$. In fact the natural examples for $p = 2$, eg., snowflakes, do not provide sharpness when $1 < p < 2$. Indeed, in [BL05, Theorem 1] we showed for certain snowflakes, Ω, fixed $p, 1 < p < 2$, and a corresponding $\hat{u}, \hat{\mu}$, that there exists $\kappa = \kappa(\Omega) > 1$, with

$$\lim_{r \to 0} \frac{\log \hat{\mu}[B(z, \hat{\mu})]}{\log r} = \kappa \text{ for } \hat{\mu} \text{ almost every } z \in \partial\Omega. \tag{1.5}$$

From (1.5) and measure theoretic arguments we see that $\hat{\mu}$ is absolutely continuous with respect to $H^{1+\epsilon}$ measure provided $\epsilon < \kappa - 1$. In particular (b) of Theorem 1 holds in this example whenever $A > 0$. Is it possible that $\hat{\mu}$ is absolutely continuous with respect to H^1 measure when $1 < p < 2$?

To outline our proof of Theorem 1, we note that using translation and dilation invariance of the p Laplace equation and arguing as in the display below (3.1) of [LNP11] one can show that it suffices to prove Theorem 1 when

$$0 \in \Omega \text{ and } d(0, \partial\Omega) = 4. \tag{1.6}$$

Thus throughout the proof of Theorem 1 we assume (1.6). Also from Lemma 2.4 in section 2 we deduce that it suffices to prove Theorem 1 for fixed $p, 1 < p < \infty$, when u is the p capacitary function for $D = \Omega \setminus \bar{B}(0,1)$ and μ is the corresponding capacitary measure. That is, u is p harmonic in D with continuous boundary values: $u \equiv 1$ on $\partial B(0,1)$ while $u \equiv 0$ on $\partial\Omega$. For this u we proved in [LNP11, Theorem 1.5], the fundamental inequality:

$$c^{-1}\frac{u(z)}{d(z,\partial\Omega)} \leq |\nabla u(z)| \leq c\frac{u(z)}{d(z,\partial\Omega)} \tag{1.7}$$

whenever $z \in D$ where c depends only on p. (1.7) was the main ingredient which allowed us to generalize our results in [BL05] for quasi-circles to simply connected domains. From (1.7) it follows (see Lemma 2.6 in section 2) that u is infinitely differentiable in $D \cap \Omega$. Using this fact and integrating (1.1) by parts we see that $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ in D. Hence u is a strong solution to the p Laplace partial differential equation in D. Differentiating this equation we obtain another key ingredient which is used in all our recent papers on the p Laplacian: If $\zeta = u$ or $\zeta = u_{x_i}, i = 1, 2$, then ζ is a solution to

$$L\zeta(z) \;=\; \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[\, b_{ij}(z)\zeta_{x_j}(z)\,\right] = 0, \tag{1.8}$$

at $z \in D$, where

$$b_{ij}(z) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](z),\ 1 \leq i, j \leq 2, \tag{1.9}$$

and δ_{ij} is the Kronecker δ. Also at $z \in D$,

$$\min\{p-1,1\}|\xi|^2\,|\nabla u(z)|^{p-2} \;\leq\; \sum_{i,j=1}^{n} b_{ij}\,\xi_i\xi_j \;\leq\; \max\{1,p-1\}|\nabla u(z)|^{p-2}\,|\xi|^2 \tag{1.10}$$

whenever $\xi \in \mathbb{R}^n \setminus \{0\}$. Furthermore, if $v = \log|\nabla u|$ then in [BL05] we showed that

$$Lv = F \text{ in } D \tag{1.11}$$

where $F \equiv 0$ if $p = 2$, and if $p \neq 2$, then for some $c = c(p) \geq 1$, bounded for $p \in (3/2, 5/2)$, we have at $z \in D$,

$$(p-2)^{-1}c^{-1}F(z) \leq |\nabla u|^{p-4}\sum_{i,j=1}^{2} u_{x_ix_j}^2 \leq (p-2)^{-1}cF(z) \tag{1.12}$$

Next we mention that in [L06] we proved Theorem 1 when $\partial\Omega$ is a k quasi-circle and k is small enough. In this case our strategy was to first show for k small that $|\nabla u|^{p-2}$ extends to an A_2 weight in $\mathbb{C}$. We could then use (1.8)-(1.10) and apply results for degenerate divergence form elliptic PDE whose degeneracy is given in terms of an A_2 weight from [FKS82], [FKJ82], [FKJ83]. Using results from these papers involving Green's functions, boundary Harnack inequalities, and Poisson integral representation formulas we were able to make estimates on v similar to those in [M85](see also [P92, ch 8] or [GM05, ch 8] for these estimates) only in D rather than the unit disk.

The emphasis in this paper is necessarily entirely different since Ω is an arbitrary bounded simply connected domain and reflects a growing philosophy of the author that to make estimates one should essentially only use integration by parts and the fact that u is a solution to (1.8)-(1.10) while v is a subsolution (supersolution) to these equations when $p > 2(1 < p < 2)$. In this respect our philosophy seems to us more akin to the arguments in [M90] and in [JW88], [W93] for arbitrary domains $\subset \mathbb{R}^2$. However the arguments in this paper also make heavy use of the fundamental inequality in (1.7). The ultimate goal though is to do away with our reliance on (1.7) in order to prove the following conjecture.

Conjecture : *The conclusion of Theorem 1 (a) remains valid with H^1 replaced by H^{n-1} when Ω is an arbitrary bounded domain $\subset \mathbb{R}^n$ and $p \geq n$.*

This conjecture is proved in [LNV] for domains $\Omega \subset \mathbb{R}^n$ whose boundaries are sufficiently flat in the Reifenberg sense. Once again however (1.7) plays a fundamental role in the arguments.

As for the plan of this paper in Lemmas 2.1 - 2.4 of section 2 we list some basic properties of p harmonic functions as well as results from [BL05], [L06] for p harmonic functions which vanish on a portion of $\partial\Omega$. In Lemmas 2.5, 2.7 we state some of the key results in [LNP11]. In Lemma 2.8 we give a slight generalization of the results in Lemma 2.7. In section 3 we prove Theorem 1 (a) while in sections 4 and 5 we prove Theorem 1 (b). Finally in the appendix (section 6) we outline the proofs of Lemma 2.2 when $p > 2$ and Lemma 2.8.

2 Results for p Harmonic Functions.

In the sequel c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1, which may depend only on $p, a_1, \dots, a_n$, not necessarily the same at each occurrence. $A \approx B$ means that A/B is bounded above and below by positive constants depending only on p. In this section, we will always assume that Ω is a bounded simply connected domain, $0 < r <$ diam $\partial\Omega$ and $w \in \partial\Omega$. We begin by stating some interior and boundary estimates for $\tilde u$, a positive weak solution to the p Laplacian in $B(w, 4r) \cap \Omega$ with $\tilde u \equiv 0$ in the Sobolev sense on $\partial\Omega \cap B(w, 4r)$. That is, $\tilde u \in W^{1,p}(B(w, 4r) \cap \Omega)$ and (1.1) holds whenever $\theta \in W_0^{1,p}(B(w, 4r) \cap \Omega)$. Also $\zeta \tilde u \in W_0^{1,p}(B(w, 4r) \cap \Omega)$ whenever $\zeta \in C_0^\infty(B(w, 4r))$. Extend $\tilde u$ to $B(w, 4r)$ by putting $\tilde u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then there exists a locally finite positive Borel measure $\tilde\mu$ with support $\subset B(w, 4r) \cap \partial\Omega$ and for which (1.2) holds with $\hat u$ replaced by $\tilde u$ and $\phi \in C_0^\infty(B(w, 4r))$. Let $\max_{B(z,s)} \tilde u$, $\min_{B(z,s)} \tilde u$ be the essential supremum and infimum of $\tilde u$ on $B(z, s)$ whenever $B(z, s) \subset B(w, 4r)$. For references to proofs of Lemmas 2.1 - 2.3 (see [BL05]). We have not been able to find a reference for Lemma 2.2 when $p > 2$. Thus we will outline the proof of Lemma 2.2 when $p > 2$ in the appendix to this paper (see section 6).

Lemma 2.1. *Fix $p, 1 < p < \infty$, and let $\Omega, w, r, \tilde{u}$, be as above. Then*

$$c^{-1} r^{p-2} \int_{B(w,r/2)} |\nabla \tilde{u}|^p \, dA \le \max_{B(w,r)} \tilde{u}^p \le c \, r^{-2} \int_{B(w,2r)} \tilde{u}^p \, dA.$$

If $B(z, 2s) \subset \Omega$, then

$$\max_{B(z,s)} \tilde{u} \le c \min_{B(z,s)} \tilde{u}.$$

Lemma 2.2. *Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 2.1. Then there exists $\alpha = \alpha(p) \in (0, 1)$ with $\alpha > \frac{p-2}{p-1}$ when $p > 2$ such that $\tilde{u}$ has a Hölder α continuous representative in $B(w, r)$ (also denoted $\tilde{u}$). Moreover if $x, y \in B(w, r)$ then*

$$|\tilde{u}(x) - \tilde{u}(y)| \le c \, (|x - y|/r)^\alpha \max_{B(w,2r)} \tilde{u}$$

Lemma 2.3. *Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 2.1 and let $\tilde{\mu}$ be the measure associated with $\tilde{u}$ as in (1.2). Then there exists c such that*

$$c^{-1} \, r^{p-2} \, \tilde{\mu}[B(w, r/2)] \le \max_{B(w,r)} \tilde{u}^{p-1} \le c \, r^{p-2} \, \tilde{\mu}[B(w, 2r)].$$

The next three lemmas are more formal statements of the discussion after (1.6). For a proof of the following lemma, see Lemma 2.4 in [LNP11].

Lemma 2.4. *Fix $p, 1 < p < \infty$, and let $\hat{u}, \Omega$ be the positive p harmonic function and bounded simply connected domain in Theorem 1. Assume (1.6) holds and that u is the p capacitary function for $D = \Omega \setminus \bar{B}(0, 1)$. defined below (1.6). Let $\mu, \hat{\mu}$, be the measures corresponding to $u, \hat{u}$, respectively. Then $\mu, \hat{\mu}$ are mutually absolutely continuous. In particular, Theorem 1 is valid for $\hat{\mu}$ if and only if it is valid for μ.*

The next lemma is Theorem 1.5 in [LNP11].

Lemma 2.5. *Let p, u, Ω, D, be as in Lemma 2.4. There exists $c_1 = c_1(p)$ such that*

$$c_1^{-1} \frac{u(z)}{d(z, \partial\Omega)} \le |\nabla u(z)| \le c_1 \frac{u(z)}{d(z, \partial\Omega)} \quad \textit{whenever } z \in D.$$

Using Lemma 2.5 and Schauder type estimates one can prove the following lemma (compare with Lemma 4.2 in [LNP11]).

Lemma 2.6 *Let p, u, Ω, D be as in Lemma 2.4. Then u is real-analytic in D, and u has a C^∞ extension to $\Omega \setminus \bar{B}(0, 3/4)$ (also denoted u). Moreover Lemma 2.5 remains valid whenever $z \in \Omega \setminus \bar{B}(0, 3/4)$ and if $\partial_k u$ denotes an arbitrary k derivative of u, then for $z \in \Omega \setminus \bar{B}(0, 3/4)$,*

$$|\partial_k u(z)| \leq \frac{\tilde{c}_k\, u(z)}{d(z, \partial\Omega)^{k+1}} \text{ where } \tilde{c}_k \text{ depends only on } k \text{ and } p.$$

From Lemma 2.6 and the maximum principle for p harmonic functions we observe that if $\Omega(t) = \{z : u(z) > t\}, t \in (0, 1)$, then $\bar{B}(0, 1) \subset \Omega(t)$ and $\partial\Omega(t)$ is a real analytic Jordan curve whenever $t \in (0, 1)$. For the next lemma see section 4.1 in [LNP11].

Lemma 2.7. *Let p, u, Ω, D be as in Lemma 2.4. Given $z_1 \in \Omega \setminus B(0, 2)$ suppose $u(z_1) > t$ and $\partial\Omega(t) \cap \bar{B}(0, 2) = \emptyset$ for some $t \in (0, 1/2)$. There exists constants $c_i = c_i(p), 2 \leq i \leq 4$, depending only on p and closed Jordan arcs γ, τ with the following properties.*

(α) *γ joins $z_2 \in \partial\Omega(t)$ to $z_3 \in \partial\Omega(t) (z_2 \neq z_3)$, $\gamma \setminus \{z_2, z_3\} \subset \Omega \setminus \bar{B}(0, 3/2)$, and $z_1 \in \gamma$.*

(β) *$H^1(\gamma) \leq c_2 d(z_1, \partial\Omega(t))$ and $u \leq c_2(u(z_1) - t)$ on γ.*

Let Ω_1 be the Jordan domain $\subset \Omega(t)$ whose boundary consists of γ and the arc of $\partial\Omega(t)$ connecting z_2 to z_3 for which $\Omega_1 \cap \bar{B}(0, 3/2) = \emptyset$.

(γ) *There is a $z_4 \in \partial\Omega_1 \cap \partial\Omega(t)$ with $c_3 d(z_4, \gamma) \geq d(z_1, \partial\Omega(t))$.*

(δ) *$\tau : [0.1] \rightarrow \Omega_1 \cup \{z_1, z_4\}$ joins z_1 to z_4 and satisfies the cigar condition,* $\min\{H^1(\tau[0, s]), H^1(\tau[s, 1])\} \leq c_4 \min\{d(z_1, \partial\Omega(t)), d(\tau(s), \partial\Omega(t))\}, s \in [0, 1]$.

We shall need the following extension of Lemma 2.7.

Lemma 2.8. *Using the same notation as in Lemma 2.7, there also exists a Jordan curve $\beta : [0, 1] \rightarrow \Omega_1 \cup \{z_5, z_6\}$, joining z_5 to z_6 where z_5 lies on the open arc of $\partial\Omega_1$ with endpoints z_2, z_4 while z_6 lies on the open arc of $\partial\Omega_1$ with endpoints z_4, z_3. Moreover, there exist constants $c_i = c_i(p), 5 \leq i \leq 7$, with*

(α') $d(z_1, \partial\Omega(t)) \leq c_5 \min\{d(\gamma, \beta), d(z_4, \beta)\} \leq c_5 \max\{d(\gamma, \beta), d(z_4, \beta)\} \leq c_5^2\, d(z_1, \partial\Omega(t))$,

(β') $\min\{H^1(\beta[0, s]), H^1(\beta[s, 1])\} \leq c_6 \min\{d(z_1, \partial\Omega(t)), d(\beta(s), \partial\Omega(t))\}, s \in [0, 1]$,

(γ') *If $s_0 \in (0, 1)$ satisfies $H^1(\beta[0, s_0]) = H^1(\beta[s_0, 1])$, then $(u(\beta(t')) - t) \leq c_7(u(\beta(s)) - t)$ for $0 \leq t' \leq s \leq s_0$, and $(u(\beta(t')) - t) \leq c_7(u(\beta(s)) - t)$ for $s_0 \leq s \leq t' \leq 1$.*

Lemma 2.8 is a straightforward extension of Lemma 2.7. However since Lemma 2.8 will play a fundamental role in our proof of Theorem 1 and for the reader's convenience we shall outline the proof of this lemma in the appendix (section 6).

3 Proof of Theorem 1(a)

To begin the proof of Theorem 1(a) we discuss the orthogonal trajectories to the levels of u. Existence and properties of these trajectories can be deduced from standard ordinary differential equation theory. Another way is to construct a local 'conjugate', h to u defined by

$$h_{x_2} = |\nabla u|^{p-2}\, u_{x_1}, \;\; h_{x_1} \,=\, -|\nabla u|^{p-2}\, u_{x_2}. \tag{3.1}$$

From Lemmas 2.5, 2.6 we see that u is a strong solution to the p Laplacian in D, so the above differential equation is exact. Thus h exists locally and is unique up to a constant. Also, it is easily checked that h is a solution to the p' Laplacian, where $p' = p/(p-1)$. Using this fact, one gets (as in Lemma 2.6) that h is real analytic, locally in D. Note that the levels of h are orthogonal to the levels of u and that the mapping $z = x_1 + ix_2 \rightarrow (u+ih)(z)$ has Jacobian equal to $|\nabla u|^p(z)$.

Given $t \in (0,1]$ let μ_t denote the measure corresponding to $u - t$ as in Lemma 2.3. From smoothness of $u, \partial\Omega(t), t \in (0,1]$ we deduce that

$$d\mu_t = |\nabla u|^{p-1} dH^1|_{\partial\Omega(t)}\;. \tag{3.2}$$

Moreover, from the divergence theorem and the fact that u is a solution to the p Laplace equation in D we get

$$\mu_t(\partial\Omega(t)) \,=\, \int_{\partial\Omega(t)} |\nabla u|^{p-1} dH^1 = \xi \neq 0, \tag{3.3}$$

where ξ is independent of $t \in (0,1]$. Fix $z_1 \in D$ and let

$$F(z) = \exp[(2\pi/\xi)(u(z) + ih(z))],$$

for z in a neighborhood of z_1. If $h(z_1) = 0$, then from our previous observation we see that F can be continued uniquely throughout D to get a sense preserving function mapping $D \rightarrow \{w : 1 < |w| < e^{2\pi/\xi}\}$. Moreover since h increases on $\partial\Omega(t)$, if this curve is traversed clockwise as viewed from the origin, it follows that F is 1 - 1 and onto the above annulus. Next choose $t_0 \in (0, 1/2]$ so small that $\partial\Omega(t_0) \cap \bar{B}(0,2) = \emptyset$. Given $\hat{z} \in \partial\Omega(t_0)$, draw the ray from the origin through $F(\hat{z})$. Let $l(F(\hat{z}),\cdot)$ be the intersection of this ray with the above annulus. Set $\sigma(\hat{z},\cdot) = F^{-1}(l(\hat{z},\cdot))$. Observe that h is constant on $\sigma(\hat{z},\cdot)$ and ∇u is tangent to $\sigma(\hat{z},\cdot)$. Thus $\sigma(\hat{z},\cdot)$ is orthogonal to $\partial\Omega(t)$ whenever $t \in (0,1)$. Since u is strictly decreasing along $\sigma(\hat{z},\cdot)$, we deduce that $\sigma(\hat{z},\cdot)$ is a Jordan arc and can be parametrized by

$$u(\sigma(\hat{z},s)) = 1 - s, \text{ for } 0 < s < 1, \; \hat{z} \in \partial\Omega(t_0). \tag{3.4}$$

Also from Lemma 2.6 we see that $\sigma(\hat{z},0) = \lim_{t\rightarrow 0} \sigma(\hat{z},t) \in \partial B(0,1)$ so (3.4) is also valid when $s = 0$. From our construction and the implicit function theorem we see that σ is infinitely differentiable in $\hat{z}, s$. Also, it is easily seen that for each $z \in D$, there is a unique $\hat{z} \in \partial\Omega(t_0)$ with z on $\sigma(\hat{z},\cdot)$. Next applying the coarea theorem (see [EG92]) to a branch of h in Ω or using F^{-1} we find that

$$\int_{\partial\Omega(t_0)} |\nabla u|^{p-1}\, H^1(\sigma(\hat{z},\cdot))\, d\mu_{t_0}(\hat{z}) = \int_{\Omega} |\nabla u|^{p-1} dA \leq C < \infty \tag{3.5}$$

where $C = C(p, \Omega)$. Here the next to last inequality follows from Lemmas 2.1, 2.5. From (3.5) and weak type estimates we deduce for given $\lambda > 0$ that if
$\Theta(\lambda) = \{\hat{z} \in \partial\Omega(t_o) : H^1(\sigma(\hat{z}, \cdot)) > \lambda\}$, then

$$\mu_{t_0}(\Theta(\lambda)) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \tag{3.6}$$

If $\Theta = \bigcap_{\lambda > 0} \Theta(\lambda)$, then from (3.6) and basic measure theory we deduce that

$$\mu_{t_0}(\Theta) = 0. \tag{3.7}$$

Also from the maximum principle for p harmonic functions and the definition of Θ we see that

$$\lim_{t \rightarrow 1} \sigma(\hat{z}, t) = \zeta \in \partial\Omega \text{ whenever } \hat{z} \in \partial\Omega(t_0) \setminus \Theta. \tag{3.8}$$

If $\hat{z} \in \partial\Omega(t_0) \setminus \Theta$ we extend $\sigma(\hat{z}, \cdot)$ continuously to [0,1] by defining $\sigma(\hat{z}, 1) = \zeta$, where ζ is as in (3.8).

For later use we also make the following observations. First note from Lemmas 2.1, 2.2, and the maximum principle for p harmonic functions that

$$\hat{h}(\partial\Omega(t), \partial\Omega) \rightarrow 0 \text{ as } t \rightarrow 0, \tag{3.9}$$

where $\hat{h}(E, F)$ denotes the Hausdorff distance between the sets E, F defined by

$$\hat{h}(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

Second, using (3.9), the definitions of μ, μ_t, and once again Lemmas 2.1, 2.2, we note that

$$\mu_t \text{ converges weakly to } \mu \text{ as } t \rightarrow 0. \tag{3.10}$$

Third, if E is a Borel subset of $\partial\Omega(t_0)$ and $T(E, s) = E$ when $s = 1 - t_0$ while otherwise $T(E, s) = \cup\{z : z = \sigma(\hat{z}, 1 - s), \hat{z} \in E\}$, then

$$\mu_{t_0}(E) = \mu_s(T(E, s)) \text{ for } s \in (0, 1]. \tag{3.11}$$

(3.11) follows easily from the divergence theorem and p harmonicity of u when E is a union of open arcs. This special case and well known measure theoretic arguments using the regularity of $\mu_s, s \in (0, 1)$, imply (3.11) for an arbitrary Borel set. The mapping of sets from $\partial\Omega(t_0)$ to $\partial\Omega(s)$ given by (3.11) is also easily seen to be onto.

To return to the proof of Theorem 1(a), we claim that it suffices to show

$$\mu(G) = 0 \text{ where } G = \{z \in \partial\Omega : \lim_{\rho \rightarrow 0} \rho^{-1}\mu(B(z, \rho)) = 0\}. \tag{3.12}$$

Indeed given m a positive integer, let E_m be the subset of $z \in \partial\Omega \setminus G$ for which

$$\limsup_{r \rightarrow 0} \frac{\mu(B(z, r))}{r} > 1/m.$$

Using a well known covering lemma, we can choose a covering $\{B(z_i, 5r_i)\}$ of $E_m \cap B(w, r)$ with $z_i \in E_m$, $r_i \leq \epsilon$, $\{B(y_i, r_i)\}$ pairwise disjoint and

$$\mu(B(y_i, r_i)) > \; r_i/m. \tag{3.13}$$

Thus

$$\sum_i 5r_i < 5m \sum_i \mu(B(y_i, r_i)) \le 5m\mu(\partial\Omega) < \infty \tag{3.14}$$

as we see from (3.13) and Lemma 2.3. Letting $\epsilon \to 0$ and using the definition of H^1 measure we conclude from (3.14) that $H^1(E_m) < \infty$. Hence $\partial\Omega \setminus G$ is σ finite so it suffices to prove (3.12).

The proof of (3.12) is by contradiction. Let $\hat{z}_1 \in \partial\Omega(t_0) \setminus \Theta$. If (3.12) is false then $\mu(G) > 0$ so from regularity of μ, there is a compact set $K \subset G \setminus (\sigma(\hat{z}_1, 1) \cup \Theta)$ with $\mu(K) > 0$. Also K can be chosen so that

$$(+) \qquad \lim_{\rho \to 0} \tfrac{\mu(K \cap B(z,\rho))}{\mu(B(z,\rho))} = 1 \text{ for every } z \in K.$$

$$(++) \qquad \liminf_{\rho \to 0} \tfrac{\mu(K \cap B(z,10\rho))}{\mu(K \cap B(z,\rho))} < 10^{20} \text{ for every } z \in K. \tag{3.15}$$

$$(+++) \quad 0 < \tfrac{\mu(B(z,\rho))}{\rho} \to 0 \text{ uniformly as } \rho \to 0 \text{ for } z \in K.$$

(3.15) (+) follow from differentiation theory while (3.15) (+ + +) is obtained from an Egoroff type argument. For the proof of (3.15)(++), see [LNP11, (2.5)]. In view of (3.15) we can find a $z_0 \in K$ with the following properties: Given $\eta > 0$ small there is an $r_0 = r_0(\eta) > 0$ such that

$$(-) \qquad \tfrac{\mu(B(z_0,10r_0)\setminus K)}{\mu(B(z_0,r_0))} \le \eta^{10}$$

$$(--) \qquad \tfrac{\mu(B(z_0,10r_0))}{\mu(B(z_0,r_0))} \le 10^{20}. \tag{3.16}$$

$$(---) \quad \tfrac{\mu(B(z,\rho))}{\rho} \le \eta^{10} \text{ whenever } z \in K \cap B(z_0, 10r_0) \text{ and } 0 < \rho \le 10^3 r_0.$$

Moreover for given $0 < \epsilon \le 1$ we deduce from (3.15) (+ + +) that there exists $r_0'(\epsilon, \eta) > 0$, with $r_0'(1, \eta) = r_0(\eta)$, $\epsilon \to r_0'(\epsilon, \eta)$, nondecreasing on $(0, 1]$, and

$$\mu(B(z, \rho)) \le \epsilon^{p-1} \eta^{10} \rho \text{ whenever } z \in K \text{ and } 0 < \rho \le 10^3 r_0'(\epsilon, \eta). \tag{3.17}$$

Finally from (3.7), (3.11), and another Egoroff type argument it follows that for fixed ϵ, η, there exists $t_1 = t_1(\epsilon, \eta), 0 < t_1 < t_0$, and a relatively open subset $U = U(t_1)$ with $\Theta \subset U \subset \partial\Omega(t_0)$ and

$$(i) \qquad H^1(\sigma(\hat{z}, [1 - t_1, 1))) \le \eta^{10} r_0'(\epsilon, \eta) \text{ whenever } \hat{z} \in \partial\Omega(t_0) \setminus U(t_1),$$

$$(ii) \qquad \mu_t(T(U(t_1), t)) \le \eta^{10} \mu(B(z_0, r_0(\eta))), t \in (0, 1). \tag{3.18}$$

$$(iii) \qquad \hat{h}(\partial\Omega, \partial\Omega(t_1)) \le \eta^{10} r_0'(\epsilon, \eta).$$

We assume, as we may that for fixed η, $\epsilon \to t_1(\epsilon, \eta)$ is nondecreasing on (0,1]. We now prove a key lemma.

Lemma 3.19 *For fixed ϵ, η, let $U_1(t_1)$ denote the set of all points $\hat{z} \in \partial\Omega(t_0) \setminus U(t_1)$ with $|\nabla u| \ge \epsilon$ at some point in $\sigma(\hat{z}, [1 - t_1, 1))$. There exists η_0, independent of ϵ, such that if*

$0 < \eta \leq \eta_0 \leq 10^{-100}$ *then*

$$\limsup_{t \to 0} \mu_t(T(U_1(t_1(\epsilon, \eta)), t) \cap B(z_0, 5r_0(\eta))) \leq \eta^5 \mu(B(z_0, r_0(\eta))).$$

Proof: Let $\tilde{U}_1 = \tilde{U}_1(\delta)$ be the set of all $\hat{z} \in \partial\Omega(t_0) \setminus U(t_1)$ with the property that $|\nabla u(z)| \geq \epsilon$ for some $z \in \sigma(\hat{z}, [1 - t_1(\epsilon, \eta), 1))$ with $d(z, \partial\Omega) \geq \delta$. Note that $U_1(t_1(\epsilon, \eta)) = \bigcup_{\delta > 0} \tilde{U}_1(\delta)$ so by the usual measure theoretic arguments it suffices to prove Lemma 3.19 for $\tilde{U}_1(\delta)$ and arbitrary small $\delta > 0$. To do this we use a Vitalli type covering argument. Fix $0 < \delta << \eta^{1000} \min(t_1, r_0')$ and let $0 < t_2 \leq \delta$ be so small that

$$\hat{h}(\partial\Omega, \partial\Omega(t_2)) \leq \delta/100. \tag{3.20}$$

Next let

$$J = \{z : |\nabla u(z)| \geq \epsilon, d(z, \partial\Omega) \geq \delta, \text{ and } z \in \bigcup_{\hat{z} \in \tilde{U}_1(\delta)} \sigma(\hat{z}, [1 - t_1, 1)) \cap \bar{B}(z_0, 6r_0(\eta))\} \tag{3.21}$$

and suppose that $J \neq \emptyset$. Let $w_1 \in J$ be a point with maximum distance from $\partial\Omega$. Existence of w_1 follows from a compactness argument. Let $\sigma(\hat{w}_1, \cdot), \hat{w}_1 \in \tilde{U}_1(\delta)$, denote the trajectory through w_1 and let $w \in \partial\Omega$ with $\rho = |w - w_1| = d(w_1, \partial\Omega)$. We note from Lemmas 2.3-2.5 that

$$\epsilon\rho \leq \rho|\nabla u(w_1)| \leq cu(w_1) \leq c[\rho^{p-2}\mu(B(w, 2\rho))]^{1/(p-1)}. \tag{3.22}$$

Solving for $\mu(B(w, 2\rho))$ in (3.22) we conclude that

$$\epsilon^{p-1}\,\rho \leq c'\mu(B(w, 2\rho)) \text{ where } c' = c'(p). \tag{3.23}$$

Let $K_1 = K \cap \bar{B}(z_0, 10r_0(\eta))$. We assert that

$$\rho < \eta^7 d(w, K_1). \tag{3.24}$$

Indeed otherwise let $r = 4\max\{d(w, K_1), \rho\}$ and observe from (3.18) (i) that $\rho < \eta^{10} r_0'(\epsilon, \eta)$ so if (3.24) is false there exists $w' \in K_1$ with $|w - w'| \leq \eta^3 r_0'(\epsilon, \eta)$. Thus by (3.17) we have

$$\mu(B(w, 2\rho)) \leq \mu(B(w', r)) \leq \epsilon^{p-1}\eta^{10} r \leq \epsilon^{p-1}\eta^2\rho.$$

This inequality contradicts (3.23) for η small enough, so (3.24) is true.

We now apply Lemmas 2.7 and 2.8 with $w_1 = z_1$ and $t = t_2$. Let $\gamma, \tau, \beta, \Omega_1$ be as in these lemmas with z_i, replaced by $\tilde{w}_i, 2 \leq i \leq 6$. Let $\hat{\phi}$ be the arc of $\partial\Omega_1 \cap \partial\Omega$ with endpoints $\tilde{w}_5, \tilde{w}_6$. Note that $\tilde{w}_4 \in \hat{\phi}$ and $\beta \cup \hat{\phi}$ is the boundary of a Jordan domain $\Omega_1' \subset \Omega_1$. Let $\hat{t} \in (0, 1)$ be such that $\tau(\hat{t}) \in \Omega_1' \cap \partial B(\tilde{w}_4, \hat{\rho}/2)$ where $\hat{\rho} = \frac{1}{8}\min\{d(\tilde{w}_4, \beta), \rho\}$. Existence of $\hat{t}$ follows from Lemmas 2.7, 2.8 , and the local connectivity of a Jordan domain. Moreover from Lemma 2.8 (α'), Lemma 2.7 (δ), and (3.20) we deduce the existence of $c = c(p)$ with $\rho \leq c\hat{\rho}$ and $c(u(\tau(\hat{t})) - t_2) \geq u(w_1) - t_2$. Let u_1 be the restriction of $u - t_2$ to Ω_1' and define $u_1 \equiv 0$ in $B(\tilde{w}_4, 2\hat{\rho}) \setminus \Omega_1'$. Using the above notes, Lemma 2.7, and Lemma 2.3 for u_1 we find that

$$u(w_1) - t_2 \approx [\hat{\rho}^{p-2}\mu_{t_2}(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho}))]^{1/(p-1)}. \tag{3.25}$$

In (3.25) $\approx$ means the ratio of the two quantities is bounded above and below by constants depending only on p.

For the moment we assume that $\sigma(\hat{w}_1, 1-t_2) \notin \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$. In this case let w_1' be the last point of intersection of τ with $\sigma(\hat{w}_1, \cdot)$. Next let $\tilde{\Omega}_1 = \tilde{\Omega}(w_1) \subset \Omega(t_2)$ denote the Jordan domain with $\tilde{\Omega}(w_1) \cap B(0,2) = \emptyset$, and whose boundary consists of

$$\begin{aligned}
&(a') \quad \text{the arc of } \sigma(\hat{w}_1, \cdot) \text{ with endpoints } \sigma(\hat{w}_1, 1-t_2),\ w_1', \\
&(b') \quad \text{the arc of } \tau \text{ with endpoints } w_1',\ \tilde{w}_4, \\
&(c') \quad \text{the arc of } \partial\Omega(t_2) \text{ with endpoints } \tilde{w}_4,\ \sigma(\hat{w}_1, 1-t_2).
\end{aligned} \tag{3.26}$$

Let $\hat{\tau}$ denote the open arc in (3.26) (b') and let $\hat{\gamma}$ denote the open arc in (3.26) (c'). We assert that

$$\mu_{t_2}(\hat{\gamma}) \leq c\mu_{t_2}(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})). \tag{3.27}$$

Indeed, using p harmonicity of u, the divergence theorem, and the fact that ∇u is tangent to each orthogonal trajectory it follows that

$$\mu_{t_2}(\hat{\gamma}) = \int_{\hat{\gamma}} |\nabla u|^{p-1} dH^1 \leq \int_{\hat{\tau}} |\nabla u|^{p-1} dH^1. \tag{3.28}$$

To estimate the righthand integral let $z \in \hat{\tau}, r' = \frac{1}{2} d(z, \partial\Omega(t_2))$. Using Lemmas 2.2, 2.5, (3.25), the cigar condition on τ, and Harnack's inequality we find that if $\alpha' = \alpha\,(p-1) + 2 - p > 0$ where α is as in Lemma 2.2, then

$$\begin{aligned}
\int_{\hat{\tau} \cap B(z, r')} |\nabla u|^{p-1} dH^1 \quad &\leq c(u(z) - t_2)^{p-1}\,(r')^{2-p} \leq\ c^2 (r')^{2-p} \left(\tfrac{r'}{\hat{\rho}}\right)^{\alpha(p-1)} (u(w_1) - t_2)^{p-1} \\
&\leq c^3 \left(\tfrac{r'}{\hat{\rho}}\right)^{\alpha'} \mu_{t_2}(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})).
\end{aligned} \tag{3.29}$$

To complete the proof of (3.27) we use a well known covering lemma to choose a covering $\{B(z_i, r_i')\}$ of $\hat{\tau}$, where $r_i' = \frac{1}{2} d(z_i, \partial\Omega(t_2))$, in such a way that the balls $\{B(z_i, \frac{1}{10} r_i')\}$ are disjoint. From the cigar condition in Lemma 2.7 we observe for a given nonnegative integer j that there are at most $c = c(p)$ positive integers i with $2^{-j-1}\hat{\rho} \leq r_i' \leq 2^{-j}\hat{\rho}$. Using this fact and (3.28) - (3.29) we get for c large enough that

$$\mu_{t_2}(\hat{\gamma})\ \leq \sum_i \int_{\hat{\tau} \cap B(z_i, r_i')} |\nabla u|^{p-1} dH^1\ \leq\ c(\sum_{j=0}^{\infty} 2^{-j\alpha'})\, \mu_{t_2}(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})). \tag{3.30}$$

Hence (3.27) is true.

We now drop the assumption that $\sigma(\hat{w}_1, 1-t_2) \notin \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$. Let $\hat{\theta} \subset \partial\Omega(t_2)$ be the largest open arc containing $\sigma(\hat{w}_1, 1-t_2)$ with the following property.

$$\text{If } z = \sigma(\hat{z}, 1-t_2) \in \hat{\theta}, \hat{z} \in \partial\Omega(t_0), \text{ then } \sigma(\hat{z}, \cdot) \cap B(w_1, \tfrac{1}{2} d(w_1, \partial\Omega(t_2)) \neq \emptyset. \tag{3.31}$$

From continuity of the mapping $\hat{z} \to \sigma(\hat{z}, 1-t_2)$ we see that $\hat{\theta}$ exists. Also joining the trajectories through the endpoints of $\hat{\theta}$ by an arc of $\partial B(w_1, \frac{1}{2} d(w_1, \partial\Omega(t_2)))$ and arguing as in the proof of (3.27) we deduce that

$$\mu(\hat{\theta}) \leq c\mu(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})). \tag{3.32}$$

Moreover, using the divergence theorem, Lemmas 2.2, 2.5, Harnack's inequality, and Lemma 2.8, as in the proof of (3.27) we deduce that

$$\mu_{t_2}(\hat{\phi}) \leq \int_\beta |\nabla u|^{p-1} dH^1 \ \leq \ c\mu_{t_2}(\partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})). \tag{3.33}$$

We put $I(w_1) = \hat{\phi} \cup \hat{\theta} \cup \hat{\gamma}$ when $\sigma(\hat{w}_1, 1-t_2) \notin \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$ and otherwise set $I(w_1) = \hat{\phi} \cup \hat{\theta}$. Let $\xi(w_1) = \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$. We note that $I(w_1)$ is an open sub arc of $\partial\Omega(t_2)$ with $\sigma(\hat{w}_1, 1-t_2) \in I(w_1)$ and $\xi(w_1) \subset I(w_1)$. Moreover from (3.33), (3.32), (3.27), and (3.24) we have

$$\mu_{t_2}(I(w_1)) \leq c\mu_{t_2}(\xi(w_1)) \ \text{ and } \delta\eta^{-6} \leq d(\bar{B}(\tilde{w}_4, \hat{\rho}), K_1). \tag{3.34}$$

Let J be as in (3.21) and suppose that whenever $w \in J \cap \sigma(\hat{w}, \cdot)$, we have $\sigma(\hat{w}, 1-t_2) \in I(w_1)$. In this case we quit. Otherwise from a compactness argument we deduce the existence of $w_2 \in J$ with maximal distance from $\partial\Omega$ among all such points $w \in J$ with the property that if $w \in \sigma(\hat{w}, \cdot)$, then $\sigma(\hat{w}, 1-t_2)$ is not in $I(w_1)$. In this case we use Lemma 2.7 with z_1, replaced by w_2, to get first $\Omega_1(w_2), \Omega_1'(w_2)$, and then $\hat{\rho}(w_2), \tilde{w}_4(w_2), I(w_2), \xi(w_2)$. Then $w_2 \in I(w_2), \xi(w_2) \subset I(w_2)$, and (3.34) holds with w_1, replaced by w_2. Continuing by induction if $I(w_k), \xi(w_k), 1 \leq k \leq m$, have been constructed we proceed as follows. First, we quit the construction if whenever $w \in J \cap \sigma(\hat{w}, \cdot)$, it is true that $\sigma(\hat{w}, 1-t_2) \in \cup_k I(w_k)$. Otherwise we choose $w_{m+1} \in J$ with maximal distance from $\partial\Omega$ among all such points $w \in J$ with the property that if $w \in \sigma(\hat{w}, \cdot)$, then $\sigma(\hat{w}, 1-t_2)$ is not in $\cup_k I(w_k)$. Proceeding as when $m = 1$ we get $\hat{\rho}_{m+1}, \tilde{w}_4(w_{m+1}), I(w_{m+1}), \xi(w_{m+1})$, with $w_{m+1} \in I(w_{m+1})$ and $\xi(w_{m+1}) \subset I(w_{m+1})$. Also (3.34) holds with w_1 replaced by w_{m+1}. By induction we get $\{w_k\}_1^N$. We claim that $N < \infty$. To prove this claim note by construction of $\hat{\theta}(w_m)$ that $I(w_m)$ contains all points $\sigma(\hat{z}, 1-t_2), \hat{z} \in \partial\Omega(t_0)$, with $\sigma(\hat{z}, \cdot) \cap B(w_m, \frac{1}{2} d(w_m, \partial\Omega)) \neq \emptyset$. Moreover $d(w_k, \partial\Omega) \geq \delta$ and $\{B(w_k, \frac{1}{10} d(w_k, \partial\Omega))\}_1^m$ are pairwise disjoint. Now to get I_{m+1} we essentially removed all trajectories from Ω that intersect $\bigcup_{k=1}^m B(w_k, \frac{1}{2} d(w_k, \partial\Omega))$. Using these facts and a volume type argument we conclude that N is finite.

Next we note that if three open arcs on $\partial\Omega(t_2)$ have a point in common then the union of two of these arcs contains the other arc. Using this fact we choose a subsequence of $I_1, \dots, I_l$ of $\{I(w_m)\}_1^N$ with $\bigcup_{j=1}^N I(w_j) \subset \bigcup_{j=1}^l I_j$ and such that each point of $\bigcup_{j=1}^l I_j$ lies in at most two of $\{I_j\}_1^l$. Given $1 \leq j \leq l$ choose $k = k(j)$ so that $I_j = I(w_k)$. Let $\xi_j = \xi(w_k)$ for $1 \leq j \leq l$. Then from (3.34) we have

$$\mu_{t_2}\left(\bigcup_{j=1}^N I(w_j)\right) \leq \bigcup_{j=1}^l \mu_{t_2}(I_j) \leq c \bigcup_{j=1}^l \mu_{t_2}(\xi_j) \leq 2c\mu_{t_2}\left(\bigcup_{j=1}^l \xi_j\right). \tag{3.35}$$

Let $W(s) = \{z \in \mathbb{C} : d(z, K_1) > s\}$. Now from (3.34) and (3.18), (3.20), (3.21), we see that

$$\bigcup_{j=1}^N \bar{B}(\tilde{w}_4(w_j), \hat{\rho}(w_j)) \subset W(2\delta) \cap B(z_0, 7r_0(\eta)) \text{ for } \eta_0 \text{ small enough.} \tag{3.36}$$

Let $0 \leq g \leq 1$ be a continuous real valued function on $\mathbb{C}$ with $g \equiv 1$ on $W(2\delta) \cap B(z_0, 7r_0(\eta))$ and compact support contained in $W(\delta) \cap B(z_0, 10r_0)$. Then from (3.36) we see that

$$\mu_{t_2}\left(\bigcup_{j=1}^{l} \xi_j\right) \leq \mu_{t_2}(W(2\delta) \cap B(z_0, 7r_0)) \leq \int g d\mu_{t_2}. \tag{3.37}$$

Now if $\sigma(\hat{z}, 1 - t_2) \in T(\tilde{U}(\delta), t_2) \cap B(z_0, 5r_0(\eta))$, then from (3.18), (3.20), (3.21), we see there exists $z \in J \cap \sigma(\hat{z}, \cdot)$. From our construction it follows that $T(\tilde{U}(\delta), t_2) \cap B(z_0, 5r_0(\eta)) \subset \cup_k I(w_k)$. Using this observation and (3.35)-(3.37) we conclude that

$$\mu_{t_2}\left(T(\tilde{U}(\delta), t_2) \cap B(z_0, 5r_0(\eta))\right) \leq \tilde{c} \int g d\mu_{t_2} \tag{3.38}$$

where $\tilde{c}$ depends only on p for η_0 sufficiently small. Letting $t_2 \rightarrow 0$ it follows from (3.38), (3.16) $(-)$ that

$$\limsup_{t_2 \rightarrow 0} \mu_{t_2}\left(T(\tilde{U}(\delta), t_2) \cap B(z_0, 5r_0(\eta))\right) \leq c \int g d\mu \leq c\mu(B(z_0, 10r_0(\eta)) \setminus K) < \eta^5 \mu(B(z_0, r_0(\eta))) \tag{3.39}$$

for η_0 sufficiently small. Letting $\delta \rightarrow 0$ we conclude from (3.39) and our earlier remarks that Lemma 3.19 is true. □

3.1 Final Proof of Theorem 1 (a)

Armed with Lemma 3.19 we are now in a position to prove Theorem 1. For this purpose fix $\epsilon > 0$ small, put $V = U_1(t_1(1, \eta_0)) \cup U(t_1(1, \eta_0))$ and choose a relatively open subset $\hat{\Lambda} \subset \partial\Omega(t_0)$ with $V \subset \hat{\Lambda}$, $\hat{z}_1 \in \hat{\Lambda} \setminus V$, and

$$\mu_{t_0}(\hat{\Lambda} \setminus V) \leq \tfrac{1}{4}\mu(B(z_0, r_0(\eta_0))).$$

This inequality and Lemma 3.l9 imply the existence of $\tilde{t} = \tilde{t}(\epsilon, \eta_0) < t_1(\epsilon, \eta_0)$ such that if $\Lambda = \hat{\Lambda} \cup U(t_1(\epsilon, \eta_0))$, then

$$\mu_t\left[T(\Lambda, t) \cap B(z_0, 5r_0(\eta_0))\right] \leq \tfrac{1}{2}\mu(B(z_0, r_0(\eta_0))) \text{ for } 0 < t \leq \tilde{t}. \tag{3.40}$$

If $\tilde{D} = D \setminus \sigma(\hat{z}_1, [0, 1])$, then $\tilde{D}$ is simply connected so a single valued branch of h in (3.1) can be defined in $\tilde{D}$. Subtracting a constant from h if necessary we see from our earlier discussion that we may assume

$$f = u(z) + ih(z) \text{ maps } \tilde{D} \text{ one to one and onto } S = \{u + ih : 0 < u < 1, 0 < h < \xi\}$$

where ξ is as in (3.3). Given $\hat{z} \in \partial\Omega(t_0) \setminus \Theta$, let $f(\sigma(\hat{z}, 1)) = ih(\sigma(\hat{z}, 1)) = \lim_{t \rightarrow 1} f(\sigma(\hat{z}, t))$. Existence of the limit follows from the fact that h is constant on each trajectory and continuity of u in $\bar{D}$. We observe that $\hat{z} \rightarrow f(\sigma(\hat{z}, 1))$ is continuous on $\partial\Omega(t_0) \setminus \hat{\Lambda}$ so

$$\hat{F} = \{f(\sigma(\hat{z}, 1)), \hat{z} \in \partial\Omega(t_0) \setminus \hat{\Lambda}\} \text{ is a compact subset of } \{is : 0 < s < \xi\}. \tag{3.41}$$

If $w = is, 0 < s < \xi$, let

$$S(w, \tau) = \{u + ih : 0 < u < \tau, |h - s| < u\}$$

and observe from (3.41) that there exists t_1', independent of $\epsilon \in (0, 1]$, with

$$0 < t_1' \leq t_1(1, \eta_0) \text{ and } \bigcup_{w \in \hat{F}} \bar{S}(w, t_1') \subset S. \tag{3.42}$$

We claim there exists a constant $\tilde{c}_0 = \tilde{c}_0(p)$ such that

$$|\nabla u(z)| \leq \tilde{c}_0 \text{ whenever } z \in f^{-1}(S(w, t_1')) \text{ and } w \in \hat{F}. \tag{3.43}$$

Indeed suppose z, w, are as in (3.43) and $|\nabla u(z)| = A$ for some large A. Then from Lemma 2.5 we have

$$u(z) \leq cAd(z, \partial\Omega). \tag{3.44}$$

Let $u_0 = u(z), h_0 = h(z)$, and let α be a parametrization of the largest arc contained in $\bar{B}(z, \frac{1}{2}d(z, \partial\Omega)) \cap \{u = u_0\}$ with parameter interval [0,1] and $h(\alpha(0)) < h_0 < h(\alpha(1))$. Using Lemma 2.5, Harnack's inequality, (3.44), and basic geometry we get for some $c \geq 1$,

$$\begin{aligned} c \min\{h(\alpha(1)) - h_0, h_0 - h(\alpha(0))\} &\geq \int_\alpha |\nabla h| dH^1 = \int_\alpha |\nabla u|^{p-1} dH^1 \\ &\geq c^{-1} A^{p-1} d(z, \partial\Omega) \geq c^{-2} A^{p-2} u_0. \end{aligned} \tag{3.45}$$

Since $u_0 + ih_0 \in S(w, t_1')$ there exists $\hat{z} \in \partial\Omega(t_0) \setminus \Lambda$ with $w = \sigma(\hat{z}, 1)$ and $\tilde{z} \in \sigma(\hat{z}, [1 - t_1', 1))$ with $u(\tilde{z}) = u_0$ and $|h(\tilde{z}) - h_0| \leq u_0$. Then for A large enough it follows from (3.45) that $\tilde{z} \in \alpha$. Since $|\nabla u(\tilde{z})| \leq 1$ we deduce from $\tilde{z} \in \alpha$ and once again Lemma 2.5, Harnack's inequality, that claim (3.43) is true.

Next fix $t, 0 < t < \min(t_1', \tilde{t})$ and let

$$O = f^{-1}\left(\bigcup_{w \in \hat{F}} S(w, t_1') \cap \{u + ih : u > t\} \right).$$

We note that if $a = \min\{s : is \in \hat{F},\ b = \max\{s : is \in \hat{F}\}$, then

$$f(O) = \{u + ih : a - t_1' < h < b + t_1', \max(t, d(ih, \hat{F})) < u < \min(t_1', d(ih, \hat{F}))\}. \tag{3.46}$$

From (3.46) and smoothness of f^{-1} we deduce that the divergence theorem can be used in O. Define L relative to u as in (1.8)-(1.10) and let $v = \log |\nabla u|$. Then from the divergence theorem, p harmonicity of u, and (1.11), (1.12), we find that

$$0 \leq \int_O (uLv - vLu)\, dA = \int_{\partial O} \sum_{i,j=1,2} b_{ij}\, \nu_i\, u v_{x_j} dH^1 - \int_{\partial O} \sum_{i,j=1,2} b_{ij}\, \nu_i v u_{x_j}\, dH^1 = J_1 + J_2, \tag{3.47}$$

where the outer unit normal to ∂O, $\nu(z) = (\nu_1(z), \nu_2(z))$, is defined H^1 almost everywhere on ∂O. To estimate the boundary integrals in (3.47) we write $\partial O = Q_1 + Q_2 + Q_3$ with

$$Q_1 = \partial O \cap \{u = t_1'\},\ Q_2 = \partial O \cap \{t < u < t_1'\},\ Q_3 = \partial O \cap \{u = t\}.$$

Then for $k = 1, 2$,

$$J_k = \int_{Q_1} \cdots + \int_{Q_2} \cdots + \int_{Q_3} \cdots = J_{k1} + J_{k2} + J_{k3}. \tag{3.48}$$

On Q_3 we have $\nu|\nabla u| = -\nabla u$. Using this fact, (1.9), (1.10), and Lemmas 2.5, 2.6 we obtain

$$J_{13} \leq c \int_{Q_3} u|\nabla u|^{p-3}\left(\sum_{i,j=1}^{2} |u_{x_i x_j}|\right) dH^1 \leq c^2 \int_{Q_3} |\nabla u|^{p-1} dH^1 = -\, c^2 \int_{Q_3} |\nabla u|^{p-2}\, u_\nu \, dH^1. \tag{3.49}$$

Also, from (1.9) and the above fact we deduce first that

$$-\sum_{i,j=1}^{2} b_{ij}\, \nu_i v u_{x_j} \; = (p-1)|\nabla u|^{p-1} v \text{ on } Q_3. \tag{3.50}$$

and thereupon from the definition of Λ, (3.50), (3.43), and (3.40) that

$$\begin{aligned} J_{23} \; &\leq c \int_{Q_3} |\nabla u|^{p-1}\, dH^1 + (p-1) \int_{T(\partial\Omega(t_0)\setminus\Lambda,\, t) \cap B(z_0, 5r_0(\eta_0))} |\nabla u|^{p-1}\, v\, dH^1 \\ &\leq -c^2 \int_{Q_3} |\nabla u|^{p-2} u_\nu \, dH^1 + \tfrac{1}{2}(p-1)\, \log(\epsilon)\, \mu(B(z_0, r_0(\eta_0))) \end{aligned} \tag{3.51}$$

provided $c = c(p) \geq 1$, is suitably large. To handle the boundary integrals over Q_2 we note that f is an open sense preserving mapping but angles can be badly distorted. Also from (3.46) we see that H^1 almost every point in $f(Q_2)$ lies on exacly one open line segment l with slope ± 1. If ∂O is oriented counterclockwise, and l has slope 1, then the tangent vector at a point on l is given by $\frac{1}{\sqrt{2}}(-1 - i)$ while the tangent vector is $\frac{1}{\sqrt{2}}(1 - i)$ when l has slope - 1. Let $\tau = f^{-1}(l)$. Then Q_2 inherits a counterclockwise orientation from $f(Q_2)$ as seen from points in components of O and on τ, u, h, are both decreasing when l has slope 1 while u is increasing, h decreasing on l when l has slope 1. Let $z \in \tau$ and let λ be the unit tangent vector to τ at z. Then $\nabla u \cdot \lambda = \pm \nabla h \cdot \lambda$, where $\cdot$ denotes the dot product, so if $\nabla u \cdot \lambda = \cos\phi\, |\nabla u|, 0 < \phi < \pi$, then from the definition of h we have $\nabla h \cdot \lambda = -|\nabla u|^{p-1} \sin\phi$. Also if l has slope 1, then $\pi/2 < \phi < \pi$ while if l has slope - 1, then $0 < \phi < \pi/2$. Since $|\nabla u| \leq c_0$ in O it follows that $|\tan\phi| = |\nabla u|^{2-p} \geq c^{-1}$ for some $c = c(p)$. Intuitively, if $|\nabla u|$ is small, then λ is nearly parallel to $-\nabla h$. From this analysis we see that $\nu = -i\lambda$ is the outer unit normal to Q_2 and for some $c = c(p) \geq 1$ that

$$-\sum_{i,j=1}^{2} b_{ij} \nu_i u_{x_j} = -(p-1)|\nabla u|^{p-2}\, \nabla u \cdot \nu \; \geq \; c^{-1} |\nabla u|^{p-1} \tag{3.52}$$

on Q_2. Using (3.52), (3.43) and arguing as in the proof of (3.49), (3.51) we get

$$J_{12} + J_{22} \leq c \int_{Q_2} |\nabla u|^{p-1} dH^1 \leq -c^2 \int_{Q_2} |\nabla u|^{p-2} u_\nu dH^1. \tag{3.53}$$

Next from Lemmas 2.3-2.6 we see that

$$J_{11} + J_{21} \leq A' \tag{3.54}$$

where A' depends on numerous quantities, eg., $u, \Omega, \mu, t_1', \eta_0$, but is independent of ϵ. Combining (3.47), (3.49), (3.51), (3.53), (3.54), we have for some $c = c(p) \geq 1$,

$$0 \leq A' - c \int_{Q_2+Q_3} |\nabla u|^{p-2} u_\nu \, dH^1 + \tfrac{1}{2}(p-1) \log \epsilon \, \mu(B(z_0, r_0)). \tag{3.55}$$

Moreover using p harmonicity of u and the divergence theorem we see that $\int_{\partial O} |\nabla u|^{p-2} u_\nu dH^1 = 0$ so

$$- \int_{Q_2+Q_3} |\nabla u|^{p-2} u_\nu \, dH^1 = \int_{Q_1} |\nabla u|^{p-2} u_\nu \, dH^1 \leq A'' \tag{3.56}$$

where A'' has the same dependence as A'. In view of (3.56) we can further simplify (3.55) to

$$0 \leq \tilde{A} + \tfrac{1}{2}(p-1) \log \epsilon \, \mu(B(z_0, r_0)) \tag{3.57}$$

where $\tilde{A}$ is independent of ϵ. Letting $\epsilon \to 0$ in (3.57) we get a contradiction to our assumption that (3.3) is false. From this contradiction and our earlier remarks we now obtain Theorem $1(a)$. $\square$

4 Proof of Theorem $1(b)$.

We continue with the same notation as in section 3 unless otherwise stated. In this section we state Proposition 4.15. This proposition is the cornerstone in our proof of Theorem $1(b)$. We then indicate how Theorem $1(b)$ follows from Proposition 4.15.

Put $\tilde{v}(z) = \max(\log |\nabla u| - c, 0), z \in D$, where c is chosen so large that $\tilde{v} \equiv 0$ on $\bar{B}(0,2) \setminus B(0,1)$. Existence of c follows from Lemma 2.6. Extend $\tilde{v}$ to Ω by defining $\tilde{v} \equiv 0$ on $\bar{B}(0,1)$. Let $f = \tilde{v}^{2m}$ where m is a positive integer and note that f has Lipschitz continuous partial derivatives. Define L relative to u as in (1.8) - (1.10) and let $\tilde{u} = u - t$ in $\Omega(t)$. From our note we find for H^2 almost every $z \in \Omega(t)$ that

$$Lf = (2m)(2m-1)\tilde{v}^{2m-2} \sum_{i,j=1}^{2} b_{ij} \tilde{v}_{x_i} \tilde{v}_{x_j} + (2m)(\tilde{v})^{2m-1} L\tilde{v} = P_1 - P_2. \tag{4.1}$$

Here we adopt the convention that $\tilde{v}^0 = 1$ even at points where $\tilde{v} = 0$. Observe that if $m = 1$ in (4.1) and $\tilde{v}(z) = 0$ we still have $Lf(z) = 0$, for H^2 almost every $z \in \Omega(t)$. If $f(z) \neq 0$, it follows from (1.10) and Lemma 2.6 that at z

$$\begin{aligned} 0 \leq P_1 = (2m)(2m-1)\tilde{v}^{2m-2} \sum_{i,j=1}^{2} b_{i,j} \tilde{v}_{x_i} \tilde{v}_{x_2} &\approx (2m)(2m-1)\tilde{v}^{2m-2} |\nabla u|^{p-4} \sum_{i,j=1}^{2} (u_{x_i x_j})^2 \\ &\leq c\,(2m)(2m-1)\tilde{v}^{2m-2} u^2 \, d(z, \partial\Omega)^{-4} \, |\nabla u|^{p-4} \end{aligned} \tag{4.2}$$

where c depends only on p. Also from (1.11), (1.12), and Lemma 2.6 we deduce for $1 < p < 2$ that

$$\begin{aligned} 0 \leq P_2 = -2m\tilde{v}^{2m-1}L\tilde{v} \ &\approx 2m(2-p)\,\tilde{v}^{2m-1}\,|\nabla u|^{p-4}\sum_{i,j=1}^{2}(u_{x_ix_j})^2 \\ &\leq\ 2m\,c\,(2-p)\tilde{v}^{2m-1}\,u^2\,d(z,\partial\Omega)^{-4}\ |\nabla u|^{p-4} \end{aligned} \tag{4.3}$$

where $c = c(p) \geq 1$ can be chosen independent of p when $p \in [3/2, 2]$. Then from the divergence theorem and p harmonicity of $\tilde{u}$ we have

$$S = \int_{\Omega(t)} [\tilde{u}\,Lf - f\,L\tilde{u}]dH^2 \ = \ -\int_{\partial\Omega(t)} f\ \left(\sum_{i,j=1}^{2} b_{ij}\,\nu_i\,\tilde{u}_{x_j}\right)\ dH^1 = K_1, \tag{4.4}$$

where ν is the outer unit normal to $\partial\Omega(t)$. Using the notation in (4.1),(4.4), we have

$$S = \int_{\Omega(t)} \tilde{u}P_1 dA - \int_{\Omega(t)} \tilde{u}P_2 dA \ = \ K_1 \ = \ (p-1)\int_{\partial\Omega(t)} f|\nabla u|^{p-1}dH^1, \tag{4.5}$$

where $P_1, P_2 \geq 0$.

Using (4.5) we show for $0 < t < 1$ that

$$K_1 = K_1(m,t) = (p-1)\int_{\partial\Omega(t)} \tilde{v}^{2m}|\nabla u|^{p-1}dH^1 \ \leq\ \tilde{c}^m m!\log^m(4/t), \tag{4.6}$$

where $\tilde{c}$ only depends on p. From (4.5), (4.2), we see that

$$K_1(m,t) \leq \int_{\Omega(t)} \tilde{u}P_1\,dA \ \leq\ c(2m)(2m-1)\int_{\Omega(t)} \tilde{v}^{2m-2}u^3\,d(z,\partial\Omega)^{-4}|\nabla u|^{p-4} = I_m(t). \tag{4.7}$$

Moreover from Lemma 2.5 and the co-area theorem we have

$$\begin{aligned} I_m(t) \ &\leq c(2m)(2m-1)\int_{\Omega(t)} u^{-1}\,|\nabla u|^p\,\tilde{v}^{2m-2}\,dA \\ &= c(2m)(2m-1)\int_t^1 \left(\int_{\partial\Omega(s)} |\nabla u|^{p-1}\tilde{v}^{2m-2}dH^1\right)\,s^{-1}ds \\ &= c(2m)(2m-1)(p-1)^{-1}\int_t^1 K_1(m-1,s)\ s^{-1}ds. \end{aligned} \tag{4.8}$$

(4.6) follows easily from (4.7), (4.8) and induction. Indeed, from (3.3) and (4.7), (4.8) we see that (4.6) is valid when $m = 1$. By induction suppose (4.6) holds for $m-1 (m \geq 2)$. Then using (4.7), (4.8) once again and the induction hypothesis we get

$$K_1(m,t) \leq c'2m(2m-1)\tilde{c}^{m-1}(m-1)!\int_t^1 \log^{m-1}(4/s)s^{-1}ds \ \leq\ \tilde{c}^m m!\,\log^m(4/t) \tag{4.9}$$

provided $\tilde{c} = \tilde{c}(p)$ is large enough. From (4.9) and induction, we obtain (4.6).

From (4.7), (4.8) we also obtain

$$\int_{\Omega(t)} \tilde{u} P_1 \, dA \,\le\, \tilde{c}^m \, m! \, \log^m(4/t) \tag{4.10}$$

and thereupon from (4.5) that

$$\int_{\Omega(t)} \tilde{u} P_2 \, dA \,\le\, \tilde{c}^m \, m! \, \log^m(4/t) \tag{4.11}$$

provided $\tilde{c}$ is chosen large enough. Next we introduce some more notation. If $z \in \partial\Omega(t)$ and $q : \partial\Omega(t) \to \mathbb{R}$ is integrable with respect to μ_t, let

$$M_t q(z) = \sup_{\zeta} \left(\frac{1}{\mu_t(\zeta)} \int_{\zeta} q \;\, d\mu_t \right) \tag{4.12}$$

where the supremum is taken over open arcs, ζ, with $z \in \zeta \subset \partial\Omega(t)$. $M_t q$ is called the Hardy - Littlewood maximal function of q on $\partial\Omega(t)$ with respect to μ_t. Next if $T(\alpha, t) = \zeta$ for some $\alpha \subset \partial\Omega_{t_0}$, set

$$G(\zeta) = \textstyle\bigcup_{t<\tau<1} T(\alpha, \tau)$$

$$g(z) = \sup_{\zeta} \left(\frac{1}{\mu_t(\zeta)} \int_{G(\zeta)} \tilde{u}\,(P_1 + P_2)\, dA \right) \tag{4.13}$$

where once again the supremum is taken over open arcs, ζ, with $z \in \zeta \subset \partial\Omega(t)$. Finally if $z \in \partial\Omega(t)$ and $z = \sigma(\hat{z}, 1-t)$ for some $\hat{z} \in \partial\Omega(t_0)$ let

$$N_t \tilde{\theta}(z) = \sup_{t \le \tau < 1} \, \tilde{\theta}(\sigma(\hat{z}, 1-\tau)) \tag{4.14}$$

whenever $\tilde{\theta}$ is a continuous function defined on $\bar{\Omega}(t)$. We note that $N_t \tilde{v}^l = (N_t \tilde{v})^l$ whenever l is a positive number. Next we state a key proposition for our proof of Theorem 1(b).

Proposition 4.15. *Given m a positive integer, let $f = \tilde{v}^{2m}$, and let P_1, P_2 be as in (4.1)-(4.3). If $z \in \partial\Omega(t)$, then there exists $c_* = c_*(p) \ge 1$ such that*

$$N_t f(z) \,\le\, c_* \,\left[M_t f(z) \,+\, g(z) \,+\, m\, M_t(N_t \tilde{v}^{(2m-1)}) \right](z).$$

The proof of Proposition 4.15 is rather involved. Thus we reserve the proof of this proposition until section 5 and indicate in this section how Proposition 4.15 implies Theorem 1(b).

4.1 Proof of Theorem 1(b) assuming Proposition 4.15

Let $\hat{z} \in \partial\Omega(t_0)$, with $\sigma(\hat{z}, 1-t) = z$. If $t < t' < 1$, we note that $\nabla u(\hat{z}, 1-t')$ is tangent to $\sigma(\hat{z}, \cdot)$ at $\sigma(\hat{z}, 1-t')$. Using this note and the chain rule we get

$$\left| \frac{d\sigma(\hat{z}, 1-t')}{dt'} \right| = \frac{1}{|\nabla u|(\sigma(\hat{z}, 1-t'))} \;. \tag{4.16}$$

From (4.16) and Lemmas 2.5, 2.6 we obtain for $t < s < 1$

$$\tilde{v}(\sigma(\hat{z}, 1-s)) \leq c \int_t^1 |\nabla \tilde{v}| |\tfrac{d\sigma(\hat{z},1-t')}{dt'}| dt' \leq c \int_t^1 dt'/t' \leq c \ln(4/t). \tag{4.17}$$

In the application of Proposition 4.15 we shall also need the inequality,

$$\int_{\partial\Omega(t)} g^{1/2} d\mu_t \leq c \left(\int_{\partial\Omega(t)} \tilde{u}(P_1 + P_2) d\mu_t \right)^{1/2} \tag{4.18}$$

where $c = c(p)$. To prove (4.18) observe for given $\eta > 0$ that if $z \in \{w \in \partial\Omega(t) : g(w) > \eta\}$, then from the definition of g, there is an arc $\zeta \subset \partial\Omega(t)$ with $z \in \zeta$ and

$$\int_{G(\zeta)} \tilde{u}\,(P_1 + P_2) dA \; > \; \eta\, \mu_t(\zeta). \tag{4.19}$$

Using the fact that if three arcs on $\partial\Omega(t)$ have a point in common, then two of the arcs contain the other arc, we see there exists a collection of open Jordan arcs $\{\zeta_j\}$ for which (4.19) holds with ζ replaced by ζ_j. Also

$$\{w : g(w) > \eta\} \subset \bigcup \zeta_j \tag{4.20}$$

and each point of the union lies in at most two of the arcs $\{\zeta_j\}$. From (4.19), (4.20), we get

$$\begin{aligned} \eta\mu_t(\{w \in \partial\Omega(t) : g(w) > \eta\}) \; &\leq \eta \sum_j \mu_t(\zeta_j) < \sum_j \int_{G(\zeta_j)} \tilde{u}\,(P_1 + P_2)\, dA \\ &\leq 2 \int_{\Omega(t)} \tilde{u}\,(P_1 + P_2)\, dA = \delta. \end{aligned} \tag{4.21}$$

Here we have used the fact that each point of $\bigcup_j G(\zeta_j)$ lies in at most two of $\{G(\zeta_j)\}$. (4.18) follows from (4.21) in a standard way. Indeed, writing the integral as a Riemann - Stieltjes integral and then integrating by parts we get

$$\int_{\partial\Omega(t)} g^{1/2} d\mu_t \leq c \int_0^\infty \eta^{-1/2} \mu_t(\{w \in \partial\Omega(t) : g(w) > \eta\}) d\eta = c \int_0^\delta ... + c \int_\delta^\infty ... \tag{4.22}$$

where δ is as in (4.21). Using (3.3) in $\int_0^\delta$ and (4.21) in $\int_\delta^\infty ...$ we obtain (4.18). Next if $q : \partial\Omega(t) \to \mathbb{R}$ is integrable with respect to μ_t, then

$$\int_{\partial\Omega(t)} (M_t q)^{1/2} d\mu_t \leq c \left(\int_{\partial\Omega(t)} |q|\, d\mu_t \right)^{1/2} \tag{4.23}$$

which follows easily from weak type estimates for $M_t q$ similar to (4.21), as in (4.22). From (4.17) and Proposition 4.15 we see that

$$N_t f(z) \; \leq \; c \left[M_t f \; + \; g \; + \; m \, \log(4/t) M_t(N_t \tilde{v}^{(2m-2)}) \right](z). \tag{4.24}$$

Integrating and using (4.18), (4.23) we find that

$$\int_{\partial\Omega(t)} (N_t f)^{1/2} d\mu_t \;\leq\; \hat{c} \int_{\partial\Omega(t)} [(M_t f)^{1/2} \;+\; g^{1/2} \;+\; (m\, \log(4/t) M_t (N_t \tilde{v}^{(2m-2)}))^{1/2}] d\mu_t$$

$$\leq \hat{c}^2 \left(\int_{\partial\Omega(t)} f d\mu_t \right)^{1/2} + \hat{c}^2 \left(\int_{\Omega(t)} (P_1 + P_2) dA \right)^{1/2} + \hat{c}^2 [m \log(4/t)]^{1/2} \left(\int_{\partial\Omega(t)} N_t \tilde{v}^{2m-2} d\mu_t \right)^{1/2}. \tag{4.25}$$

Using (4.25) we show there exists $c = c(p) \geq 1$ such that

$$\left(\int_{\partial\Omega(t)} (N_t f)^{1/2} d\mu_t \right)^2 \leq c^m \, m! \log^m(4/t). \tag{4.26}$$

Indeed for $m = 1$ this inequality follows from (4.6), (4.10), (4.11) and the fact that in this case $N_t \tilde{v}^{2m-2} \equiv 1$. By induction if (4.26) holds for $m - 1$ $(m \geq 2)$ then using (4.6), (4.10), (4.11), for m and the induction hypotheses we get (4.26) provided c is large enough. Thus (4.26) is true for $m = 1, 2, ...$

Let $\theta(t) = \log(4/t), 0 < t \leq t_0$, and put

$$\psi(z) = (u(z))^{-1} \, [\theta \circ u(z)]^{-1} \, [1 - \; \theta \circ \theta \circ u(z)]^{-(1+2/m)}, \; z \in D.$$

Note from (3.4) for $\hat{z} \in \partial\Omega(t_0)$, $0 < s < 1$, that

$$m \, \tfrac{d}{dt} \, [1 - (\theta \circ \theta)(t)]^{-2/m} \;=\; -\, 2 \, \psi(\sigma(\hat{z}, 1 - t)). \tag{4.27}$$

Thus if $\hat{z} \in \partial\Omega(t_0)$, then

$$\int_0^1 \; \psi(\sigma(\hat{z}, 1 - t)) \, dt \;\leq\; c\, m. \tag{4.28}$$

Using (4.28), (4.26), and the change of variables or co-area formulas as in section 3, we find that

$$\begin{aligned} I \;&=\; \int_\Omega \; |\nabla u|^p(z) \, (N_t f)^{1/2}(z) \, \theta(u(z))^{-m/2} \, \psi(z) dA \\ &\leq\; c^{m/2} \, (m!)^{1/2} \int_0^1 \; \psi[\sigma(\hat{z}, 1 - t)] \, dt \;\leq\; c^m \, (m!)^{1/2}. \end{aligned} \tag{4.29}$$

On the other hand using the change of variables formula, and the Tonelli theorem to interchange the order of integration, we see that

$$I \;=\; \int_{\partial\Omega(t_0)} \; |\nabla u|^{p-1}(\hat{z}) \; \left(\int_0^1 \; (N_t f)^{1/2}(\sigma(\hat{z}, 1 - t)) \, \theta(t)^{-m/2} \, \psi(\sigma(\hat{z}, 1 - t)) \, dt \right) \; dH^1 \hat{z} \tag{4.30}$$

Let

$$q(\hat{z}) = \int_0^1 \; (N_t f)^{1/2} [\sigma(\hat{z}, 1 - t)] \, \theta(t)^{-m/2} \, \psi[\sigma(\hat{z}, 1 - t)] \, dt, \;\; \hat{z} \in \partial\Omega(t_0) \tag{4.31}$$

and let E_m be the set of $\hat{z} \in \partial\Omega(t_0)$ with

$$q(\hat{z}) \;\geq\; c^{2m} \, (m!)^{1/2}$$

where c as in (4.29). From (4.30), (4.31) and weak type estimates we get

$$\mu_{t_0}(E_m) \leq c^{-m} \text{ for } m \geq 10. \tag{4.32}$$

From (4.31) and the fact that

$$-\tfrac{d}{dt}\left([\theta(t)]^{-m/2} \cdot [1-(\theta\circ\theta)(t)]^{-(1+2/m)}\right) \leq cm\,[\theta(t)]^{-m/2}\,\psi[\sigma(\hat{z},1-t)]$$

for $\tau \in (0,1], \hat{z} \in \partial\Omega(t_0)$, we find

$$N_\tau f(\sigma(\hat{z},1-\tau))^{1/2}\left([\theta(\tau)]^{-m/2} \cdot [1-(\theta\circ\theta)(\tau)]^{-(1+2/m)}\right) \leq c\,m\,q(z). \tag{4.33}$$

Here we have also used the fact that $t \rightarrow N_t f(\sigma(\hat{z},1-t))$ is nonincreasing as a function of t. If $\hat{z} \notin E_m$, it follows from (4.33) that

$$N_\tau \tilde{v}^m[\sigma(\hat{z},1-\tau)]\left([\theta(\tau)]^{-m/2} \cdot [1-(\theta\circ\theta)(\tau)]^{-(1+2/m)}\right) \leq c^{3m}\,(m!)^{1/2} \tag{4.34}$$

for $0 < \tau < 1$.

Now suppose that

$$m \geq \log\log[-\log(\tau)] = \kappa(\tau) > m-1$$

for τ near 1 so $m \geq l$. Taking $1/m$ roots of the above inequality, using Stirling's formula, and doing some arithmetic, it follows that

$$\frac{\tilde{v}[\sigma(\hat{z},1-\tau)]}{\sqrt{\theta(\tau)\,\kappa(\tau)}} \leq \hat{c}, \tag{4.35}$$

where $\hat{c}$ depends only on p and is bounded on (3/2, 2). We conclude from (4.35) for $\hat{z} \notin \bigcup_{m=l}^{\infty} E_m$ that

$$\limsup_{\tau\rightarrow 0}\left(\frac{\tilde{v}[\sigma(\hat{z},1-\tau)]}{\sqrt{-\log(\tau)\cdot\log\log[-\log\tau]}}\right) \leq \hat{c}. \tag{4.36}$$

Since l is arbitrary we see from (4.32) that (4.36) holds μ_{t_0} almost everywhere.

Theorems 1(b) follows from (4.36). Indeed let

$$\lambda(r) = r\,\exp\left[A\sqrt{\log 1/r\ \log\log\log 1/r}\right],\ 0 < r < 10^{-6}, \tag{4.37}$$

and suppose $F \subset \partial\Omega$ is a Borel set with $H^\lambda(F) = 0$. For the moment we allow $A \geq 2$ to vary but shall later fix it to be a large constant depending only on p which is bounded in (3/2, 2). Fix $p, 1 < p < 2$, and let $K \subset \partial\Omega$ be a Borel set with $H^\lambda(K) = 0$. Let K_1 be the subset of all $z \in K$ with

$$\limsup_{r\rightarrow 0}\frac{\mu(B(z,r))}{\lambda(r)} < \infty.$$

Then from the definition of λ and a covering argument (see [Mat95, sec 6.9]), it is easily shown that $\mu(K_1) = 0$. Thus to prove $\mu(K) = 0$, it suffices to show

$$\mu(U) = 0 \text{ where } U = \left\{z \in \partial\Omega : \limsup_{r\rightarrow 0}\frac{\mu(B(z,r))}{\lambda(r)} = \infty\right\}. \tag{4.38}$$

Given $0 < \hat{r}_0 < 10^{-100}$ and a positive constant $b \geq 10^8$ we first show for each $z \in U$ that there exists $s = s(z), 0 < bs < \hat{r}_0$, such that

$$t^{-1}\,\mu(B(z,t)) \leq s^{-1}\mu(B(z,s)) \text{ and } \lambda(t) \leq (t/s)\mu(B(z,s)) \text{ for } s \leq t \leq bs. \tag{4.39}$$

In fact let s be the first point starting from $\hat{r}_0/b$ where

$$\frac{\mu(B(z,s))}{\lambda(s)} \geq 2 \max_{r \in [\hat{r}_0/b, \hat{r}_0]} \left\{ \frac{\mu(B(z,r))}{\lambda(r)}, 1 \right\}.$$

From (4.38) we see that s exists. Using $\lambda(r') \leq (r'/r)\lambda(r), r < r'$, and our choice of s we get

$$\mu(B(z,t)) \leq \frac{\lambda(t)\mu(B(z,s))}{\lambda(s)} \leq (t/s)\mu(B(z,s)) \text{ for } s \leq t \leq bs.$$

The last inequality in (4.39) follows in a similar manner. From (4.39) and a covering argument, we get $\{B(z_i, r_i)\}$ with $z_i \in U, 0 < br_i < \hat{r}_0$, and the property that

$$\begin{aligned} &(a) \qquad (4.39) \text{ holds with } z = z_i, s = r_i, \text{ for each } i, \\ &(b) \qquad U \subset \bigcup_i B(z_i, br_i), \\ &(c) \qquad B(z_i, br_i/10) \cap B(z_j, br_j/10) = \emptyset \text{ when } i \neq j. \end{aligned} \tag{4.40}$$

We claim there exists $w_i \in \Omega \cap B(z_i, 5r_i)$ with

$$\begin{aligned} &(\alpha) \qquad u(w_i) = t_i \text{ and } d(w_i, \partial\Omega) \approx r_i \\ &(\beta) \qquad \mu[B(z_i, r_i)]/r_i \approx [u(w_i)/d(w_i, \partial\Omega)]^{p-1} \approx |\nabla u(w)|^{p-1} \\ &\qquad\quad \text{whenever } w \in B(w_i, d(w_i, \partial\Omega)/2). \end{aligned} \tag{4.41}$$

To prove (4.41) choose $w_i \in \partial B(z_i, 2r_i)$ with $u(w_i) = \max_{\bar{B}(z_i, 2r_i)} u$. Then $d(w_i, \partial\Omega) \approx r_i$, since otherwise, it would follow from Lemma 2.2 that $u(w_i)$ is small in comparison to $\max_{\bar{B}(z_i, 5r_i)} u$. However from (4.40) (a) and Lemma 2.3, these two maximums are proportional with constants depending only on p. Thus $d(w_i, \partial\Omega) \approx r_i$ where all proportionality constants depend only on p and can be chosen independent of $p \in [3/2, 2]$. Using this fact, Lemma 2.5, (4.40) (a), and Lemma 2.3, we get (4.41). Also we have

$$H^1[B(w_i, d(w_i, \partial\Omega)/2) \cap \{z : u(z) = t_i\}] \geq d(w_i, \partial\Omega)/2 \tag{4.42}$$

as we see from the maximum principle for p harmonic functions, a connectivity argument and basic geometry. Using (4.40)-(4.42) we find that

$$\mu[B(z_i, 10r_i)] \,\leq\, c \int_{\partial\Omega(t_i) \cap B(w_i, d(w_i, \partial\Omega)/2)} |\nabla u|^{p-1}\, dH^1. \tag{4.43}$$

Choose N a positive integer so large that

$$\mu\left(\bigcup_i B(z_i, br_i)\right) \,\leq\, 2\mu\left(\bigcup_{i=1}^N B(z_i, br_i)\right). \tag{4.44}$$

Existence of N is a consequence of (4.39), (4.40) (c), and finiteness of μ. Summing (4.43) and using (4.44), (4.40) (b), it follows that

$$\begin{aligned}\mu(U) &\leq 2\mu\left(\bigcup_{i=1}^{N} B(z_i, br_i)\right) \leq 100b \sum_{i=1}^{N} \mu[B(z_i, 10r_i)] \\ &\leq c\, b\sum_{i=1}^{N} \int_{\partial\Omega(t_i)\cap B(w_i, d(w_i,\partial\Omega)/2)} |\nabla u|^{p-1} dH^1. \end{aligned} \tag{4.45}$$

To estimate the last integral in (4.45) we note from (4.40), (4.41), that for some $c = c(p) \geq 2$,

$$v(z) = \log|\nabla u(z)| \geq (A/c)\sqrt{-\log r_i \cdot \log\log[-\log r_i]} \text{ on } B(w_i, d(w_i, \partial\Omega)/2)\,. \tag{4.46}$$

Also, we can use (4.40) (a) and (4.41) (β) to estimate t_i below in terms of r_i and Lemma 2.2 to estimate t_i above in terms of r_i. Doing this we find for some $a = a(p), 0 < a < 1, \bar{c} = \bar{c}(p)$, that

$$r_i \leq \bar{c}\, t_i^a \leq \bar{c}^2\, r_i^{a^2}\,. \tag{4.47}$$

From (4.46), (4.47) we conclude, for some $\tilde{c} = \tilde{c}(p) \geq 2$ bounded for $p \in [3/2, 2]$ that

$$v(z) \geq (A/\tilde{c})\sqrt{-\log t_i \cdot \log\log[-\log t_i]} \text{ on } B(w_i, d(w_i, \partial\Omega)/2) \tag{4.48}$$

provided $\hat{r}_0$ is small enough.

Next choose $0 < t << \min\{t_i : 1 \leq i \leq N\}$ so that

$$d(\partial\Omega(t), \partial\Omega) \leq 10^{-8}\ \hat{h}\left(\partial\Omega, \bigcup_{i=1}^{N} \partial\Omega(t_i)\right) \tag{4.49}$$

where $\hat{h}$ denotes Hausdorff distance. We show there exist Borel sets $\alpha_i \subset \partial\Omega(t_0), 1 \leq i \leq N$, $b_0 = b_0(p), c' = c'(p)$ such that if $b \geq b_0$, then

$(\breve{a})$ $T(\alpha_i, t_i) \subset B(z_i, b/10)$

$(\breve{b})$ $\mu_{t_i}[\partial\Omega(t_i) \cap B(w_i, d(w_i, \partial\Omega)/2)] \leq c'\mu_t(T(\alpha_i, t))$

$(\breve{c})$ If $\hat{z} \in \alpha_i, A = A(p)$ is large enough, and $\hat{r}_0 > 0$, small enough, then $v(\sigma(\hat{z}, 1 - t_i)) \geq 2\hat{c}\sqrt{-\log t_i \cdot \log\log[-\log t_i]}$, where $\hat{c}$ is the constant in (4.36).

(4.50)

Also c', b_0 can be chosen independent of $p \in [3/2, 2]$. For notational purposes we prove (4.50) only when $i = 1$ and shall use much of the same notation as in displays (3.25) - (3.33). Recall from (4.41) that $u(w_1) = t_1$ and choose $\hat{w}_1 \in \partial\Omega(t_0)$ with $w_1 = \sigma(\hat{w}_1, 1-t_1)$. As in the paragraph following (3.24) we apply Lemma 2.7 with $w_1 = z_1$. Let $\gamma, \tau, \beta, \Omega_1$, be as in Lemmas 2.7 and 2.8, with z_i, replaced by $\tilde{w}_i, 2 \leq i \leq 6$, and define Ω_1' as above (3.25). Let $\hat{\phi}$ be the arc of $\partial\Omega_1 \cap \partial\Omega$ with endpoints $\tilde{w}_5, \tilde{w}_6$. As in (3.31) let $\hat{\theta} \subset \partial\Omega(t)$ be the largest open arc containing $\sigma(\hat{w}_1, 1 - t)$ with the following property.

$$\text{If } z = \sigma(\hat{z}, 1 - t) \in \hat{\theta}, \hat{z} \in \partial\Omega(t_0), \text{ then } \sigma(\hat{z}, \cdot) \cap B(w_1, \tfrac{1}{2}d(w_1, \partial\Omega(t_1))) \neq \emptyset. \tag{4.51}$$

Also if $\sigma(\hat{w}_1, 1-t) \notin \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$, we let $\hat{\gamma}$ denote the arc of $\partial\Omega(t)$ with endpoints $\tilde{w}_4$, $\sigma(\hat{w}_1, 1-t)$. Let $I(w_1) = \hat{\phi} \cup \hat{\theta} \cup \hat{\gamma}$ in this case and otherwise let $I(w_1) = \hat{\phi} \cup \hat{\theta}$. Set $\xi(w_1) = \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$ where $\hat{\rho} = \frac{1}{8} d(\tilde{w}_4, \beta)$. Then $\xi(w_1) \subset I(w_1)$ and $I(w_1)$ is an open arc of $\partial\Omega(t)$ with $z = \sigma(\hat{w}_1, 1-t) \in I(w_1)$. Moreover as in (3.34) we have

$$\mu_t(I(w_1)) \leq c\mu_t(\xi(w_1)) \approx t_1^{p-1} d(w_1, \partial\Omega)^{2-p} \tag{4.52}$$

where the last inequality follows from Lemmas 2.3 and 2.7. All constants depend only on p and can be chosen independent of $p \in [3/2, 2)$. Choose the open arc $\alpha \subset \partial\Omega(t_0)$, with $T(\alpha, t) = I(w_1)$. Observe from the definition of $\hat{\theta}$ that $w_1 \in T(\alpha, t_1)$ and $\partial B(w_1, d(w_1, \partial\Omega)/2) \cap \bar{T}(\alpha, t_1) \neq \emptyset$ We claim that

$$H^1(T(\alpha, t_1)) \leq c d(w_1, \partial\Omega) \tag{4.53}$$

for some $c = c(p) \geq 1$ bounded on [3/2, 2]. To prove this claim note that if $w \in T(\alpha, t)$ then from Lemmas 2.5, 2.6, and arguments akin to those used in the proof of the implicit function theorem, we deduce the existence of $c'' = c''(p)$ satisfying

$$B(w, d(w, \partial\Omega)/c'') \cap \{\zeta : u(\zeta) = t_1\} = G \text{ is contained in a graph and } H^1(G) \leq c'' d(w, \partial\Omega). \tag{4.54}$$

To prove (4.53) choose a covering $\{B(\zeta_j, s_j)\}$ of $T(\alpha, t_1)$ with $s_j = d(\zeta_j, \partial\Omega)/c'', \zeta_j \in T(\alpha, t_1), \zeta_1 = w_1$, and $B(\zeta_k, s_k/10) \cap B(\zeta_l, s_l/10) = \emptyset$ when $l \neq k$. To keep the constant in (4.53) from blowing up as $p \to 2$ we consider two cases, say $1 < p \leq p_0$ and $p_0 < p < 2$. To handle the case $p_0 < p < 2$ we remark once again that the constants in (4.52), (4.54), and Lemmas 2.1 - 2.5 can be chosen independent of $p \in [3/2, 2]$. In fact retracing the proofs one sees that the constants essentially only depend on the Harnack constant and the boundary Hölder exponent of continuity for u in Lemmas 2.1, 2.2. The latter constants are classical and so easily checked. (see [BL] for references). From the previous remark, (4.54), Harnack's inequality, Lemma 2.5, and the argument following (4.42) we see for some $c' = c'(p) \geq 1$, that

$$H^1(T(\alpha, t_1) \cap B(\zeta_j, s_j)) \leq c' d(\zeta_j, \partial\Omega) \text{ and } c' \int_{T(\alpha,t_1) \cap B(\zeta_j, s_j/10)} |\nabla u|^{p-1} \, dH^1 \geq t_1^{p-1} \, d(\zeta_j, \partial\Omega)^{2-p}. \tag{4.55}$$

where c' is independent of $p \in [3/2, 2)$. Given a positive integer k we deduce from properties of the distance function that either there are $< k$ balls in $\{B(\zeta_j, s_j)\}$ or there exists a subsequence $\{B(\zeta_j', s_j')\}_1^k$ of this sequence with $s_j' > (100)^{-k} s_1$ for $1 \leq j \leq k$. In the latter case we get from the second display in (4.55) that

$$\mu_{t_1}(T(\alpha, t_1)) \geq \sum_{j=1}^{k} \mu_{t_1}(T(\alpha, t_1) \cap B(\zeta_j', s_j'/10)) \geq c_0^{-1} t_1^{p-1} k (100)^{-k(2-p)} d(w_1, \partial\Omega)^{2-p}, \tag{4.56}$$

where c_0 is independent of $p \in [3/2, 2)$. On the other hand from (4.52) and (3.11) we see that

$$\mu_{t_1}(T(\alpha, t_1)) \leq c_0' t_1^{p-1} d(w_1, \partial\Omega)^{2-p} \tag{4.57}$$

where c_0' can be chosen independent of $p \in [3/2, 2)$. Combining (4.56), (4.57), we see that if $p = p(k)$ is so near 2 that $(100)^{-k(2-p)} \geq 1/2$ then $k \leq c_0''$ where c_0'' is independent of $p \in [3/2, 2)$. Since all steps are reversible it follows that there exists $p_0 \in (1, 2)$ and a positive integer k_0

such that the sequence $\{B(\zeta_j, s_j)\}$ has at most k_0 members whenever $p_0 < p < 2$. In this case we see from properties of the distance function that

$$s_j \leq (100)^{k_0} s_1 \leq c d(w_1, \partial\Omega).$$

Using this inequality and (4.55) we conclude that (4.53) is true when $p_0 < p < 2$.

If $1 < p \leq p_0$, then from (4.55) and the fact that $2 - p < 1$ we obtain

$$H^1(T(\alpha, t_1))^{2-p} \leq c \left(\sum_j d(\zeta_j, \partial\Omega) \right)^{2-p} \leq c \sum_j d(\zeta_j, \partial\Omega)^{2-p} \leq c\, t_1^{1-p} \mu_{t_1}(T(\alpha, t_1)). \quad (4.58)$$

In view of (4.57), (4.58), we deduce that (4.53) holds for $1 < p \leq p_0$.

Finally choose $\alpha_1 \subset \alpha$ with $T(\alpha_1, t) = \xi(w_1) = \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$. Then (4.50) $(\breve{a})$ follows from (4.53). To prove (4.50)$(\breve{b})$ we note from (4.54) and a covering argument that

$$H^1[\partial\Omega(t_i) \cap B(w_1, d(w_1, \partial\Omega)/2)] \leq \tilde{c} d(w_1, \partial\Omega). \quad (4.59)$$

Using (4.59) and Lemma 2.5 it follows that

$$\mu_{t_1}[\partial\Omega(t_1) \cap B(w_1, d(w_1, \partial\Omega)/2)] \leq c t_1^{p-1} d(w_1, \partial\Omega)^{2-p} \leq c^2 \mu_t(T(\alpha_1, t)) \quad (4.60)$$

where the last inequality is a consequence of (4.52). To prove (4.50) $(\breve{c})$ we point out that if $z \in T(\alpha_1, t_1)$, then from (4.53) we have $d(z, \partial\Omega) \leq c d(w_1, \partial\Omega)$. From Lemma 2.5 it follows that $|\nabla u(w_1)| \leq c^* |\nabla u(z)|$ for some $c^* = c^*(p)$. In view of this observation and (4.47), (4.48), we see for $A = A(p)$ large enough and $r_0 > 0$ small enough that (4.50) $(\breve{c})$ holds. Thus (4.50) is true.

With $A = A(p), b = b(p)$, now fixed and bounded on [3/2, 2) we complete the estimate in (4.45) and get (4.38), Theorem $1(b)$ in the following way. Using (4.50) and (4.40) (c), we deduce

$$\sum_{i=1}^{N} \int_{\partial\Omega(t_i) \cap B(w_i, d(w_i, \partial\Omega)/2)} |\nabla u|^{p-1} dH^1 \leq c \sum_{i=1}^{N} \mu_t(T(\alpha_i, t)) = c \mu_t \left(\bigcup_{i=1}^{N} T(\alpha_i, t) \right) \quad (4.61)$$

Also from (4.50) $(\breve{c})$, (4.49), (4.36), and measure theoretic arguments we see that

$$\mu_t \left(\bigcup_{i=1}^{N} T(\alpha_i, t) \right) \to 0 \text{ as } \hat{r}_0 \to 0. \quad (4.62)$$

In view of (4.61), (4.62), and (4.45) we deduce that

$$\mu(U) \to 0 \text{ as } \hat{r}_0 \to 0.$$

Since $\hat{r}_0$ can be arbitrarily small we conclude that (4.38) and Theorem $1(b)$ are valid . □

5 Proof of Proposition 4.15

To begin the proof of Proposition 4.15 let $z \in \partial\Omega(t)$ and choose $\hat{z} \in \partial\Omega(t_0)$ with $z = \sigma(\hat{z}, 1-t)$. Also choose $s \in [t, 1)$ with $N_t f(z) = f(\sigma(\hat{z}, 1-s))$. Let $w_1 = \sigma(\hat{z}, 1-s)$ and observe that if $w_1 \in B(z, \frac{1}{2} d(z, \partial\Omega))$, then Proposition 4.15 follows from Lemma 2.5 and Harnack's inequality for u. Also $w_1 = \sigma(\hat{z}, 1-s) \notin \bar{B}(0, 2)$ since $f \equiv 0$ on $\bar{B}(0, 2)$. Thus we assume $t < s < 1$ and $w_1 = \sigma(\hat{z}, 1-s) \notin (\bar{B}(0, 2) \cup B(z, \frac{1}{2} d(z, \partial\Omega)))$. Also from (4.16), (4.42) with t_i replaced by t, Lemma 2.5, and Harnack's inequality, we have for some $c = c(p) \geq 1$, that

$$s - t = \int_t^s |\nabla u| |\tfrac{d\sigma(\hat{z}, \cdot)}{dt'}| dt' \geq t/c. \tag{5.1}$$

As in the paragraph following (3.24) we apply Lemmas 2.7, 2.8, with z_1 replaced by $w_1 = \sigma(\hat{z}, 1-s)$. Let $\gamma, \tau, \beta, \Omega_1$, be as in these lemmas with z_i, replaced by $\tilde{w}_i, 2 \leq i \leq 6$. Moreover we let $\hat{\phi}$ be the arc of $\partial\Omega_1 \cap \partial\Omega(t)$ with endpoints $\tilde{w}_5, \tilde{w}_6$ (see (3.25) - (3.34)) and define Ω_1' as above (3.25). Likewise (see (3.31)) let $\hat{\theta} \subset \partial\Omega(t)$ be the largest arc with $z = \sigma(\hat{z}, 1-t) \in \hat{\theta}$ and the property that if $\sigma(\hat{x}, 1-t) \in \hat{\theta}$ for some $\hat{x} \in \partial\Omega(t_0)$, then $\sigma(\hat{x}, 1-s) \in B(w_1, \frac{1}{2} d(w_1, \partial\Omega(t))$. Also if $z = \sigma(\hat{z}, 1-t) \notin \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$, we let $\hat{\gamma}$ denote the arc of $\partial\Omega(t)$ with endpoints $\tilde{w}_4$, $\sigma(\hat{z}, 1-t) = z$. Let $I(w_1) = \hat{\phi} \cup \hat{\theta} \cup \hat{\gamma}$ in this case and otherwise let $I(w_1) = \hat{\phi} \cup \hat{\theta}$. Set $\xi(w_1) = \partial\Omega_1' \cap B(\tilde{w}_4, \hat{\rho})$ where $\hat{\rho} = \frac{1}{8} d(\tilde{w}_4, \beta)$. Then $\xi(w_1) \subset I(w_1)$ and $I(w_1)$ is an open arc of $\partial\Omega(t)$ with $z = \sigma(\hat{z}, 1-t) \in I(w_1)$. Moreover as in (3.34) and (4.52) we deduce that

$$\mu_t(I(w_1)) \leq c\mu_t(\xi(w_1)) \approx s^{p-1} d(w_1, \partial\Omega)^{2-p}. \tag{5.2}$$

Let $\alpha \subset \partial\Omega(t_0)$ be the arc with $T(\alpha, t) = I(w_1)$. From (5.2) and (3.11) it follows first that

$$\mu_s(T(\alpha, s)) \approx s^{p-1} d(w_1, \partial\Omega(t))^{2-p} \tag{5.3}$$

and then as in the proof of (4.53) that

$$H^1(T(\alpha, s)) \approx d(w_1, \partial\Omega) \tag{5.4}$$

where all proportionality constants depend on p and can be chosen to be bounded (as a function of p) on (3/2, 2). As in the proof of (4.53) we will need to consider two cases, $1 < p < p_0'$ and $p_0' \leq p < 2$, in order to keep the constants in Proposition 4.15 from exploding as $p \to 2$. To this end we show that p_0' can be chosen so that if $\hat{y} \in \alpha$, and $p_0' \leq p < 2$, then

$$\int_t^s |\nabla u|^{p-1}(\sigma(\hat{y}, 1-t')) dH^1 t' \leq c s^{p-1} d(w_1, \partial\Omega(t))^{2-p} \tag{5.5}$$

where $c = c(p)$ is bounded on $[p_0', 2)$. To prove (5.5) let $\zeta \in \sigma(\hat{y}, (1-s, 1-t))$, and $r = \frac{1}{100} d(\zeta, \partial\Omega(t))$. We note as in (5.1) that if $\tilde{u} = u - t$, then

$$\min_{\partial B(\zeta, r) \cap \sigma(\hat{y}, \cdot)} \tilde{u} \leq b\, \tilde{u}(\zeta) \tag{5.6}$$

for some $b = b(p)$ with $0 < b < 1$ where $1 - b$ is bounded below by a positive constant on [3/2, 2). Since $\tilde{u}$ decreases along the arc from $\sigma(\hat{y}, 1-s)$ to $\sigma(\hat{y}, 1-t)$ it follows from (5.6) that there exists a covering $\{B(\zeta_j,\rho_j)\}$ of $\sigma(\hat y,[1-s,1-t])$ with $\zeta_1=\sigma(\hat y,1-s), \rho_j=\frac{1}{10}d(\zeta_j,\partial\Omega(t)), \zeta_j\in\sigma(\hat y,(1-t,1-s])$ and $B(\zeta_k,\rho_k/10)\cap B(\zeta_l,\rho_l/10)=\emptyset$ when $l\neq k$. Moreover,

$$\tilde u(\zeta_j)\le c\,b^j\,s. \tag{5.7}$$

Using properties of the distance function we may also assume that

$$d(\zeta_j,\partial\Omega(t))\le 100^j\,d(\zeta_1,\partial\Omega(t)). \tag{5.8}$$

From (5.7), (5.8), (4.16), lemma 2.5, and Harnack's inequality, we see there exists $p_0'\in(1,2)$ such that if $p_0'\le p<2$, then

$$\int_{B(\zeta_j,\rho_j)\cap\sigma(\hat y,\cdot)}|\nabla\tilde u|^{p-1}dH^1\le c\tilde u(\zeta_j)^{p-1}\rho_j^{2-p}\le cs^{p-1}\,b^{j(p-1)/2}d(\zeta_1,\partial\Omega(t))^{2-p}. \tag{5.9}$$

where $c=c(p)$ is bounded on [3/2,2]. Summing this inequality we get (5.5) with $d(w_1,\partial\Omega(t))$ replaced by $d(\zeta_1,\partial\Omega(t))$ However from (5.4) we have $d(\zeta_1,\partial\Omega(t))\le cd(w_1,\partial\Omega(t))$ so this inequality implies (5.5).

We do not know if (5.5) holds for all $\hat y\in\alpha$ when $1<p<p_0'$. Given our lack of this knowledge we will have to settle for showing (5.5) holds on a ‘ large set ’ $\subset\alpha$ when $1<p<p_0'$. We begin by choosing $\hat y_1,\hat y_2\in\partial\Omega(t_0)$ with $\sigma(\hat y_1,1-t)=\tilde w_5, \sigma(\hat y_2,1-t)=\tilde w_6$. If $1<p<p_0'$ we claim there exists $c=c(p)\ge 1$, with

$$|\sigma(\hat y_j,1-t')-\tilde w_4|\le cd(w_1,\partial\Omega(t)) \text{ for } j=1,2, \text{ and } t\le t'\le s. \tag{5.10}$$

We first prove (5.10) when $j=1$ and $t'=u(\beta(\hat s))$ for some $\hat s\le s_0$ (s_0 as in Lemma 2.8 (γ')) . Let O^* be the open set with $O^*\cap B(0,2)=\emptyset$ and whose boundary consists of

$$\begin{array}{ll} (i) & \text{The arc of } \beta \text{ joining } \tilde w_5 \text{ to } \beta(\hat s),\\ (ii) & \text{The arc of } \{w: u(w)=t'\} \text{ joining } \beta(\hat s) \text{ to } \sigma(\hat y_1,1-t'),\\ (iii) & \text{The arc } \sigma(\hat y_1,(1-t',1-t)) \text{ from } \sigma(\hat y_1,1-t') \text{ to } \sigma(\hat y_1,1-t).\end{array} \tag{5.11}$$

Let $\delta_1,\delta_2,\delta_3$, denote the arcs in $(i),(ii),(iii)$, respectively. From the divergence theorem and p harmonicity of u we see that

$$\sum_{i=1}^3\int_{\delta_i}|\nabla u|^{p-2}u_\nu\,dH^1\;=\;0 \tag{5.12}$$

where ν is the outer unit normal to O^*. Since ∇u is tangent to $\sigma(\hat y_1,\cdot)$ we have

$$\int_{\delta_3}|\nabla u|^{p-2}u_\nu\,dH^1=0. \tag{5.13}$$

Next If $\zeta\in\delta_1$, and $r=\frac{1}{100}d(\zeta,\partial\Omega(t))$ then from Lemma 2.8 $(\beta'),(\gamma')$, and Lemma 2.5, we get

$$\int_{B(\zeta,r)\cap\delta_1}|\nabla\tilde u|^{p-1}dH^1\le c\,\tilde u(\zeta)^{p-1}\,r^{2-p}\;\le c^2\tilde u(\beta(\hat s))^{p-1}\,r^{2-p}. \tag{5.14}$$

Summing (5.14) over a covering of δ_1 and using the cigar condition on β we obtain

$$\int_{\delta_1} |\nabla u|^{p-1} dH^1 \leq\, c\,(u(\beta(\hat{s})) - t)^{p-1}\, d(\beta(\hat{s}), \partial\Omega(t))^{2-p} \leq\, c^2\,(t')^{p-1} d(\beta(\hat{s}), \partial\Omega(t))^{2-p}. \tag{5.15}$$

Putting (5.15) in (5.12) and using (5.13) we find that

$$\left| \int_{\delta_2} |\nabla u|^{p-2} u_\nu \, dH^1 \right| = \mu_{t'}(\delta_2) \leq c\,(t')^{p-1}\, d(\beta(\hat{s}), \partial\Omega(t))^{2-p}. \tag{5.16}$$

Using (5.16) we can now repeat the argument leading to (4.58) (since $1 < p < p_0'$). We get

$$H^1(\delta_2)^{2-p} \leq c\,(t')^{1-p} \mu_{t'}(\delta_2). \tag{5.17}$$

In view of (5.17), (5.16), and Lemma 2.8 (α'), we conclude that

$$H^1(\delta_2) \leq c\, d(\beta(\hat{s}), \partial\Omega(t)) \leq c d(w_1, \partial\Omega(t)) \text{ for } 0 < \hat{s} \leq s_0 \text{ and } 1 < p < p_0'. \tag{5.18}$$

Thus (5.10) is valid when $t' = \beta(\hat{s}), j = 1$, and $0 \leq \hat{s} \leq s_0$.

Next we observe for some $\tilde{c} = \tilde{c}(p)$ that

$$\tilde{c}(u(\beta(s_0)) - t) \geq s = u(w_1). \tag{5.19}$$

In fact from the definition of s_0 and the cigar condition on β we have

$$c d(\beta(s_0), \partial\Omega(t)) \geq d(w_1, \partial\Omega(t)). \tag{5.20}$$

Also if $w' \in \tau \cap \beta$, then $c d(w', \partial\Omega(t)) \geq d(w_1, \partial\Omega(t))$ as follows from Lemma 2.8 (α') and Lemma 2.7 (δ). Using this fact, Harnack's inequality for $u - t$, the cigar condition on τ, β, and (5.1) we see there exists $c = c(p)$ such that

$$s/c \leq u(w_1) - t = s - t \leq c\,(u(w') - t) \leq c^2 (u(\beta(s_0)) - t)$$

which implies (5.19). From (5.19) and the earlier assumption on t' we deduce that it remains only to prove (5.10) for $j = 1$ when $s/\tilde{c} \leq t' \leq s$ where $\tilde{c}$ is as in (5.19). To do this one shows as in the proof of (4.53) and similar to (5.18) that

$$H^1(T(\alpha, t')) \leq c d(w_1, \partial\Omega(t))$$

which clearly implies (5.10) for $s/\tilde{c} \leq t' \leq s$, since $\sigma(\hat{y}_1, 1 - t) \in T(\alpha, t) = I(w_1)$. Thus (5.10) is valid when $j = 1$. The proof of (5.10) when $j = 2$ is essentially the same so we omit the details.

From (5.10) and (5.4) we see that if $1 < p < p_0'$, then there exists $c_*, 1 \leq c_* \leq c(p)$, with

$$c_* d(w_1, \partial\Omega) = \sup\{|\zeta - \tilde{w}_4| : \text{ either } \zeta \in \sigma(\hat{y}_i, [t, s]), i = 1, 2, \text{ or } \zeta \in T(\alpha, s)\}. \tag{5.21}$$

Let $\alpha' \subset \partial\Omega(t_0)$ satisfy

$$T(\alpha', t) = \hat{\phi} = \partial\Omega_1' \cap \partial\Omega_1$$

and put

$$U = \bigcup_{t<t'<s} T(\alpha', t') \ .$$

Note from the definition of U that $u \le s$ in U. In our proof of Proposition 4.15 for $1 < p < p_0'$ we shall need several more lemmas. The next lemma gives a decay estimate for $\tilde{u} = u - t$ far away from $\tilde{w}_4$.

Lemma 5.22. *Let $1 < p < p_0', w \in U$, and c_* as in (5.21). There exists $c = c(p)$ and $\gamma > 0, 0 < \gamma < 1$, such that if $w \in U \setminus B(\tilde{w}_4, 2c_* d(w_1, \partial\Omega(t)))$, then*

$$\tilde{u}(w) \le cs \left(\frac{|w - \tilde{w}_4|}{d(w_1, \partial\Omega(t))} \right)^{\frac{p-2-\gamma}{p-1}} .$$

Proof: We prove Lemma 5.22 by an iterative type argument. More specifically let k be a positive integer, $\rho_k = 2^k c_* d(w_1, \partial\Omega(t)))$, and $U_k = U \setminus B(\tilde{w}_4, \rho_k)$. Set

$$\lambda_k \rho_k^{(p-2)/(p-1)} = \max_{U_k} \tilde{u} \, .$$

We show there exists $\eta \in (0, 1)$ such that

$$\lambda_{k+1} \le (1 - \eta) \, \lambda_k \text{ for } k = 1, 2, \dots \tag{5.23}$$

Iterating this inequality for $k = 1, \dots n$ and using $s \ge \lambda_1 \rho_1^{(p-2)/(p-1)}$ we then get

$$\max_{U_{n+1}} \tilde{u} \le 16 s \, (1 - \eta)^n \, (\rho_n / \rho_1)^{(p-2)/(p-1)}$$

for $n = 1, 2, \dots$ which is easily seen to imply Lemma 5.22. Therefore to complete the proof of Lemma 5.22 we only need to prove (5.23). To this end let

$$g(w) = \lambda_k |w - \tilde{w}_4|^{(p-2)/(p-1)}, \, w \in U_k,$$

$$q(w) = g(w) - \tilde{u}(w), w \in U_k,$$

$$V_k = U_k \cap \partial B(\tilde{w}_4, 2^{1/2} \rho_k \,),$$

for $k = 1, 2, \dots$ To estimate q on V_k we consider two cases. First if $w \in V_k, 0 < \delta < 1/2$, and

$$d(w, \partial U_k) \le \delta \rho_k, \tag{5.24}$$

then from Lemma 2.2 and the fact that $\tilde{u} = 0$ on $\partial U \cap \partial U_1$ we have

$$u(w) \le c \delta^\alpha \lambda_k \rho_k^{(p-2)/(p-1)} \le \lambda_k (1 - \epsilon) |w - \tilde{w}_4|^{(p-2)/(p-1)} \tag{5.25}$$

where $0 < \epsilon \le 1/4$ provided δ is chosen small enough. With δ now fixed we conclude from (5.25) that

$$q(w) \ge \epsilon \lambda_k |w - \tilde{w}|^{(p-2)/(p-1)} \text{ for } w \in V_k \cap \{w' \in U_k : d(w', \partial U_k) \le \delta \rho_k\} \, . \tag{5.26}$$

If $w \in V_k$ and (5.24) is false, we note for $j = 1, 2$, and $w', \zeta' \in \mathbb{C} \setminus \{0\}$, that

$$\begin{aligned} |\zeta'|^{p-2}\, \zeta'_j - |w'|^{p-2}\, w'_j \;\; &= \int_0^1 \tfrac{d}{d\theta}\{|\theta\zeta + (1-\theta)w'|^{p-2}\,[\theta\zeta'_i + (1-\theta)w'_j]\}d\theta \\ &= \sum_{l=1}^{2} (\zeta' - w')_j \left(\int_0^1 a_{lj}[\theta\zeta' + (1-\theta)w']d\theta \right), \end{aligned} \tag{5.27}$$

where, for $1 \le l, j \le 2$, and $\xi \in \mathbb{C} \setminus \{0\}$,

$$a_{lj}(\xi) = |\xi|^{p-4}\,[(p-2)\xi_l\xi_j \;+\; \delta_{lj}\,|\xi|^2]. \tag{5.28}$$

In this display δ_{lj}, denotes the Kronecker delta. Using (5.27), (5.28), and the fact that $g, \tilde{u}$ are p harmonic on U_k with non vanishing gradients we see that if

$$A_{lj}(w') = \int_0^1 a_{lj}[\theta\nabla g(w') + (1-\theta)\nabla\tilde{u}(w')]\, d\theta,$$

whenever $w' \in U_k$, and $1 \le l, j \le 2$, then

$$\tilde{L}\, q(w') = \nabla \cdot \left(|\nabla g|^{p-2}\nabla g - |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\right)(w') \;=\; \sum_{l,j=1}^{2} \frac{\partial}{\partial w'_l}[A_{lj}(w')\, q_{w'_j}] = 0\,. \tag{5.29}$$

Next observe from (5.28), Lemma 2.5, as well as the definiton of A_{lj}, that for $w' \in U_k$ with $d(w', \partial U_k) \ge \delta\rho_k/8$ and $\xi \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} &(*) \quad c^{-1}\delta^{2-p}\lambda_k^{p-2}\rho_k^{(2-p)/(p-1)}\,|\xi|^2 \le (|\nabla g(w')| + |\nabla\hat{u}(w')|)^{p-2}\,|\xi|^2 \;\le\; \sum_{l,j=1}^{2} A_{lj}(w')\xi_i\xi_j, \\ &(**) \quad \sum_{l,j=1}^{2} |A_{ij}(w')| \le c(|\nabla g(w')| + |\nabla\hat{u}(w')|)^{p-2} \;\le\; c^2\lambda_k^{p-2}\rho_k^{(2-p)/(p-1)} \end{aligned} \tag{5.30}$$

where c depends only on p since δ is fixed. Dividing both inequalities by $\lambda_k^{p-2}\rho_k^{(2-p)/(p-1)}$ we conclude that q satisfies a uniformly elliptic equation with uniformly bounded coefficients when $w' \in U_k$ and $d(w', \partial U_k) \ge \delta\rho_k/8$. It follows from this discussion and Harnack's inequality for partial differential equations as above that if $w' \in U_k$ and $d(w', \partial U_k) \ge \delta\rho_k/2$, then

$$0 < \max_{B(w', \delta\rho_k/8)} q \;\le\; c \min_{B(w', \delta\rho_k/8)} q \tag{5.31}$$

where $c \ge 1$ depends only on p. Using this fact we deduce that if (5.24) is false, then w can be connected to a point w'' in $\{w' : \delta\rho_k/2 \le d(w', \partial U_k) \le \delta\rho_k\}$ by a Harnack chain of balls with radii $\ge \delta\rho_k/8$ and centers in $\{w' : d(w', \partial U_k) \ge \delta\rho_k/2\}$ Applying (5.31) at most c/δ times beginning with w'' we get for some $\epsilon_1 = \epsilon_1(p) > 0$ that

$$q(w) \ge \epsilon_1\lambda_k|w - \tilde{w}_4|^{(p-2)/(p-1)} \text{ on } V_k \cap \{w : d(w, \partial U_k) \ge \delta\rho_k\}. \tag{5.32}$$

If $\eta = \min(\epsilon, \epsilon_1)$ then in view of (5.32), (5.26), and the minimum principle for p harmonic functions that

$$q(w) \geq \eta |w - \tilde{w}_4|^{(p-2)/(p-1)} \text{ for } w \in V_k$$

or equivalently $\tilde{u} \leq (1-\eta) g$ in $V_k \supset U_{k+1}$ Thus (5.23) is true and the proof of Lemma 5.22 is now complete. □

Lemma 5.33 *If $1 < p < p_0'$, then for some $c = c(p)$,*

$$\int_U \frac{|\nabla u|^{p-1}}{|w - \tilde{w}_4|} \, dA \; \leq \; c \, s^{p-1} \, d(w_1, \partial\Omega(t))^{2-p}.$$

Proof: Let $U_k, k = 1, 2, ...$ be as in Lemma 5.22 and put $U_k' = U \cap B(\tilde{w}_4, 2^{-k} d(\tilde{w}_1, \partial\Omega(t)), k = -1, 0, 1, 2, \ldots$. We write

$$\int_U \frac{|\nabla u|^{p-1}}{|w - \tilde{w}_4|} \, dA \; = \; \int_{U \setminus U_1} \ldots dA + \int_{U_1} \ldots dA = I_1 \; + \; I_2 \, . \tag{5.34}$$

To estimate I_1 we note from Lemmas 2.1 and 2.2 applied to $\tilde{u}$ that for $c = c(p)$ large enough

$$\begin{aligned} \int_{U_k' \setminus U_{k+1}'} \tfrac{|\nabla u|^{p-1}}{|w - \tilde{w}_4|} \, dA \; & \leq \; c \left(\int_{U_k' \setminus U_{k+1}'} |\nabla u|^p \, dA \right)^{1-1/p} (2^{-k} d(w_1, \partial\Omega(t))^{(2-p)/p} \\ & \leq c \, s^{p-1} \, d(w_1, \partial\Omega(t))^{2-p} \, 2^{-k(2-p)} \, . \end{aligned} \tag{5.35}$$

Summing this inequality from -1 to ∞ we find that

$$I_1 \leq c \, s^{p-1} \, d(w_1, \partial\Omega(t))^{2-p} \; . \tag{5.36}$$

Also using Lemma 5.22 we have

$$\begin{aligned} \int_{U_k \setminus U_{k+1}} \; \tfrac{|\nabla u|^{p-1}}{|w - \tilde{w}_4|} \, dA \leq \; c & \left(\int_{U_k \setminus U_{k+1}} |\nabla u|^p \, dA \right)^{1-1/p} [2^k d(w_1, \partial\Omega(t))]^{(2-p)/p} \\ \leq c \; d(w_1, \partial\Omega(t))^{2-p} \, 2^{k(2-p)} & (\max_{U_k} \tilde{u})^{p-1} \; \leq \; c^2 \, s^{p-1} \, d(w_1, \partial\Omega(t))^{2-p} \, 2^{-\gamma k} \, . \end{aligned} \tag{5.37}$$

Summing (5.37) from 1 to ∞ we obtain first (5.36) with I_1 replaced by I_2 and then Lemma 5.33. □

Next we state

Lemma 5.38. *Let z, α' be as in the definition of U and fix $p, 1 < p < 2$. Then there is a compact set $E \subset \alpha' \subset \partial\Omega(t_0)$ and $c = c(p) \geq 1$ such that*

$$(a) \quad 2\,\mu_t(T(E,t)) \geq \mu_t(T(\alpha',t))$$

$$(b) \quad \text{if } w \in T(E,t), \text{ then } N_t(\tilde{v}^{2(m-1)})(w) \leq c\,M_t(N_t\tilde{v}^{2(m-1)})(z)$$

$$(c) \quad \text{if } \hat{w} \in E, \text{ then } \int_{\sigma(\hat{w},[1-s,1-t])} |\nabla u|^{p-1}\,dH^1 \leq c\mu_t(T(\alpha',t)) \approx s^{p-1}d(w_1,\partial\Omega(t))^{2-p}.$$

Proof: Observe from the usual weak type estimates that for each $a > 0$,

$$\begin{aligned} a\,\frac{\mu_t(\{w \in T(\alpha,t) : N_t(\tilde{v}^{2(m-1)})(w) > a\})}{\mu_t(T(\alpha,t))} &\leq c\,\frac{\int_{T(\alpha,t)} N_t(\tilde{v}^{2(m-1)})(w')d\mu_t(w')}{\mu_t(T(\alpha,t))} \\ &\leq cM_t(N_t\tilde{v}^{2(m-1)})(z) \end{aligned} \tag{5.39}$$

since $z \in I(w_1) = T(\alpha,t)$. From (5.39), (5.2), (5.3), and the fact that $\xi(w_1) \subset T(\alpha',t)$, we deduce the existence of a compact set $\hat{E}_1 \subset \alpha'$ with $\mu_t(T(\hat{E}_1,t)) \geq \frac{3}{4}\mu_t(T(\alpha',t))$. Also Lemma 5.38 (b) holds for $w \in T(\hat{E}_1,t)$ provided c is suitably large.

To prove (c) of Lemma 5.38 first suppose $1 < p < p_0'$. We observe from either the co-area or the change of variables theorem, as in (3.5) and Lemma 5.33 that

$$\begin{aligned} &\int_{\alpha'} \left(\int_{\sigma(\hat{w},[1-s,1-t))} \tfrac{|d\sigma/dt'(\sigma(\hat{w},1-t'))|}{|\sigma(\hat{w},1-t')-\tilde{w}_4\,|} dt' \right) d\sigma_{t_0}(\hat{w}) = \int_U \tfrac{|\nabla u|^{p-1}}{|w-\tilde{w}_4|}\,dA \\ &\leq cs^{p-1}\,d(w_1,\partial\Omega(t))^{2-p} \approx \mu_t(T(\alpha',t)). \end{aligned} \tag{5.40}$$

To estimate the second integral on the far lefthand side of (5.40) observe for a positive integer $n > 1$, that

$$c\int_{\sigma(\hat{w},[1-s,1-t))\cap U\setminus U_n} \frac{|d\sigma/dt'(\sigma(\hat{w},1-t'))|}{|\sigma(\hat{w},1-t')-\tilde{w}_4\,|}dt' \geq \frac{H^1[\sigma(\hat{w},[1-s,1-t))\cap U\setminus U_n]}{2^n d(w_1,\partial\Omega(t))}. \tag{5.41}$$

Also if $\alpha'' = \{\hat{w} \in \alpha' : \sigma(\hat{w},[1-s,1-t)) \cap U_n \neq \emptyset\}$, and $\hat{w} \in \alpha''$, then for $1 \leq k \leq n$,

$$\int_{\sigma(\hat{w},[1-s,1-t))\cap U_k\setminus U_{k-1}} \frac{|d\sigma/dt'(\sigma(\hat{w},1-t'))|}{|\sigma(\hat{w},1-t')-\tilde{w}_4\,|}dt' \;\geq\; 1/2. \tag{5.42}$$

Summing (5.42) from $k = 1$ to n it follows that

$$\int_{\sigma(\hat{w},[1-s,1-t))\cap U\setminus U_n} \frac{|d\sigma/dt'(\sigma(\hat{w},1-t'))|}{|\sigma(\hat{w},1-t')-\tilde{w}_4\,|}dt' \geq (1/2)n. \tag{5.43}$$

From (5.43), (5.40), and weak type estimates we deduce

$$\mu_{t_0}(\alpha'') \leq cn^{-1}\mu_{t_0}(\alpha')$$

for $1 < p < p_0'$, and thereupon from (5.41) for $n = n(p)$ large enough that there exists a compact set $\hat{E}_2 \subset \alpha' \setminus \alpha''$ with

$$\mu_{t_0}(\hat{E}_2) \geq 3\mu_{t_0}(\alpha')/4 \text{ and } H^1[\sigma(\hat{w}, [1-s, 1-t))] \leq c d(w_1, \partial\Omega(t)) \text{ for } \hat{w} \in \hat{E}_2 \,. \tag{5.44}$$

We now prove (5.5) when $1 < p \leq p_0'$ and $\hat{y} \in \hat{E}_2$. Indeed let $\{B(\zeta_j, \rho_j)\}$ be the covering of $\sigma(\hat{y}, [1-s, 1-t))$ defined above (5.7). Arguing as in (5.6) - (5.9) and using (5.44) we obtain for some $b = b(p), 0 < b < 1$, and $c = c(p) \geq 1$, that

$$\int_{B(\zeta_j, \rho_j)} |\nabla \tilde{u}|^{p-1} dH^1 \leq c \tilde{u}(\zeta_j)^{p-1} \rho_j^{2-p} \leq c s^{p-1} \, b^{j(p-1)} \, d(w_1, \partial\Omega(t))^{2-p} \,. \tag{5.45}$$

Summing this inequality we get (5.5) and Lemma 5.38 (c) when $\hat{w} \in \hat{E}_2$ and $1 < p < p_0'$. Lemma 5.38 (c) for $p_0' \leq p < 2$, and $\hat{w} \in \hat{E}_2$ follows directly from (5.5). Let $E = \hat{E}_1 \cap \hat{E}_2$. Lemma 5.38 is a consequence of the above remarks, (5.44), and the definition of $\hat{E}_1$ below (5.39). □

5.1 Proof of Proposition 4.15.

To begin the proof of Proposition 4.15 let $z(t'), 0 \leq t' \leq 1$, be a parametrization of $\hat{\phi} = T(\alpha', t)$ with $z(0) = \tilde{w}_5, z(1) = \tilde{w}_6$. Let

$$t_1 = \min\{t' : z(t') \in T(E, t)\} \text{ and } t_2 = \max\{t' : z(t') \in T(E, t)\}. \tag{5.46}$$

Choose $\hat{x}, \hat{y} \in E$ with $\sigma(\hat{x}, 1-t) = z(t_1), \sigma(\hat{y}, 1-t) = z(t_2)$. Let $\tilde{\alpha} \subset \alpha' \subset \alpha$ be the arc of $\partial\Omega(t_0)$ containing E and with endpoints $\hat{x}, \hat{y}$. Put $\tilde{U} = \bigcup_{t<t'<s} T(\tilde{\alpha}, t')$. Using the divergence theorem as in (4.4), (4.5) we obtain

$$\begin{aligned} \hat{S} = & \int_{\tilde{U}} [\tilde{u}\, Lf - f\, L\tilde{u}] dA = \int_{\tilde{U}} \tilde{u}\, (P_1 - P_2) dA \\ & = - \int_{\partial\tilde{U}} \sum_{i,j=1}^{2} b_{ij}\, \nu_i f\, \tilde{u}_{x_j}\, dH^1 + \int_{\partial\tilde{U}} \sum_{i,j=1}^{2} b_{ij}\, \nu_i f_{x_j}\, \tilde{u}\, dH^1 = \hat{K}_1 + \hat{K}_2 \end{aligned} \tag{5.47}$$

where $\nu = (\nu_1, \nu_2)$ is the outer unit normal to $\partial\tilde{U}$. Moreover,

$$\hat{K}_1 = (p-1) \int_{T(\tilde{\alpha}, t)} |\nabla u|^{p-1} \, f \, dH^1 - (p-1) \int_{T(\tilde{\alpha}, s)} |\nabla u|^{p-1} \, f \, dH^1 \,. \tag{5.48}$$

Here we have used the fact that $\nabla u(\sigma(\cdot, 1-t'))$ is tangent to $\sigma(\cdot, 1-t')$. Let $\phi_i, i = 1, 2$, denote the arcs, $\sigma(\hat{x}, 1-\tau), \sigma(\hat{y}, 1-\tau), t < \tau < s$, respectively. Then

$$\hat{K}_2 = \left(\sum_{i,j,l=1}^{2} \int_{\phi_l} \tilde{u}\, b_{ij}\, \nu_i\, f_{x_j}\, dH^1 \right) + \left(\int_{T(\tilde{\alpha}, s)} \tilde{u} \sum_{i,j=1}^{2} b_{ij}\, \nu_i\, f_{x_j}\, dH^1 \right) = K_{21} + K_{22}. \tag{5.49}$$

From Lemma 2.5 we note that there exists $c_- = c_-(p) \geq 1$ such that if

$$c_- d(w, \partial\Omega(t)) \leq d(w_1, \partial\Omega(t)) \tag{5.50}$$

for all $w \in T(\tilde{\alpha}, s)$, then $N_t f(z) = f(w_1) \leq f(w)$ for all $w \in T(\tilde{\alpha}, s)$ and

$$N_t f(z)\mu_s(T(\tilde{\alpha}, s)) = \mu_s(T(\tilde{\alpha}, s))f(w_1) \leq \int_{T(\tilde{\alpha},s)} |\nabla u|^{p-1} f(w) dH^1 w. \tag{5.51}$$

Otherwise we consider two cases. If (5.50) holds for some points in $T(\tilde{\alpha}, s)$ and fails for other points on this arc then by continuity of $d(\cdot, \partial\Omega(t))$ there exists a point w' in $T(\tilde{\alpha}, s)$ for which equality holds in (5.50). If $w \in B(w', d(w', \partial\Omega(t))/4) \cap T(\tilde{\alpha}, s) = Q$ we deduce from the fundamental theorem of calculus and Lemma 2.6 that

$$\begin{aligned} |f(w) - f(w')| &\leq (c\, m/d(w_1, \partial\Omega(t)) \int_Q \tilde{v}^{(2m-1)} dH^1 \\ &\leq \tfrac{c' m}{\mu_s(Q)} \int_Q |\nabla u|^{p-1} \tilde{v}^{(2m-1)} dH^1 \end{aligned} \tag{5.52}$$

for some $c' = c'(p) \geq 1$. From basic geometry, Lemma 2.5, and (5.2) we also have

$$c^2 \mu_s(Q) \geq c s^{p-1} d(w_1, \partial\Omega(t))^{p-2} \geq \mu_s(T(\alpha, s)). \tag{5.53}$$

Combining (5.52), (5.53), and using $N_t f(z) = f(w_1) \leq f(w')$ as well as Lemma 2.5 we find for $\bar{c} = \bar{c}(p)$ large enough that

$$\begin{aligned} N_t f(z)\mu_s(T(\tilde{\alpha}, s)) &\leq \bar{c} \int_{T(\tilde{\alpha},s)} |\nabla u|^{p-1} f\, dH^1 \;+\; \bar{c}m \int_{T(\alpha,s)} |\nabla u|^{p-1} \tilde{v}^{(2m-1)} dH^1 \\ &\leq \bar{c} \int_{T(\tilde{\alpha},s)} |\nabla u|^{p-1} f\, dH^1 \;+\; \bar{c}m\, M_t(N_t \tilde{v}^{2m-1})(z)\, \mu_s(T(\alpha, s)). \end{aligned} \tag{5.54}$$

If (5.50) is false for all $w \in T(\tilde{\alpha}, s)$, and $w_1 \in \bar{T}(\tilde{\alpha}, s)$ we can argue as above with $w' = w_1$ to obtain (5.54). If (5.50) is false and $w_1 \notin \bar{T}(\tilde{\alpha}, s)$, we let $\Gamma \subset T(\alpha, s)$ denote the arc joining an endpoint of $T(\tilde{\alpha}, s)$ to w_1 with $T(\tilde{\alpha}, s) \cap \Gamma = \emptyset$. Let w' be either the first point on Γ (proceeding from $T(\tilde{\alpha}, s)$) where (5.50) holds with equality or if no such point exists let $w' = w_1$. From (5.2) and falseness of (5.50) we see that $\Gamma \cup T(\tilde{\alpha}, s)$ can be covered by a finite number (depending only on p) of balls with radii $\approx d(w_1, \partial\Omega(t))$. Thus (5.52), (5.53) hold in this case with $Q = \Gamma \cup T(\tilde{\alpha}, s)$. Using (5.52), (5.53), and Lemma 2.5, we conclude that (5.54) also holds in this case. Thus (5.54) is true in general.

From (5.54) and (5.48) we have

$$\begin{aligned} N_t f(z)\, \mu_s(T(\tilde{\alpha}, s)) &\leq c \int_{T(\tilde{\alpha},s)} |\nabla u|^{p-1} f dH^1 + cm\, M_t(N_t \tilde{v}^{2m-1})(z)\, \mu_s(T(\alpha, s)) \\ &\leq c(p-1)^{-1}|\hat{K}_1| + c \int_{T(\tilde{\alpha},t)} |\nabla u|^{p-1} f\, dH^1 + cm\, M_t(N_t \tilde{v}^{2m-1})(z)\, \mu_s(T(\alpha, s)) \\ &\leq c(p-1)^{-1}|\hat{K}_1| + c\mu_t(T(\alpha, t)) M_t f(z) \;+\; cm\, M_t(N_t \tilde{v}^{2m-1})(z)\, \mu_s(T(\alpha, s)) \end{aligned} \tag{5.55}$$

From (5.47) we deduce

$$|\hat{K}_1| \leq |\hat{K}_2| + \int_U \tilde{u}(P_1 + P_2) dA \leq |\hat{K}_2| + \mu_t(T(\alpha, t)) g(z). \tag{5.56}$$

Using (5.56) in (5.55), (3.11), and $c\mu(T(\tilde{\alpha}, t)) \geq \mu(T(\alpha, t))$, which follows from (5.2), (3.11), and Lemma 5.38 (a), we get after some arithmetic,

$$N_t f(z) \;\leq\; c \left[M_t f(z) + g(z) + \frac{|\hat{K}_2|}{\mu_t(T(\alpha, t))} \;+\; m\, M_t(N_t \tilde{v}^{2m-1})(z) \right] \tag{5.57}$$

where $c = c(p) \geq 1$ can be chosen independent of p on (3/2, 2).

To estimate $|\hat{K}_2|$ we use (1.10) and Lemmas 2.5, 2.6 as in (4.2), (4.3), to find that

$$\begin{aligned} |K_{22}| \;&\leq\; c\, m \int_{T(\tilde{\alpha}, s)} |\nabla u|^{p-1}\, \tilde{v}^{2m-1}\, dH^1 \;\;\leq\; c\, m \int_{T(\tilde{\alpha}, t)} |\nabla u|^{p-1}\, N_t \tilde{v}^{2m-1}\, dH^1 \\ &\leq\; cm M_t(N_t(\tilde{v}^{2m-1}))(z)\, \mu_t(T(\alpha, t))\,. \end{aligned} \tag{5.58}$$

K_{21} is estimated similarly only now we also use Lemma 5.38. We obtain

$$|K_{21}| \;\leq\; c\, m \sum_{j=1}^{2} \int_{\phi_j} |\nabla u|^{p-1}\, \tilde{v}^{2m-1}\, dH^1 \;\leq\; cm M_t(N_t(\tilde{v}^{2m-1}))(z)\, \mu_t(T(\alpha, t))\,. \tag{5.59}$$

Constants in (5.58), (5.59), can be chosen independent of p when $p \in (3/2, 2)$. Using $K_2 = K_{21} + K_{22}$ and (5.58), (5.59) in (5.57) we conclude the validity of Proposition 4.15. □

From Proposition 4.15 and our earlier work in section 4 we also get Theorem 1(b). □

6 Appendix

In this section we sketch proofs of Lemma 2.2 when $p > 2$ and Lemma 2.8. We begin with Lemma 2.2.

6.1 Proof of Lemma 2.2 when $p > 2$.

Proof: Fix $p > 2$ and let $\Omega, \tilde{u}, r, w$ be as in Lemma 2.2. The proof of Hölder continuity in Lemma 2.2 for $p > 2$ and some $\alpha > 0$ is a consequence of Morrey's lemma for Sobolev functions or classical theory for partial differential equations. Classical theory shows in fact that $\min_{p \in (2,3)} \alpha(p) > 0$. Also using Lemma 2.5 one can argue that it suffices to prove Hölder continuity of $\tilde{u}$ for some $\alpha > p - 2$ when one point lies in $B(w, r) \cap \partial\Omega$. Finally using an iterative argument as in Lemma 5.22 one can further reduce the proof of Lemma 2.2 when $p > 2$ to an inequality similar to (5.23). Thus if $\rho_k = 2^{-k} r$ and $B_k = B(w, \rho_k) \cap \Omega$, then it suffices to show for $k = 1, 2, \ldots$ that if

$$\lambda_k \rho_k^{(p-2)/(p-1)} = \max_{B_k} \tilde{u}, k = 1, 2, \ldots, \text{ then there exists } \eta > 0 \text{ for which } \lambda_{k+1} \leq (1 - \eta)\lambda_k. \tag{6.1}$$

To this end one defines as below (5.23)

$$g(w) = \lambda_k |w - \tilde{w}_4|^{(p-2)/(p-1)}, \; w \in U_k,$$

$$q(w) = g(w) - \tilde{u}(w), w \in U_k,$$

$$V_k = B_k \cap \partial B(\tilde{w}_4, 2^{-1/2}\rho_k\,),$$

for $k = 1, 2, \ldots$ To estimate q on V_k as in Lemma 5.22 one considers two cases. First if $w' \in \partial V_k, 0 < \delta < 1/2$, and

$$d(w', \partial\Omega) \leq \delta\rho_k, \tag{6.2}$$

then from the above remarks we deduce that

$$u(w') \leq c\delta^\alpha \lambda_k \rho_k^{(p-2)/(p-1)} \leq \lambda_k(1-\epsilon)|w' - w|^{(p-2)/(p-1)} \tag{6.3}$$

where $0 < \epsilon \leq 1/4$ provided δ is chosen small enough and $w' \in \partial V_k \cap \{\bar{w} : d(\bar{w}, \partial\Omega) \leq \delta\rho_k\}$. Moreover arguing as in (5.27)-(5.32) one deduces that (6.3) is valid for some $\epsilon = \epsilon(p) > 0$, whenever $w' \in \partial V_k$. Using the maximum principle for p harmonic functions we deduce first that (6.3) holds in V_k and thereupon from (6.3) that (6.1) is valid. This concludes our sketch of the proof of Lemma 2.2 when $p > 2$.

6.2 Proof of Lemma 2.8

To outline the proof of Lemma 2.8 we first indicate some results from [LNP11]. For this purpose let f be the Riemann mapping function from $\mathcal{H} = \{z = x_1 + ix_2 : x_2 > 0\}$ onto $\Omega(t)$ with $f(i) = 0$ and$f(a) = z_1$ where $a = is$ for some $0 < s < 1$. Then f extends continuously to $\mathcal{H}$, since $\partial\Omega(t)$ is a Jordan curve. If $b = b_1 + ib_2 \in \mathcal{H}$, let $I(b) = \{x \in \mathbb{R} : b_1 - b_2 \leq x \leq b_1 + b_2\}$. Next we state lemma 4.7 in [LNP11] using a slightly different notation.

Lemma 6.4 *If $b \in \mathcal{H}$, then there exists a compact set $K = K(b) \subset I(b)$ with the following properties. If $L \subset I(b)$ is an interval with $H^1(L) \geq b_2/100$, then $H^1(L \cap K) \geq b_2/1000$ and if $x \in L \cap K$, then*

(*a*) $\displaystyle\int_0^{b_2} |f'(x+iy)|dy \leq \hat{c}_1 d(f(b), \partial\Omega(t))$ *for some absolute* $\hat{c}_1$,

(*b*) *If* $0 < \delta < 10^{-100}$, *and* $\delta_* =$ *exp [-* $\hat{c}_1^2/\delta$ *], then* $\displaystyle\int_0^{\delta_* b_2} |f'(x+iy)|dy \leq \hat{c}_1 d(f(b), \partial\Omega(t))$,

(*c*) *If* $\{\tau_1, \ldots, \tau_m\}$ *is a set of points in* $I(b)$, *then there exists* $\tau_{m+1} \in K \cap L$ *with*

$|f(\tau_{m+1}) - f(\tau_j)| \geq \frac{d(f(b),\partial\Omega(t))}{10^{10}m^2}, 1 \leq j \leq m.$

To outline the proof of Lemmas 2.7 and Lemma 2.8 we first apply Lemma 6.4 with $b = a = is$, to deduce for given $\delta > 0$, sufficiently small (depending only on p), that there exists $x_i, 1 \leq i \leq 5$, with $-s < x_1 < -4s/5$, $-\frac{3}{5}s < x_4 < -\frac{2}{5}s$, $-\frac{1}{5}s < x_3 < \frac{1}{5}s$, $\frac{2}{5}s < x_5 < \frac{3}{5}s$, $\frac{4}{5}s < x_2 < s$, satisfying Lemma 6.4 $(a), (b)$ with x replaced by $x_j, 1 \le j \le 5$. Moreover (c) holds with $m = 5$ and τ_j replaced by $x_j, 1 \le j \le 5$. Let $z_{j+1} = f(x_i), j = 1, 2$, and let ξ consist of the horizontal line segment from $x_1 + is$ to $x_2 + is$, together with the vertical line segments from x_j to $x_j + is$, for $j = 1, 2$. Then $\lambda = f(\xi)$. Also $\tau = f(\eta)$ where $\eta = \sum_{k=1} \eta_k$. If $a_k = t_k + is_k, k = 0, 1, \ldots,$ then η_k joins a_{k-1} to a_k and consists of a horizontal line segment followed by a vertical line segment pointing down. Here $s_0 = s, t_0 = 0, t_1 = x_3$ and $s_k = \delta^* s_{k-1}$. $\{t_k\}$ is chosen inductively using Lemma 6.4 (b) so that $t_k \in K(t_{k-1} + is_{k-1})$ with

$$|t_k - t_{k-1}| \le s_{k-1}/4 \text{ and } \int_0^{s_k} |f'(t_k + it)|dt \le \delta d(f(a_{k-1}, \partial\Omega(t)). \tag{6.5}$$

From (6.5) and the choice of $\{s_k\}$ one can show that $\lim_{t \to 1} \eta(t) = x_3'$ and $|x_3' - x_3| \le c\delta_*$ where c is an absolute constant. Then $f(x_3') = z_4$. This completes our outline of the construction of λ, τ in Lemma 2.7.

The construction of β is similar. we write $\beta = f(\theta)$ where $\theta = \theta_0 \cup \theta_4 \cup \theta_5$. Let θ_0 be the horizontal line segment from $x_4 + is/2$ to $x_5 + is/2$. Next let $s_0' = s/2$, $t_{4,0}' = x_4$, $t_{5,0}' = x_5$, and $s_k' = \delta_* s_{k-1}'$, for $k = 1, 2, \ldots$ Then $\theta_j = \sum_{k=1}^{\infty} \theta_{j,k}, j = 4, 5$, where $\theta_{j,k}$ consists of a downward pointing vertical segment followed by a horizontal segment joining $a_{j,k-1}' = t_{j,k-1}' + is_{k-1}'$ to $a_{j,k}'$ for $j = 4, 5$, and $k = 1, 2, \ldots$ Using Lemma 6.4 one can choose $t_{j,k}'$ so that (6.5) is valid with $\{t_k\}, \{s_k\}$ replaced by $\{t_{j,k}'\}, \{s_{j,k}'\}$ for $j = 4, 5$. Also, $\lim_{t \to 1} \theta_j(t) = x_j'$ where $|x_j' - x_j| \le c\delta_*$ and $|f(x_j') - f(x_j)| \le c\delta d(f(a), \partial\Omega(t))$. Put $z_{j+1} = f(x_j')$ for $j = 4, 5$.

Finally to prove Lemma 2.8 (γ') note from Lemma 2.7 (β), Harnack's inequality, and Lemma 2.8 (β') that it suffices to consider the case when $\beta(s) \in f(\theta_{j,k})$ for $j = 4, 5$, and $k \ge 1$. Suppose for example $j = 4$ and $f(b) = \beta(s)$ with $b = b_1 + ib_2$. Then from our construction the arc of θ_5 from b to x_4' is contained in the rectangle, $\{(y_1, y_2) : 0 \le y_2 \le b_2, |b_1 - y_1| \le b_2/2\}$. Using Lemma 6.4 it follows that there exists $b_1', b_2', b_1' < b_1 < b_1'' \in K(b)$ with

$$\tfrac{3}{4} b_2 < |\bar{b}_1 - b_1| < b_2 \text{ wnenever } \bar{b}_1 \in \{b_1', b_1''\}.$$

Moreover,

$$\min\{|f(b_1') - f(b_1'')|, |f(b_1'') - f(b_1)|, |f(b_1') - f(b_1)|\} \ge c^{-1} d(\beta(s), \partial\Omega(t))$$

for some absolute $c \ge 1$. Let $Q(b) \subset \mathcal{H}$ be the rectangle whose sides in $\mathcal{H}$ are a horizontal side from b' to b'' and vertical sides joining b', b'' to b_1', b_1'', respectively. Then $Q(b)$ has the same properties as $Q(a)$ in [LNP11]. Thus we can apply the argument in section 5.2 of this paper to conclude that $u \le cu(\beta(s))$ in $f(Q(b)) \supset \beta(0, s]$. Hence Lemma 2.8 (γ') is valid when $s \le s_0$. The proof for $s > s_0$ is similar. This completes our sketch of the proof of Lemma 2.8.

References

[BL05] B. Bennewitz and J. Lewis, *On the dimension of p-harmonic measure,* Ann. Acad. Sci. Fenn. Math., 30 (2005), no. 2, 459-505.

[EG92] L. Evans and R. Gariepy, *Measure theory and fine properties of functions,* Studies in Advanced Mathematics, CRC Press, 1992.

[FKS82] E. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions to degenerate elliptic equations,* Comm. Partial Differential Equations, **7** (1982), no. 1, 77 - 116.

[FJK82] E. Fabes, D. Jerison, and C. Kenig, *The Wiener test for degenerate elliptic equations,* Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 151-182.

[FJK83] E. Fabes, D. Jerison, and C. Kenig, *Boundary behavior of solutions to degenerate elliptic equations.* Conference on harmonic analysis in honor of Antonio Zygmund, Vol I, II Chicago, Ill, 1981, 577-589, Wadsworth Math. Ser, Wadsworth Belmont CA, 1983.

[GM05] J. Garnett and D. Marshall, *Harmonic Measure*, Cambridge University Press, 2005.

[HK07] H. Hedenmalm and I. Kayamov, *On the Makarov law of the iterated logarithm,* Proc. Amer. Math. Soc. **135** (2007), no. 7, 2235-2248.

[HKM93] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, 1993.

[JW88] P. Jones and T. Wolff, *Hausdorff dimension of harmonic measure in the plane*, Acta Math. 161 (1988), 131-144.

[L06] J. Lewis, *Note on p harmonic measure,* Computational Methods in Function Theory **6** (2006), No.1, 109-144.

[LNP11] J. Lewis, K. Nyström, P. Poggi Corradini, *p harmonic measure in simply connected domains* Ann. Inst. Fourier Grenoble **61**, 2 (2011), 689-715.

[LNV] J. Lewis, K. Nyström, A. Vogel, *p harmonic measure in space,* to appear in JEMS.

[M85] N. Makarov, *Distortion of boundary sets under conformal mapping,* Proc. London Math. Soc. **51** (1985), 369-384.

[M90] N. Makarov, *Probability methods in conformal mappings,* Leningrad Math. J **1** (1990), 1-56.

[Ma95] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces,* Cambridge University Press, 1995.

[P92] C. Pommerenke, *Boundary behaviour of conformal maps,* Grundlehren der mathematischen Wissenschaften **299**, Springer - Verlag, 1992.

[W93] T. Wolff, *Plane harmonic measures live on sets of finite linear length*, Ark. Mat. **31** (1993), no. 1, 137-172.