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Metric properties in the mean of polynomials on compact isotropy irreducible homogeneous spaces

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Abstract

Let $M = G/H$ be a compact connected isotropy irreducible Riemannian homogeneous manifold, where G is a Lie group (may be, disconnected) acting on M by isometries. This class includes all compact irreducible Riemannian symmetric spaces and, for example, the tori $\mathbb{R}^n/\mathbb{Z}^n$ with the natural action on itself extended by the finite group generated by all transpositions of coordinates and inversions in circle factors. We say that u is a polynomial on M if it belongs to some G-invariant finite dimensional subspace $\mathcal{E}$ of $L^2(M)$. We compute or estimate from above the averages over the unit sphere $\mathcal{S}$ in $\mathcal{E}$ for some metric quantities such as Hausdorff measures of level set and norms in $L^p(M)$, $1 \le p \le \infty$, where M is equipped with the invariant probability measure. For example, the averages over $\mathcal{S}$ of $\|u\|_{L^p(M)}$, $p \ge 2$, are less than $\sqrt{\frac{p+1}{e}}$ independently of M and $\mathcal{E}$.

Contents

Introduction

Let M be a compact connected Riemannian manifold, G be a compact Lie group acting on M transitively by isometries, and H be the stable subgroup of a base point $o \in M$. We assume M isotropy irreducible. This means that the natural action of H in T_oM has no proper invariant subspaces. This class of homogeneous spaces is rather wide — it includes all irreducible Riemannian symmetric spaces, in particular, real spheres, Grassman manifolds, and simple

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Lie groups. The mentioned spaces are strongly isotropy irreducible, i.e., the connected component of H is irreducible in T_oM. A torus T considered as a homogeneous space of the semidirect product of T and a finite group F of its isometrical automorphisms is not strongly isotropy irreducible but it can be isotropy irreducible (this happens if and only if F is irreducible on T_oM). The circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is also contained in this class.

We say that a function u on M is a polynomial if the linear span of its translates $u \circ g$, $g \in G$, is finite dimensional. The polynomials are real analytic functions on M. Let $\mathcal{E}$ be a finite dimensional G-invariant linear subspace of $L^2(M)$ and $\mathcal{S}$ be the unit sphere in it. (In the notation $L^2(M)$, the probability invariant measure on M is assumed.) In this paper, we compute or estimate the averages over $\mathcal{S}$ of some metric quantities, such as the Hausdorff measures of level sets and their intersections or norms in $L^p(M)$ for polynomials $u \in \mathcal{S}$.

The strongly isotropy irreducible homogeneous spaces were classified first by O.V. Manturov ([20]–[22]) in 1961, independently by J.A. Wolf ([37]) in 1968, and by M. Krämer ([17]) in 1975. Their structure was clarified in papers by M. Wang and W. Ziller ([34]), E. Heintze and W. Ziller ([13]). This class of homogeneous spaces is closely connected with the symmetric spaces.

Due to Schur's lemma, M admits the unique up to a scaling factor G-invariant Riemannian metric. It follows from the uniqueness that it is a quotient of some bi-invariant metric on G. We fix these metrics and denote by Δ and $\tilde{\Delta}$ the corresponding Laplace–Beltrami operators on M and G, respectively.

Let $\mathfrak{g}$ be the Lie algebra of G realized by right invariant vector fields on G. The Lie algebra of their projections onto M is its homomorphic image whose kernel is trivial if the action is virtually effective. We shall assume this. Thus, we may identify these Lie algebras. If $\xi_1, \dots, \xi_l$ is an orthonormal base in $\mathfrak{g}$, then $\tilde{\Delta} = \xi_1^2 + \dots + \xi_l^2$, where $l = \dim G$, and

$$\Delta = \xi_1^2 + \dots + \xi_l^2, \tag{1}$$

where ξ_j denotes a vector field on G as well as its projection onto M. Hence, any G-invariant finite dimensional linear subspace $\mathcal{E}$ of $L^2(M)$ is Δ-invariant.

Throughout the paper,

$$m = \dim M.$$

For a real function u on M and $t \in \mathbb{R}$, set

$$\begin{aligned} L_u^t &= \{p \in M : u(p) = t\}, \\ U_u^t &= \{p \in M : u(p) \geq t\}. \end{aligned} \tag{2}$$

If u is an eigenfunction of Δ, then L_u^0 is called *a nodal set*; we shall also use the notation N_u for it. The Hausdorff measure of dimension k is denoted as $\mathfrak{h}^k$. We assume t fixed and consider Hausdorff measures of these sets as functions of u, and, for example, $\mathfrak{h}^m(U_{u_1}^{t_1} \cap \dots \cap U_{u_k}^{t_k})$ as functions of $u_1, \dots, u_k$. Fixing a probability measure in $\mathcal{E}$ and $t \in \mathbb{R}$, we get random variables $\mathfrak{h}^{m-1}(L_u^t)$, $\mathfrak{h}^m(U_u^t)$, etc.. Their distributions contain essential information on the polynomials. To the best of my knowledge, investigations in this direction were initiated by papers [4] by Bloch and Polya, [18], [19] by Littlewood and Offord. They considered the number of real zeroes of algebraic equations with various types of random coefficients. There is a lot of papers in this area now; we describe briefly only the results which are close to this article.

For real zeroes of random polynomials of one variable M. Kac in [15] proved an exact integral formula for the expectation and found its asymptotic. Edelman and Kostlan in the paper [8] noted that the expectation may be treated as the length of some curve in a sphere due to a Crofton type formula in spheres. They used this approach in some other situations.

For the Laplace-Beltrami eigenfunctions on compact manifolds, Berard found the asymptotic of expectations of $\mathfrak{h}^{m-1}(N_u)$, where u runs over the linear span of eigenfunctions corresponding to the eigenvalues of $-\Delta$ which are less than λ, as $\lambda \to \infty$ (see [3] for more details).

The case of $\mathbb{T}$ is classical. We mention only papers [29], [5], [12]. For the trigonometric polynomials

$$u = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt),$$

where a_k, b_k are Gaussian standard (i.e., with zero mean and variance 1) random coefficients, Qualls ([29]) found the expectations E_n of the number of zeroes $Z_n(u) = \mathfrak{h}^0(N_u)$:

$$E_n = 2\sqrt{\frac{1}{n}\sum\nolimits_{k=1}^{n} k^2} \sim \frac{2}{\sqrt{3}} n. \tag{3}$$

Bogomolny, Bohigas and Leboeuf conjectured in [5] that the variance of Z_n is cn for some $c > 0$. In [12], Granville and Wigman gave an affirmative answer, moreover, they proved that $\frac{Z_n(u)-E_n}{\sqrt{cn}}$ converges weakly to the standard Gaussian distribution, where c is an absolute constant which is equal to the value of some complicated explicitly written definite integral.

In [27], Oravecz, Rudnick, and Wigman considered the standard tori $\mathbb{R}^m/\mathbb{Z}^m$, a suitably normalized Gaussian measure in the space $\mathcal{E}_\lambda$ of λ-eigenfunctions, and the Leray measure of a nodal set

$$\mathfrak{l}(N_u) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathfrak{h}^m \left(U_u^{-\varepsilon} \setminus U_u^{\varepsilon} \right). \tag{4}$$

Note that the space $\mathcal{E}_\lambda$ of all λ-eigenfunctions on $\mathbb{R}^m/\mathbb{Z}^m$ is always invariant with respect to the permutations of the coordinates and changes of their signs, in other words, it is an invariant subspace of the semidirect product G of the torus $\mathbb{R}^n/\mathbb{Z}^n$ and the finite irreducible group BC_n, which is described above. In fact, the results of [27] were obtained for G-invariant subspaces of $\mathcal{E}_\lambda$. They calculated the expectation of $\mathfrak{l}(N_u)$, which appears to be equal to $\frac{1}{\sqrt{2\pi}}$ independently of m and λ, and proved for $m = 2$ and $m \geq 5$ that

$$\operatorname{var} \mathfrak{l}(N_u) \sim \frac{1}{4\pi \dim \mathcal{E}_\lambda}$$

as $\lambda \to \infty$. In [30], Rudnick and Wigman proved that the expectation of $\mathfrak{h}^{m-1}(N_u)$ is equivalent to $C\sqrt{\lambda}$ and

$$\operatorname{var}\left(\lambda^{-\frac{1}{2}}\mathfrak{h}^{m-1}(N_u)\right) = O\left(\lambda^{-\frac{1}{2}}\right)$$

for the Gaussian distributions of $u \in \mathcal{E}_\lambda$ on the tori $\mathbb{R}^m/\mathbb{Z}^m$, $m \geq 2$, assuming $\dim \mathcal{E}_\lambda \to \infty$.

We denote by S^m the unit sphere in $\mathbb{R}^{m+1}$. The nth eigenspace $\mathcal{E}_{\lambda_n} = \mathcal{H}^m_n$ corresponds to the eigenvalue $\lambda_n = n(n+m-1)$ and consists of traces of harmonic homogeneous of degree n polynomials on S^m. This case was considered in the papers [26], [11], [35], [36], [23], [24], where u was subject either to the Gaussian distribution in $\mathcal{E}_\lambda$ or to the uniform one in $\mathcal{S}$.

Remark 1. Both distributions mentioned above are rotation invariant. Since $\mathfrak{h}^{m-1}(N_u)$ and $\mathfrak{h}^m\left(U^0_u\right)$ are homogeneous of degree 0 on u, the resulting distributions of $\mathfrak{h}^{m-1}(N_u)$ and $\mathfrak{h}^m\left(U^0_u\right)$ are identical. In particular, the expectations and variances in the Gaussian and the uniform cases are equal. For the level sets and for the Leray measures, this is not true but the results for any of the two types of distributions can be deduced from the results on the other one (for instant, we compute the expectations for radial measures in Proposition 1). In the papers cited above, except for the first and the second, the authors work with the Gaussian distribution.

Neuheisel proved that the normalized Hausdorff and Leray measures on the nodal sets almost surely converges weakly to the probability invariant measure as $\lambda \to \infty$ (for the precise statement, see [26]). He found the expectations of $\mathfrak{h}^{m-1}(N_u)$ and $\mathfrak{l}(N_u)$ and estimated their variances as $O\left(n^{-\frac{(m-1)^2}{3m+1}}\right)$ and $O\left(n^{-\frac{m-1}{2}}\right)$, $n \to \infty$, respectively. In [35], Wigman refined this: he proved that $\operatorname{var}(\mathfrak{l}(N_u)) = \frac{c}{N}$, where c depends only on m, $N = \dim \mathcal{H}^n_m$, and $\operatorname{var}\left(\mathfrak{h}^{m-1}(N_u)\right) = O\left(\frac{\lambda}{\sqrt{N}}\right)$. For S^2 he proved that $\operatorname{var}\left(\mathfrak{h}^1(N_u)\right) = c\ln n + O(1)$ in [36] (there was an error in calculation of c which had been corrected later). Marinucci and Wigman studied the random area $\mathfrak{h}^2(S^2 \setminus U^t_u)$. In [23], they show that for a fixed $t \in \mathbb{R}$

$$\operatorname{var}\left(\mathfrak{h}^2(S^2 \setminus U^t_u)\right) = \frac{t^2\phi(t)}{n} + O\left(\frac{\log n}{n^2}\right)$$

as $n \to \infty$, where ϕ is the standard Gaussian distribution function. For $t = 0$, it is proved in [24] that $\operatorname{var}\left(U^0_u\right) = \frac{C}{n^2}(1+o(1))$, where n is even,

$$C = 8\pi \int_0^\infty (\arcsin J_0(\tau) - J_0(\tau))\tau \, d\tau,$$

J_0 is the Bessel function of the first kind and zero index; the integral converges conditionally. (Actually, the authors considered the difference between measures of sets of positivity and negativity of u (defect); their result differs by the factor 4 from that of above.)

In [11], estimates for some metric quantities of the nodal sets in spheres (in particular, the sharp upper bound for lengths of the nodal sets of spherical harmonics on S^2) and the expectations of Hausdorff measures of their intersections for the uniform distributions were found, including the mean number of common zeroes of m independent random eigenfunctions on S^m.

Surveys [14] and [38] describe the current state of this area. They contain many known facts as well as methods and open problems.

In this paper, we consider an arbitrary compact connected isotropy irreducible homogeneous manifold M and expectations of some metric quantities for polynomials. Throughout the text it is assumed that

(E) $\mathcal{E}$ is a finite dimensional G-invariant linear subspace of $L^2(M)$ such that $\mathbf{1} \perp \mathcal{E}$, where $\mathbf{1}(p) = 1$ for all $p \in M$, and $\mathcal{S}$ is the unit sphere in $\mathcal{E}$.

Unless the contrary is not explicitly stated, the expectations relate to the uniform distribution in $\mathcal{S}$ (i.e., to the $\mathrm{SO}(\mathcal{E})$-invariant probability measure).

We formulate below some useful observations.

1. Let N be a Riemannian G-manifold and $\iota : M \to N$ be an equivariant nonconstant smooth map. Then ι is a local diffeomorphism onto its image since M is isotropy irreducible (hence $\ker d_p\iota = 0$ for all $p \in M$). Therefore, ι is a finite covering.

2. The restriction of the Riemannian metric in N onto $\iota(M)$ is proportional to the Riemannian metric in M since the invariant Riemannian metric in M is unique up to a scaling factor.

3. Let s be the coefficient of proportionality. If γ is a path in M of length l, then the path $\iota \circ \gamma$ has length sl. It follows that the inner distances locally are also multiplied by s (i.e., ι is a local metric homothety) and the same is true for the Hausdorff measure $\mathfrak{h}^k$, with the coefficient s^k.

4. There is a natural equivariant immersion $\iota : M \to \mathcal{S}$. For $p \in M$, let $\phi_p \in \mathcal{E}$ be such that $u(p) = \langle \phi_p, u \rangle$ and set $\iota(p) = \frac{\phi_p}{|\phi_p|}$.

5. For the immersion ι we have $s = \sqrt{\frac{|\operatorname{Tr} \Delta|}{\dim M \dim \mathcal{E}}}$. If $\mathcal{E}$ is an eigenspace of Δ, then $s = \sqrt{\frac{\lambda}{m}}$, where λ is the eigenvalue of $-\Delta$ in $\mathcal{E}$.

6. Using a Crofton type formula of Integral Geometry in spheres, one can compute averages (expectations) of $\mathfrak{h}^{k-1}(L_u^t \cap X))$, $\mathfrak{h}^k(U_u^t \cap X))$ for suitable subsets X of M, and some other functions of u, with respect to the probability invariant measure on the sphere $\mathcal{S}$.

All of them were already used (may be, except for the third and the fifth; the coefficient s appeared implicitly in several papers, for example, in [26]).

This scheme was realized in the paper [11] for $M = S^m = \mathrm{SO}(m+1)/\mathrm{SO}(m)$, the unit sphere in $\mathbb{R}^{m+1}$ and the spaces of spherical harmonics on them (which can be characterized as eigenspaces of Δ or irreducible components of $L^2(A^m)$). Clearly, S^m is isotropy irreducible, moreover, the stable subgroup $H = \mathrm{SO}(m)$ acts transitively on the unit sphere in $T_o S^m$. It turns out that the assumption that $\mathcal{E}$ is an eigenspace of Δ is not essential for the computation of expectations, the results depends (at most) on the coefficient s. (However, for variances and upper or lower bounds this is not true usually.)

We compute the expectations of Hausdorff measures of level sets and their intersections (Theorem 2). In particular, for the sets L_u^t and U_u^t we get

$$\mathsf{M}(\mathfrak{h}^{m-1}(L_u^{ct})) = \varpi \frac{\varpi_{m-1}}{\varpi_m} s \left(1 - t^2\right)^{\frac{d-1}{2}}, \tag{5}$$

$$\mathsf{M}(\mathfrak{h}^{m}(U_u^{ct})) = \varpi \frac{\varpi_{d-1}}{\varpi_d} \int_t^1 \left(1 - \tau^2\right)^{\frac{d}{2}-1} d\tau, \tag{6}$$

where $m = \dim M$, $\mathcal{E}, s$ are as above, $c^2 = d + 1 = \dim \mathcal{E}$, ϖ, ϖ_k are volumes of M, S^k, respectively, $t \in \mathbb{R}$ is fixed, and

$$\mathsf{M}(f) = \int_{\mathcal{S}} f(u)\, du$$

for a function f on $\mathcal{S}$ (du corresponds to the invariant probability measure on $\mathcal{S}$).

The formulas above hold for all isotropy irreducible homogeneous spaces, in particular, for spheres $\mathrm{SO}(m+1)/\mathrm{SO}(m)$ and for the standard tori $T_m = \mathbb{R}^m/\mathbb{Z}^m$ considered as a homogeneous space of T_m extended by the finite group BC_m of all compositions of permutations and componentwise inversions in T_m. This is equivalent to the assumption that the T_m-invariant space $\mathcal{E}$ (equivalently, its spectrum) is BC_m-invariant; in fact, it was assumed in the paper [27] (clearly, the spectrum of any eigenspace on T_m is BC_m-invariant). For $M = G = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, (5) with $t = 0$ is equivalent to Qualls's formula (3) since $\varpi_0 = 2$, $\varpi = \varpi_1 = 2\pi$, and $s = \sqrt{\frac{1}{n}\sum_{k=1}^n k^2}$ according to Lemma 1. Formula (5) can be applied to spaces $\mathcal{E}$ with arbitrary spectra, e.g., to the space of trigonometric polynomials of the type

$$\sum_{i=1}^n (a_{k_i} \cos k_i t + b_{k_i} \sin k_i t),$$

where $0 < k_1 < \cdots < k_n$ are integer. By (5), the expectation equals $2s$, where $s = \sqrt{\frac{1}{n}\sum_{k=1}^n k_i^2}$ by Lemma 1.

A similar formulas can be derived for the intersections of sets L_u^t and U_u^t, for the natural extension of the Leray measure onto all level sets (see (15) for the definition), and for quantities of the type $\int_M f(u(p))\, dp$. The results are stated in Theorem 2. For $f(t) = |t|^a$ we derive the explicit formula (45) (Theorem 3), which holds for all $a > -1$; its right-hand side is independent of M. If $a = 1$, then we get the expectation of L^1-norm:

$$\mathsf{M}\left(\|u\|_1\right) = \sqrt{\frac{d+1}{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}$$

where $d = \dim \mathcal{S} = \dim \mathcal{E} - 1$. The right-hand side decreases on d. For $d = 1$ it is equal to $\frac{2\sqrt{2}}{\pi} \approx 0.9$ and has the limit $\sqrt{\frac{2}{\pi}} \approx 0.8$ at infinity. For any $a > -1$ we have

$$E(a,d) := \int_{\mathcal{S}} \left(\int_M |u(p)|^a \, dp \right) du \to \; 2^{\frac{a}{2}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}},$$

as $d \to \infty$, where $E(a,d)$ increases with d if $a > 2$ or $a \in (-1,0)$ and decreases if $a \in (0,2)$ (for $a = 0$ and $a = 2$ the equality holds). As $a \to -1$,

$$E(a,d) \sim \frac{A}{a+1},$$

where A depends only on d. The integral $\int_M |u(p)|^a \, dp$ may diverge for arbitrary small negative a, for example, if $M = \mathbb{T}$ and $u(t) = (\sin t)^{2k+1}$, but the averages are finite if $a > -1$.

The computation of $E(a,d)$ makes it possible to estimate from above the expectation of norms $\|u\|_p$ in $L^p(M)$, $1 \le p < \infty$ (Theorem 4). If $p \ge 2$, then

$$\mathsf{M}\left(\|u\|_p\right) < \sqrt{\frac{p+1}{e}}. \tag{7}$$

For $p \in [1,2)$ the same inequality holds if $\dim \mathcal{E}$ is sufficiently large. Note that the estimate is independent of $\mathcal{E}$ and M. Inequalities like $\|u\|_p \le C\sqrt{p}\,\|u\|_2$ are known for the trigonometric lacunary series. Perhaps, this means that the upper bound above cannot be improved essentially. Of course, the uniform estimate for the expectations does not yield a similar estimate for individual eigenfunctions. For example, if $M = S^2 \subset \mathbb{R}^3$, then for any $p > 2$ and spherical harmonics $\varphi_n(x,y,z) = c \operatorname{Re}(x+yi)^n$ of degree n such that $\|\varphi_n\|_2 = 1$ we have $\|\varphi_n\|_p \to \infty$ as $n \to \infty$ (see [32]). On the other hand, if $M = \mathbb{R}^2/\mathbb{Z}^2$, then $\|u\|_4 \le C\|u\|_2$ for all eigenfunctions u by a result of Zygmund ([39]). Similar problems for L^4 norms of elements of a random orthogonal base in the space $\mathcal{H}_n$ of spherical harmonics on S^2 of degree n were considered in [33]; in particular, this paper contains a sketch of the proof of a bound for the average of the functional $\sum_{j=1}^{2n+1} \|u_k\|_4^4$ of such a base.

Estimates for $\mathsf{M}\left(\|u\|_\infty\right)$ cannot be derived from the results on $\mathsf{M}\left(\|u\|_p\right)$ directly while for any fixed $u \in \mathcal{E}$ we have $\|u\|_p \to \|u\|_\infty$ as $p \to \infty$. Indeed, the upper bound (7) for $\mathsf{M}\left(\|u\|_p\right)$ is independent of $\dim \mathcal{E}$ but it may happen that $\mathsf{M}\left(\|u\|_\infty\right) \to \infty$ as $\dim \mathcal{E} \to \infty$, for example, this is true if $\mathcal{E}$ is contained in a subspace of $L^2(\mathbb{T})$ with a lacunary spectrum. There is the evident sharp upper bound $\sqrt{\dim \mathcal{E}}$ for $\|u\|_\infty$ (see (11)) which is attained on $u = \iota(q) \in \mathcal{S}$ for any $q \in M$. We get the estimate

$$\mathsf{M}\left(\|u\|_\infty\right) < K\sqrt{\ln s}, \tag{8}$$

where $K > 0$ is independent of $\mathcal{E}$ and s is sufficiently large. Similar upper bounds for the random variable $\|u\|_\infty$ appeared in various problems. For instant, the analogous inequalities of weak type for $\|u\|_\infty$ are contained in Kahane's book [16, Ch. 6]; in [26], Neuheisel proved for spherical harmonics on S^m that $\|u_n\|_\infty = O(\sqrt{\ln n})$ almost surely as $n \to \infty$, where $u_n \in \mathcal{S}_n \subseteq \mathcal{E}_n$ and $\mathcal{E}_n$ is the space of spherical harmonics of degree n. In these cases, the bounds $\sqrt{\ln s}$, $\sqrt{\ln \dim \mathcal{E}}$, and $\sqrt{\ln n}$ are equivalent. Furthermore, Lemma 1 implies the inequality

$$s \leq \sqrt{\frac{\|\Delta\|}{m}},$$

where $\|\Delta\|$ is the norm of Δ as an operator in $\mathcal{E}$ (since Δ is symmetric, $\|\Delta\|$ is equal to the maximal eigenvalue of $-\Delta$). Hence, s may be replaced with $\|\Delta\|$ in (8). The inequality (8) says nothing if $K^2 \ln s > \dim \mathcal{E}$. This happens, for example, for the spaces of trigonometrical polynomials with the spectrum $\{1, 2, \ldots, n, n!\}$ for sufficiently large n. Thus, (8) should be refined. Probably, the right-hand side of (8) has the sharp order of growth for strictly isotropy irreducible homogeneous spaces if $\mathcal{E}$ is an eigenspace.

1 Preparatory material

In what follows, we keep the notation of the introduction; $|\ |$ and $\langle\ ,\ \rangle$ denote the norm and inner product in Euclidean spaces, respectively, $\|\ \|_p$ is the norm in $L^p(M)$. We write $\int_M f(p)\,dp$, $\int_G f(g)\,dg$, etc., for integration over the invariant probability measures on M, G, and other homogeneous spaces of compact groups; also, these measures are assumed in the notation $L^2(M)$, $L^2(G)$, etc.. Thus, we consider on M two finite invariant measures: the probability one and $\mathfrak{h}^m$, where $m = \dim M$. Set

$$\varpi = \mathfrak{h}^m(M);$$

then $d\mathfrak{h}^m = \varpi\,dp$. Functions are assumed to be real-valued unless the contrary is not explicitly stated. The space $\mathcal{E}$ is always assumed to satisfy (E). Since we consider real functions and M is compact, (E) implies

$$\dim \mathcal{E} > 1.$$

Being finite dimensional, $\mathcal{E}$ consists of real analytic functions. For any $p \in M$ there exists the unique $\phi_p \in \mathcal{E}$ which realizes the evaluation functional at p:

$$\langle u, \phi_p \rangle = u(p)$$

for all $u \in \mathcal{E}$. Set

$$\phi(p, q) = \langle \phi_p, \phi_q \rangle, \quad p, q \in M.$$

Then $\phi(p, q) = \phi_p(q) = \phi(q, p)$; moreover, $\phi(x, y)$ is the reproducing kernel for $\mathcal{E}$ (i.e., the mapping $u(x) \to \int_M \phi(x, y)u(y)\,dy$ is the orthogonal projection onto

$\mathcal{E}$ in $L^2(M)$). Due to the homogeneity of M, $|\phi_p| = |\phi_o| \neq 0$ for all $p \in M$. Since the trace of the projection is equal to $\int_M \phi(x,x)\,dx$, we have

$$\phi(o,o) = |\phi_o|^2 = \dim \mathcal{E}.$$

Let $\mathcal{S}$ be the unit sphere in $\mathcal{E}$. There is a natural equivariant mapping $\iota : M \to \mathcal{S}$, which is an embedding if $\mathcal{E}$ separates points of M:

$$\iota(p) = \frac{\phi_p}{|\phi_p|}, \tag{9}$$

where the denominator is independent of p. Note that ι is a local diffeomorphism since M is isotropy irreducible; hence, it is a finite covering. Moreover, the restriction of the Euclidean metric in $\mathcal{E}$ onto $\iota(M)$ coincides with the Riemannian metric on M up to the scaling factor

$$s = s(\mathcal{E}) = \frac{|d_p\iota(v)|}{|v|}, \tag{10}$$

where the right-hand side is independent of $p \in M$ and $v \in T_pM \setminus \{0\}$. Thus, the length of curves and (locally) the inner distance in $\iota(M)$ defined by the Riemannian metric in $\mathcal{S}$ (equivalently, by the Euclidean metric in $\mathcal{E}$) are proportional to that of M. For short, we shall denote

$$\iota(p) = \bar{p}, \qquad \iota(M) = \bar{M}.$$

Thus $\bar{o} = \iota(o)$ is the base point of $\mathcal{S}$. The following notation will be used throughout the paper:

$$\begin{aligned} d &= \dim \mathcal{E} - 1 = \dim \mathcal{S}, \\ c &= |\phi_o| = \sqrt{\phi(o,o)} = \sqrt{d+1}, \\ \mathcal{S}_u^t &= \{x \in \mathcal{S} : \langle x, u\rangle = t\}, \\ \mathcal{U}_u^t &= \{x \in \mathcal{S} : \langle x, u\rangle \geq t\}, \end{aligned}$$

where $u \in \mathcal{E}$ and $t \in \mathbb{R}$. Clearly, for all $u \in \mathcal{S}$, $p \in M$

$$|u(p)| \leq c, \tag{11}$$

where the equality holds only for $u = \bar{p}$. If $t \in [-1,1]$, then we obviously have

$$\iota\left(L_u^{ct}\right) = \mathcal{S}_u^t \cap \bar{M}, \tag{12}$$

$$\iota\left(U_u^{ct}\right) = \mathcal{U}_u^t \cap \bar{M}. \tag{13}$$

A set which can be realized as a Lipschitz image of a bounded subset of $\mathbb{R}^k$ is called *k-rectifiable* (we consider only countable unions of compact sets). If ι is one-to-one on an r-rectifiable set $X \subseteq M$, then, due to (10),

$$\mathfrak{h}_{\mathcal{S}}^r(\iota(X)) = s^r \mathfrak{h}_M^r(X). \tag{14}$$

In this equality and in the sequel, the Hausdorff measures corresponds to the metrics in the related spaces. We shall drop the lower index usually. Let S^k be the unit sphere in $\mathbb{R}^{k+1}$. Set

$$\varpi_k = \mathfrak{h}^k(S^k) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)},$$
$$\varkappa_d(t) = \mathfrak{h}^d\left(\mathcal{U}_u^t\right) = \varpi_{d-1}\int_t^1 \left(1-\tau^2\right)^{\frac{d}{2}-1}\,d\tau,$$

where $-1 \le t \le 1$. Note that the second term is independent of $u \in \mathcal{S}$ and that $\varkappa_d(-1) = \varpi_d$, $\varkappa_d(0) = \frac{\varpi_d}{2}$. Also, we assume that $\varkappa(t) = \varpi_d$ if $t \le -1$ and $\varkappa(t) = 0$ for $t \ge 1$. The definition (4) may be extended onto all level sets:

$$\mathfrak{l}(L_u^t) = \limsup_{\varepsilon\to+0}\frac{1}{2\varepsilon}\mathfrak{h}^m\left(U_u^{t-\varepsilon}\setminus U_u^{t+\varepsilon}\right). \tag{15}$$

We assume that $\mathfrak{l}(L_u^t)$ is defined on $\mathbb{R}$ (as well as $\mathfrak{h}^k(L_u^t)$, $\mathfrak{h}^m\left(U_u^t\right)$), and so on); since $\mathfrak{h}^m\left(U_u^t\right)$) is piecewise real analytic (see the beginning of the next section), $\mathfrak{l}(L_u^t)$ is real analytic outside a finite subset of $\mathbb{R}$. If $t = 0$, then $\mathfrak{l}(L_u^t)$ is called the *Leray measure* of the nodal set $N_u = L_u^0$. Since $\mathfrak{h}^m\left(U_u^\tau\right)$ is non-increasing on τ,

$$\mathfrak{l}(L_u^t) = -\frac{d}{d\tau}\mathfrak{h}^m\left(U_u^\tau\right)\big|_{\tau=t}. \tag{16}$$

where the derivative exists almost everywhere. If t is regular, then

$$\mathfrak{h}^m(U_u^t) = \int_t^\infty \Big(\int_{L_u^\tau}\frac{d\mathfrak{h}^{m-1}(p)}{|\nabla u(p)|}\Big)\,d\tau$$

due to the coarea formula and almost everywhere on $[-c, c]$

$$\mathfrak{l}(L_u^t) = \int_{L_u^t}\frac{d\mathfrak{h}^{m-1}(p)}{|\nabla u(p)|}. \tag{17}$$

Let us fix an orthogonal decomposition of $\mathcal{E}$ into a sum of G-invariant subspaces:

$$\mathcal{E} = \sum_{j=1}^{l}\oplus\,\mathcal{E}^j, \tag{18}$$

where $\Delta u = -\lambda_j u$ for $u \in \mathcal{E}^j$. We do not assume that $\lambda_j \ne \lambda_k$ if $j \ne k$. It follows from (18) that

$$\phi_p = \sum_{j=1}^{l}\phi_p^j, \tag{19}$$

where $\phi_p^j \in \mathcal{E}^j$ represents the evaluation functional at $p \in M$ on $\mathcal{E}^j$, $j = 1, \dots, l$.

The following theorem is a simplified version of Theorem 3.2.48 in [9].

Theorem 1. *Let $A, B \subseteq S^d$ be compact, A be k-rectifiable, B be j-rectifiable, and φ, ψ be continuous functions on A, B, respectively. Set $r = k + j - d$. Suppose $r \geq 0$. Then*

$$\int_{\mathrm{O}(d+1)} \int_{A \cap gB} \varphi(x)\psi(g^{-1}x)\, d\mathfrak{h}^r(x)\, dg = K \int_A \varphi(x)\, d\mathfrak{h}^k(x) \int_B \psi(x)\, d\mathfrak{h}^j(x),$$

where $K = \dfrac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)^d \Gamma\left(\frac{r+1}{2}\right)} = \dfrac{\varpi_r}{\varpi_k \varpi_j}$. □

In particular, for $\varphi = \psi = 1$ we have

$$\int_{\mathrm{O}(d)} \mathfrak{h}^r(A \cap gB)\, dg = K \mathfrak{h}^k(A)\mathfrak{h}^j(B). \tag{20}$$

Let each of $\mathcal{E}_1, \dots, \mathcal{E}_l$ be as $\mathcal{E}$ above; for $i \in \{1, \dots, l\}$, we equip the notations of the objects related to $\mathcal{E}_i$ with the lower index i (for example, $\mathcal{E}_i^j, \phi_{i,a}^j, \lambda_{i,j}$). Further, we write $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$,

$$\begin{array}{rclrcl} \mathbf{E} & = & \mathcal{E}_1 \times \cdots \times \mathcal{E}_l, & \mathbf{S} & = & \mathcal{S}_1 \times \cdots \times \mathcal{S}_l, \\ \mathbf{u} & = & (u_1, \dots, u_l) \in \mathbf{E}, & d\mathbf{u} & = & du_1 \dots du_l, \\ L_{\mathbf{u}}^{\mathbf{t}} & = & L_{u_1}^{t_1} \cap \dots \cap L_{u_l}^{t_l}, & U_{\mathbf{u}}^{\mathbf{t}} & = & U_{u_1}^{t_1} \cap \dots \cap U_{u_l}^{t_l}, \end{array} \tag{21}$$

and so on. The mean value (expectation) of a function f on $\mathbf{S}$ or $\mathcal{S}$ is denoted as $\mathsf{M}(f)$:

$$\mathsf{M}(f) = \int_{\mathbf{S}} f(\mathbf{u})\, d\mathbf{u}, \quad \mathsf{M}(f) = \int_{\mathcal{S}} f(u)\, du.$$

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H respectively, realized as Lie algebras of vector fields on M (this implies that the action of G on M has finite kernel; we do not assume that it is effective), G be equipped with a bi-invariant Riemannian metric such that the projection $\eta : G \to G/H = M$ is a metric submersion. Then $d\eta : \mathfrak{g} \to T_o M$ is isometric on $\mathfrak{h}^\perp$ and vanishes on $\mathfrak{h}$ and Δ may be defined by (1).

In the following two lemmas, we perform some calculations of [11] in a more general setting. First, we find the coefficient $s = s(\mathcal{E})$ of the local metric homothety $\iota : M \to \bar{M}$ for $\mathcal{E}$ in (18) (the case of $M = S^m$ and one eigenspace was considered in [11]). Set

$$d_j = \dim \mathcal{E}^j - 1, \quad \alpha_j = \frac{d_j + 1}{d + 1} = \frac{\dim \mathcal{E}^j}{\dim \mathcal{E}} = \frac{c_j^2}{c^2},$$

$j = 1, \dots, l$. Thus, $\sum_{j=1}^l \alpha_j = 1$.

Lemma 1. *Let $s_j = s(\mathcal{E}^j)$, $j = 1, \dots, l$, and α_j be as above. Then*

$$s^2 = \alpha_1 s_1^2 + \dots \alpha_l s_l^2 = \frac{|\operatorname{Tr} \Delta|}{m \dim \mathcal{E}}, \tag{22}$$

where $\operatorname{Tr} \Delta$ *is the trace of* Δ *in* $\mathcal{E}$.

Proof. Since ι is equivariant, for all $\xi \in \mathfrak{g}$

$$d_o\iota(\xi(o)) = \frac{1}{c}\xi\phi_o. \tag{23}$$

We may choose the base in (1) such that $\xi_{m+1}, \ldots, \xi_k \in \mathfrak{h}$, where $k = \dim \mathfrak{g}$. Then remaining ξ_j are orthogonal to $\mathfrak{h}$. Since $\eta : G \to G/H = M$ is a metric submersion, we have $|\xi_i(o)| = 1$ for $i = 1, \ldots, m$; clearly, $\xi_i(o) = 0$ for $i = m+1, \ldots, k$. Using consequently (10), (23), (9), (1), and (19), we get

$$ms^2 = s^2 \sum_{i=1}^{k} |\xi_i(o)|^2 = \sum_{i=1}^{k} |d_o\iota(\xi_i(o))|^2 = \frac{1}{c^2}\sum_{i=1}^{k} |\xi_i\phi_o|^2 = -\frac{1}{c^2}\sum_{i=1}^{k} \left\langle \xi_i^2\phi_o, \phi_o \right\rangle$$
$$= -\frac{1}{c^2}\left\langle \Delta\phi_o, \phi_o \right\rangle = \frac{1}{c^2}\left\langle \sum_{j=1}^{l} \lambda_j\phi_o^j, \phi_o \right\rangle = \frac{1}{c^2}\sum_{j=1}^{l} \lambda_j|\phi_o^j|^2 = \sum_{j=1}^{l} \alpha_j\lambda_j.$$

If $l = 1$, then $s = \sqrt{\frac{\lambda_1}{m}}$; hence, $s_j = \sqrt{\frac{\lambda_j}{m}}$. This proves the first equality in (22); the second is true since $|\phi_o^j|^2 = \dim \mathcal{E}^j$. □

The second lemma is similar to Lemma 6 of the paper [11], where (24) was proved for $t = 0$.

Lemma 2. *Let $|t| \le 1$ and $X \subseteq M$ be $(r+1)$-rectifiable, where $r \le m-1$. Then*

$$\int_{\mathcal{S}} \mathfrak{h}^r(L_u^{ct} \cap X)\, du = \frac{\varpi_r}{\varpi_{r+1}} s\mathfrak{h}^{r+1}(X)\left(1 - t^2\right)^{\frac{d-1}{2}}, \tag{24}$$

$$\int_{\mathcal{S}} \mathfrak{h}^{r+1}(U_u^{ct} \cap X)\, du = \mathfrak{h}^{r+1}(X)\frac{\varkappa_d(t)}{\varpi_d}. \tag{25}$$

Proof. Since both sides of (24) are additive on X and ι is a finite covering, it is sufficient to prove (24) assuming that ι is injective on X. For each $t \in [-1, 1]$, the group $O(\mathcal{E})$ acts transitively on the family of spheres $\{\mathcal{S}_u^t\}_{u\in\mathcal{S}}$. Due to (12), we may apply Theorem 1 to $\mathcal{S}$ setting $A = \mathcal{S}_o^t$, $k = d-1$, $B = \iota(X)$, $j = r+1$ in (20). Since the Euclidean radius of $\mathcal{S}_o^t$ is equal to $\sqrt{1-t^2}$, we have

$$\mathfrak{h}^{d-1}\left(\mathcal{S}_o^t\right) = \varpi_{d-1}\left(1 - t^2\right)^{\frac{d-1}{2}}.$$

Using (12), (14), replacing integration over $\mathcal{S}$ with averaging over $\mathrm{O}(\mathcal{E})$, and applying (20), we get (24):

$$\int_{\mathcal{S}} \mathfrak{h}^r(L_u^{ct} \cap X)\, du = \frac{1}{s^r}\int_{\mathcal{S}} \mathfrak{h}^r(\iota(L_u^{ct} \cap X))\, du = \frac{1}{s^r}\int_{\mathcal{S}} \mathfrak{h}^r(\mathcal{S}_u^t \cap \iota(X))\, du$$
$$= \frac{1}{s^r}\int_{\mathrm{O}(\mathcal{E})} \mathfrak{h}^r(g\mathcal{S}_o^t \cap \iota(X))\, dg = \frac{1}{s^r}K\mathfrak{h}^{d-1}\left(\mathcal{S}_o^t\right)\mathfrak{h}^{r+1}(\iota(X))$$
$$= \frac{\varpi_r}{s^r\varpi_{r+1}}\left(1 - t^2\right)^{\frac{d-1}{2}}\mathfrak{h}^{r+1}(\iota(X)) = s\left(1 - t^2\right)^{\frac{d-1}{2}}\frac{\varpi_r}{\varpi_{r+1}}\mathfrak{h}^{r+1}(X).$$

To prove (25), set $A = \mathcal{U}_o^t$, $k = d$, $B = \iota(X)$, $j = r+1$ in (20). Then

$$\int_{\mathcal{S}} \mathfrak{h}^{r+1}(U_u^{ct} \cap X)\, du = \frac{1}{s^{r+1}} \int_{\mathrm{O}(\mathcal{E})} \mathfrak{h}^{r+1}(g\mathcal{U}_o^t \cap \iota(X))\, dg$$
$$= \frac{1}{s^{r+1}} \frac{\varpi_{r+1}}{\varpi_d \varpi_{r+1}} \varkappa_d(t) \mathfrak{h}^{r+1}(\iota(X)) = \frac{\varkappa_d(t)}{\varpi_d} \mathfrak{h}^{r+1}(X),$$

where we omit some steps since they are similar to that of above. □

2 Computation of expectations

We formulate below two Łojasiewicz's inequalities following [2] but in a weaker form (cf. Theorem 6.4, Remark 6.5, and Proposition 6.8 of [2]). Let $\mathrm{CV}_N(f)$ denote the set of critical values of a smooth function f on a manifold N (we drop the index if no confusion can occur). Suppose f real analytic in a domain $\mathcal{D} \subseteq \mathbb{R}^n$, $N_f = f^{-1}(0)$. Let Q be a compact subset of $\mathcal{D}$. Then there exist $\nu, \eta > 0$ such that

$$|f(q)| \geq \eta \, \mathrm{dist}(q, N_f)^\nu \tag{26}$$

for all $q \in Q$, where dist denotes the Euclidean distance. Furthermore, for any $x \in N_f$ there are its neighborhood U and $\eta > 0$, $\theta \in (0,1)$ such that for all $q \in U$

$$|\nabla f(q)| \geq \eta |f(q)|^\theta, \tag{27}$$

where ∇ stands for the Euclidean gradient.

Lemma 3. *For all $u \in \mathcal{S}$*

(i) *the set* $\mathrm{CV}(u)$ *is finite,*

(ii) $\mathfrak{h}^{m-1}(L_u^t)$ *and* $\mathfrak{l}(L_u^t)$ *are smooth on* $u(M) \setminus \mathrm{CV}(u)$ *as functions of* t*,*

(iii) *for any $t_0 \in \mathrm{CV}(u)$ there are $\theta \in (0,1)$ and $\eta > 0$ such that*

$$\mathfrak{l}(L_u^t) \leq \eta \mathfrak{h}^{m-1}\left(L_u^t\right) |t - t_0|^{-\theta}, \tag{28}$$

where $t \in u(M) \setminus \mathrm{CV}(u)$.

Proof. (i). The set of critical points of u is defined by the equation $|\nabla u(x)|^2 = 0$. Hence, it has a finite number of components being an analytic set in a compact manifold (moreover, ι maps this set onto a real algebraic set in $\mathcal{E}$ since $\bar{M}$ can be distinguished in $\mathcal{E}$ by G-invariant polynomials). On the other hand, the set $\mathrm{CV}(u)$ has zero Lebesgue measure in $\mathbb{R}$ by Sard's theorem since u is sufficiently smooth. Hence u is constant on each component.

(ii). By (i), the set $u(M) \setminus \mathrm{CV}(u)$ is the union of a finite family of disjoint open intervals. Let I be a compact subinterval in $u(M) \setminus \mathrm{CV}(u)$ and let $t \in I$.

By a basic theorem of Morse Theory, $u^{-1}(I)$ is diffeomorphic to $I \times N$, where $N = u^{-1}(t)$ is a smooth submanifold of M (see [25, Theorem 3.1] or [28, 9.3.3]). Together with the coarea formula and (17), this implies (ii).

(iii). Set $f(p) = u(p) - t_0$. Every point in the critical level set $L_u^{t_0}$ has a neighborhood in M where (27) holds. Standard compactness arguments and (i) show that (27) is true in some neighborhood of $L_u^{t_0}$ in M (we may assume that $|f(p)| < 1$ in every neighborhood, then we may increase θ keeping the inequality (27) and the inclusion $\theta \in (0,1)$). Applying (17) with $t = u(p)$, we get (28) in some neighborhood of t_0; it admits an extension onto $u(M)$ with, may be, a smaller η. $\square$

In the following theorem, we use the notation of (21). For a function f on $[-c, c]$ set

$$I_f(u) = \int_M f(u(p))\, dp. \tag{29}$$

Theorem 2. *Let all factors in* $\mathbf{E}$ *satisfy* (E), $X \subseteq M$ *be* r*-rectifiable for some* $r \leq m$, $l \in \mathbb{N}$, $\mathbf{t} = (t_1, \dots, t_l)$, *and* $t_i \in [-c_i, c_i]$ *for all* $i = 1, \dots, l$.

(1) *If* $l \leq r$, *then*

$$\mathsf{M}\left(\mathfrak{h}^{r-l}\left(L_{\mathbf{u}}^{\mathbf{t}} \cap X\right)\right) = \frac{\varpi_{r-l}}{\varpi_r}\mathfrak{h}^r(X) \prod_{i=1}^{l} s_i \left(1 - \frac{t_i^2}{c_i^2}\right)^{\frac{d_i - 1}{2}}, \tag{30}$$

where s_i *is subject to* (22) *with* $\mathcal{E} = \mathcal{E}_i$, $i = 1, \dots, l$.

(2) *For any* $l \in \mathbb{N}$

$$\mathsf{M}\left(\mathfrak{h}^r\left(U_{\mathbf{u}}^{\mathbf{t}} \cap X\right)\right) = \mathfrak{h}^r(X) \prod_{i=1}^{l} \frac{\varkappa_{d_i}\left(\frac{t_i}{c_i}\right)}{\varpi_{d_i}}. \tag{31}$$

(3) *For almost all* $t \in [-c, c]$

$$\mathsf{M}\left(\mathfrak{l}(L_u^t)\right) = \frac{\varpi \varpi_{d-1}}{c \varpi_d}\left(1 - \frac{t^2}{c^2}\right)^{\frac{d}{2} - 1}. \tag{32}$$

Moreover, if f *is a piecewise continuous function on* $[-c, c]$, *then*

$$\mathsf{M}\left(I_f\right) = \frac{\varpi \varpi_{d-1}}{c \varpi_d} \int_{-c}^{c} f(t) \left(1 - \frac{t^2}{c^2}\right)^{\frac{d}{2} - 1} dt. \tag{33}$$

Proof. (1). Applying (24) repeatedly, we get (30):

$$\begin{aligned}
&\int_{\mathbf{S}} \mathfrak{h}^{r-l}\left(L_{\mathbf{u}}^{\mathbf{t}} \cap X\right) d\mathbf{u} \\
&\qquad = \int\limits_{\mathcal{S}_2\times\cdots\times\mathcal{S}_l} \left(\int_{\mathcal{S}_1} \mathfrak{h}^{r-l}\left(L_{u_1}^{t_1} \cap \left(L_{u_2}^{t_2} \cap \ldots \cap L_{u_l}^{t_l} \cap X\right)\right) du_1\right) du_2 \ldots du_l \\
&= \frac{\varpi_{r-l}}{\varpi_{r-l+1}} s_1 \left(1 - \frac{t_1^2}{c_1^2}\right)^{\frac{d_1-1}{2}} \int\limits_{\mathcal{S}_2\times\cdots\times\mathcal{S}_l} \mathfrak{h}^{r-l+1}\left(L_{u_2}^{t_2} \cap \ldots \cap L_{u_l}^{t_l} \cap X\right) du_2 \ldots du_l \\
&\qquad\qquad = \cdots = \frac{\varpi_{r-l}}{\varpi_r} \mathfrak{h}^r(X) \prod_{i=1}^{l} s_i \left(1 - \frac{t_i^2}{c_i^2}\right)^{\frac{d_i-1}{2}}.
\end{aligned}$$

(2). Similarly, by (25),

$$\begin{aligned}
\int_{\mathbf{S}} \mathfrak{h}^m\left(U_{\mathbf{u}}^{\mathbf{t}} \cap X\right) d\mathbf{u} &= \frac{\varkappa_{d_1}\left(\frac{t_1}{c_1}\right)}{\varpi_{d_1}} \int\limits_{\mathcal{S}_2\times\cdots\times\mathcal{S}_l} \mathfrak{h}^m\left(U_{u_2}^{t_2} \cap \ldots \cap U_{u_l}^{t_l}\right) du_2 \ldots du_l = \ldots \\
&= \varpi \prod_{i=1}^{l} \frac{\varkappa_{d_i}\left(\frac{t_i}{c_i}\right)}{\varpi_{d_i}}.
\end{aligned}$$

This proves (31).

(3). The equality (6) is a particular case of (31) for $r = m$ and $X = M$. By (6), the right-hand side of (32) is equal to $-\frac{d}{dt}\mathsf{M}\left(\mathfrak{h}^m(U_u^t)\right)$. According to (15), we have to check the equality

$$\int_{\mathcal{S}} \frac{d}{dt}\mathfrak{h}^m(U_u^t)\, du = \frac{d}{dt}\mathsf{M}(\mathfrak{h}^m(U_u^t)). \tag{34}$$

for almost all t. We claim that for any fixed $u \in \mathcal{S}$ the function $\mathfrak{h}^m\left(U_u^t\right)$ is absolutely continuous on t in the interval $u(M)$. Indeed, on the set $u(M)\backslash\mathrm{CV}(u)$ this is true according to Lemma 3, (ii); since this function is non-increasing and $\mathrm{CV}(u)$ is finite, it is sufficient to prove that it is continuous. We have

$$\lim_{\varepsilon\to 0} \mathfrak{h}^m\left(U_u^{t-\varepsilon} \setminus U_u^{t+\varepsilon}\right) = \mathfrak{h}^m(L_u^t) = 0,$$

where the first equality is evident and the second holds because u is real analytic on M: $\mathfrak{h}^m(L_u^t) > 0$ implies $u = t$, contradictory to the assumption in (E) that $\mathcal{E}$ is orthogonal to constant functions since $u \neq 0$ due to the inclusion $u \in \mathcal{S}$. (The implication is obvious if $m = 1$; for $m > 1$ one can use the immersion ι to prove that a set of positive $\mathfrak{h}^m$-measure in M intersects sufficiently many real analytic curves (preimages of the big circles) in sets of positive $\mathfrak{h}^1$-measure.)

Thus, $\mathfrak{h}^m\left(U_u^t\right)$ is absolutely continuous on $[-c, c]$ and we may apply the Newton–Leibnitz formula on t to $\mathfrak{l}(L_u^t)$ on any subinterval $[a, b]$ of $[-c, c]$. Since it is nonnegative and has variation ϖ on $[-c,c]$, $\mathfrak{l}(L_u^t)$ is summable on $\mathcal{S}\times[-c,c]$. In particular, $\mathsf{M}(\mathfrak{l}(L_u^t))$ is well defined for almost all t. By Fubini's theorem,

$$\int_a^b \mathsf{M}\left(\mathfrak{l}(L_u^t)\right)\,dt = \mathsf{M}\Big(\int_a^b \mathfrak{l}(L_u^t)\,dt\Big) = \mathsf{M}(\mathfrak{h}^m(U_u^a)) - \mathsf{M}(\mathfrak{h}^m(U_u^b)). \tag{35}$$

Thus, the integrals of the left-hand and the right-hand parts of (34) over any subinterval in $[-c,c]$ coincide; hence, (34) holds almost everywhere on t. This proves (32).

Since $\mathfrak{h}^m\left(U_u^t\right)$ is absolutely continuous, we may apply Fubini's theorem to the equality

$$\mathsf{M}(I_f) = \int_{\mathcal{S}}\Big(\int_{-c}^{c} f(t)\mathfrak{l}(L_u^t)\,dt\Big)\,du.$$

Thus, (33) follows from (32). □

Remark 2. Note that the right-hand side of (6), as well as (25) and (31), depends only on d and ϖ. Thus, for isotropy irreducible homogeneous spaces, the expectations of $\mathfrak{h}^m(U_u^t)$ are independent of their topology and of the spectrum of Δ in $\mathcal{E}$. According to (15), the same is true for the Leray measure. The corollary below shows that the asymptotic behavior as $\dim\mathcal{E}\to\infty$ of the expectations is independent of the choice of subspaces $\mathcal{E}\subset L^2(M)$, as well as of M, except for the left-hand side of (36). □

Corollary 1. *Let $\mathcal{E}_n$ be a sequence of subspaces of $L^2(M)$ which satisfy* (E). *Suppose $d_n\to\infty$ as $n\to\infty$. Then for any $t\in\mathbb{R}$ and r-rectifiable $X\subseteq M$, where $r\le m$, we have*

$$\lim_{n\to\infty}\frac{1}{s_n}\mathsf{M}(\mathfrak{h}^{r-1}(L_u^t\cap X)) = \frac{\varpi_{r-1}}{\varpi_r}\,\mathfrak{h}^r(X)\,e^{-\frac{t^2}{2}}, \tag{36}$$

$$\lim_{n\to\infty}\mathsf{M}(\mathfrak{h}^r(U_u^t\cap X)) = \frac{\mathfrak{h}^r(X)}{\sqrt{\pi}}\,\mathrm{erfc}\left(\frac{t}{\sqrt{2}}\right), \tag{37}$$

$$\lim_{n\to\infty}\mathsf{M}(\mathfrak{l}(L_u^t)) = \frac{\varpi}{\sqrt{2\pi}}\,e^{-\frac{t^2}{2}}, \tag{38}$$

where $\mathrm{erfc}(t)=\int_t^\infty e^{-\tau^2}\,d\tau$. *If f is a piecewise continuous function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty}|f(t)|e^{-\frac{t^2}{2}}\,dt<\infty$, then*

$$\lim_{n\to\infty}\mathsf{M}\left(I_f\right) = \frac{\varpi}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\frac{t^2}{2}}\,dt. \tag{39}$$

Proof. Since $c_n^2 = d_n+1$, we have $\lim_{n\to\infty}\left(1-\frac{t^2}{c_n^2}\right)^{\frac{d_n-1}{2}} = e^{-\frac{t^2}{2}}$. Together with the theorem for $l=1$, this implies (30) and (31). Taking in account the equality

$$\lim_{n\to\infty}\frac{\varpi_{d_n-1}}{c_n\varpi_{d_n}} = \lim_{n\to\infty}\frac{\Gamma\left(\frac{d_n+1}{2}\right)}{\sqrt{\pi}\,c_n\,\Gamma\left(\frac{d_n}{2}\right)} = \frac{1}{\sqrt{2\pi}},$$

we get (38) and (39). □

The results above make it possible to find expectations for radial distributions on $\mathcal{E}$. Let α be a nonnegative measurable function on $[0,\infty)$ such that $\alpha \neq 0$ and

$$a_k = \int_0^\infty r^k \alpha(r)\, dr < \infty$$

for all $k \in \mathbb{N}$. It defines a probability measure $\alpha_d(x)\, dx$ on $\mathcal{E}$, where dx stands for the Lebesgue measure on $\mathcal{E}$, with the density

$$\alpha_d(x) = \frac{1}{a_d \varpi_d} \alpha(|x|).$$

We denote the mean value of a function f on $\mathcal{E}$ with respect to a probability measure $\eta(x)\, dx$ as

$$\mathsf{M}^\eta(f) = \int_{\mathcal{E}} f(x)\, \eta(x)\, dx.$$

Since $U_u^t \cup U_{-u}^{-t} = M$, $U_u^t \cap U_{-u}^{-t} = L_u^t$, and u is real analytic, for $u \neq 0$ we have

$$\mathfrak{h}^m \left(U_u^t\right) + \mathfrak{h}^m \left(U_{-u}^{-t}\right) = \varpi.$$

Hence we may assume $t \geq 0$.

Proposition 1. *Let α, a_d, and α_d be as above and $t \geq 0$. Then*

$$\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^{m-1}\left(L_u^{ct}\right)\right) = \frac{\varpi \varpi_{m-1} s}{a_d \varpi_m} \int_t^\infty (r^2 - t^2)^{\frac{d-1}{2}} r \alpha(r)\, dr, \tag{40}$$

$$\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^m (U_u^{ct})\right) = \frac{\varpi \varpi_{d-1}}{2 a_d \varpi_d} \int_t^\infty \left(\int_0^\infty \tau^{\frac{d}{2}-1} \alpha\left(\sqrt{\tau + \xi^2}\right) d\tau \right) d\xi, \tag{41}$$

$$\mathsf{M}^{\alpha_d}\left(\mathfrak{l}(L_u^{ct})\right) = \frac{\varpi \varpi_{d-1}}{2 c a_d \varpi_d} \int_0^\infty \tau^{\frac{d}{2}-1} \alpha\left(\sqrt{\tau + t^2}\right) d\tau. \tag{42}$$

Proof. Let $\mathcal{S}_r$ denote the sphere of radius r centered at zero (thus $\mathcal{S} = \mathcal{S}_1$) and du be the invariant probability measure on $\mathcal{S}_r$. We have

$$\int_{\mathcal{E}} \mathfrak{h}^{m-1}\left(L_u^{ct}\right) \alpha(|x|)\, dx = \varpi_d \int_0^\infty \left(\int_{\mathcal{S}_r} \mathfrak{h}^{m-1}\left(L_u^{ct}\right)\, du \right) r^d \alpha(r)\, dr. \tag{43}$$

Clearly,

$$\int_{\mathcal{S}_r} f(u)\, du = \int_{\mathcal{S}} f(ru)\, du \tag{44}$$

for any continuous function f on $\mathcal{E}$. Furthermore, $L_u^t = L_{ru}^{rt}$ for all $r > 0$. If $|u| < t$, then $L_u^{ct} = \varnothing$ by (11). Thus, the integral in the right-hand side of (43) is equal to

$$\int_t^\infty \left(\int_{\mathcal{S}_r} \mathfrak{h}^{m-1}\left(L_u^{ct}\right)\, du \right) r^d \alpha(r)\, dr = \int_t^\infty \left(\int_{\mathcal{S}} \mathfrak{h}^{m-1}\left(L_u^{\frac{ct}{r}}\right)\, du \right) r^d \alpha(r)\, dr$$
$$= \int_t^\infty \mathsf{M}\left(\mathfrak{h}^{m-1}(L_u^{\frac{ct}{r}})\right) r^d \alpha(r)\, dr.$$

Using (5), we obtain (40) by a straightforward calculation. The expectations of $\mathfrak{h}^m\left(U_u^{ct}\right)$ and $\mathfrak{l}\left(L_u^{ct}\right)$ can be calculated similarly: since $U_u^t = U_{ru}^{rt}$ for any $r > 0$ and $U_u^{ct} = \varnothing$ if $|u| < t$, (44) and (6) imply

$$\begin{aligned}\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^m(U_u^{ct})\right) &= \frac{1}{a_d}\int_t^\infty \Big(\int_{\mathcal{S}} \mathfrak{h}^m(U_u^{\frac{ct}{r}})\, du\Big) r^d \alpha(r)\, dr \\ &= \frac{\varpi\varpi_{d-1}}{a_d\varpi_d}\int_t^\infty \Big(\int_{\frac{t}{r}}^1 \left(1-\eta^2\right)^{\frac{d}{2}-1} d\eta\Big) r^d\alpha(r)\, dr.\end{aligned}$$

Let us change the order of integration and substitute $\xi = \frac{\eta}{r}$:

$$\int_t^\infty \Big(\int_t^r \left(r^2-\xi^2\right)^{\frac{d}{2}-1} d\xi\Big) r\alpha(r)\, dr = \int_t^\infty \Big(\int_\xi^\infty \left(r^2-\xi^2\right)^{\frac{d}{2}-1} r\alpha(r)\, dr\Big) d\xi.$$

Using the change of variable $\tau = r^2 - \xi^2$, we get (41) and, by differentiation of (41) on t, (42). □

Corollary 2. *If α_d is a Gaussian measure with the density $\alpha(t) = G^\sigma(t) = e^{-\frac{t^2}{\sigma^2}}$, then*

$$\begin{aligned}\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^{m-1}\left(L_u^{ct}\right)\right) &= \varpi\frac{\varpi_{m-1}}{\varpi_m} s e^{-\frac{t^2}{\sigma^2}}, \\ \mathsf{M}^{\alpha_d}\left(\mathfrak{h}^m\left(U_u^{ct}\right)\right) &= \frac{\varpi}{\sqrt{\pi}}\operatorname{erfc}\left(\frac{t}{\sigma}\right), \\ \mathsf{M}^{\alpha_d}\left(\mathfrak{l}\left(L_u^{ct}\right)\right) &= \frac{\varpi}{c\sigma\sqrt{\pi}} e^{-\frac{t^2}{\sigma^2}},\end{aligned}$$

where $\operatorname{erfc}(t) = \int_t^\infty e^{-\tau^2}\, d\tau$.

Proof. We have $a_d = \frac{\sigma^{d+1}}{2}\Gamma\left(\frac{d+1}{2}\right)$. Hence

$$\begin{aligned}\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^{m-1}\left(L_u^{ct}\right)\right) &= \frac{\varpi\varpi_{m-1}s}{a_d\varpi_m}\int_t^\infty \left(r^2-t^2\right)^{\frac{d-1}{2}} re^{-\frac{r^2}{\sigma^2}}\, dr \\ &= \frac{\varpi\varpi_{m-1}s}{\Gamma\left(\frac{d+1}{2}\right)\varpi_m} e^{-\frac{t^2}{\sigma^2}}\int_0^\infty \tau^{\frac{d-1}{2}} e^{-\tau}\, d\tau = \varpi\frac{\varpi_{m-1}}{\varpi_m} s e^{-\frac{t^2}{\sigma^2}},\end{aligned}$$

where $\tau = \frac{r^2-t^2}{\sigma^2}$. Further, $\frac{\varpi_{d-1}}{a_d\varpi_d} = \frac{2}{\sqrt{\pi}\sigma^{d+1}\Gamma\left(\frac{d}{2}\right)}$; therefore,

$$\begin{aligned}\mathsf{M}^{\alpha_d}\left(\mathfrak{h}^{m-1}\left(U_u^{ct}\right)\right) &= \frac{\varpi}{\sigma^{d+1}\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}\int_t^\infty \Big(\int_0^\infty \tau^{\frac{d}{2}-1} e^{-\frac{\tau}{\sigma^2}}\, d\tau\Big) e^{-\frac{\xi^2}{\sigma^2}}\, d\xi \\ &= \frac{\varpi}{\sigma\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}\int_t^\infty \Big(\int_0^\infty \eta^{\frac{d}{2}-1} e^{-\eta}\, d\eta\Big) e^{-\frac{\xi^2}{\sigma^2}}\, d\xi = \frac{\varpi}{\sqrt{\pi}}\operatorname{erfc}\left(\frac{t}{\sigma}\right).\end{aligned}$$

Differentiating on t, we get the last equality of the corollary. □

Remark 3. According to (38), $\lim_{d\to\infty} \mathsf{M}(\mathfrak{l}(N_u)) = \frac{\varpi}{\sqrt{2\pi}}$. In the papers [27] and [35], the expectations of $\mathfrak{l}(N_u)$ were computed for tori $\mathbb{R}^n/\mathbb{Z}^n$ and spheres S^m with the Gaussian distribution in $\mathcal{E}$ normalized by the condition that for any fixed $p \in M$ the average of $|u(p)|^2$ is equal to 1. In both cases, the expectation is independent of $\dim \mathcal{E}$ and equals to $\frac{\varpi}{\sqrt{2\pi}}$, where $\varpi = 1$ for $\mathbb{R}^n/\mathbb{Z}^n$ and $\varpi = \varpi_m$ for S^m. By a direct computation one can check that this is equivalent to the relation $\sigma c = \sqrt{2}$ (in the notation of Corollary 2). It follows from Corollary 2 that the same is true for all isotropy irreducible homogeneous spaces: the expectation of $\mathfrak{l}(N_u)$ for the Gaussian distribution with this normalization is independent of $\dim \mathcal{E}$. For the uniform distribution on spheres the expectation depends on $\dim \mathcal{E}$ but mildly since $\frac{\varpi_{d-1}}{c\varpi_d} \to \frac{1}{\sqrt{2\pi}}$ as $d \to \infty$. □

3 Upper bounds for the expectations of L^p norms

We use the symbol a instead of the standard p in $\|u\|_a$ (the norm in the spaces $L^a(M)$, $1 \le a \le \infty$). There are two reasons for it: first, p denotes a point of M in the text above, and second, we do not exclude the cases $a \in (0,1)$ and even $a \in (-1,0)$ in the calculation below.

Theorem 3. *Let $a > -1$. The function $|u|^a$ is integrable on M for almost all $u \in \mathcal{S}$. Moreover, $\int_M |u(p)|^a \, dp \in L^1(\mathcal{S})$ and*

$$\mathsf{M}\left(\int_M |u(p)|^a \, dp\right) = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)(d+1)^{\frac{a}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{a+d+1}{2}\right)}. \tag{45}$$

If $a > 2$ or $a \in (-1,0)$, then for all $d \in \mathbb{N}$

$$\mathsf{M}\left(\int_M |u(p)|^a \, dp\right) < 2^{\frac{a}{2}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}},$$

the reverse inequality holds for $a \in (0,2)$, and the equality is true if $a = 0$ or $a = 2$.

Proof. If $a > 0$, then we may apply (33) to $f(t) = |t|^a$:

$$\int_{\mathcal{S}} \int_M |u(p)|^a \, dp \, du = \int_M \int_{\mathcal{S}} |u(p)|^a \, du \, dp = \int_{\mathcal{S}} |u(p)|^a \, du = c^a \int_{\mathcal{S}} |\langle u, \bar{p}\rangle|^a \, du$$
$$= c^a \frac{\varpi_{d-1}}{\varpi_d} \int_{-1}^{1} |t|^a \left(1-t^2\right)^{\frac{d}{2}-1} dt = c^a \frac{\varpi_{d-1}}{\varpi_d} \int_0^1 \tau^{\frac{a-1}{2}} (1-\tau)^{\frac{d}{2}-1} d\tau$$
$$= c^a \frac{\varpi_{d-1}}{\varpi_d} B\left(\frac{a+1}{2}, \frac{d}{2}\right) = c^a \frac{\varpi_{d-1}}{\varpi_d} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+d+1}{2}\right)} = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a+d+1}{2}\right)} c^a.$$

The equalities in the second and the third lines hold true for $a \in (-1,0)$. Hence, we get the same result for such a approximating $|t|^a$ by the functions

$f_n(t) = \min\{|t|^a, n\}$. Indeed, the sequence $I_{f_n}(u)$ increases for any $u \in \mathcal{S}$ and $f_n(t)$ converges to $|t|^a$ if $t \neq 0$. Hence

$$\frac{c\varpi_d}{\varpi\varpi_{d-1}}\mathsf{M}\left(I_{f_n}\right) = \int_{-c}^{c} f_n(t)\left(1-\frac{t^2}{c^2}\right)^{\frac{d}{2}-1} dt \to \int_{-c}^{c} |t|^a \left(1-\frac{t^2}{c^2}\right)^{\frac{d}{2}-1} dt$$

as $n \to \infty$. It follows from Levy's and Lebesgue's theorems that $I_{f_n}(u) \to \int_M |u(p)|^a\, dp$ and $|u|^a \in L^1(M)$ for almost all $u \in \mathcal{S}$. (Hence, $\mathfrak{h}^m(N_u) = 0$ for such $u \in \mathcal{S}$; actually, it was shown in the proof of Theorem 2, (3), that this is true for all $u \in \mathcal{S}$). Thus, $\int_M |u(p)|^a\, dp \in L^1(\mathcal{S})$. This verify the calculation. The inequalities follows from Lemma 4 below, the cases $a = 0$ and $a = 2$ are clear. □

For the sake of completeness, we give a proof of some properties of Euler's function Γ.

Lemma 4. *Set* $\varphi_b(t) = \frac{t^b\Gamma(t)}{\Gamma(t+b)}$, $f(t) = \ln\left(\left(\frac{e}{t}\right)^{t-\frac{1}{2}}\Gamma(t)\right)$.

(a) *The function* φ_b *decreases on* $(0,\infty)$ *if* $0 < b < 1$ *and increases if* $b > 1$. *Moreover, if* $b < 0$, *then* φ_b *increases on* $(-b,\infty)$. *For any* $b \in \mathbb{R}$

$$\lim_{t\to\infty} \varphi_b(t) = 1. \tag{46}$$

(b) *The function* f *is convex on* $(0,\infty)$ *and* $\lim_{t\to\infty} f(t) = \ln\sqrt{\frac{2\pi}{e}}$.

(c) *For all* $t \in (\frac{1}{2},\infty)$

$$1 > \left(\frac{e}{t}\right)^{t-\frac{1}{2}} \frac{\Gamma(t)}{\sqrt{\pi}} > \sqrt{\frac{2}{e}}. \tag{47}$$

Proof. The equality (46) follows from the standard asymptotic formula for Γ. To prove the first and the second assertions in (a), let us consider $\Psi(x) = \frac{d}{dx}\ln\Gamma(x)$. We have

$$\Psi''(x) = -2\sum_{k=0}^{\infty}\frac{1}{(x+k)^3} < 0$$

for all $x > 0$. Hence the function

$$\eta_t(b) = \frac{d}{dt}\ln\varphi_b(t) = \frac{b}{t} + \Psi(t) - \Psi(t+b)$$

is strictly convex on $(-t,\infty)$ for any fixed $t > 0$. The evident equalities

$$\eta_t(0) = \eta_t(1) = 0$$

imply $\eta_t(b) < 0$ for $b \in (0,1)$ and $\eta_t(b) > 0$ if $b \in (1,\infty)$ or $b \in (-t,0)$. This proves (a).

Computation of the limit in (b) is standard. Differentiating f we get $f''(t) = \Psi'(t) - \frac{1}{t} - \frac{1}{2t^2}$, where $\Psi'(t) = \frac{d^2}{dt^2}\ln\Gamma(t) = \sum_{n=0}^{\infty}\frac{1}{(t+n)^2}$. Hence

$$\Psi'(t) = \frac{1}{2t^2} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(t+n)^2} + \frac{1}{(t+n+1)^2}\right) > \frac{1}{2t^2} + \sum_{n=0}^{\infty}\int_n^{n+1}\frac{d\tau}{(t+\tau)^2}$$
$$= \frac{1}{2t^2} + \int_0^{\infty}\frac{d\tau}{(t+\tau)^2} = \frac{1}{t} + \frac{1}{2t^2},$$

where the inequality holds since the function $\frac{1}{t^2}$ is convex. It follows that $f'' > 0$ on $(0,\infty)$. Thus, (b) is true.

The function f decreases since it is convex and has a finite limit at infinity. Therefore,

$$f\left(\frac{1}{2}\right) = \ln\sqrt{\pi} > f(t) > \ln\sqrt{\frac{2\pi}{e}} = \lim_{t\to\infty} f(t)$$

for all t in $(\frac{1}{2},\infty)$. This proves (c). □

Let $\mathfrak{h}_p$ denote the Lie algebra of the stable subgroup of $p \in M$ and π_p be the orthogonal projection in $\mathcal{E}$ onto $T_{\bar{p}}\bar{M} = d_p\iota\,(T_pM)$.

Lemma 5. *All functions in $\mathcal{S}$ are Lipschitz with the coefficient cs. Moreover, this coefficient is attained if and only if $u = \frac{1}{cs}\xi\phi_p$ for some $p \in M$, $\xi \in \mathfrak{g}$, $\xi \perp \mathfrak{h}_p$, and $|\xi| = 1$.*

Proof. For any $u \in \mathcal{S}$ and $p \in M$ we have $u(p) = c\,\langle u, \bar{p}\rangle$. Since the mapping $p \to \bar{p}$ is a local metric homothety with the coefficient s and the linear function $\ell_u(v) = \langle u, v\rangle$ is Lipschitz with the coefficient 1, the first assertion follows. Further, the gradient of the restriction of ℓ_u onto $\bar{M}$ may be identified with $\pi_p u$. Hence, $\max_{q\in M}|\nabla u(q)| = cs$ if and only if $u \in T_{\bar{p}}\bar{M}$ for some $p \in M$. This happens if and only if u is proportional to $\xi\phi_p$ for some $\xi \in \mathfrak{g}$ and $|u| = 1$. A description of such u is given in the statement of the lemma. □

Let $B(p,r) = \{q \in M : \rho(q,p) < r\}$ be the ball with respect to the Riemannian distance ρ in M. Clearly, there exist $b > 0$ and $r_0 > 0$, which depend only on the geometry of M, such that

$$\mathfrak{h}^m(B(p,r)) > b\varpi r^m \tag{48}$$

for all $r \in (0, r_0)$.

Lemma 6. *Let $u \in L^a(M)$ be k-Lipschitz and b, r_0 be as above. Then*

$$\|u\|_\infty \le b^{-\frac{1}{a}}\|u\|_a r^{-\frac{m}{a}} + kr. \tag{49}$$

for all $r \in (0, r_0)$.

Proof. We may assume that $\|u\|_\infty = \sup_{p\in M} u(p)$ replacing u with $-u$ if necessary. According to the Chebyshev inequality, $\frac{1}{\varpi}\mathfrak{h}^m(U_u^t) \le \frac{\|u\|_a^a}{t^a}$ for all $t > 0$. Thus, if $p \in M$ and

$$\frac{\|u\|_a^a}{t^a} < \frac{1}{\varpi}\mathfrak{h}^m(B(p,r)), \tag{50}$$

then the ball $B(p,r)$ contains a point $q \notin U_u^t$. We have $\rho(p,q) < r$ and $u(q) < t$; it follows that $u(p) < t + kr$, moreover, $\|u\|_\infty < t + kr$ since p is independent of t and r. If $\|u\|_a^a t^{-a} = br^m$ and $r < r_0$, then (50) is true due to (48). This proves (49). □

Theorem 4. *Let $\mathcal{E}_n$ be a sequence of subspaces of $L^2(M)$ which satisfy* (E). *Suppose $d_n \to \infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \mathsf{M}\left(\int_M |u(p)|^a\, dp\right) = \frac{2^{\frac{a}{2}}}{\sqrt{\pi}}\,\Gamma\left(\frac{a+1}{2}\right), \qquad a > -1, \tag{51}$$

$$\limsup_{n\to\infty} \mathsf{M}\left(\|u\|_a\right) \le \sqrt{2}\pi^{-\frac{1}{2a}}\,\Gamma\left(\frac{a+1}{2}\right)^{\frac{1}{a}} < \sqrt{\frac{a+1}{e}}, \qquad a \ge 1. \tag{52}$$

Moreover, for any space $\mathcal{E}$ satisfying (E)

$$\mathsf{M}\left(\|u\|_a\right) < \sqrt{\frac{a+1}{e}}, \qquad a \ge 2, \tag{53}$$

$$\mathsf{M}\left(\|u\|_\infty\right) < K(\sqrt{\ln s} + 1), \tag{54}$$

where $K > 0$ is independent of $\mathcal{E}$.

Proof. The equality (51) follows from Theorem 3 and Lemma 4, (a), with $t = \frac{d+1}{2}$ and $b = \frac{a}{2}$.

If $a > 2$, then $\varphi_b(t) < 1$ due to Lemma 4, (a); this proves the inequality

$$\mathsf{M}\left(\|u\|_a^a\right) < \frac{2^{\frac{a}{2}}}{\sqrt{\pi}}\,\Gamma\left(\frac{a+1}{2}\right).$$

Set $t = \frac{a+1}{2}$. Then $t - \frac{1}{2} = \frac{a}{2}$. By (47), for all $a > 0$ we have

$$\left(\frac{2}{e}\right)^{\frac{1}{2a}}\sqrt{\frac{a+1}{e}} < \left(\frac{2^{\frac{a}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{a+1}{2}\right)\right)^{\frac{1}{a}} < \sqrt{\frac{a+1}{e}}. \tag{55}$$

Since $\mathsf{M}\left(\|u\|_2\right) = 1 < \sqrt{\frac{3}{e}}$, we get (53). Moreover, (55) and (51) imply (52) for $a \ge 1$:

$$\limsup_{n\to\infty} \mathsf{M}(\|u\|_a) \le \limsup_{n\to\infty} \mathsf{M}(\|u\|_a^a)^{\frac{1}{a}} = \left(\frac{2^{\frac{a}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{a+1}{2}\right)\right)^{\frac{1}{a}} < \sqrt{\frac{a+1}{e}},$$

where the equality in the first inequality holds if and only if the function $\|u\|_a$ is constant on $\mathcal{S}$ but this is true if and only if $a = 2$.

Due to Lemma 5, we may use Lemma 6 with $k = cs$. Setting $r = \frac{1}{k}$ and assuming k sufficiently large, we get

$$\|u\|_\infty \leq b^{-\frac{1}{a}} \|u\|_a k^{\frac{m}{a}} + 1.$$

Set $a = \ln k$. Then $k^{\frac{m}{a}} = e^m$ and $b^{-\frac{1}{a}} \leq \max\{1, b^{-1}\}$. Integrating over $\mathcal{S}$ and using (53), we get $\mathsf{M}(\|u\|_\infty) \leq K\sqrt{\ln k} + 1$ with some $K > 0$. The inequality

$$c \leq \alpha s, \tag{56}$$

where $\alpha > 0$ depends only on M, implies (54) (may be, with another K). To prove (56), note that ϕ_o takes value 0 due to the assumption $\mathbf{1} \perp \mathcal{E}$. Since $\frac{1}{c}\phi_o(o) = c$, we get $s \geq \frac{c}{\operatorname{diam} M}$. Also, it follows from (56) that $s \to \infty$ as $d \to \infty$. Since the choice of K depends only on d and M, this completes the proof of the theorem. □

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