THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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J. Brennan and C. Mattingly

REPORT No. 21, 2011/2012, fall

ISSN 1103-467X
ISRN IML-R- -21-11/12- -SE+fall

Approximation by Rational Functions on Compact Nowhere Dense Subsets of the Complex Plane

J. E. Brennan[1] and C. N. Mattingly

In Memory of Boris Korenblum

Abstract Let X be a compact nowhere dense subset of the complex plane, and let dA denote two-dimensional or area measure on X. Let $R(X)$ denote the uniform closure of the rational functions having no poles on X, and for each p, $1 \le p < \infty$, let $R^p(X)$ be the closure of $R(X)$ in the $L^p(X, dA)$-norm. Since X has no interior $R^p(X) = L^p(X)$ whenever $1 \le p < 2$, but for $p = 2$ a kind of phase transition occurs that can be quite striking at times. Our main goal here is to study the manner in which similar phase transitions can occur at any value of p, $2 \le p \le \infty$.

1 Introduction

Let X be a compact set in the complex plane $\mathbb{C}$, and let $|X|$ stand for the area (that is dA or two-dimensional Lebesgue measure) of X. Define $C(X)$ to be the space of all continuous functions on X endowed with the uniform norm, and let $R(X)$ be the closure in $C(X)$ of the rational functions with poles off X. It is an old problem to determine conditions on X so that $R(X) = C(X)$. An obvious necessary condition is that X have no interior, and so we shall adopt that hypothesis as a standing assumption.

In 1931, Hartogs and Rosenthal [32] proved that $R(X) = C(X)$ whenever $|X| = 0$, leaving open the question as to whether the rational functions are dense in $C(X)$ for every compact nowhere dense set X. By the end of the decade, Alice Roth [49] (cf. [17], [27], [28]) settled that question by producing an example of a compact nowhere dense set X so that $R(X) \ne C(X)$, the so-called Swiss cheese, constructed as follows: From inside the open unit disk $D = \{z : |z| < 1\}$ choose a sequence of open disks $D_j, j = 1, 2, 3 \dots,$ having mutually disjoint closures in D in such a way that

(a) $\overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ has no interior;

(b) $\sum_{j=1}^{\infty} \text{length}(\partial D_j) < \infty$.

Then, $X = \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ is a compact set with no interior, and moreover $R(X) \ne C(X)$. To verify the non density assertion define a measure μ on X as follows: Let $\mu = dz$ on $|z| = 1$ and let $\mu = -dz$ on each ∂D_j. If f is any rational function whose poles lie outside X, then by Cauchy's theorem

$$\int_X f d\mu = \int_{|z|=1} f dz - \sum_{j=1}^{\infty} \int_{\partial D_j} f dz = 0.$$

Hence, μ annihilates $R(X)$ and therefore $R(X) \ne C(X)$. However, it wasn't until 1958 that Vitushkin (cf. [57]) established necessary and sufficient conditions for $R(X) = C(X)$ in terms of

[1] J. E. Brennan wishes to express his gratitude to the Institut Mittag-Leffler for support during the fall of 2011 when work on this paper was begun.

analytic capacity.

In the 1960's, a renewed interest developed in a another aspect of rational approximation, due in part to its connection with the invariant subspace problem for subnormal operators on a Hilbert space (cf. [9], [11]). For $p \geq 1$, let $L^p(X, dA)$ (or more succinctly $L^p(X)$) be the usual space of functions on X which are $p-$integrable with respect to the area measure dA. Let $R^p(X, dA)$ (or more succinctly $R^p(X)$) be the closure in the $L^p(dA)$ norm of the rational functions with poles off X. Since X has no interior it is well-known that if $1 \leq p < 2$, then $R^p(X) = L^p(X)$. In general $R(X) \subset R^p(X)$, and so if $R(X) = C(X)$, then $R^p(X) = L^p(X)$ for all $p \geq 1$. Two questions immediately arise:

(i) What conditions are necessary and sufficient in order that $R^p(X) = L^p(X)$;

(ii) Can it happen that $R^p(X) = L^p(X)$ for all $p \geq 1$, but $R(X) \neq C(X)$?

With regard to question (ii), Sinanjan [52] (cf. [9], [14]) constructed a Swiss cheese X for which $R(X) \neq C(X)$, but nevertheless $R^p(X) = L^p(X)$ for all p, $1 \leq p < \infty$. His argument, however, depends on a construction of Mergeljan [47, p. 315], and the reader is referred to an earlier paper [51] for many of the computational details. Here we shall describe an argument from [14] which is more transparent and yields an entire family of compact nowhere dense sets X having a locally nonrectifiable perimeter such that $R(X) \neq C(X)$, and still $R^p(X) = L^p(X)$ for all $p < \infty$. Later we shall see that given any $p^* \geq 2$ there exists a compact nowhere dense set X such that $R^p(X) = L^p(X)$ if $1 \leq p < p^*$, but not if $p \geq p^*$.

There is an obvious obstruction to the possibility that $R^p(X) = L^p(X)$. There may exist a point x_0 with the property that

$$|f(x_0)| \leq C \, \|f\|_{L^p(X)}$$

for every rational function f with poles off X, and some fixed constant C. Such a point x_0 is referred to as a *bounded point evaluation* (or *bpe*) for $R^p(X)$. In that case, the map $f \to f(x_0)$ extends from $R(X)$ to a bounded linear functional on $R^p(X)$, and the Hahn-Banach theorem guarantees the existence of a function $k \in L^q(X)$, where $1/p + 1/q = 1$, such that

$$f(x_0) = \int_X fk \; dA$$

for all $f \in R(X)$. Thus, $(z - x_0)k(z)dA$ is a nontrivial annihilating measure for $R^p(X)$, and therefore $R^p(X) \neq L^p(X)$. In this way, when $p \geq 2$ it is possible to construct a compact nowhere dense set X such that $R^p(X) \neq L^p(X)$, but as noted above there will always be equality when $p < 2$ (cf. [9]). In order that $R^p(X) = L^p(X)$ when $p > 2$ it is both necessary and sufficient that $R^p(X)$ have no bpe's, but as Fernström [21] has shown this is not sufficient if $p = 2$. Earlier, Hedberg [34] had obtained a necessary and sufficient condition in terms of $q-$capacity for a point $x_0 \in X$ to be a bpe for $R^p(X)$ whenever $p > 2$, and also a corresponding necessary condition when $p = 2$ subsequently employed by Fernström in [21]. Later, Hedberg's necessary criterion in the case $p = 2$ was shown to be sufficient as well by Fernström and Polking [23].

As indicated above, when considering approximation in the L^p-norm by rational functions on sets without interior points there is a kind of phase transition which occurs at $p = 2$, a transition that in fact can be quite striking. Although $R^p(X) = L^p(X)$ when $1 \leq p < 2$, it can happen that X is nevertheless so massive that $R^2(X)$ retains the uniqueness property of the analytic functions in the sense that: if any two functions in $R^2(X)$ agree on a set of positive dA measure, then they agree almost everywhere. The idea of extending the uniqueness property to certain large classes of functions defined on sets without interior points was first suggested by Borel during the last decade of the 19-th century, and finally brought to fruition more than twenty years later in his seminal work [8] on the theory of *monogenic functions*. We shall return to this topic later in Section 6.

For a more extensive discussion of the history and current state of many of the problems described herein see [12] and [46]. Whenever possible the page numbers on individual citations to Russian articles will refer to the English translation.

2 Early Results on Rational Approximation

2.1 The Cauchy transform

In order to deal with approximation questions it is often most convenient to argue by duality. If, for example, we wish to prove that $R(X) = C(X)$ it is enough to show that if μ is any measure of finite total variation supported on X, and if $\mu \perp R(X)$ in the sense that

$$\int_X f \, d\mu = 0$$

for all $f \in R(X)$, then $\mu = 0$ as a measure. That will be the inescapable conclusion whenever it can be shown that the corresponding *Cauchy transform*

$$\widehat{\mu}(z) = \int \frac{d\mu_\zeta}{\zeta - z}$$

vanishes a.e$-dA$ in $\mathbb{C}$. Similar remarks are valid for approximation in $L^p(X)$ with $d\mu = k\,dA$ and $k \in L^q(X)$, where $1/p + 1/q = 1$. In this context we will need the following two lemmas:

Lemma 2.1. *Let μ be any complex measure on X. Then the* Newtonian potential

$$\widetilde{\mu}(z) = \int_X \frac{d|\mu_\zeta|}{|\zeta - z|}$$

is finite a.e.-dA in $\mathbb{C}$.

Lemma 2.2. *Let μ be a measure of finite total variation on X. If $\widehat{\mu} = 0$ a.e.$-dA$ in $\mathbb{C}$, then $\mu = 0$ as a measure.*

Proof. The proof presented here is due to Beurling (cf. [59, p. 75]). Let R be any rectangle in $\mathbb{C}$ such that $|\mu| = 0$ on ∂R: Then,

$$\int_{\partial R} \widehat{\mu} \, dz = \int_{\partial R} \int_X \frac{d\mu_\zeta}{\zeta - z} dz = \int_X \int_{\partial R} \frac{dz}{\zeta - z} d\mu_\zeta = 0.$$

However, by Cauchy's theorem

$$\frac{1}{2\pi i} \int_{\partial R} \frac{dz}{z - \zeta} = \chi_R(\zeta)$$

and so

$$\frac{-1}{2\pi i}\int_X\int_{\partial R}\frac{dz}{\zeta - z}d\mu_\zeta = \int_X \chi_R(\zeta)d\mu_\zeta = \mu(R\cap X) = 0.$$

This can be done with enough rectangles to conclude that $\mu = 0$ as a measure. □

As a corollary, let us recall the theorem of Hartogs and Rosenthal [32] from 1931 in which we can illustrate the use of the Cauchy transform.

Theorem 2.3 (Hartogs and Rosenthal)**.** *If* $|X| = 0$*, then* $R(X) = C(X)$*.*

Proof. Let μ be a measure on X with $\mu \perp R(X)$. By assumption, it follows that

$$\widehat{\mu}(z) = \int_X \frac{d\mu_\zeta}{\zeta - z} = 0$$

whenever $z \in \mathbb{C}\setminus X$, and so $\widehat{\mu} = 0$ a.e.$-dA$. Hence $\mu = 0$ as a measure by Corollary 2.2. Thus μ not only annihilates the rational functions, but all continuous functions as well, and so $R(X) = C(X)$. □

In order to study approximation in $L^p(X)$ we can argue along similar lines. In this case, let $k \in L^q(X)$ where $1/p + 1/q = 1$, and assume that $\int_X fk\, dA = 0$ for all $f \in R(X)$. Hence,

$$\widehat{k}(z) = \int_X \frac{k(\zeta)}{\zeta - z}\, dA_\zeta = 0$$

whenever $z \in \mathbb{C}\setminus X$. Our problem is to determine whether $\widehat{k}$ enjoys sufficient continuity at points of X to ensure that $\widehat{k} = 0$ a.e.$-dA$ on X. If so we can conclude that $R^p(X) = L^p(X)$.

Theorem 2.4. *If* $1 \le p < 2$ *then* $R^p(X) = L^p(X)$ *for any compact nowhere dense set* X*.*

Here the theorem is a consequence of the fact that X has no interior and $\widehat{k}$ is a continuous function whenever $k \in L^q(X)$ for $q > 2$. The continuity of $\widehat{k}$ follows easily from the fact that translation is a continuous operator on L^q (cf. [50, p.3]). A more precise description, however, of the degree of continuity enjoyed by $\widehat{k}$ in this case is contained in the following, a proof of which is included here since there appears to be no convenient reference:

Lemma 2.5. *If* $k \in L^q(X)$ *for* $q > 2$*, then* $|\widehat{k}(z_1) - \widehat{k}(z_2)| \le C|z_1 - z_2|^{1-2/q}$*.*

Proof of lemma. Let $k \in L^q(X)$ for $q > 2$ and let x_1, x_2 be any pair of points in the plane. Then

$$\left|\widehat{k}(x_1) - \widehat{k}(x_2)\right| \le |x_1 - x_2| \int \frac{|k(z)|}{|z - x_1||z - x_2|}\, dA.$$

Define $R = \frac{1}{2}|x_1 - x_2|$, and let D_1 and D_2 be the disks of radius R centered at x_1 and x_2 respectively. We will proceed in two parts: first by considering z in either D_1 or D_2, and then by considering z outside $D = D_1 \cup D_2$.

Case 1: Without loss of generality, assume $z \in D_1$. We have $|z - x_2| \geq \frac{1}{2}|x_1 - x_2|$ on D_1, and so

$$\begin{aligned} |x_1 - x_2| \int_{D_1} \frac{|k(z)|}{|z - x_1||z - x_2|} \, dA &\leq |x_1 - x_2| \int_{D_1} \frac{2|k(z)|}{|z - x_1||x_1 - x_2|} \, dA \\ &\leq 2\|k\|_q \left(\int_{D_1} \frac{1}{|z - x_1|^p} \, dA \right)^{1/p}. \end{aligned}$$

Using polar coordinates centered at x_1 inside the parenthesis,

$$\int_{D_1} \frac{1}{|z - x_1|^p} \, dA = \int_0^{2\pi} \int_0^R \frac{1}{r^p} \, r \, dr \, d\theta = \frac{2\pi}{2 - p} \, R^{2-p}.$$

Recall that $1/p = 1 - 1/q$ and $R = \frac{1}{2}|x_1 - x_2|$ which gives

$$|x_1 - x_2| \int_{D_1} \frac{|k(z)|}{|z - x_1||z - x_2|} \, dA \leq C|x_1 - x_2|^{1-2/q},$$

where C is a constant that depends only on q. Similar reasoning gives the same bound for the contribution from integrating over D_2.

Case 2: Consider what happens when $z \notin D$. Since $ab \leq \frac{1}{2}(a^2 + b^2)$ for all real numbers a, b, we have:

$$|x_1 - x_2| \int_{\mathbb{C} \setminus D} \frac{|k(z)|}{|z - x_1||z - x_2|} \, dA \leq |x_1 - x_2| \int_{\mathbb{C} \setminus D} \left(\frac{|k(z)|}{|z - x_1|^2} + \frac{|k(z)|}{|z - x_2|^2} \right) \, dA$$

For the first term, we estimate that

$$\int_{\mathbb{C} \setminus D} \frac{|k(z)|}{|z - x_1|^2} \, dA \leq \int_{\mathbb{C} \setminus D_1} \frac{|k(z)|}{|z - x_1|^2} \, dA \leq \|k\|_q \left(\int_{\mathbb{C} \setminus D_1} \frac{1}{|z - x_1|^{2p}} \, dA \right)^{1/p}.$$

Again, we can use polar coordinates to estimate the integral inside the parenthesis yielding

$$\int_{\mathbb{C} \setminus D_1} \frac{1}{|z - x_1|^{2p}} \, dA = \int_0^{2\pi} \int_R^\infty \frac{1}{r^{2p}} \, r \, dr = \frac{2\pi}{2p - 2} \, R^{2-2p}.$$

Using similar reasoning for the second integral, we obtain

$$|x_1 - x_2| \int_{\mathbb{C} \setminus D} \frac{|k(z)|}{|z - x_1||z - x_2|} \, dA \leq 2R(CR^{2/p-2}) = C|x_1 - x_2|^{1-2/q},$$

where C is a constant that depends only on q.

Combining the two cases gives a bound for the integral over the entire plane, and completes the proof of the lemma. □

When $p = 2$ the situation changes rather abruptly, as the following theorem clearly indicates. The striking disparity here was first noticed by Sinanjan [51] in 1965, and subsequently rediscovered by the first author [10]. The proof presented here is from [10].

Theorem 2.6. *There exists a compact set having no interior and positive dA measure such that* $R^2(X) \neq L^2(X)$.

Proof. In order to obtain a compact nowhere dense set X for which $R^2(X) \neq L^2(X)$ we can take advantage of the manner in which the Bergman kernel varies under certain deformations of the underlying region (cf. [2], Theorem 1, p.362 and [10], Lemma 2, p.312). Choose a countable dense set of points $S = \{\alpha_1, \alpha_2, \alpha_3 \dots\}$ in $D \setminus \{0\}$. Let $a_1 = \alpha_1$ and remove from D an open disk $D_1 = \{z : |z - a_1| < r_1$ so that $0 \in \Omega_1 = \overline{D} \setminus D_1$. For any rational function f having no poles on Ω_1,

$$|f(0)|^2 \leq K_1(0,0) \int_{\Omega_1} |f|^2 dA,$$

where $K_1(z, \zeta)$ is the Bergman kernel for Ω_1. Next, let $a_2 = \alpha_{j_2}$ be the first point of S not contained in the closure of D_1. Remove a second disk $D_2 = \{z : |z - a_2| < r_2\}$ in such a way that

(1) D_1 and D_2 have disjoint closures;

(2) $0 \in \Omega_2 = \overline{D} \setminus (D_1 \cup D_2)$;

(3) $K_2(0,0) - K_1(0,0) < 1/2$, where $K_j(z, \zeta)$ is the Bergman kernel for Ω_j, $j = 1, 2$.

To ensure that property (3) is satisfied one has only to choose the radius of D_2 sufficiently small (cf. [10], p. 301). Continue in this way to obtain a sequence of disks D_j, $j = 1, 2, \dots,$ and a sequence of closed regions $\Omega_n = \overline{D} \setminus \bigcup_{j=1}^{n} D_j$, $n = 1, 2, \dots$, such that

(4) the disks D_j have mutually disjoint closures;

(5) $\bigcap_{n=1}^{\infty} \Omega_n$ has no interior and contains the point $z = 0$;

(6) $K_{n+1}(0,0) - K_n(0,0) < 1/2^n$, where $K_j(z, \zeta)$ is the Bergman kernel for Ω_j, $j = 1, 2, \dots$

The set $X = \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j = \bigcap_{j=1}^{\infty} \Omega_j$ is compact, has no interior and $0 \in X$. If f is a rational function with no poles on X, then f has no poles on Ω_n whenever $n \geq n_0$, and so

$$|f(0)|^2 \leq K_n(0,0) \int_{\Omega_n} |f|^2 dA$$

as soon as $n \geq n_0$. It follows from the monotone convergence theorem for integrals that

$$|f(0)|^2 \leq \left(K_1(0,0) + \sum_{n=1}^{\infty} \frac{1}{2^n} \right) \int_X |f|^2 dA$$

for all $f \in R(X)$. Thus, $R^2(X)$ has a bpe at 0 , and so $R^2(X) \neq L^2(X)$. □

3 Sobolev Spaces and Capacity

3.1 Sobolev Spaces

In order to prove that $R^p(X) = L^p(X)$ when $p \geq 2$ we can, in principle, proceed along the same lines outlined in our proof of the Hartogs-Rosenthal theorem. More specifically, if $k \in L^q(X)$ where

$1/p + 1/q = 1$, and $k \perp R^p(X)$ we must prove that

$$\widehat{k}(\zeta) = \int_X \frac{k(\zeta)}{\zeta - z} dA = 0$$

a.e.$-dA$ in $\mathbb{C}$. It is clear, of course, that $\widehat{k} \equiv 0$ in $\mathbb{C} \setminus X$. Our task is to determine whether $\widehat{k}$ enjoys a sufficient residual continuity to conclude that $\widehat{k} = 0$ a.e.$-dA$ on X. Throughout this discussion p and q will denote conjugate indices as above. The differential operators ∂ and $\overline{\partial}$ are defined as follows:

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It is an easy matter to check that $\overline{\partial}\widehat{k} = -\pi k$ as a distribution. Since, by assumption, $k \in L^q(X)$ for some $q > 1$ and $\widehat{k}$ has compact support, it follows from the Calderon-Zygmund theoem on the continuity of singular integral operators that $\partial\widehat{k}$ also exists as a distribution and belongs to $L^q(X)$ (cf. [15]; [53, p. 35]; and [55, p. 72, Thm 1.36]). As a result, the real partial derivatives of $\widehat{k}$ exist as distributions and

$$\|\nabla\widehat{k}\|_q \leq C\|\overline{\partial}\,\widehat{k}\|_q = C\pi\|k\|_q,$$

provided $q > 1$ and $k \perp R^p(X)$. Therefore $\widehat{k}$ belongs to the Sobolev space W_1^q which consists of all functions in L^q whose first-order real partial derivatives are also in L^q.

In Lemma 2.5 we saw that if $k \in L^q(X)$ for $q > 2$, then $\widehat{k}$ is Hölder continuous. In fact, for $q > 2$, every element $f \in W_1^q$ admits a precise Hölder continuous representative with exponent $1 - 2/q$ (cf. [63, p. 61]. On the other hand, $\widehat{k} \in L^q$ for any $q \geq 1$ and is therefore approximately continuous a.e.$-dA$. That is, for a.e.$-dA$ point $x_0 \in X$ there exists an exceptional set E with the property that

$$\frac{|B_r(x_0) \cap E|}{|B_r(x_0)|} \to 0$$

as $r \to 0$ and so that

$$f(x_0) = \lim_{z \to x_0,\ z \notin E} f(z)$$

(cf. [20]). Here, $B_r(x_0)$ denotes the disk with center at x_0 and radius r. However, we need a finer measure of the continuity enjoyed by $\widehat{k}$ when $q \leq 2$, and that continuity is best described in terms of capacity. Sobolev spaces and their associated capacities are treated extensively in [1], [20], [41], [43], and [63]. We shall assume throughout that $1 < q \leq 2$.

3.2 Sobolev and Potential Theoretic Capacities

For $1 < q \leq 2$, define the *Sobolev q-capacity* of a compact set $X \subset \mathbb{C}$ by

$$\Gamma_q(X) = \inf \int |\nabla u|^q \, dA,$$

where the infimum is taken over all infinitely differentiable functions u of compact support with $u \equiv 1$ on X. For an arbitrary set E, define

$$\Gamma_q(E) = \sup \Gamma_q(X),$$

where the supremum is taken over all compact sets $X \subset E$. All Borel sets are capacitable in the sense that it is also true that

$$\Gamma_q(E) = \inf \Gamma_q(G),$$

the infimum being taken over all open sets $G \supset E$. We say that a property holds $q-$*quasieverywhere* if it holds everywhere except on a set of $q-$capacity zero.

The Sobolev capacity Γ_q was introduced by Maz'ja [39] in 1961, and since then it has been studied in various contexts by a number of authors (see, e.g., [3], [4], [11], [29], [33], [34], [35], [40], [42], [44], [58], [62]). It is often useful, however, to have a different, but equivalent, definition of capacity. The *potential theoretic q-capacity* of a Borel set E is defined by

$$C_q(E)^{1/q} = \sup_{\nu} \nu(E),$$

where the supremum is taken over all positive measures ν concentrated on E for which $\|\widetilde{\nu}\|_p \leq 1$.

These two capacities are equivalent in that there exists a constant $A > 0$ so that

$$A^{-1}\Gamma_q(E) \leq C_q(E) \leq A\Gamma_q(E)$$

for every E (cf. [35, p. 312]). This (and similar equivalences) will be denoted by writing $C_q \approx \Gamma_q$. More information on these capacities, as well of proofs of the following can be found in the books [1] and [37] (cf. also [11], [13], [33], [35]):

(1) C_q is countably subadditive

(2) If B_r is any disk of radius r, then

$$C_q(B_r) \approx C_q(\mathrm{diam} B_r) \approx r^{2-q} \quad \textit{if } 1 < q < 2 \quad \text{and} \quad C_2(B_r) \approx \left(\log \frac{1}{r}\right)^{-1}$$

(3) If Φ is a contraction, $C_q(\Phi E) \leq AC_q(E)$ where A is a constant depending only on q (cf. [1, p. 140])

For any $\lambda > 0$, we have a weak-type inequality similar to Tchebyschev's inequality for L^1 functions:

$$\Gamma_q\{z \in \mathbb{C} : |\widehat{k}(z)| > \lambda\} \leq \frac{1}{\lambda^q} \int |\nabla \widehat{k}|^q \, dA$$

and this is key to obtaining the substitute for approximate continuity promised above. If $\widehat{k}_j = \widehat{k} * \chi_j$ is a sequence of mollifiers obtained by convolving $\widehat{k}$ with a C^∞ approximate identity χ_j, $j = 1, 2, 3, \ldots$, it is well-known that

$$\|\widehat{k}_j - \widehat{k}\|_q \to 0 \text{ and } \|\nabla\widehat{k}_j - \nabla\widehat{k}\|_q \to 0.$$

Passing to a subsequence if necessary, we can arrange that $\widehat{k}_j \to \widehat{k}$ uniformly off open sets of arbitrarily small q-capacity (cf. [19, p. 354] and [62, p. 124]). Hence, given any $\varepsilon > 0$ there exists an open set U so that $\Gamma_q(U) < \varepsilon$ and $\widehat{k}$ is continuous in the complement of U. Functions with this property are said to be $q-$*quasicontinuous.* Every W_1^q function agrees a.e.$-dA$ with a quasicontinuous representative. If $q > 2$, then $\widehat{k}$ is actually continuous as we have seen.

In addition to quasicontinuity, there is a pointwise notion more closely resembling approximate continuity which is also enjoyed by W_1^q functions, called *fine continuity.* A function h that is defined $q-$q.e. is said to be $q-$*finely continuous* at x_0 if there exists a set E that is thin in a potential theoretic sense at x_0 and

$$\lim_{z\to x_0,\ z\notin E} h(z) = h(x_0).$$

The precise sense in which E is understood to be thin is this: If $1 < q < 2$ a set E is $q-$*thin* at x_0 if and only if

$$\int_0 \left(\frac{\Gamma_q(E \cap B_r(z_0))}{r^{2-q}} \right)^{p-1} \frac{dr}{r} < \infty.$$

If E is not thin at x_0, then it is said to be *thick* there. It can be shown that every $q-$quasicontinuous function is $q-$finely continuous $q-$q.e. (cf. [1, p. 177]). Because C_q is countably subadditive ([1, p. 126]) and $\Gamma_q \approx C_q$, it follows that E is thick at x_0 whenever

$$\limsup_{r\to 0} \frac{\Gamma_q(E \cap B_r(x_0))}{r^{2-q}} > 0,$$

which is more in line with the aforementioned condition describing approximate continuity.

3.3 Analytic Capacity

Sobolev and potential theoretic $q-$capacities are set functions designed to measure the size of the exceptional sets associated with functions in the Sobolev space W_1^q, and therefore to measure the size of those associated with the Cauchy integral $\widehat{k}$ for a function $k \in L^q$. For this reason, $q-$capacity is especially useful in studying questions of approximation in the $L^p(dA)$ norm. However, in order to be able to present an accurate picture of the differences between the results in [14] and our work in section 4 we need to have a corresponding understanding of the exceptional sets for the Cauchy integral $\widehat{\mu}$ of an arbitrary measure μ. And for this, we need to consider analytic capacity, a concept introduced by Ahlfors in 1947.

The *analytic capacity* of a compact set X, denoted $\gamma(X)$ is defined as

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is taken over all functions f analytic in $\widehat{\mathbb{C}} \setminus X$, where $\|f\|_\infty = \sup_{\widehat{\mathbb{C}}\setminus X} |f| \leq 1$ and $f(\infty) = 0$. For a general set E, we define $\gamma(E) = \sup \gamma(X)$ where this supremum is taken over all compact sets $X \subset E$. For an extensive survey of the properties of analytic capacity and its relation to problems in approximation theory, the reader is referred to [28] and [61] (cf. also [57]). Two of the more basic properties of analytic capacity are these:

(i) $\gamma(B_r) = r$ for every disk B_r of radius r

(ii) $\gamma(K) \leq \mathrm{diam}(K) \leq 4\gamma(K)$ if K is compact and connected.

There is, however, an equivalent capacity γ^+ which is more directly linked to the Cauchy integral. For a compact set X, let

$$\gamma^+(X) = \sup_\nu \nu(X),$$

where the supremum is over all positive measures ν supported on X so that $\widehat{\nu} \in L^\infty(\mathbb{C})$ and $\|\widehat{\nu}\|_\infty \leq 1$. Since $\widehat{\nu}$ is analytic in $\widehat{\mathbb{C}} \setminus X$ and $\widehat{\nu}'(\infty) = \nu(X)$, the function $\widehat{\nu}$ which also vanishes at ∞ is admissible in the definition of γ and so

$$\gamma^+(X) \leq \gamma(X).$$

Again, if E is an arbitrary set in $\mathbb{C}$, we let

$$\gamma^+(E) = \sup_X \gamma^+(X),$$

where X is compact and $X \subset E$. Moreover, Tolsa [54] has shown that there exists an absolute constant $C > 0$ such that

$$\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$$

for all planar sets E, and therefore $\gamma \approx \gamma^+$. It follows that γ and γ^+ share the properties:

(1) If $E_1, E_2, \ldots$ are Borel sets then

$$\gamma\left(\bigcup_n E_n\right) \leq C \sum_n \gamma(E_n),$$

with C being an absolute constant; that is, γ is *countably semiadditive.*

(2) If μ is a complex measure and $\widehat{\mu}(x)$ is taken in the principal value sense, then for any $\lambda > 0$,

$$\gamma\{x \in \mathbb{C} : |\widehat{\mu}| > \lambda\} \leq \frac{C}{\lambda}|\mu|,$$

where $|\mu|$ denotes the total variation of the measure μ.

Perhaps the major difference between analytic capacity and $q-$capacity, in-so-far as we are concerned, is that if Φ is a contraction and $1 < q \leq 2$, then

$$C_q(\Phi E) \leq AC_q(E),$$

where A is a constant depending only on q (cf. [1, p. 140] and [26]). But, in the case of analytic capacity no such constant A exists. In fact, Garnett [30] and Vitushkin [56] have constructed compact sets X with the property that $\gamma(X) = 0$, but $\gamma(\Phi X) > 0$. This phenomenom played a key role in [14].

3.4 Instability of Capacity

Let E be an arbitrary Borel measurable subset of the complex plane. It is a well-known fact and a classic theorem (cf. [20]) that Lebesgue measure is unstable in the sense that for almost every $x \in \mathbb{C}$, either

$$\lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1 \qquad \text{or} \qquad \lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 0.$$

In the late 1960's, Vitushkin [57] was able to show that analytic capacity enjoys a similar instability. He proved that for almost every $x \in \mathbb{C}$, either

(i) $\lim_{r\to 0} \dfrac{\gamma(B_r(x)\cap E)}{r} = 1$, or

(ii) $\lim_{r\to 0} \dfrac{\gamma(B_r(x)\cap E)}{r^2} = 0$.

Contrasting this with the case of Lebesgue density, one might have expected the $\gamma-$capacitary density to either be 0 or 1. However, since $\gamma(B_r) = r$, the second conclusion is a stronger statement.

Around the same time that Vitushkin's work appeared in [57], Lysenko and Pisarevskiĭ [38] proved that a similar instability holds for harmonic capacity (ie. $2-$capacity), albeit in $\mathbb{R}^3$. On the other hand, Hedberg [35] discovered that each of the $q-$capacities considered here are unstable in the sense that the following two relations are equivalent for every Borel set $E \subset \mathbb{C}$:

(a) $C_q(E\cap\Omega) = C_q(\Omega)$ for every open set Ω

(b) $\limsup_{r\to 0} \dfrac{C_q(B_r(x)\cap E)}{r^2} > 0$ for a.e. $x \in \mathbb{C}$.

Shortly thereafter, Fernström [22] obtained the correct analogue of Vitushkin's theorem by showing that the limit as $r \to 0$ in (b) actually exists, and by also proving that for almost every $x \in \mathbb{C}$, either

(i) $\lim_{r\to 0} \dfrac{C_q(B_r(x)\cap E)}{r^{2-q}} = 1$

(ii) $\lim_{r\to 0} \dfrac{C_q(B_r(x)\cap E)}{r^2} = 0$

Here again, the conclusion in (ii) is stronger than what might be expected. We shall take full advantage of that fact for the construction in Section 4.

3.5 Rational Approximation

Necessary and sufficient conditions for the rational functions to be dense in either $C(X)$ or in $L^p(X)$ were first obtained by Vitushkin (cf. [57]) in the case of uniform approximation, and later by Hedberg [35] for L^p approximation. In both cases, the condition is expressed in terms of an appropriate capacity:

Theorem 3.1 (Vitushkin). *For a compact set X, the following are equivalent:*

(a) $R(X) = C(X)$

(b) $\limsup_{r\to 0} \dfrac{\gamma(B_r(x)\setminus X)}{r} > 0$ *for almost every* $x \in X$.

Theorem 3.2 (Hedberg). *For a compact set X and $2 < p < \infty$, the following are equivalent:*

(a) $R^p(X) = L^p(X)$

(b) $\limsup_{r\to 0} \dfrac{C_q(B_r(x)\setminus X)}{r^{2-q}} > 0$ *for almost every* $x \in X$.

In both theorems, the implication (b) $\Rightarrow$ (a) depends largely on the continuity of the Cauchy transform of an annihilator. In Hedberg's theorem, for example, suppose that $k \in L^q(X)$ and that $k \perp R^p(X)$. By our earlier discussion, $\widehat{k}$ is $q-$finely continuous q.e., and by assumption vanishes identically off X. Since (b) ensures that $\mathbb{C} \setminus X$ is $q-$thick at a.e. point of X, it follows that $\widehat{k} = 0$ a.e. on X. Thus by Thm. 2.2, $k = 0$ a.e.$-dA$ and so $R^p(X) = L^p(X)$.

In Vitushkin's theorem, the implication (b) $\Rightarrow$ (a) can be obtained from the following lemma, which gives a kind of *lower semicontinuity* to the Cauchy transform $\widehat{\mu}$ of a compactly supported measure μ. The proof, which can be found in [13], depends on Tolsa's theorem that $\gamma \approx \gamma^+$, and the lemma itself can be viewed as a generalization of an earlier idea of Carleson [16, Lemma 1].

Lemma 3.3 (Brennan)**.** *Let μ be a finite, complex, compactly supported measure in $\mathbb{C}$, and let x_0 be any point where $\widetilde{\mu}(x_0) < \infty$. Suppose that E is a set with the property that for each $r > 0$ there is a relatively large subset $E_r \subset (E \cap B_r(x_0))$ on which $\widetilde{\mu}$ is bounded, that is*

(1) $\widetilde{\mu} \leq M_r < \infty$ *on* E_r,

(2) $\gamma(E_r) \geq \varepsilon\gamma(E \cap B_r(x_0))$ *for some absolute constant* ε.

If E is thick at x_0 in the sense that

$$\limsup_{r\to 0} \frac{\gamma(E \cap B_r(x_0))}{r} > 0,$$

then $|\widehat{\mu}(x_0)| \leq \limsup_{z\to x_0,\ z\in E} |\widehat{\mu}(z)|$.

Going back to Vitushkin's theorem, let ν be any measure on X so that $\nu \perp R(X)$. Then, $\widehat{\nu} \equiv 0$ in $\mathbb{C} \setminus X$ and since (b) gives sufficient thickness, the lemma implies that for a.e. $x_0 \in X$

$$|\widehat{\nu}(x_0)| \leq \limsup_{z\to x_0,\ z\in\mathbb{C}\setminus X} |\widehat{\mu}(z)| = 0.$$

So $\widehat{\nu} = 0$ a.e.$-dA$ on X, and hence $R(X) = C(X)$.

In the next section, we will present several examples whose constructions depend on the fact that the capacitary density condition (b) in both the Vitushkin and Hedberg theorems can be replaced using the instability of capacity by a stronger condition. In particular, for Hedberg's theorem, the instability of $q-$capacity allows us to conclude that if for a.e. $x \in \mathbb{C} \setminus X$

$$\limsup_{r\to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} > 0$$

then $R^p(X) = L^p(X)$, and conversely, even when $p = 2$ (cf. [35, p. 316]).

4 Construction of Compact Nowhere Dense Sets

From the preceding discussion, there are compact nowhere dense sets X for which each one of following two possibilities have been realized:

(1) $R^p(X) = L^p(X)$ for $1 \leq p < 2$, but $R^p(X) \neq L^p(X)$ if $p \geq 2$

(2) $R^p(X) = L^p(X)$ for $1 \leq p < \infty$, but $R(X) \neq C(X)$

As was shown in Theorem 2.4, for a compact nowhere dense set, density is guaranteed for $1 \leq p < 2$. To ensure that property (1) is satisfied, it is sufficient to construct a Swiss cheese X which has a bpe for $R^2(X)$ at some point $x_0 \in X$ (cf. [9, p. 301]). In the second case, (2), the difficulties are more subtle (cf. [52, p.114] and [14, p. 49]), but together these two examples provide motivation for the main theorem of this section.

Theorem 4.1. *Fix* p^* *with* $2 < p^* < \infty$. *There exists a compact nowhere dense set* X *in the plane so that*

(i) $R^p(X) = L^p(X)$ *for* $1 \leq p < p^*$

(ii) $R^p(X) \neq L^p(X)$ *if* $p \geq p^*$.

In their 2011 paper, Brennan and Militzer [14] constructed a set which satisfies (2). There are some important differences between the construction in [14] and the construction of the set promised in Theorem 4.1. For example, the argument in [14] depends in an essential way on the fact that $q-$capacity C_q and analytic capacity γ behave in fundamentally different ways under a contraction. In order to provide some background and to contrast the arguments involved, we shall first recall the line of reasoning in [14] and later return to the proof of Theorem 4.1.

The argument in [14] begins with the construction of a planar Cantor set as follows: Let Q be the closed unit square, split Q into sixteen congruent squares of side length $1/4$ and choose the four corner squares, that is those squares which contain a vertex of Q. Apply the same procedure to each of the four squares obtained in the first step, and continue in this manner. At the $n-$th stage, there are 4^n closed squares Q_j^n, $j = 1, 2, \dots 4^n$, each having side length $1/4^n$. For each n, define

$$E_n = \bigcup_{j=1}^{4^n} Q_j^n$$

and let

$$K = \bigcap_{n=1}^{\infty} E_n.$$

The set K is known as the *corner quarters Cantor set.* The orthogonal projection of K onto the line $2y = x$ covers an interval of length $3/\sqrt{5}$, and therefore of length greater than $\frac{1}{2}\text{diam}(Q)$. Garnett [30] and Ivanov [36, pp. 346-348] have shown that $\gamma(K) = 0$. A similar, but more complicated example of this kind was first obtained by Vitushkin [56].

Now iterate the procedure outlined above. Decompose Q into 4 congruent squares S_j^1, $j = 1, 2, 3, 4$. In each square S_j^1, construct another Cantor set K_j^1 similar to K with a scaling factor of $1/4$. Let $K_1 = \cup_j K_j^1$. Continue the process by decomposing Q into 4^n congruent squares S_j^n, in each of which a Cantor set K_j^n similar to K is contructed. Thus we obtain a sequence of Cantor sets $K_1, K_2, \dots$ with $K_n = \cup_j K_j^n$ and

(i) $\gamma(K_n) = 0$

(ii) $E = \cup K_n$ is dense in Q

(iii) $\Lambda_1\left(\text{proj}(K_j^n)\right) > \frac{1}{2}\text{diam}(S_j^n)$.

where $\text{proj}(K_j^n)$ denotes the orthogonal projection of K_j^n onto the line $2y = x$, and $\Lambda_1\left(\text{proj}(K_j^n)\right)$ denotes the 1−dimensional Hausdorff measure or length of the projection. It follows from Tolsa's theorem on the countable semiadditivity of analytic capacity that $\gamma(E) = 0$, and so $|E| = 0$ also.

Choose a compact set X_0 lying in the interior of Q so that $|X_0| > 0$ and $E \cap X_0 = \emptyset$. Let r_1 be small enough that $\{z : \text{dist}(z, X_0) < r_1\}$ lies inside Q. Since K_1 is a compact totally disconnected set with $\gamma(K_1) = 0$, it is possible to cover K_1 by finitely many open rectangles with sides parallel to the coordinate axes, having mutually disjoint closures, and so that their union Ω_1 satisfies $\gamma(\Omega_1) < \frac{1}{2}r_1$. Next, choose $r_2 < r_1$ so that $\{z : \text{dist}(z, X_0) < r_2\}$ does not meet $\overline{\Omega}_1$. In a completely analogous fashion, cover $K_2 \setminus \overline{\Omega}_1$ by open rectangles whose union Ω_2 satisfies

(i) $\gamma(\Omega_2) < \dfrac{1}{2^2}r_2$

(ii) $\gamma(\Omega_1 \cup \Omega_2) < C\left(\dfrac{r_1}{2} + \dfrac{r_2}{2^2}\right) < Cr_1$,

where C is an absolute constant guaranteed by Tolsa's theorem. Continuing in this way, we arrive at a sequence of numbers $r_j \downarrow 0$ and a sequence of open sets $\Omega_1, \Omega_2, \ldots$ so that

(a) $E \subset \bigcup\limits_j \Omega_j$

(b) $X_0 \subset Q \setminus (\bigcup\limits_j \Omega_j)$

(c) $\gamma(\Omega_j) < \dfrac{1}{2^j}r_j$

(d) $\gamma(\Omega_1 \cup \ldots \cup \Omega_j) < \dfrac{C}{2^{j-1}}r_j$ for all $j = 1, 2, \ldots$

Setting $X = Q \setminus (\cup_j \Omega_j)$ we obtain a compact nowhere dense set with the desired properties, that is $R(X) \neq C(X)$, but $R^p(X) = L^p(X)$ for all p, $1 \leq p < \infty$.

For each point $x \in X_0$, we have

$$\frac{\gamma(B_{r_j}(x) \setminus X)}{r_j} \leq \frac{C}{2^{j-1}}$$

for all $j = 1, 2, \ldots$ with C an absolute constant. Thus, at each point of X_0 the lower capacitary density of $\mathbb{C} \setminus X$ is zero. By the instability of capacity,

$$\lim_{r \to 0} \frac{\gamma(B_r(x) \setminus X)}{r} = 0$$

at a.e.$-dA$ point of X_0, and so by Vitushkin's theorem (Thm. 3.1), $R(X) \neq C(X)$.

Again, for a.e.$-dA$ point $x \in X$ and r sufficiently small,

$$\Lambda_1\left(\text{proj}(B_r(x) \setminus X)\right) \geq Cr,$$

where C is an absolute constant. Since $q-$capacity decreases modulo a multiplicative constant under a contraction (cf. [1, p. 140] and [26]), for a fixed $q < 2$ this implies that $C_q(B_r(x) \setminus X) \geq Cr^{2-q}$. Therefore by Hedberg's theorem (Thm. 3.2), it follows that $R^p(X) = L^p(X)$ for all p.

The preceding discussion highlights the subtleties involved in ensuring that property (2) holds. However, as we return to our proof of Theorem 4.1, it should be pointed out that a different approach is required to cut off the density in L^p at some specific value greater than 2.

Proof of Theorem 4.1. Begin with a constant p^* where $2 < p^* < \infty$. Let q^* be the dual exponent to p^*, that is $q^* = p^*/(p^* - 1)$. We shall construct a compact set X with the property that either

$$\limsup_{r\to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} > 0 \qquad \text{or} \qquad \lim_{r\to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} = 0$$

for a.e. $x \in X$, depending on whether $q > q^*$ or $q \leq q^*$, respectively; or equivalently whether $p < p^*$ or $p \geq p^*$. The desired result will then be an immediate consequence of Hedberg's Theorem 3.2.

Start with the closed unit square $Q = [0,1] \times [0,1]$. We shall place a grid of squares inside of Q consisting of lines parallel to the coordinate axes. Let δ_1 be the side length of a generic square grid in Q. At each vertex of the grid in the interior of Q, remove a much smaller disk Δ_{α_1} of radius $\delta_1^{\alpha_1}$ whose closure does not meet ∂Q, and where $\alpha_1 > 0$ has yet to be determined. Form the set

$$X_1 = Q \setminus \bigcup \Delta_{\alpha_1},$$

where the union is taken over the entire family of deleted disks. Since any disk B_{δ_1} of radius δ_1 meets at least one, and at most four Δ_{α_1}'s, it follows from the subadditivity of $q-$capacity that

$$\frac{C_q(B_{\delta_1} \setminus X_1)}{\delta_1^2} \approx \frac{(\delta_1^{\alpha_1})^{2-q}}{\delta_1^2} = \delta_1^{\alpha_1(2-q)-2},$$

provided B_{δ_1} lies entirely inside Q. If $2 > q_1 > q^*$ and α_1 is chosen so that

$$\frac{2}{2-q^*} < \alpha_1 < \frac{2}{2-q_1},$$

then we can choose δ_1 sufficiently small so that

(1) $\dfrac{C_{q_1}(B_{\delta_1}(x) \setminus X_1)}{\delta_1^2} > 1/2$ for all $x \in X_1$.

(2) $\dfrac{C_{q^*}(B_{\delta_1}(x) \setminus X_1)}{\delta_1^2} < \varepsilon/2$ for all $x \in X_1$ with $\mathrm{dist}(x, \partial Q) \geq \delta_1$.

for an arbitrary, but fixed, $\varepsilon > 0$.

Now continue the process. Pick a sequence

$$2 > q_1 > q_2 > \cdots > q^*$$

so that $q_j \downarrow q^*$, or equivalently, $p_j \uparrow p^*$. Choose a second generation grid of side length δ_2 and fix

α_2 so that

$$\frac{2}{2-q^*} < \alpha_2 < \frac{2}{2-q_2}.$$

We may assume that δ_2 is sufficiently small to ensure that by subadditivity the total q^*-capacity of the union of all disks Δ_{α_2} of radius $\delta_2^{\alpha_2}$ at points of the new grid does not exceed

$$\frac{1}{\delta_2^2}(\delta_2^{\alpha_2})^{2-q^*} = \delta_2^{\alpha_2(2-q^*)-2} < \frac{\varepsilon}{4}\ \delta_1^2.$$

Now remove from X_1 those disks Δ_{α_2} centered at points of the new grid lying in the interior of X_1 whose closures do not meet ∂X_1 and set

$$X_2 = X_1 \setminus \bigcup \Delta_{\alpha_2},$$

where again the union is over all deleted disks.

Taking δ_2 even smaller if necessary, we can arrange that the inequalities

$$(3)\ \frac{C_{q_2}(B_{\delta_2}(x) \setminus X_2)}{\delta_2^2} > 1/2 \quad \text{and} \quad \frac{C_{q_1}(B_{\delta_2}(x) \setminus X_2)}{\delta_2^2} > 1/2 \quad \text{for all } x \in X_2.$$

$$(4)\ \frac{C_{q^*}(B_{\delta_2}(x) \setminus X_2)}{\delta_2^2} < \varepsilon/4 \quad \text{for all } x \in X_2 \text{ with } \operatorname{dist}(x, \partial X_1) \geq \delta_2$$

are also satisfied simultaneously. At this stage, inequalities (1) and (2) are essentially preserved, except that (2) is now replaced by

$$(2')\ \frac{C_{q^*}(B_{\delta_1}(x) \setminus X_2)}{\delta_1^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \quad \text{for all } x \in X_2 \text{ with } \operatorname{dist}(x, \partial X_1) \geq \delta_1.$$

Continuing in this manner, we obtain a descending sequence of compact sets $X_1 \supset X_2 \supset \cdots$ together with sequences $q_j \downarrow q^*$ and $\delta_j \downarrow 0$ so that whenever $k \geq j$

$$(5)\ \frac{C_{q_n}(B_{\delta_j}(x) \setminus X_k)}{\delta_j^2} > 1/2, \ n = 1, 2, \ldots, j, \quad \text{for all } x \in X_k.$$

$$(6)\ \frac{C_{q^*}(B_{\delta_j}(x) \setminus X_k)}{\delta_j^2} < \frac{\varepsilon}{2^j} + \cdots + \frac{\varepsilon}{2^k} \quad \text{for all } x \in X_k \text{ with } \operatorname{dist}(x, \partial X_j) \geq \delta_j. \text{ are both satisfied.}$$

Now, define the set

$$X = \bigcap_{n=1}^{\infty} X_n.$$

Since C_q is a capacity in the Choquet sense and $(B \setminus X_k) \uparrow (B \setminus X)$ for any disk B,

$$C_q(B \setminus X) = \lim_{n\to\infty} C_q(B \setminus X_n)$$

for any q, $1 \leq q < 2$, (cf. [48, p. 262] and [1, p. 29]). In particular, it follows that

$$(7)\ \frac{C_{q_n}(B_{\delta_j}(x) \setminus X)}{\delta_j^2} > 1/2 \quad \text{whenever } j \ge n \text{ and } x \in X.$$

$$(8)\ \frac{C_{q^*}(B_{\delta_j}(x) \setminus X)}{\delta_j^2} < \frac{\varepsilon}{2^j} + \frac{\varepsilon}{2^{j+1}} + \cdots = \frac{\varepsilon}{2^{j-1}} \text{ for all } x \in X \text{ with } \operatorname{dist}(x, \partial X_j) \ge \delta_j.$$

Letting $j \to \infty$ it follows from the instability of capacity that

$$\text{(a) } \limsup_{r\to 0} \frac{C_{q_n}(B_r(x) \setminus X)}{r^2} > 1/2,\ n = 1, 2, \ldots$$

$$\text{(b) } \lim_{r\to 0} \frac{C_{q^*}(B_r(x) \setminus X)}{r^2} = 0$$

for almost every $x \in X$, since $\delta_j \downarrow 0$.

In view of property (a), there is a sequence $p_n \uparrow p^*$ for which $R^{p_n}(X) = L^{p_n}(X)$, $n = 1, 2, \ldots$, and therefore $R^p(X) = L^p(X)$ for all $p < p^*$. Property (b), on the other hand, implies that $R^p(X) \ne L^p(X)$ for any $p \ge p^*$. □

5 Bounded Point Evaluations and Rational Approximation

It is well known that bounded point evaluations (or more properly unbounded point evaluations) play a role in L^p-approximation similar to that of peak points in uniform rational approximation. By definition a point $x \in X$ is a *peak point* for $R(X)$ if there exists a function $f \in R(X)$ such that $f(x) = 1$, but $|f(y)| < 1$ whenever $y \ne x$. The source of the analogy is unmistakable in the following theorem:

Theorem 5.1. *If X is a compact nowhere dense subset of the complex plane, then*

(1) $R(X) = C(X)$ *if and only if almost every point of X is a peak point for $R(X)$.*

(2) $R^p(X) = L^p(X)$ *if and only if almost every point of X fails to be a bpe for $R^p(X)$, whenever $p \ne 2$.*

Part (1) is a theorem of Bishop [7], and part (2) is from [9]. From our remarks in Section 2 it is easy to see that when $p < 2$ the only bpe's for $R^p(X)$ are interior points, but when $p \ge 2$ there may well be bpe's on ∂X. What is most intriguing here is the exclusion of the case $p = 2$ in the statement of the theorem, underscoring once again the special nature of the situation when $p = 2$. Our objective in this section is to sketch a proof along the lines outlined in Theorem 4.1 of a theorem of Fernström [21] which deals with the anomalous situation.

Theorem 5.2 (Fernström)**.** *There exists a compact nowhere dense subset X of the complex plane with the property that*

(1) $R^2(X)$ *has no bounded point evaluations, but*

(2) $R^2(X) \ne L^2(X)$.

In order to proceed we will need an effective way to determine whether a given point $x_0 \in X$ is a bpe for $R^p(X)$, or not. Such a criterion was obtained by Hedberg [34], and is analogous to

Mel'nikov's criterion [45] for peak points in $R(X)$. An earlier necessary condition can be found in [9]. The theorem we need is this:

Theorem 5.3. *Let X be a compact nowhere dense subset of the complex plane, let $x_0 \in X$, and for each $n = 1, 2, 3, \ldots$, let $A_n(x_0) = \{z : 2^{-(n+1)} < |z - x_0| \le 2^{-n}\}$. Then,*

(1) *x_0 is a peak point for $R(X)$ if and only if*

$$\sum_{n=1}^{\infty} 2^n \, \gamma \left(A_n(x_0) \setminus X \right) = \infty.$$

(2) *x_0 is a bpe for $R^p(X)$ when $p \ge 2$ if and only if*

$$\sum_{n=1}^{\infty} 2^{nq} \, \Gamma_q \left(A_n(x_0) \setminus X \right) < \infty.$$

Initially, when $p = 2$ Hedberg [34] was only able to show that the the condition in part (2) of the theorem is necessary for the existence of a bpe at x_0, and at the same time he obtained a slightly weaker sufficient condition. Subsequently, Fernström and Polking [23] were able to show that it is both necessary and sufficient. Although $\Gamma_q \approx C_q$, Hedberg bases his argument on the Sobolev capacity Γ_q and so for easy reference we have retained that approach here.

Proof of Theorem 5.2. Our goal here is to construct a compact set X such that the following two conditions are satisfied simultaneously:

(i) $\displaystyle\sum_{n=1}^{\infty} 2^{2n} \, \Gamma_2(A_n(x) \setminus X) = \infty$ for every $x \in X$;

(ii) $\displaystyle\lim_{n \to \infty} 2^{2n} \, \Gamma_2(B_n(x) \setminus X) = 0$ for almost every $x \in X$.

According to part (2) of Theorem 5.3 property (i) implies that $R^2(X)$ has no bounded point evaluations. And, from our remarks at the end of Section 3 property (ii) ensures that $R^2(X) \ne L^2(X)$.

Once again, we begin with the unit square $Q = [0, 1] \times [0, 1]$. Place a grid of squares inside Q having side length $\delta_1 = 1/2^{n_1}$, and consisting of lines parallel to the coordinate axes. At each vertex of the grid in the interior of Q remove a much smaller disk Δ_{α_1} of radius $\delta_1^{\alpha_1}$ whose closure does not meet ∂Q, where $\alpha_1 > 0$ has yet to be determined. Form the set

$$X_1 = Q \setminus \bigcup \Delta_{\alpha_1},$$

the union being taken over the entire family of deleted disks. Since any disk B_{δ_1} of radius δ_1 meets at least one, and at most four of the Δ_{α_1}'s, it follows from the subadditivity of 2 − capacity that

$$\frac{\Gamma_2(B_{\delta_1} \setminus X_1)}{\delta_1^2} \approx \frac{\Gamma_2(A_{\delta_1} \setminus X_1)}{\delta_1^2} \approx \frac{1}{\delta_1^2} \left(\log \frac{1}{\delta_1} \right)^{-1} = \left(\frac{2^{\,2n_1}}{n_1 \alpha_1} \right) \frac{1}{\log 2},$$

whenever $A_{\delta_1} = B_{\delta_1} \setminus B_{\delta_1/2}$ and B_{δ_1} lie entirely inside Q. Here, and throughout the current discussion, the symbol $\approx$ will denote equality up to multiplication by an absolute constant. By choosing $\alpha_1 \approx 2^{\,2n_1}/n_1$ we can arrange that

(1) $2^{\,2n_1}\Gamma_2\,(A_{n_1}(x) \setminus X_1) \geq 1$ for every $x \in X_1$;

(2) $2^{\,2n_1}\Gamma_2\,(B_{n_1}(x) \setminus X_1) < A$ for every $x \in X_1$ with $\operatorname{dist}(x, \partial Q) \geq \delta_1$, and some absolute constant A.

Now continue the process, choosing a second generation grid of squares having individual side lengths $\delta_2 = 1/2^{n_2}$ with $n_2 >> n_1$. At each vertex of the new grid lying inside X_1 remove a much smaller disk Δ_{α_2} of radius $\delta_2^{\alpha_2}$ whose closure does not meet ∂X_1, where n_2 and α_2 have not as yet been specified, and set

$$X_2 = X_1 \setminus \bigcup \Delta_{\alpha_2},$$

the union being taken over all deleted disks.

Once again, from the subadditivity of capacity it follows that

$$\frac{\Gamma_2(B_{\delta_2} \setminus X_2)}{\delta_2^2} \approx \frac{\Gamma_2(A_{\delta_2} \setminus X_2)}{\delta_2^2} \approx \frac{1}{\delta_2^2}\left(\log\frac{1}{\delta_2}\right)^{-1} = \left(\frac{2^{\,2n_2}}{n_2\alpha_2}\right)\frac{1}{\log 2},$$

whenever $A_{\delta_2} = B_{\delta_2} \setminus B_{\delta_2/2}$ and B_{δ_2} lie entirely inside X_1. By choosing $\alpha_2 \approx 2^{\,2n_2}/n_2$ we can arrange that

(3) $2^{\,2n_2}\,\Gamma_2\,(A_{n_2}(x) \setminus X_2) \geq 1/2$ for every $x \in X_2$, and

(4) $2^{\,2n_2}\,\Gamma_2\,(B_{n_2}(x) \setminus X_2) < A/2$ for every $x \in X_2$ with $\operatorname{dist}(x, \partial X_1) \geq \delta_2$

are satisfied simultaneously. Taking n_2 sufficiently large, and therefore $\delta_2^{\alpha_2}$ sufficiently small, (2) can be replaced by

(2′) $2^{\,2n_1}\,\Gamma_2\,(B_{n_1}(x) \setminus X_2) < A + \dfrac{A}{2}$ for every $x \in X_1$ with $\operatorname{dist}(x, \partial X_1) \geq \delta_1$.

Continuing in this manner, we obtain a descending sequence of compact sets $X_1 \supset X_2 \supset \cdots$ together with a sequence $\delta_j \downarrow 0$ so that whenever $k \geq j$

(5) $2^{\,2n_k}\,\Gamma_2\,(A_{n_k}(x) \setminus X_k) \geq 1/k$ for every $x \in X_k$

(6) $2^{\,2n_j}\,\Gamma_2\left(B_{n_j}(x) \setminus X_k\right) < \dfrac{A}{2^j} + \cdots + \dfrac{A}{2^k} < \dfrac{A}{2^{j-1}}$ for all $x \in X_k$ with $\operatorname{dist}(x, \partial X_j) \geq \delta_j$

are both satisfied. In particular, if we define

$$X = \bigcap_{k=1}^{\infty} X_k$$

it follows from our construction, the fact that $\delta_j \downarrow 0$, and the instability of capacity, that

(i) $\displaystyle\sum_{n=1}^{\infty} 2^n\,\Gamma_2(A_n(x) \setminus X) \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty$ for every $x \in X$;

(ii) $\lim_{n\to\infty} 2^{2n}\,\Gamma_2(B_n(x)\setminus X) = 0$ for almost every $x \in X$.

Therefore, $R^2(X)$ has no bpe's, but $R^2(X) \neq L^2(X)$. □

Let us suppose briefly that $R^p(X) \neq L^p(X)$ for some $p > 2$. It follows from Theorem 3.2 and the instability of capacity that there exists some subset $E \subset X$ of positive area such that

$$\lim_{r\to 0} \frac{\Gamma_q(B_r(x)\setminus X)}{r^2} = 0$$

for all $x \in E$, where q is the index conjugate to p. At any such point x the inequality $\Gamma_q(B_r(x)\setminus X) \leq Cr^2$ is satisfied for some constant C depending on x. Setting $r = 1/2^n$ we conclude that

$$\sum_{n=1}^{\infty} 2^{nq}\,\Gamma_q(A_n(x)\setminus X) \leq C\sum_{n=1}^{\infty} 2^{n(q-2)} < \infty,$$

since $q < 2$. Thus, every point $x \in E$ is a bpe for $R^p(X)$. This is consistent with, and was predicated by, Theorem 5.1.

6 The Uniqueness Property on Sets Without Interior Points

At his thesis defense in 1894, at which Poincaré was the rapporteur, Émile Borel suggested that it should be possible to extend the theory of analytic functions to larger classes of functions defined on sets without interior points in such a way that the distinctive property of unique continuation is preserved. Poincaré, having constructed certain examples with the aim of disproving any possibility of such an extension, took a negative point of view. At that time, however, Borel was not yet able to exhibit the kind of extension he envisioned, and so the matter remained unresolved for many years. But, eventually Borel was able to put his ideas on a more solid foundation. To that end he introduced the notion of a monogenic function. By definition a function f defined on a set E is *monogenic* at a point $x_0 \in E$ if it has a derivative at x_0 in the sense that

$$\lim_{x\to x_0,\, x\in E} \frac{f(x)-f(x_0)}{x-x_0}$$

exists through points of E (cf. [6, p. 123]) Thus, if Ω is an open set then a function f is monogenic at each point of Ω if and only if it is analytic in Ω. Borel's main result (cf. [8]), published almost a quarter of a century after his confrontation with Poincaré, was to construct a compact set X having no interior and containing a large dense subset E such that every function f monogenic on X is uniquely determined by its value and the values of all its derivatives at any fixed point of E. In this setting he also obtained an integral representation of Cauchy type for monogenic functions, and he provided the basis and inspiration for the subsequent development of the theory of *quasianalytic classes* by Denjoy, Carleman, Bernštein, Beurling, Vol'berg and others. A nice survey of the early results in this area can be found in the articles by Bilodeau [6] and Fuglede [24], [25].

Our principle objective in this section is to give a brief indication of how the uniqueness property of the analytic functions can be extended to various classes of functions defined on closed sets without interior points. To that end we will first sketch a proof of the following theorem from [10]:

Theorem 6.1. *There exists a compact set X_0 having no interior and positive dA measure such that whenever any two functions in $R^p(X_0)$, $p \geq 2$, coincide on a set of positive measure in X_0 they coincide almost everywhere.*

The first example of this kind in connection with L^p approximation was constructed by Sinanjan [51] in 1965. At that time he was able to establish the existence of a compact nowhere dense set X with the property that whenever two functions in $R^p(X)$, $p \geq 2$, agree on a *relatively open* subset of X, they agree almost everywhere. Theorem 6.1 strengthens that result. The history of the various attempts to extend the uniqueness property of the analytic functions to broader classes of functions defined on sets without interior points is not, however, entirely clear. There is an oblique reference by Mergeljan (cf. [47, pp. 317-318]) to the existence of a compact nowhere dense set X such that the functions in the uniform closure $R(X)$ are monogenic in the sense described above, and so if any two of them coincide on a *arbitrary portion* of X, then they are identical everywhere on X. Sinanjan [51, p. 1365] and [46, p. 765] credits this result to M. V. Keldysh, but no specific reference is given. Presumably, an arbitrary portion refers to an arbitrary relatively open subset of X. Unfortunately, in [46] the Russian expression for "arbitrary portion" is erroneously translated as "derivative portion". A. A. Gonchar has exhibited what is perhaps a qualitatively definitive result in the case of the uniform metric by constructing a compact nowhere dense set X with the property that if any two functions in $R(X)$ coincide on a set of positive one-dimensional Hausdorff measure (ie. positive length), then they coincide everywhere on X. Gonchar's result, however, appears only in the survey article of Mel'nikov and Sinanjan [46, p. 746-748]. His argument is similar in spirit to the proof of Theorem 6.1 outlined below in that it avoids lengthy and cumbersome computations.

Before sketching the proof of Theorem 6.1 two results need to be mentioned. The first is due to W. K. Allard and furnishes additional information on the surprising underlying geometric nature of a Swiss cheese $E = \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ (cf.[9, p. 304] and [14, p. 43]). The second is a theorem of Denjoy [18] for quasianalytic classes on the real line.

Lemma 6.2. *Let E be a Swiss cheese, and for each $x \in [-1,1]$ let $E_x = \{z \in E : Re\, z = x\}$. For almost every $x \in [-1,1]$ the set E_x is the union of finitely many disjoint non degenerate intervals.*

It follows from the lemma that almost every pair of points in a Swiss cheese E can be joined by finitely many line segments in E and finitely many subarcs of ∂D and ∂D_j, $j = 1, 2, \ldots$. It is this property of E that will be of critical importance in the proof of Theorem 6.1. Since the proof of the lemma is quite short and is related to another important idea from geometric measure theory, namely to the *Banach indicatrix theorem* [5], we present it here in its entirety.

Proof of Lemma 6.2. Set $D_0 = D$, and for each $x \in [-1,1]$ let $N_j(x)$, $j = 1, 2, \ldots$ denote the number of points in $E_x \cap \partial D_j$. Thus, $N_j(x) = 0, 1, \text{or } 2$, and it follows from the monotone convergence theorem for integrals that

$$\int_{-1}^{1} \sum_{j=0}^{\infty} N_j(x)\, dx = \sum_{j=0}^{\infty} \int_{-1}^{1} N_j(x)\, dx \leq \sum_{j=0}^{\infty} \text{length}(\partial D_j) < \infty.$$

Hence, $\sum_{j=0}^{\infty} N_j(x) < \infty$ for almost every $x \in [-1,1]$. For any such x all but finitely many $N_j(x) = 0$, and the Lemma follows. $\square$

Theorem 6.3 (Denjoy) *Let f be a function of one real variable defined and having derivatives of all orders on a closed interval* [a,b]. *Assume that* $\sup_{[a,b]} |f^{(n)}| \le M_n$, $n = 0, 1, 2, \dots$. *Then, f is uniquely determined by its value and the values of all its derivatives at any fixed point $x_0 \in [a, b]$ if*

$$\sum_{n=0}^{\infty} \left(\frac{1}{M_n}\right)^{1/n} = \infty.$$

Let X be a compact set and fix a point $\zeta \in X$. We shall say that $R^p(X)$ has a *bounded point derivation of order n* at ζ if there exists a constant $C > 0$ such that

$$|f^{(n)}(\zeta)| \le C \, \|f\|_{L^p(X)}$$

for every rational function f having no poles on X. Evidently, $R^p(X)$ has bounded point derivations of all orders at each interior point of X. In case $p \ge 2$, Theorem 6.1 suggests that this is not the exclusive property of points in the interior of X. The study of point derivations for $R(X)$, relative to the uniform norm, was initiated by Wermer [60] in 1967. Included among Wermer's results is a example of a Swiss cheese X with the property that $R(X)$ admits a bounded point derivation of order 1 at almost every point of X. Hallstrom [31] subsequently generalized this by producing a Swiss cheese X with the added feature that $R(X)$ admits bounded point derivations of all orders at almost every point of X, which incidentally is implied by the argument presented below in the proof of Theorem 6.1. To accomplish this, however, he first obtained a necessary and sufficient condition in terms of analytic capacity for $R(X)$ to have a bounded point derivation of a given order at a fixed point of X. That condition can be viewed as an extension of Mel'nikov's peak point criterion for $R(X)$ (cf. [45] and [28, p. 205]). Hedberg [34] later obtained analogous criteria in terms of the Sobolev capacity Γ_q to test for bounded point derivations on $R^p(X)$ when $p \ge 2$ (cf. also [23]).

Proof of Theorem 6.1. We may assume that $p = 2$. The idea is to construct a Swiss cheese X_0 in such a way that it contains an increasing sequence of compact sets $E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots$ with the following properties:

(1) Each E_k is a Swiss cheese whose complementary components are bounded by polygonal arcs;

(2) $\text{meas}\{X_0 \setminus E_k\} \to 0$ as $k \to \infty$;

(3) For each k there are positive constants A_n, $n = 0, 1, 2, \dots$, such that

$$|f^{(n)}(\zeta)| \le A_n \, \|f\|_{L^2(X_0)}$$

for all rational functions f having no poles on X_0, and all $\zeta \in E_k$;

(4) $\displaystyle\sum_{n=0}^{\infty} (1/A_n)^{1/n} = \infty.$

The crucial properties (3) and (4) are easily arranged by taking advantage of the fact that, not only the values of the Bergman kernel, but also those of all its derivatives can be controlled under a deformation of the underlying region in the manner described here in Theorem 2.6. For the details of the construction see [10].

Suppose now that X_0 has been constructed as indicated, and that a function $f \in R^2(X_0)$ vanishes on a set of positive dA measure in X_0. By assumption f is the limit in the $L^2(X_0)$ norm of a sequence of rational functions f_j, $j = 1, 2, \ldots$, having no poles on X_0, and so by (3) the sequence of derivatives $f_j^{(n)}, j = 1, 2, \ldots$, converges uniformly on E_k, $k = 1, 2, \cdots$ for each n. Hence, the limit function f has directional derivatives of all orders along any line segment lying in E_k, $k = 1, 2, \ldots$. That is, the restriction of f to such a line segment can be viewed as a C^∞ function of one real variable. Since $f = 0$ on a set of positive dA measure in X_0, it follows from (2) that $f = 0$ on a set of positive dA measure in E_k if k is sufficiently large. We can infer from Lemma 6.2 and Fubini's theorem that $f = 0$ on a set of positive linear measure on some line segment l lying in E_k. Moreover, $\sup_l |f^{(n)}| \leq A_n \|f\|_{L^2(X_0)}$, and so by property (4), f satisfies the conditions of Denjoy's theorem with $M_n = A_n \|f\|_{L^2(X_0)}$. If x_0 is a point of linear density in l for the zero set of f it follows that $f^{(n)}(x_0) = 0$, $n = 0, 1, 2, \ldots$. Here, of course, $f^{(n)}$ denotes differentiation along l. Therefore, by quasianalyticity, f vanishes identically on l. On the other hand, by Lemma 6.2 and the accompanying remark, we may assume that x_0 can be joined to almost every other point of E_k by a polygonal arc in E_k whose initial segment belongs to l. Again, by quasianalyticity, f vanishes identically along any such arc, since at each vertex its derivatives coincide in the appropriate directions. Therefore, $f = 0$ almost everywhere on E_k for all k, $k = 1, 2, \ldots$, and so by property (2), almost everywhere on X_0. □

With the proof of Theorem 6.1 and the constructions in Sections 4 an 5 as background it is now possible to sketch a proof of Gonchar's theorem which is more transparent in certain of its technical aspects, and closer in spirit to Borel's original presentation in [8] (cf. also [46]).

Theorem 6.4 (Gonchar) *There exists a compact nowhere dense set X_0 with the property that each function in $R(X_0)$ is completely determined by its values on any subset of positive one-dimensional Hausdorff measure.*

Proof. Fix a small closed line segment $l = \{z : Re\, z = 0 \;\; and \;\; |Im\, z| \leq \varepsilon\}$ containing the origin and lying in the open unit disk D. The idea here is to remove small open disks D_j, $j = 1, 2, \ldots$, not meeting l and centered at the vertices of successively finer dyadic grids in such a way that the resulting set $X_0 = \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ contains an increasing sequence of compact sets E_k, $k = 1, 2, \ldots$, such that $l \subset E_k$ for each k, and properties (1) − (4) in the proof of Theorem 6.1 are once again satisfied. Now, set $\Omega_n = \overline{D} \setminus \bigcup_{j=1}^{n} D_j$ and let ω_n^x denote harmonic measure on $\partial\Omega_n$ representing any fixed point $x \in l$. As before, Λ_1 will denote one-dimensional Hausdorff measure. By construction (cf. [10]), each intermediate set E_k has the form $E_k = \Pi_k \setminus \bigcup_j S_j$, where Π_k is a region bounded by finitely many polygonal curves and the S_j's are squares lying inside Π_k and having mutually disjoint closures. Then, by carefully selecting the squares S_j and disks $D_j \subset S_j$ we can arrange as in [10] that length $(\partial D_j) <$ length (∂S_j), and moreover with $A_n(x) = \{z : 2^{-(n+1)} < |z - x| \leq 2^{-n}\}$ that

(5) $\omega_k^x(A) \geq C_n\, \Lambda_1(A)$ uniformly for all $k \geq n$ and all $x \in l$ whenever $A \subset \partial\Omega_n$ and $\Lambda_1(A) > 1/n$, since

$$\omega_{n+1}^x(A) \geq \omega_n^x(A) - \omega_{n+1}^x(\partial D_{n+1}),$$

and the second term on the right can be made arbitrarily and uniformly small for $x \in l$ by choice of D_{n+1}. Here, C_n depends only on n.

(6) For each $x \in E_k$

$$\sum_n 2^{(n+1)} \,\text{length}\,(\partial S_j \cap A_n(x)) < \infty,$$

where the sum is over all bounded complementary components S_j of E_k.

(7) For each point $x \in E_k$ and all $\rho > 0$

$$\sum_n 2^{n\rho} \,\text{length}\,(\partial^* X_0 \cap A_n(x)) < C_k(\rho) < \infty,$$

where $C_k(\rho)$ is a constant depending only on k and ρ, while $\partial^* X_0$ stands for the *outer boundary* of X_0; that is, the collection of all points in X_0 that are also boundary points of one of its complementary components.

Since Ω_n consists of only finitely many smooth curves at a positive distance from l, property (5) is clearly satisfied at the n-th stage. By choosing subsequent deleted disks sufficiently small the inequality is preserved through future generations. In this case property (6) can also be guaranteed by deleting small squares as in [10], but selecting them along the lines outlined in Theorem 5.2. Suppose now that x_0 is a fixed point in some E_k, and let I_θ be a ray extending out from x_0 and making an angle θ with the positive real axis. Denote by $N_j(\theta)$ the number of points in $I_\theta \cap S_j$, where S_j is a bounded complementary component of E_k, and contained in the annulus $A_n(x_0)$. Then, $N_j(\theta) = 0, 1, 2$ for almost all θ, and

$$\int_0^{2\pi} N_j(\theta)\, d\theta \;\leq\; 2^{(n+1)} \,\text{length}\,(\partial S_j).$$

A factor of $2^{(n+1)}$ arises since the integration is taken over the projection of ∂S_j onto the unit circle, which introduces a distortion of corresponding magnitude. Hence, by virtue of property (6)

$$\int_0^{2\pi} \sum_j N_j(\theta)\, d\theta = \sum_j \int_0^{2\pi} N_j(\theta)\, d\theta \leq \sum_{n,\,j} 2^{(n+1)} \,\text{length}\,(\partial S_j \cap A_n(x_0)) < \infty,$$

where the sum here is over all bounded complementary components S_j of E_k. Consequently, $\sum_j N_j(\theta) < \infty$ for almost every θ, and therefore for any such θ the ray I_θ meets E_k in finitely many non degenerate intervals.

Suppose now that $f \in R(X_0)$ and that $f = 0$ on a set of positive Λ_1 measure on $\partial^* X_0$, and hence on a set G of positive Λ_1 measure on $\partial\Omega_n$ for some n. By assumption f is the uniform limit on X_0 of a sequence of rational functions f_j, $j = 1, 2, \dots$, where each f_j is analytic in a neighborhood of some $\overline{\Omega}_{n_j}$, $n_j \geq n$, and by property (5), $\omega^x_{n_j}(G) \geq C_n\, \Lambda_1(G)$ uniformly for all $n_j \geq n$ and all $x \in l$, provided n is sufficiently large. Since $\log|\, f_j\,|$ is subharmonic in Ω_{n_j} it follows that

$$\log|\, f_j\,(x)\,| \leq \int_{\partial\Omega_{n_j}} \log|\, f_j\,|\, d\,\omega^x_{n_j}$$

for all $x \in l$. Since the right hand side tends to $-\infty$ uniformly for $x \in l$ as $j \to \infty$, we must conclude that $f \equiv 0$ on l. On the other hand, by property (3) $f \in C^\infty$ on the line segment l, which lies in every E_k, $k = 1, 2, \ldots$, and so $f^{(n)} \equiv 0$ on l for all $n = 0, 1, 2, \ldots$. At any point $x_0 \in l$ we can choose a ray I_θ extending out from x_0 and meeting E_k in finitely many non degenerate intervals. Then, because the derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ satisfy the growth estimates in (3), the Denjoy criteria in (4), and since x_0 can be joined to almost every other point of E_k by a polygonal line whose initial segment belongs to l, it follows from quasianalyticity that $f \equiv 0$ on every E_k, and therefore by continuity $f \equiv 0$ on all of X_0.

In order to deal with functions vanishing off $\partial^* X_0$ we first observe that for any $f \in R(X_0)$ and any $x_0 \in E_k$ we have the generalized Cauchy formula

$$f(x_0) = \frac{1}{2\pi i} \int_{\partial^* X_0} \frac{f(z)}{z - x_0}\, dz,$$

since it is clearly valid for any sequence of rational functions tending uniformly to f on X_0 and $(z - x_0)^{-1} dz$ is a finite measure on $\partial^* X_0$ by property (7). From here it follows that f is *monogenic* in the sense of Borel at each point $x_0 \in E_k$, that is,

$$\lim_{x \to x_0,\, x \in E_k} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0,\, x \in E_k} \frac{1}{2\pi i} \int_{\partial^* X_0} \frac{f(z)}{(z - x)(z - x_0)}\, dz = \frac{1}{2\pi i} \int_{\partial^* X_0} \frac{f(z)}{(z - x_0)^2}\, dz,$$

the limit being taken through points of E_k. To verify this assertion fix $\varepsilon > 0$ and choose $\delta > 0$ so that as in the proof of Lemma 2.5 and here by property (7)

$$\int_{\partial^* X_0 \cap (|z - x_0| < \delta)} \frac{|f(z)|}{|z - x||z - x_0|}\, |dz| \le \|f\|_\infty \int_{\partial^* X_0 \cap (|z - x_0| < \delta)} \left(\frac{1}{|z - x|^2} + \frac{1}{|z - x_0|^2} \right) |dz| < \varepsilon,$$

uniformly for $|x - x_0| < \delta$. Therefore,

$$\limsup_{x \to x_0,\, x \in E_k} \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{1}{2\pi i} \int_{\partial^* X_0 \cap (|z - x_0| > \delta)} \frac{f(z)}{(z - x_0)^2}\, dz \right| < \varepsilon,$$

and since the inequality is valid for an arbitrary ε and corresponding δ we can infer that

$$f'(x_0) = \lim_{x \to x_0,\, x \in E_k} \frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{2\pi i} \int_{\partial^* X_0} \frac{f(z)}{(z - x_0)^2}\, dz.$$

The same reasoning implies that all derivatives $f^{(n)}(x)$, $n = 0, 1, 2, \ldots$, exist in this sense on each E_k, $k = 1, 2, \ldots$.

By the construction in [10] we are assured that $X_0 \setminus \partial^* X_0 = \bigcup_{k=1}^\infty E_k$, and so if $f = 0$ on a set of positive Λ_1 measure in $X_0 \setminus \partial^* X_0$, then $f = 0$ on some subset of positive Λ_1 measure on E_k for k sufficiently large. At any point $x_0 \in E_k$ of positive Λ_1 density $f^{(n)}(x_0) = 0$ for all $n = 0, 1, 2, \ldots$, and since x_0 can again be joined to almost every other point of E_k by a polygonal arc lying in E_k we may conclude that $f \equiv 0$ on every E_k, and therefore by continuity $f \equiv 0$ on X_0. ☐

At this juncture we are able to offer as a complement to Theorem 4.1 the following result:

Theorem 6.5. *Fix p^* with $2 < p^* < \infty$. There exists a compact nowhere dense subset X of the complex plane with the property that*

(i) $R^p(X) = L^p(X)$ *whenever* $1 \le p < p^*$;

(ii) $R^{p^*}(X)$ *admits bounded point derivations of all orders at almost every point of* X.

The proof depends in an essential way on Hedberg's criterion [34] for $R^p(X)$ to have a bounded point derivation of a given order at a fixed point of X, and that can be viewed as an extension of Hallstrom's criteria [31] for bounded point derivations on $R(X)$.

Theorem 6.6 (Hedberg). *Let $2 < p < \infty$. In order for $R^p(X)$ to have a bounded point derivation of order s at a fixed point $x_0 \in X$ it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} 2^{(s+1)nq}\, \Gamma_q(A_n(x_0) \setminus X) < \infty.$$

Here, as usual, q denotes the index dual to p.

In view of the material presented heretofore the question naturally arises:

Question. *Does there exist a compact nowhere dense subset X of the complex plane with the property that*

(i) $R^p(X) = L^p(X)$ *for all* $p < \infty$, *but nevertheless,*

(ii) *each function in $R(X)$ is uniquely determined by its values on any subset of positive one-dimensional Hausdorff measure?*

Unfortunately, we do not know the answer to that question. In our proof of Theorem 6.4 use was made of certain L^2 properties of the Bergman kernel, and so any approach along those lines will ensure that $R^2(X) \neq L^2(X)$. Moreover, there is no known necessary and sufficient condition in terms of analytic capacity, for example, in order for $R(X)$ to retain the uniqueness property of the analytic functions. Perhaps a modified approach along the lines of that in Gonchar's example (cf. [46]) which is not so directly tied to L^2 estimates could be used to shed additional light on the problem.

Proof of Theorem 6.5. Our task is to construct a compact set X for which the following two conditions are satisfied simultaneously:

(i) $\displaystyle\limsup_{r\to 0} \frac{\Gamma_q(B_r(x) \setminus X)}{r^2} > 0$ for almost every $x \in X$, and all $q > q^*$;

(ii) $\displaystyle\sum_{n=1}^{\infty} 2^{(s+1)nq^*} \Gamma_{q^*}(A_n(x) \setminus X) < \infty$ for almost every $x \in X$, and all $s \ge 1$.

Here, of course, q and q^* are the exponents dual to p and p^*, respectively.

As in the proof of Theorem 4.1, let $Q = [0, 1] \times [0, 1]$ and place a grid of squares inside of Q consisting of lines parallel to the coordinate axes. Let δ_1 be the side length of a typical square in the grid. At each vertex of the grid lying in the interior of Q, remove a small disk Δ_{α_1} of radius

$\delta_1^{\alpha_1}$ whose closure does not meet ∂Q, and where $\alpha_1 > 0$ has yet to be determined. Form the set

$$X_1 = Q \setminus \bigcup \Delta_{\alpha_1},$$

the union being taken over the entire family of deleted disks. Next, choose a sequence

$$2 > q_1 > q_2 > \cdots > q^*$$

so that $q_j \downarrow q^*$, or equivalently, $p_j \uparrow p^*$, and a sequence of positive integers

$$N_1 < N_2 < \cdots < N_k < \cdots$$

so that $\dfrac{N_k}{k} \uparrow +\infty$ as $k \uparrow +\infty$. If α_1 is selected in such a way that

$$\frac{2}{2-q^*} < \alpha_1 < \frac{2}{2-q_1},$$

then by an appropriate choice of δ_1 we can arrange that

(1) $\dfrac{\Gamma_{q_1}(B_{\delta_1}(x) \setminus X_1)}{\delta_1^2} > 1/2$ for all $x \in X_1$;

(2) $2^{N_1 q^*} \Gamma_{q^*}(\bigcup \Delta_{\alpha_1}) < \dfrac{\epsilon_1}{N_1^3}$,

where the union is over all deleted disks of radius $\delta_1^{\alpha_1}$. Since any disk B_{δ_1} of radius δ_1, meets at least one, and at most four, Δ_{α_1}'s of radius $\delta_1^{\alpha_1}$, it follows from the subadditivity of q-capacity that

$$\frac{\Gamma_{q_1}(B_{\delta_1} \setminus X_1)}{\delta_1^2} \approx \frac{\delta_1^{\alpha_1(2-q_1)}}{\delta_1^2} = \delta_1^{\alpha_1(2-q_1)-2}.$$

On the other hand, by assumption, the exponent $\alpha_1(2-q_1) - 2 < 0$, and property (1) is satisfied by taking δ_1 sufficiently small. In the same way $\Gamma_{q^*}(\cup \Delta_{\alpha_1})$ can be made arbitrarily small and (2) is also satisfied.

Next, choose a second generation grid of side length δ_2 and fix α_2 so that

$$\frac{2}{2-q^*} < \alpha_2 < \frac{2}{2-q_2},$$

Remove from X_1 those disks Δ_{α_2} of radius $\delta_2^{\alpha_2}$ centered at points of the new grid lying in the interior of X_1 whose centers do not meet ∂X_1, and set

$$X_2 = X_1 \setminus \bigcup \Delta_{\alpha_2},$$

the union again being taken over all deleted disks Δ_{α_2}. In this way (1) is unaltered, and by taking δ_2 sufficiently small we can ensure that

(3) $\dfrac{\Gamma_{q_2}(B_{\delta_2}(x) \setminus X_2)}{\delta_2^2} > 1/2$ for all $x \in X_2$;

(4) $2^{N_2 q^*}\Gamma_{q^*}(\bigcup \Delta_{\alpha_2}) < \frac{\epsilon_2}{N_2^3}$;

(2′) $2^{N_1 q^*}\Gamma_{q^*}(\bigcup_{j=1}^{2} \Delta_{\alpha_j}) < (\epsilon_1 + \frac{\epsilon_1}{2})\frac{1}{N_1^3}$.

Continuing in this manner we obtain a decreasing sequence of compact sets $X_1 \supset X_2 \supset \cdots$ together with sequences $q_j \downarrow q^*$, $\delta_j \downarrow 0$, and $N_k \uparrow +\infty$ so that at the k-th stage

(5) $\frac{\Gamma_{q_k}(B_{\delta_k}(x) \setminus X_k)}{\delta_k^2} > 1/2$ for all $x \in X_k$;

(6) $2^{N_k q^*}\Gamma_{q^*}(\bigcup \Delta_{\alpha_k}) < \frac{\epsilon_k}{N_k^3}$;

and, moreover, whenever $l < k$

(7) $2^{N_l q^*}\Gamma_{q^*}(\bigcup_{j=1}^{l} \Delta_{\alpha_j}) < (\epsilon_l + \ldots + \frac{\epsilon_l}{2^{k-l}})\frac{1}{N_l^3}$.

Now, consider the set X defined by setting

$$X = \bigcap_{k=1}^{\infty} X_k,$$

and for each $k = 1, 2, \ldots$ let $E_k = \{x \in X_k : \mathrm{dist}(x, \partial X_k) \geq \delta_k\}$. We may assume that the δ_k's have been chosen small enough so that $\mathrm{meas}\{X_k \setminus E_k\} \to 0$ as $k \to \infty$, and in particular so that $\mathrm{meas}\{X \setminus \bigcup_k E_k\} = 0$. If s is a fixed positive integer and $x \in E_k$ for some k, then as soon as k_0 sufficiently large $(s+1)k \leq N_k$ for $k \geq k_0 - 1$ and we may assume that $A_n(x) \subset X_k$ for $n \geq k_0 - 1$ as well. Hence, according to property (7)

$$\sum_{n=1}^{\infty} 2^{(s+1)nq^*}\Gamma_{q^*}(A_n(x) \setminus X) < C(s) + 2\sum_{k=k_0}^{\infty}\sum_{n=N_{k-1}}^{N_k} 2^{N_k q^*}\frac{\epsilon_k}{N_k^3} < 2\sum_{k=k_0}^{\infty} 2^{N_k q^*}\frac{\epsilon_k}{N_k^2},$$

where $C(s)$ is a constant depending only on s. If at each stage we are careful to choose ϵ_k so that $k^2 2^{N_k q^*}\epsilon_k < N_k^2$, then

$$\sum_{n=1}^{\infty} 2^{(s+1)nq^*}\Gamma_{q^*}(A_n(x) \setminus X) < \infty,$$

and it follows that $R^{p^*}(X)$ admits bounded point derivations of all orders at almost every point of X. At the same time, as in the proof of Theorem 4.1,

$$\limsup_{r \to 0} \frac{\Gamma_{q_j}(B_r(x) \setminus X)}{r^2} > \frac{1}{2} \quad \text{for all} \quad j = 1, 2, \ldots,$$

and every $x \in X$, from which it also follows that $R^p(X) = L^p(X)$ for all $p < p^*$. □

In much the same way we are able to generalize a theorem of Sinanjan [52] (cf. also [14]) described earlier in Section 4.

Theorem 6.7. There exists a compact nowhere dense subset X of the complex plane with the property that

(i) $R^p(X) = L^p(X)$ for all p, $1 \le p < \infty$;

(ii) $R(X)$ admits bounded point derivations of all orders at almost every point of X.

The proof here is essentially the same as in that of Theorem 6.5. In this case, however, we need only make use of Hallstrom's criterion [31] to the effect that $R(X)$ admits a bounded point derivation of order s at $x_0 \in X$ if, and only if,

$$\sum_{n=1}^{\infty} 2^{(s+1)n} \gamma(A_n(x_0) \setminus X) < \infty,$$

and in place of the subadditivity of q-capacity make use of the semiadditivity of analytic capacity as established by Tolsa [54].

In closing we wish to reiterate that we have not been able to determine whether there exists a compact nowhere dense set X with the property that

(i) $R^p(X) = L^p(X)$ for $1 \le p < \infty$, and yet

(ii) $R(X)$ enjoys the property of unique continuation.

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J. E. Brennan, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: brennan@ms.uky.edu

C. N. Mattingly, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: cmattingly@ms.uky.edu