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ON GAP CONJECTURE CONCERNING GROUP GROWTH

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Abstract. We discuss some new results concerning the Gap Conjecture on group growth and present a reduction of the Gap Conjecture to several special classes of groups. Namely we show that its validity for the classes of simple groups and residually finite groups will imply the Gap Conjecture in full generality. A similar type reduction holds if the Conjecture is valid for residually polycyclic groups and just-infinite groups. The cases of residually solvable groups and right orderable groups are considered as well.

1. Introduction

Growth functions of finitely generated groups were introduced by A.S. Schvarz [36] and independently by J. Milnor [26], and remain popular subject of geometric group theory. Growth of a finitely generated group can be polynomial, exponential or intermediate between polynomial and exponential. The class of groups of polynomial growth coincides with the class of virtually nilpotent groups as was conjectured by Milnor and confirmed by M. Gromov [21]. Milnor's problem on existence of groups of intermediate growth was solved by the author in [11, 12], where for any prime p an uncountable family of 2-generated torsion p-groups $\mathcal{G}_\omega^{(p)}$ with different types of intermediate growth is constructed. Here ω is a parameter of construction taking values in the space of infinite sequences over the alphabet on $p+1$ letters. All $\mathcal{G}_\omega^{(p)}$ satisfy the following lower bound on growth function

$$\gamma_{\mathcal{G}_\omega}(n) \succeq e^{\sqrt{n}}, \tag{1.1}$$

where $\gamma_G(n)$ denotes the growth function of a group G and $\succeq$ is a natural comparison of growth functions (see the next section for definition). The inequality (1.1) just indicates that growth of a group is not less than the growth of the function $e^{\sqrt{n}}$.

All groups from families $\mathcal{G}_\omega^{(p)}$ are residually finite-p groups. In [13] the author proved that the lower bound (1.1) is universal for all residually finite-p groups and this fact has straightforward generalization to residually nilpotent groups, as is indicated in [25].

The paper [12] also contains an example of a torsion free group of intermediate growth, which happened to be right orderable group, as is shown in [17]. For this group the lower bound (1.1) also holds.

In the ICM Kyoto paper [20] the author raised a question if the function $e^{\sqrt{n}}$ gives a universal lower bound for all groups of intermediate growth. Moreover, later he conjectured that indeed this is the case. The corresponding conjecture is now called the *Gap Conjecture* on group growth. In this note we collect known facts related to the Conjecture of it and present some new results.

Our exposition starts with the case of residually solvable groups where basically we present recent results of J.S. Wilson. Then we consider the case of right orderable groups, and the final part contains two reductions of the Conjecture to the classes of residually finite

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groups and simple groups (theorem 7.6), and to the class of just-infinite groups modulo its correctness for residually polycyclic groups (theorem 7.7).

Observe that gap type conjectures related to other asymptotic characteristics of groups were formulated by P. Pansu and author around 2000 (for more on this see [19]).

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2. Preliminary facts

Let G be a finitely generated group with a system of generators $A = \{a_1, a_2, \ldots, a_m\}$ (throughout the paper we consider only infinite finitely generated groups and only finite systems of generators). The *length* $|g| = |g|_A$ of an element $g \in G$ with respect to A is the length n of the shortest presentation of g in the form

$$g = a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} \ldots a_{i_n}^{\pm 1},$$

where a_{i_j} are elements in A. This depends on the set of generators, but for any two systems of generators A and B there is a constant $C \in \mathbb{N}$ such that the inequalities

$$|g|_A \leq C|g|_B, \qquad |g|_B \leq C|g|_A. \tag{2.1}$$

hold.

The *growth function* of a group G with respect to a generating set A is the function

$$\gamma_G^A(n) = \big|\{g \in G : |g|_A \leq n\}\big|,$$

where $|E|$ denotes the cardinality of a set E, and n is a natural number.

If $\Gamma = \Gamma(G, A)$ is the Cayley graph of a group G with respect to a generating set A, then $|g|$ is the combinatorial distance between vertices g and e (the identity element in G), and $\gamma_G^A(n)$ counts the number of vertices at combinatorial distance $\leq n$ from e (i.e., it counts the number of elements in the ball of radius n with center at the identity element).

It follows from (2.1) that the growth functions $\gamma_G^A(n), \gamma_G^B(n)$ satisfy the inequalities

$$\gamma_G^A(n) \leq \gamma_G^B(Cn), \qquad \gamma_G^B(n) \leq \gamma_G^A(Cn). \tag{2.2}$$

The dependence of the growth function on the generating set is an inconvenience and it is customary to avoid it by using the following trick due to Milnor. Two functions on the naturals γ_1 and γ_2 are called *equivalent* (written $\gamma_1 \sim \gamma_2$) if there is a constant $C \in \mathbb{N}$ such that $\gamma_1(n) \leq \gamma_2(Cn)$, $\gamma_2(n) \leq \gamma_1(Cn)$ for all $n \geq 1$. Then according to (2.2), the growth functions constructed with respect to two different systems of generators are equivalent. The class of equivalence $[\gamma_G^A]$ of the growth function is called the *degree of growth*, or the *rate of growth* of a group G. It is an invariant of the group not only up to isomorphism but also up to a weaker equivalence relation called *quasi-isometry*.

We will also consider a preoder $\preceq$ on the set of growth functions:

$$\gamma_1(n) \preceq \gamma_2(n) \tag{2.3}$$

if there is an integer $C > 1$ such that $\gamma_1(n) \leq \gamma_2(Cn)$ for all $n \geq 1$. This makes a set $\mathcal{W}$ of growth degrees of finitely generated groups a partially ordered set. The notation $\prec$ will be used in this article to indicate a strict inequality.

Let us remind some basic facts about the main examples of growth rates that will be used in the paper.

- The power functions n^α belong to different equivalence classes for different $\alpha \geq 0$.
- The polynomial function $P_d(n) = c_d n^d + \cdots + c_1 n + c_0$, where $c_d \neq 0$ is equivalent to the power function n^d.
- All exponential functions $\lambda^n, \lambda > 1$ are equivalent and belong to the class $[e^n]$.
- All functions of *intermediate type* $e^{n^\alpha}, 0 < \alpha < 1$ belong to different equivalence classes.

This is not a complete list of rates of growth that a group may have. Much more is provided in [11] and [3].

It is easy to see that the growth of a group coincides with the growth of a subgroup of finite index, and that the growth of a group is not smaller than the growth of a finitely generated subgroup or of a factor group. Since a group with m generators can be presented as a quotient group of a free group of rank m, the growth of a finitely generated group cannot be faster than exponential (i.e., it can not be superexponential). Therefore we can split the growth types into three classes:

- *Polynomial* growth. A group G has a *polynomial* growth is there are constants $C > 0$ and $d > 0$ such that $\gamma(n) < Cn^d$ for all $n \geq 1$. Minimal d with this property is called the degree of polynomial growth.
- *Intermediate* growth. A group G has *intermediate* growth if $\gamma(n)$ grows faster than any polynomial but slower than any exponent function $\lambda^n, \lambda > 1$ (i.e. $\gamma(n) \prec e^n$).
- *Exponential growth.* A group G has *exponential* growth if $\gamma(n)$ is equivalent to e^n.

The question of existence of groups of intermediate growth was raised in 1968 by Milnor [27]. It was answered by the author in 1984 [9, 11, 18], where it was shown that there are uncountably many 2-generated torsion groups of intermediate growth. Moreover, it was shown in [11, 12, 18] that the partially ordered set of growth degrees of finitely generated torsion p-groups contains a chain of the cardinality of continuum and contains an anti-chain of the cardinality of continuum. An immediate consequence of this result is the existence of uncountably many quasi-isometry equivalence classes of finitely generated groups (in fact 2-generated groups) [11].

Below we will use several times the following lemma ([21, page 59]).

Lemma 2.1 (Splitting lemma)**.** *Let G be a finitely generated group of polynomial growth of degree d and $H \lhd G$ be a normal subgroup with quotient G/H an infinite cyclic group. Then H has polynomial growth of degree $\leq d - 1$.*

3. Gap Conjecture and its modifications

We will say that a group is *virtually nilpotent* (resp. virtually solvable) if it contains a nilpotent (solvable) subgroup of finite index. Around 1968 it was observed by Milnor, Wolf, Hartly and Guivarc'h that a nilpotent group has polynomial growth and hence a virtually nilpotent group also has polynomial growth. In his remarkable paper [21], Gromov established the converse.

Theorem 3.1. *(Gromov 1981) If a finitely generated group G has polynomial growth, then G contains a nilpotent subgroup of finite index.*

In fact Gromov obtained a stronger result about polynomial growth.

Theorem 3.2. *For any positive integers d and k, there exist positive integers R, N and q with the following property. If a group G with a fixed system of generators satisfies the* *inequality* $\gamma(n) \leq kn^d$ *for* $n = 1, 2, \dots, R$ *then* G *contains a nilpotent subgroup* H *of index at most* q *and whose degree of nilpotence is at most* N.

The above theorem implies the existence of a function v growing faster than any polynomial and such that if $\gamma_G \prec v$, then the growth of G is polynomial.

Indeed, taking a sequence $\{k_i, d_i\}_{i=1}^{\infty}$ with $k_i \to \infty$ and $d_i \to \infty$ when $i \to \infty$ and the corresponding sequence $\{R_i\}_{i=1}^{\infty}$, whose existence follows from Theorem 3.2, one can build a function $v(n)$ which coincides with the polynomial $k_i n^{d_i}$ on the interval $[R_{i-1}+1, R_i]$. Therefore there is a *Gap* in the scale of rates of growth of finitely generated groups and a big problem is to find the optimal function (or at least the rate of its growth) which separates polynomial growth of groups from intermediate. The best known result in this direction is the function $n^{(\log\log n)^c}$ (c some positive constant) which appears in the paper of Shalom and Tao [33, Corollary 8.6].

The lower bound of the type $e^{\sqrt{n}}$ for all groups $\mathcal{G}_\omega^{(p)}$ of intermediate growth established in [9, 11, 12, 18] allowed to guess that the equivalence class of function $e^{\sqrt{n}}$ could be a good candidate for a "border" between polynomial and exponential growth in groups. This guess was further strengthened in 1988 when the author obtained the result published in [13] (see Theorem 5.1). For the first time the Gap Conjecture was formulated in print as a question in the author's Kyoto ICM paper [20].

Conjecture 1. (Gap Conjecture) If the growth function $\gamma_G(n)$ of a finitely generated group G is strictly bounded from above by $e^{\sqrt{n}}$ (i.e. if $\gamma_G(n) \prec e^{\sqrt{n}}$), then the growth of G is polynomial.

The question of independent interest is whether there is a group, or more generally a cancellative semigroup, with growth equivalent to $e^{\sqrt{n}}$ (for the role of cancellative semigroups see [19].

Conjecture 2. (Gap Conjecture with parameter β). There exists β, $0 < \beta < 1$, such that, if the growth function $\gamma_G(n)$ of a finitely generated group G is strictly bounded from above by e^{n^β} (i.e. if $\gamma(n) \prec e^{n^\beta}$) then the growth of G is polynomial.

Thus the Gap Conjecture with parameter $1/2$ is just the Gap Conjecture 1. If $\beta < 1/2$ then the Gap Conjecture with parameter β is weaker than the Gap Conjecture, and if $\beta > 1/2$ then it is stronger than the Gap Conjecture.

Conjecture 3. (Weak Gap Conjecture). There is a $\beta, \beta < 1$ such that if $\gamma_G(n) \prec e^{n^\beta}$ then the Gap Conjecture with parameter β holds.

4. Growth and elementary amenable groups

Amenable groups were introduced by von Neumann in 1929 [35] and play extremely important role in many branches of mathematics. Let AG denotes the class of amenable groups. By a theorem of Adelson-Velskii [1], each finitely generated group of subexponential growth belongs to the class AG. This class contains finite groups and commutative groups and is closed under the following operations:

(1) taking a *subgroup*,
(2) taking a *quotient group*,
(3) *extensions*,
(4) *unions* (i.e. if for some net $\{\alpha\}, G_\alpha \in AG, G_\alpha \subset G_\beta$ if $\alpha < \beta$ then $\cup_\alpha G_\alpha \in AG$).

Let EG be the class of *elementary* amenable groups i.e., the smallest class of groups containing finite groups, commutative groups which is closed with respect to the operations (1)-(4). For instance virtually nilpotent and more generally virtually solvable groups belong to the class EG. This concept defined by M. Day in [6] got a further development in the article [5] of Chou who suggested the following approach to study elementary amenable groups.

For each ordinal α, define a subclass EG_α of EG. EG_0 consists of finite groups and commutative groups. If α is a limit ordinal then

$$EG_\alpha = \bigcup_{\beta \preceq \alpha} EG_\beta.$$

Further, $EG_{\alpha+1}$ is defined as the set of groups which are extensions of groups from the set EG_α by groups from the same set. The *elementary complexity* of a group $G \in EG$ is the smallest α such that $G \in EG_\alpha$.

It was shown in [5] that the class EG does not contain groups of intermediate growth, groups of Burnside type (i.e. finitely generated infinite torsion groups), and finitely generated infinite simple groups. A further study of elementary groups and its generalizations was done by D. Osin [30].

A larger class SG of subexponentially amenable groups was (implicitly) introduced in [8], explicitly in [14] and studied in [7] and other papers.

A useful fact about groups of intermediate growth is due to S. Rosset [32].

Theorem 4.1. *If G is a finitely generated group which does not grow exponentially and H is a normal subgroup such that G/H is solvable, then H is finitely generated.*

We propose the following generalization of this result.

Theorem 4.2. *Let G be a finitely generated group with no free subsemigroup on two generators and let the quotient G/N be an elementary amenable group. Then the kernel N is a finitely generated group.*

Proof. The basic tool for the proof of this type of theorems is analogous of the following lemma by Milnor [28].

Lemma 4.3. *If G is a finitely generated group with subexponential growth, and if $x, y \in G$, then the group generated by the set of conjugates $y, xyx^{-1}, x^2yx^{-2}, \dots$ is finitely generated.*

We will apply induction on elementary complexity α of the quotient group $H = G/N$. If complexity is 0 then the group is either finite or abelian. In the first case N is finitely generated for obvious reasons. In the second case we apply the following statements from the paper of P. Longobardi and A. Rhemtulla [24, Lemmas 1,2].

Lemma 4.4. *If G has no free subsemigroups, then for all $a, b \in G$ the subgroup $\langle a^{b^n}, n \in \mathbb{Z}\rangle$ is finitely generated.*

Lemma 4.5. *Let G be a finitely generated group. If $H < G, G/H$ is cyclic, and $\langle a^{b^n}, n \in \mathbb{Z}\rangle$ is finitely generated for all $a, b \in G$, then H is finitely generated.*

Assume that the statement is correct for groups with complexity $\alpha \leq \beta - 1$ for some ordinal $\beta, \beta \geq 1$. The subgroups H, being finitely generated, allows a short exact sequence

$$\{1\} \to A \to H \to B \to \{1\},$$

where $A, B \in EG_{\beta-1}$. Let $\varphi : G \to G/N$ be the canonical homomorphism and $M = \varphi^{-1}(A)$. Then M is a normal subgroup of G and $G/M \simeq G/N/M/N \simeq H/A \simeq B$. By inductive assumption M is finitely generated. As $M/N \simeq A$, again by induction, N is finitely generated and we are done.

□

We will discuss in detail just-infinite groups in the last section. But let us prove first a preliminary result which will be used later. Recall that a group is called just-infinite if it is infinite, but every proper quotient is finite (i.e. every nontrivial normal subgroup is of finite index). A group G is called hereditary just-infinite if it is residually finite and every subgroup $H < G$ of finite index (including G itself) is just infinite. Observe that a subgroup of finite index of a hereditary just-infinite group is hereditary just-infinite.

We learned the following fact from Y. de Cournulier. We provide a proof here as there is no one in the literature.

Theorem 4.6. *Let G be a finitely generated hereditary just-infinite group, and suppose that G belongs to the class EG of elementary amenable groups. Then G is isomorphic either to the infinite cyclic group $\mathbb{Z}$ or to the infinite dihedral groups D_∞.*

Proof. If $G \in EG_0$ then G is abelian and hence $G \simeq \mathbb{Z}$. Assume that the statement is correct for all groups from classes $EG_\alpha, \alpha < \beta$ for some ordinal β. Let us prove it for β. Assume $G \in EG_\beta$ and β is smallest with this property. β can not be a limit ordinal because G is finitely generated. Therefore G is the extension of a group A by the group $B = G/A$, where $A, B \in EG_{\beta-1}$. In fact B is finite group (as G is just-infinite). As a subgroup of finite index in hereditary just-infinite groups, A is hereditary just-infinite and moreover finitely generated (as a subgroup of finite index in a finitely generated group). By inductive assumption A is isomorphic either to an infinite cyclic group $\mathbb{Z}$ or to an infinite dihedral groups D_∞. In particular G admits a subgroup of finite index H isomorphic to $\mathbb{Z}$.

Let G act on H by conjugation. Then we get a homomorphism $\psi : G \to Aut(H) \simeq \mathbb{Z}_2$. If $\psi(G) = \{1\}$, then H is a central subgroup. It is a standard fact in group theory (see for instance [22, Proposition 2.4.4]) that if there is a central subgroup of finite index in G then the commutator subgroup G' is finite. But as G is just-infinite, $G' = \{1\}$ and so G is abelian, hence $G \simeq \mathbb{Z}$ in this case.

If $\psi(G) = Aut(H)$ then $N = ker\psi$ is a centralizer $C_G(H)$ of H in G. Subgroup N has index 2 in G is just-infinite and hence by the same reason as above $N' = \{1\}$, so N is abelian. Being finitely generated and just-infinite $N \simeq \mathbb{Z}$.

Let $x \in G, x \notin N$. The element x acts on N by conjugation mapping each element to its inverse. In particular, $x^{-1}(x^2)x = x^{-2}$, so $(x^2)^2 = 1$. But $x^2 \in N$. Since N is torsion free $x^2 = 1$. Therefore

$$G = \langle x, N \rangle = \langle x, y : x^2 = 1, x^{-1}yx = y^{-1} \rangle \simeq D_\infty,$$

where y is a generator of N.

□

5. Gap Conjecture for residually solvable groups

Recall that a group G is said to be a residually finite-p group (sometimes also called residually finite p-group) if it is approximated by finite p-groups, i.e., for any $g \in G$ there is a finite p-group H and a homomorphism $\phi : G \to H$ with $\phi(g) \neq 1$. This class is, of course, smaller than the class of residually finite groups, but it is pretty large. For instance, Golod-Shafarevich groups, p-groups $\mathcal{G}_\omega$ from [11, 12], and many other groups belong to this class.

Theorem 5.1. *([13]) Let G be a finitely generated residually finite-p group. If $\gamma_G(n) \prec e^{\sqrt{n}}$ then G has polynomial growth.*

As was established by the author in a discussion with A. Lubotzky and A. Mann during the conference on profinite groups in Oberwolfach in 1990, the same arguments as given in [11] combined with Lemma 1.7 from [25] allow one to prove a stronger version of the above theorem (see also the remark after Theorem 1.8 in [25].

Theorem 5.2. *Let G be a residually nilpotent finitely generated group. If $\gamma_G(n) \prec e^{\sqrt{n}}$ then G has polynomial growth.*

Lemma 5.3 (Lemma 1.7, [25])**.** *Let G be a finitely generated residually nilpotent group. Assume that for every prime p the pro-p-closure $G_{\hat{p}}$ of G is p-adic analytic. The G is linear.*

To be linear means to be isomorphic to a subgroup of the linear group $GL_n(\mathbb{K})$ for some field $\mathbb{K}$. As by Tits alternative [34] every finitely generated linear group either contains a free subgroup on two generators or is virtually solvable, the above lemma immediately reduces theorem 5.2 to the theorem 5.1. Theorem 1.8 from [25] contains an interesting approach to polynomial growth type theorems in the case of residually nilpotent groups. In fact, as is mentioned in [25] in the remark after the theorem, their proof yields the same conclusion under the weaker assumption that $\gamma_G(n) \prec 2^{2^{\sqrt{\log_2 n}}}$.

Surprisingly, in his first paper on the gap type problem [39] Wilson used a similar upper bound $\gamma_G(n) \prec e^{e^{(1/2)\sqrt{\ln n}}}$ to prove existence of a gap for residually solvable groups. Wilson's approach is quite different from those that were used before and is based on exploring self-centralizing chief factors in finite solvable groups.

Recall that a chief factor of a group G is a (nontrivial) minimal normal subgroup of some quotient G/N, and that L/M is a self-centralizing chief factor of a group G if M is normal in G, L/M is a minimal normal subgroup of G/M, and $L/M = C_{G/M}(L/M)$. One of the results of [39] is

Theorem 5.4 (Wilson)**.** *Let G be a residually solvable group of subexponential growth whose finite self-centralizing chief factors all have rank at most k. Then G has a residually nilpotent normal subgroup whose index is finite and bounded in terms of k and $\gamma_G(n)$.*

If, in addition $\gamma_G(n) \prec e^{\sqrt{n}}$, then G has a nilpotent normal subgroup whose index is finite and bounded in terms of k and $\gamma_G(n)$.

The proof of this result is based on the following lemma.

Lemma 5.5 (Lemma 2.1, [39])**.** *Let k be a positive integer and $\alpha : \mathbb{N} \to \mathbb{R}_+$ a function such that $\alpha(n)/n \to 0$ as $n \to \infty$. Suppose that G is a finite solvable group having (i) a self-centralizing minimal normal subgroup V of rank at most k and (ii) a generating set A such that $\gamma_G^A(n) \le e^{\alpha(n)}$ for all n. Then $|G/V|$ is bounded in terms of k and α alone.*

The proof if this lemma uses ultraproducts of groups in a way perhaps never used before. One of the immediate corollaries of the technique developed in [39] is the following fact which we would like to mention.

Recall that a group is called supersolvable if it has a finite normal descending chain of subgroups with cyclic quotients. Every finitely generated nilpotent group is supersolbavle [31], and the symmetric group $Sym(4)$ is the simplest example of a solvable but not supersolvable group. The next theorem slightly improves Theorem 5.2.

Theorem 5.6. *The Gap Conjecture holds for residually supersolvable groups.*

Developing his technique and using known facts about maximal primitive solvable subgroups of $GL_n(p)$ (p prime) Wilson in [37] proved the following result.

Theorem 5.7. *The Gap Conjecture with parameter* $1/6$ *holds for residually solvable groups.*

There is a hope that eventually Gap Conjecture will be proved for residually solvable groups, or at least for residually polycyclic groups (which is the same as to prove it for groups approximated by finite solvable groups, because polycyclic groups are residually finite [31]). If the latter is done, then we will have a complete reduction of the Gap Conjecture to just-infinite groups (more on this in the last section).

6. Gap Conjecture for right orderable groups

Recall that a group is called right orderable if there is a linear order on the set of its elements invariant with respect to multiplication on the right. In a similar way are defined left orderable groups. A group is bi-orderable (or totally orderable) if there is a linear order invariant with respect to multiplication on the left and on the right. Every right orderable group is left orderable and vise versa but there are right orderable groups which are not totally orderable (see [23] for examples). As was shown by A. Machi and the author the class of finitely generated right orderable groups of intermediate growth is nonempty [17]. The corresponding group $\hat{\mathcal{G}}$ was earlier constructed in [14] as an example of torsion free group of intermediate growth. It was implicitly observed in [17] that the class of countable right orderable groups coincides with the class of groups acting faithfully by homeomorphisms on the line $\mathbb{R}$ (or, what is the same, on the interval $[0,1]$). Recently A. Erschler and L. Bartholdi managed to compute the growth of $\hat{\mathcal{G}}$ which happens to be $n^{\log(n)n_0^{\alpha}}$ where $\alpha_0 = \log 2/\log(2/\rho) \approx 0.7674$, where ρ is the real root of the polynomial x^3+x^2+x-2. The question if there exists a finitely generated, totally orderable group of intermediate growth is still open.

Gap Conjecture and it modifications stated in section 3 are interesting problems even for the class of right orderable groups. Our next result gives some contribution to this topic. The methods of [39, 37] combined with theorems of Morris [29] and Rosset [32] can be used to prove the following statement.

Theorem 6.1. *(i) The Gap Conjecture with parameter* $1/6$ *holds for left orderable groups.*

(ii) The Gap Conjecture holds for left orderable groups if it holds for residually polycyclic groups.

Proof. (i) Let G be a finitely generated right orderable group with growth $\prec e^{n^{1/6}}$. In [29] D. Morris proved that every finitely generated right orderable amenable group is indicable (i.e. can be mapped onto $\mathbb{Z}$). As by Adelson-Velskii theorem [1] a group of intermediate growth is amenable, we conclude that the abelianization $G_{ab} = G/[G,G]$ is infinite and hence has a decomposition $G_{ab} = G_{ab}^{-} \oplus G_{ab}^{+}$ where $G_{ab}^{-} \simeq \mathbb{Z}^d, d \geq 1$ is a torsion free part and G_{ab}^{+} is a torsion part. Let $N \lhd G$ be a normal subgroup such that $G/N = G_{ab}^{-}$. Since the commutator subgroup of a group is a characteristic group and the torsion free part of abelian group also is a characteristic group we conclude that N is a characteristic subgroup of G (recall that an automorphism of a group G induces an automorphism of a quotient group G/H where $H < G$ is a characteristic subgroup).

By theorem 4.1 N is a finitely generated group. Therefore we can proceed with N as we did with G. This allows us to get a descending chain

$$G > G_1 > G_2 > \dots \tag{6.1}$$

(where $G_1 = N$ etc) of characteristic subgroups with the property that $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ if $G_{i+1} \neq \{1\}$, for some sequence $d_i \in \mathbb{N}, i = 1, 2, \dots$.

If the sequence (6.1) is finite then G is solvable and by the results of Milnor and Wolf [28, 40] it is virtually nilpotent.

Suppose that the chain (6.1) is infinite and consider the intersection $G_\omega = \bigcap_{i=1}^{\infty} G_i$. If $G_\omega = \{1\}$, then the group G is residually solvable (in fact residually polycyclic), and, because of restriction on growth, by the Wilson's theorem 5.7, G is virtually nilpotent and hence has polynomial growth of some degree d. But this contradicts Splitting Lemma 2.1. Therefore $G_\omega \neq \{1\}$. G/G_ω is residually polycyclic, has growth not greater than the growth of G and by previous argument is virtually nilpotent. If the degree of polynomial growth of G/G_ω is l then again by Splitting Lemma the length of chain (6.1) can not be larger than l, and we again get a contradiction. The part (i) of the theorem is proven.

Now the proof of part (ii) follows immediately. If we assume that G has growth $\prec e^{\sqrt{n}}$, but that Gap Conjecture holds in the class of residually polycyclic groups then the arguments from previous part (i) can be applied in the same manner. The only difference is that instead of Theorem 5.7 we should use the assumption that Gap Conjecture holds for residually polyciclic groups.

□

7. Gap Conjecture and just-infinite groups

There is some evidence based on considerations presented below that the Gap Conjecture can be reduced to consideration of three classes of groups: *simple* groups, *branch* groups and *hereditary just-infinite* groups. These three types of groups appear in a natural division of the class of just-infinite groups into three subclasses described in Theorem 7.7. The following statement is an easy application of Zorn's lemma.

Proposition 7.1. *Let G be a finitely generated infinite group. Then G has a just-infinite quotient.*

Corollary 7.2. *Let $\mathcal{P}$ be a group theoretical property preserved under taking quotients. If there is a finitely generated group satisfying the property $\mathcal{P}$ then there is a just-infinite group satisfying this property.*

Although the property of a group to have intermediate growth is not preserved when passing to a quotient group (the image may have polynomial growth), by theorems of Gromov [21] and Rosset [32], if the quotient G/H of a group G of intermediate growth is a virtually nilpotent group then H is a finitely generated group of intermediate growth and one may look for a just-infinite quotient of H and iterate this process in order to represent G as a consecutive extension of a chain of groups that are virtually nilpotent or just-infinite groups. This observation is the base of the arguments for statements given in Theorems 6.1,5.6, 7.6 and 7.7. It was used in the previous section and will be used again in this section.

Recall that hereditary just-infinite groups were already defined in section 4. A hint for a definition of branch groups will be given after the next statement (see also [16, 4]). We call a just infinite group near simple if contains a subgroup of finite index which is a direct product of finitely many copies of a simple group. The next theorem was derived by the author from a result of Wilson [38].

Theorem 7.3. [16] *The class of just-infinite groups naturally splits into three subclasses: (B) branch just-infinite groups, (H) hereditary just-infinite groups, and (S) near-simple just-infinite groups.*

Branch groups are groups that have a faithful level transitive action on an infinite spherically homogeneous rooted tree $T_{\bar{m}}$ defined by a sequence $\{m_n\}_{n=1}^{\infty}$ of natural numbers $m_n \geq 2$ (determining the branching number for vertices of level n) with the property that the rigid stabilizer $rist_G(n)$ has finite index in G for each $n \geq 1$. Here by $rist_G(n)$ we mean the subgroup $\prod_{v \in V_n} rist_G(v_n)$ which is a product of rigid stabilizers $rist_G(v_n)$ of vertices v_n taken over the set V_n of all vertices of level n, and $rist_G(v)$ is a subgroup of G consisting of elements fixing the vertex v and acting trivially outside the full subtree with the root at v. For a more detailed discussion of this notion see [16, 4]. This is a geometric definition. It follows immediately from the definition that branch groups are infinite. The definition of algebraically branch group can be found in [15, 4]. Every geometrically branch group is algebraically branch but not vice versa. If G is algebraically branch then it has a quotient G/N which is a geometrically branch. The difference between the two versions of the definitions is not large but still there is no complete understanding how much the two classes differ (more precisely what can be said about the kernel N, it is believed that it should be central in G). Not every branch group is just-infinite, but every proper quotient of a branch group is virtually abelian [16]. Therefore branch groups are "almost just-infinite" and most of known finitely generated branch groups are just-infinite. From what has been written, it is clear that if G is algebraically branch and just infinite, then it is geometrically branch. We will need a small generalization of the notion of a branch group. We call a group G slightly (geometrically) branch if it has a faithful action by automorphisms on some infinite spherically homogeneous rooted tree $T_{\bar{m}}$ with the property that the number of orbits of the action on each level of the tree is uniformly bounded and for each n the rigid stabilizer $rist_G(n)$ has finite index in G. So we slightly weakened the condition that the action should be level transitive (as in the case of definition of a branch group).

Proposition 7.4. *Let G be a finitely generated slightly branch just-infinite group. Then it does not belong to the class EG of elementary amenable groups.*

Proof. We begin with

Lemma 7.5. *A subgroup of finite index of a slightly branch group is slightly branch group.*

Proof. Let G be a slightly branch group acting on $T_{\bar{m}}$ in the way prescribed by the definition of slightly branch group, and let $H < G$ be a subgroup of finite index. Then the number of orbits for the action of H on $T_{\bar{m}}$ is uniformly bounded by a constant depending on the index $[G : H]$ and on the constant C bounding the number of orbits for the action of G on levels.

Let $rist_G(n) = \prod_{v \in V_n} rist_G(v_n)$ and consider the intersection $H \cap rist_G(n)$. As H has finite index in G, for each vertex $v \in V_n$ the subgroup $L_v = H \cap rist_G(v_n)$ has finite index in $rist_G(v_n)$ and $rist_H(n) \geq \prod_{v \in V_n} L_v$. Therefore $rist_H(n)$ has finite index in H for any n, and hence H is slightly branch.

□

Now suppose G is slightly branch and $G \in EG_{\alpha-1}$. Let α be a minimal ordinal with the property $G \in EG_\alpha$. Then G is an extension of a subgroup $N \lhd G, N \in EG_{\alpha-1}$ with the finite quotient G/N (G can not be presented as a union of subgroups of smaller elementary complexity as G is finitely generated).

Theorem 4 from [16] states that for each nontrivial normal subgroup K of (geometrically) branch group G there is n such that $K \geq (rist_G(n))'$ where $'$ means taking the commutator subgroup. A similar fact with the same proof (using small modifications) works for slightly branch groups, one just needs to "decompose" the tree $T_{\bar{m}}$ on which group acts into finitely many invariant subtrees on each of which the group acts level transitive (and so the restriction of the action of group onto each component is a branch group) and apply the arguments from [16]. As each class EG_β is closed with respect to taking of a subgroup or a quotient, the group $(rist_G(n))'$ belongs to the class $EG_{\alpha-1}$. From now on let $N = (rist_G(n))'$. As G is just-infinite N has finite index and hence is slightly branch by previous lemma.

Consider the decomposition $rist_N(n) = \prod_{v \in V_n} rist_N(v)$. For each $v \in V_n$ the corresponding group $M_v = rist_N(v)$ is a slightly branch group acting on a rooted subtree T_v of $T = T_{\bar{m}}$. Indeed, for each level k of T_v the number of orbits for the action of $rist_N(v)$ is uniformly bounded by the same constant which bounds the number of orbits of the action of N on $T_{\bar{m}}$.

The rigid stabilizer $rist_{M_v}(k)$ is a subgroup of finite index in M_v as it contains the product $\prod_{u \in V_k(T_v)} rist_H(u)$ where $V_k(T_v)$ denotes the set of vertices of level k in subtree T_v.

Moreover, each M_v is a just-infinite group. Indeed, let us suppose that $P_v \lhd M_v$ is a normal subgroup and consider the group $Q = \prod_{w \in V_n} P_v^{g_w}$, where elements $g_w \in G$ are chosen in such a way that $P_v^{g_w}$ is a subgroup of $rist_G(w)$. Then the group Q is normal not only in N, but also in G and has infinite index. Contradiction, as G is supposed to be just-infinite.

Therefore M_v is a finitely generated (it is a quotient of N_v which is quotient of finitely generated group $rist_N(n)$) slightly branch just-infinite group belonging to the class $EG_{\alpha-1}$ and we get a final contradiction.

□

We already know that there are finitely generated branch groups of intermediate growth. For instance, groups $\mathcal{G}_\omega$ of intermediate growth from articles [10, 12] are of this type. In fact, all known examples of groups of intermediate growth are of branch type or are constructions on the base of groups of branch type. The question on the existence of non-elementary amenable hereditary just-infinite groups is still open (observe that the only elementary amenable hereditary just-infinite groups are $\mathbb{Z}$ and D_∞), theorem 4.6).

Problem 1. *Are there finitely generated hereditary just-infinite groups of intermediate growth?*

Problem 2. *Are there finitely generated simple groups of intermediate growth?*

Theorem 7.6. *If the Gap Conjecture holds for the classes of residually finite groups and simple groups, then it holds for the class of all groups.*

Proof. Assume that the Gap Conjecture is correct for residually finite groups and for simple groups. Let G be a finitely generated group with growth $\prec e^{\sqrt{n}}$. By Proposition 7.1 it has just-infinite quotient $\bar{G} = G/N$. By Theorem 7.7 $\bar{G}$ belongs to one of the three types of groups listed in the statement of the theorem. The rate of growth of $\bar{G}$ is also less than the rate of growth of $e^{\sqrt{n}}$. $\bar{G}$ can not be near simple because in this case it will have a subgroup H of finite index which has an infinite finitely generated simple quotient P whose rate of growth is $\prec e^{\sqrt{n}}$, which is impossible as virtually nilpotent groups can not be simple.

So we can assume that $\bar{G}$ is branch or hereditary just infinite. Both these classes are subclasses of the class of residually finite groups. Therefore, using our hypothesis, we conclude that $\bar{G}$ is virtually nilpotent, and therefore elementary amenable. A branch group cannot belong to the class EG (Proposition 7.4). Therefore, by Lemma 7.5 $\bar{G}$ can only be hereditary just-infinite elementary amenable. By Theorem 4.6 $\bar{G}$ is isomorphic either to an infinite cyclic group or to an infinite dihedral group D_∞. By theorem 4.1 the kernel N is finitely generated. As the rate of growth of N is less than $e^{\sqrt{n}}$ we can apply to N the same arguments as for G in order to get a surjective homomorphism either onto $\mathbb{Z}$ or onto D_∞.

Now we go back to the arguments used for proving Theorem 6.1. If $G/N \simeq \mathbb{Z}$, then we just repeat the first step of arguments from the proof of Theorem 6.1 replacing N by a finitely generated characteristic subgroup $N_1 \lhd G$ with quotient $G/N_1 \simeq \mathbb{Z}^d$ for some $d \geq 1$. If $G/N_1 \simeq D_\infty$ then we slightly modify the first step. Namely, in this case G has an indicable subgroup H of index 2. Let H_1 be an intersection of groups $H^\phi, \phi \in Aut(G)$. As there are only finitely many subgroups of index 2 in G this intersection in fact involves only finitely many groups and H_1 is a characteristic subgroup of G of some finite index of type 2^k for some $k \in \mathbb{N}$. Moreover, $G/H_1 \simeq \mathbb{Z}_2^{t_i}$ for some $t_i \in \mathbb{N}$ as this quotient is isomorphic to a subgroup of direct product of finitely many copies of the group $\mathbb{Z}_2$. H_1, being a subgroup of index 2^{k-1} in H, is indicable and we can apply to it the arguments of the first step of the proof of Theorem 6.1 getting a finitely generated subgroup $H_2 \unlhd H_1$ characteristic in G with quotient $H_1/H_2 \simeq \mathbb{Z}^d$ for some $d \in \mathbb{N}$.

Let $G_1 \lhd G$ be a subgroup N,H_1 or H_2 depending on the case. Proceed with G_1 in a similar fashion as we did with G etc. We get a descending chain $\{G_i\}_{i\geq 1}$ of finitely generated subgroups characteristic in G. There are three possibilities.

1) After finitely many steps we will get a group G_i which is a branch group or a non-elementary amenable hereditary just-infinite group. But since a finitely generated branch group can not be elementary amenable (Proposition 7.4) this is in contradiction with the hypothesis of the theorem because in both cases the group G_i is residually finite (and has growth $\prec e^{\sqrt{n}}$).

2) After finitely many steps we will get a group G_i which is hereditary just-infinite and elementary amenable, and hence infinite cyclic or D_∞ (theorem 4.6). In this case G is polycyclic and we are done in view of the result of Milnor and Wolf on growth of solvable groups.

3) The process of construction of the chain of subgroups will continue forever. In this case we get a chain with the property that G_i/G_{i+1} is isomorphic either to (i) $\mathbb{Z}_i^d, d_i \in \mathbb{N}$ or to (ii) $\mathbb{Z}_2^{t_i}, t_i \in \mathbb{N}$. Moreover, each step of type (ii) is immediately followed by a step of type (i).

The end of the proof basically is the same as the end of the proof of Theorem 6.1.

Let us show that this is impossible. Let G_ω be the intersection $\bigcap_{i\geq 1} G_i$. Then G/G_ω is residually polycyclic and hence residually finite as every polycyclic group is residually finite [31]. The growth of G/G_ω is less than $e^{\sqrt{n}}$ and by the hypothesis G/G_ω is virtually nilpotent with the rate of polynomial growth of degree d for some $d \in \mathbb{N}$. But this contradicts to splitting lemma as for infinitely many i the quotients G_i/G_{i+1} are isomorphic to $\mathbb{Z}^{t_i}$.

□

In fact the above theorem follows from the main result of [2]. We preferred to provide independent proof because it also works (with slight modifications) for proving our main result stated in the next theorem. This theorem gives the reduction of the Gap Conjecture to the class of just-infinite groups, namely the subclasses of just-infinite branch groups, hereditary just-infinite groups and simple groups under the assumption that it holds for residually polycyclic groups.

Theorem 7.7. *(i) If the Gap Conjecture with parameter* $1/6$ *holds for just-infinite groups then it holds for all groups.*

(ii) If the Gap Conjecture hold for residually polycyclic groups and for just-infinite groups then it holds for all groups.

Proof. (i) The proof follows the same strategy as the proof of theorem 7.6. There can be three possibilities. 1) G has a finite descending chain $\{G_i\}_{i=1}^k$ of finitely generated characteristic subgroups G_i with the property that G_k has a just-infinite non-elementary abelian quotient. But this is impossible by the hypothesis of the theorem.

2) G has a finite descending chain $\{G_i\}_{i=1}^k$ of finitely generated characteristic in G groups with consecutive quotients $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ or $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$, for $i < k$ and $G_k = \{1\}$. In this case G is polycyclic and hence virtually nilpotent

3) G has an infinite descending chain $\{G_i\}_{i=1}^{\infty}$, with the property that $G_i/G_{i+1} \simeq \mathbb{Z}^{d_i}$ or $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$, and if $G_i/G_{i+1} \simeq \mathbb{Z}_2^{t_i}$ then $G_{i+1}/G_{i+2} \simeq \mathbb{Z}_2^{d_{i+2}}$. The group G/G_ω where $G_\omega = \bigcap_{i\geq 1} G_i$ is residually polycyclic with growth $\prec e^{n^{1/6}}$. Apply in this case result of Wilson stated in theorem 5.4 concluding that G/G_ω is virtually nilpotent which is impossible by splitting lemma.

(ii) Proceed as in (i) with the only difference that in the subcase 3) one should apply the hypothesis that the Conjecture holds for residually polycyclic groups to conclude that subcase 3) is impossible.

□

References

[1] G. M. Adel′son-Vel′skiĭ and Yu. A. Šreĭder. The Banach mean on groups. *Uspehi Mat. Nauk (N.S.)*, 12(6(78)):131–136, 1957. 4, 8

[2] B. Bajorska and O. Macedońska. A note on groups of intermediate growth. *Comm. Algebra*, 35(12):4112–4115, 2007. 13

[3] L. Bartholdi and A. Erscler. Groups of given intermediate word growth, 2011. (available at *http://arxiv.org/abs/1110.3650*). 3

[4] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunik. Branch groups. In *Handbook of algebra, Vol. 3*, pages 989–1112. North-Holland, Amsterdam, 2003. 10

[5] Ching Chou. Elementary amenable groups. *Illinois J. Math.*, 24(3):396–407, 1980. 5

[6] Mahlon M. Day. Amenable semigroups. *Illinois J. Math.*, 1:509–544, 1957. 5

[7] P. de la Harpe, R. I. Grigorchuk, and T. Chekerini-Sil′berstaĭn. Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces. *Tr. Mat. Inst. Steklova*, 224(Algebra. Topol. Differ. Uravn. i ikh Prilozh.):68–111, 1999. 5

[8] Michael H. Freedman and Peter Teichner. 4-manifold topology. I. Subexponential groups. *Invent. Math.*, 122(3):509–529, 1995. 5

[9] R. I. Grigorchuk. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR*, 271(1):30–33, 1983. 3, 4

[10] R. I. Grigorchuk. Construction of p-groups of intermediate growth that have a continuum of factor-groups. *Algebra i Logika*, 23(4):383–394, 478, 1984. 11
[11] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984. 1, 3, 4, 7
[12] R. I. Grigorchuk. Degrees of growth of p-groups and torsion-free groups. *Mat. Sb. (N.S.)*, 126(168)(2):194–214, 286, 1985. 1, 3, 4, 7, 11
[13] R. I. Grigorchuk. On the Hilbert-Poincaré series of graded algebras that are associated with groups. *Mat. Sb.*, 180(2):207–225, 304, 1989. 1, 4, 7
[14] R. I. Grigorchuk. An example of a finitely presented amenable group that does not belong to the class EG. *Mat. Sb.*, 189(1):79–100, 1998. 5, 8
[15] R. I. Grigorchuk. Branch groups. *Mat. Zametki*, 67(6):852–858, 2000. 10
[16] R. I. Grigorchuk. Just infinite branch groups. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 121–179. Birkhäuser Boston, Boston, MA, 2000. 10, 11
[17] R. I. Grigorchuk and A. Machi. On a group of intermediate growth that acts on a line by homeomorphisms. *Mat. Zametki*, 53(2):46–63, 1993. 1, 8
[18] R.I. Grigorchuk. *Groups with intermediate growth function and their applications.* Habilitation, Steklov Institute of Mathematics, 1985. 3, 4
[19] Rostislav Grigorchuk. Milnor's problem on the growth of groups and its consequences, 2011. (available at *http://arxiv.org/abs/1111.0512*). 2, 4
[20] Rostislav I. Grigorchuk. On growth in group theory. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 325–338, Tokyo, 1991. Math. Soc. Japan. 1, 4
[21] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981. 1, 3, 9
[22] Gregory Karpilovsky. *The Schur multiplier*, volume 2 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1987. 6
[23] Ali Ivanovič Kokorin and Valeriĭ Matveevič Kopytov. *Fully ordered groups.* Halsted Press [John Wiley & Sons], New York-Toronto, Ont., 1974. Translated from the Russian by D. Louvish. 8
[24] P. Longobardi, M. Maj, and A. H. Rhemtulla. Groups with no free subsemigroups. *Trans. Amer. Math. Soc.*, 347(4):1419–1427, 1995. 5
[25] Alexander Lubotzky and Avinoam Mann. On groups of polynomial subgroup growth. *Invent. Math.*, 104(3):521–533, 1991. 1, 7
[26] J. Milnor. A note on curvature and fundamental group. *J. Differential Geometry*, 2:1–7, 1968. 1
[27] J. Milnor. Problem 5603. *Amer. Math. Monthly*, 75:685–686, 1968. 3
[28] John Milnor. Growth of finitely generated solvable groups. *J. Differential Geometry*, 2:447–449, 1968. 5, 9
[29] Dave Witte Morris. Amenable groups that act on the line. *Algebr. Geom. Topol.*, 6:2509–2518, 2006. 8
[30] D. V. Osin. Algebraic entropy of elementary amenable groups. *Geom. Dedicata*, 107:133–151, 2004. 5
[31] Derek J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996. 8, 12
[32] Shmuel Rosset. A property of groups of non-exponential growth. *Proc. Amer. Math. Soc.*, 54:24–26, 1976. 5, 8, 9
[33] Yehuda Shalom and Terence Tao. A finitary version of Gromov's polynomial growth theorem. *Geom. Funct. Anal.*, 20(6):1502–1547, 2010. 4
[34] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972. 7
[35] John von Neumann. Zurr allgemeinen theorie des masses. *Fund.Math.*, 13:73–116, 1929. 4
[36] Albert Švarc. A volume invariant of covering. *Dokl. Akad. Nauj SSSR*, 105:32–34, 2001955. 1
[37] J. Wilson. The gap in the growth of residually soluble groups. *Bull. London Math. Soc.* 8
[38] J. S. Wilson. Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.*, 69:373–391, 1971. 10
[39] John S. Wilson. On the growth of residually soluble groups. *J. London Math. Soc. (2)*, 71(1):121–132, 2005. 7, 8
[40] Joseph A. Wolf. Growth of finitely generated solvable groups and curvature of Riemanniann manifolds. *J. Differential Geometry*, 2:421–446, 1968. 9

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