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TOPOLOGY AND DYNAMICS OF LEVI-FLATS IN SURFACES OF GENERAL TYPE

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To the memory of Marco Brunella

ABSTRACT. We focus on the topology and dynamics of minimal sets and Levi-flats in surfaces of general type. Our method relies on the ergodic theory of Riemann surface laminations: we use harmonic measures and Lyapunov exponents. Our first result establishes that minimal sets have large Hausdorff dimension when a leaf is simply connected. Our second result shows that the class of Anosov Levi-flats does not occur in surfaces of general type. In particular, by using rigidity results, we obtain that Levi-flats are not be diffeomorphic to $\Gamma \backslash G$, where G is either Sol or $\mathrm{PSL}(2, \mathbb{R})$ and Γ is a cocompact lattice in G .

Key Words : Riemann surface laminations, surfaces of general type, harmonic measure, Lyapunov exponent.

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1. INTRODUCTION

In a complex surface S , a Levi-flat is a smooth real hypersurface M whose Cauchy-Riemann distribution $TM \cap iTM$ is integrable. More generally, a minimal set M is a closed set consisting of leaves of a singular holomorphic foliation on S , each of them being dense in M . Classical examples of minimal sets and Levi-flats appear e.g. in flat $\mathbb{C}\mathbb{P}^1$ -bundles (saturated of the limit set of the monodromy), in complex tori (linear hypersurface) and in fibrations by holomorphic curves (preimage of a simple closed curve in the base). There are other examples in elliptic surfaces [44] and Kummer surfaces [46]. Levi-flats in non Kählerian surfaces appear e.g. in Hopf surfaces (Reeb components) and in Inoue surfaces (hyperbolic torus bundles).

In the present work we study minimal sets and Levi-flats in surfaces of general type. These complex surfaces generalize hyperbolic Riemann surfaces. Examples are given by hypersurfaces of degree ≥ 5 in \mathbb{P}^3 , quotients of unit balls of \mathbb{C}^2 , etc. Contrary to the case of the projective plane, there are plenty of examples of minimal sets and Levi-flat hypersurfaces, e.g. by taking ramified coverings of previous examples, see section 2. Our goal is to explore their topological and dynamical properties.

1.1. Minimal sets. The ramified construction mentionned above provides the most general and complicated dynamics that a Kleinian group can produce (Cantor set, quasi-circle, dendrite, etc), in particular, the Hausdorff dimension can be arbitrarily close to 2. Observe that the topology of the

leaves becomes very complicated by using that construction: their fundamental group is infinitely generated. Our first result shows that the dimension must be large if a leaf has no topology.¹

Theorem 1.1. *Let M be a minimal set in a surface of general type with no transverse invariant measure and with a simply connected leaf. Then the Hausdorff dimension of M is > 3 , unless M is an analytic Levi-flat hypersurface. In the latter case, the holonomy pseudo-group is not discrete and the foliation is ergodic with respect to the Lebesgue measure on M .*

The proof uses the stationary measure for the leafwise Brownian motions introduced by Garnett [22], commonly called harmonic measure. We obtain the following lower bound for the transverse Hausdorff dimension when the holonomy pseudogroup is discrete

$$\text{Hausdorff dimension} \geq \text{Kaimanovich entropy} / |\text{Lyapunov exponent}|.$$

When the holonomy pseudogroup is not discrete, the minimal set is Levi-flat by combining results of [15] and [40]. Recall that the Kaimanovich entropy is the growth of separated Brownian motions [32], the Lyapunov exponent is the rate at which leaves get closer along a Brownian path [15]. For discrete random walks, the ratio entropy/exponent is equal to the dimension of the harmonic measure [34, 35]. We stress that this gives a new method to prove that a pseudo-group is non discrete. Up to now, the only known criterion to ensure such a property is to use explicit elements close to identity [24, 40, 43].

Let us see how the theorem follows in surfaces of general type. We first prove (see proposition 4.1) that the leaves of minimal sets are hyperbolic, i.e. covered by the unit disc. The arguments rely on the negative curvature of S , the foliated adjunction formula and Ahlfors currents. We then endow the leaves with the Poincaré metric. On one hand, from the presence of a simply connected leaf, the Kaimanovich entropy is equal to 1. On the other hand, the Lyapunov exponent belongs to $] - 1, 0[$. The exponent is indeed very related to the geometry of the surface: it is equal to the cohomological intersection between the harmonic current and the normal bundle of the foliation [13]. In surfaces of general type, the negative curvature of the tangent bundle forces the leaves to stay appart from each other.

1.2. Topology of Levi-flats. We call a Levi-flat *Anosov* if its CR foliation is topologically conjugated to the weak stable foliation of a 3-dimensional Anosov flow, see section 6. Classical examples are the weak stable foliation of the geodesic flow on unitary tangent bundles of negatively curved surfaces, and the weak stable foliation of suspensions of linear hyperbolic diffeomorphisms. Many other constructions hold on graph manifolds and hyperbolic manifolds, see [3, 20, 27, 28]. We prove

Theorem 1.2. *A C^1 Levi-flat in a surface of general type is not Anosov.*

Our method covers the more general case of Levi-flats with a *point at infinity* (in any complex surface): we prove that the Lyapunov exponent

¹Anosov conjectured in the case of the complex projective plane that the leaves of generic singular holomorphic foliations are biholomorphic to the unit disc, appart a countable number which are annulus.

of their ergodic harmonic measures is ≤ -1 . A contradiction arises in surfaces of general type since the Lyapunov exponent belongs to $] -1, 0]$ (as for minimal sets). A cornerstone to prove that Anosov flows support a point at infinity is to stretch their trajectories in the hyperbolic uniformizations. The upper bound ≤ -1 for the exponent is established in the point at infinity context, it can be viewed as a Margulis-Ruelle type inequality.

We are able to use theorem 1.2 to determine the possible geometries carried by atoroidal Levi-flats in surfaces of general type. (Toroidal manifolds are not covered by our approach.) Let us recall the eight 3-dimensional geometries of Thurston, see [50]:

$$\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \text{Nil}, \widetilde{\text{SL}(2, \mathbb{R})}, \text{Sol}.$$

Not surprisingly, there exist Levi-flats in surfaces of general type modelled on \mathbb{H}^3 and $\mathbb{H}^2 \times \mathbb{R}$: we can use singular fibrations or ramified coverings. The fact that $\widetilde{\text{SL}(2, \mathbb{R})}$ does appear is more involved, see proposition 2.2. Recall that this Lie group provides non trivial Seifert bundles over surfaces of genus ≥ 2 , see [50]. We precisely construct Levi-flat circle bundles with $|e/\chi|$ arbitrary close to $1/5$, where e is the Euler class and χ is the Euler characteristic of the base. The other geometries can not be realized:

Theorem 1.3. *A C^2 Levi-flat in a surface of general type can not carry one of the geometries $\mathbb{R}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \text{Nil}, \text{PSL}(2, \mathbb{R})$ or Sol.*

We also observe (see section 4.3) that Levi-flats must have a trivial second fundamental group. We adopt the terminology: a 3-manifold carries the geometry of $\text{PSL}(2, \mathbb{R})$ if it is a quotient of this group by a lattice. Up to finite coverings, those manifolds are unitary tangent bundles of orientable closed compact surfaces of genus ≥ 2 .

The proof of theorem 1.3 relies on topological and dynamical methods. The first four cases are ruled out by the exponential growth of the fundamental group: that property comes from the hyperbolicity and the tautness of the CR foliation, we conclude by applying classical Novikov's theory. To eliminate the remaining two cases, i.e. $\text{PSL}(2, \mathbb{R})$ and Sol, we establish for such Levi-flats the Anosov property, and we apply theorem 1.2. That step relies on deep theorems by Thurston [52], Ghys-Sergiescu [26] and Matsumoto [42]. We note that our results (subsections 1.1 and 1.2) actually hold for immersed minimal sets and immersed Levi-flats, we write all the proofs in the embedded case for sake of simplicity.

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2. DEFINITIONS AND EXAMPLES

We recall the definition of surfaces of general type. We also present examples of minimal sets and Levi-flat hypersurfaces in those surfaces, with various dynamical and topological properties.

2.1. Surface of general type. We refer to the books [4], [21] and [33] for a general account. Let S denote a compact complex surface and K_S its canonical line bundle. The surface S is of *general type* if the dimension of the space of holomorphic sections of nK_S grows like n^2 . Surfaces of general type are algebraic, and the general type property is a birational invariant, stable by blowup. If S is minimal, the pluricanonical map associated to the linear system $|nK_S|$ is birational onto its image for n large. Basic examples are given by products of curves of genus larger than 2, general complete intersections in projective spaces (e.g. hypersurfaces of degree ≥ 5 in \mathbb{P}^3) and cocompact quotients of the unit ball in \mathbb{C}^2 .

Other important examples are obtained by ramified covering. Let us explain the construction starting with an algebraic surface X (see [4], section I.17), we shall use it in the next subsections. Let $D \subset X$ be a smooth reduced effective divisor and s be a holomorphic section of $\mathcal{O}(D)$ such that $D = \{s = 0\}$. Let $d \geq 1$ and assume that $\mathcal{O}(D) = dE$ for some line bundle E . Then $S := \{z \in E, z^d = s\}$ is a smooth algebraic surface whose canonical bundle is the pull-back of $K_X + (d-1)E$ by the projection $\pi : S \rightarrow X$. In particular, S is of general type if $K_X + (d-1)E$ is ample.

2.2. Minimal sets. Let \mathcal{F} be a holomorphic foliation on S and \mathcal{E} denote its possible finite singular set. A *minimal set* M of \mathcal{F} is a compact union of leaves such that M does not intersect \mathcal{E} and every leaf of M is dense in M .

Proposition 2.1. *There exist minimal sets in surfaces of general type with arbitrarily small transverse Hausdorff dimension.*

Proof. Let M be a minimal set in an algebraic surface X with arbitrarily small transverse Hausdorff dimension. For instance X can be a flat \mathbb{P}^1 -bundle whose monodromy is a Kleinian group whose limit set has small Hausdorff dimension. Let E be a line bundle such that $2E$ is very ample and $K_X + E$ is ample, and let s be a generic section of $2E$. Then, the zero divisor $(s)_0$ is transverse to the foliation at each point of M . Let $c : S \rightarrow X$ be the double covering defined in subsection 2.1 and $M' = c^{-1}(M)$. This is a minimal set in S with same Hausdorff dimension than M . \square

Observe that the same arguments show that any minimal set with a transverse structure modelled on the limit set of a Kleinian group does exist in surfaces of general type. Moreover, that allows to construct minimal Levi-flat hypersurfaces in surfaces of general type with transverse invariant measures. This can be done by considering a ramified covering of a flat \mathbb{P}^1 -bundle whose monodromy is dense in $\text{PSO}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$.

2.3. Levi-flat hypersurface. A real hypersurface M of class C^1 in a complex surface S is *Levi-flat* if the distribution $p \mapsto T_p M \cap iT_p M$ is integrable in the sense of Frobenius. We denote by \mathcal{F} its Cauchy-Riemann (CR) foliation. If M is of class C^k ($k \geq 1$), then the CR foliation is also of class C^k ,

see [3]. We do not assume that the leaves are dense in M neither that M is a minimal set (a priori its CR foliation can not be extended to a singular holomorphic foliation on S). We call the Levi-flat M *minimal* when every leaf is dense in M .

Proposition 2.2. *There exists a Levi-flat hypersurface in a surface of general type which is diffeomorphic to a circle bundle such that the ratio Euler class/Euler characteristic is arbitrarily close to $1/5$.*

We do not know any example of a Levi-flat circle bundle in a surface of general type such that Euler class/Euler characteristic $\geq 1/5$.

Proof. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a cocompact and torsion free Fuchsian group. Let X be the quotient of $\mathbb{H} \times \mathbb{P}^1$ by the action of Γ given by $\gamma(p, z) = (\gamma(p), \gamma(z))$. This is a flat \mathbb{P}^1 -bundle over $\Sigma := \Gamma \backslash \mathbb{H}$. We denote by $M \subset X$ the Levi-flat hypersurface which is the quotient of $\mathbb{H} \times \mathbb{P}^1(\mathbb{R})$, this circle bundle is diffeomorphic to the unitary tangent bundle of $\Sigma = \Gamma \backslash \mathbb{H}$.

We consider the section σ which is the quotient of the curve $\{(p, p) \in \mathbb{H} \times \mathbb{H} \mid p \in \mathbb{H}\}$. Observe that σ is transverse to the \mathbb{P}^1 -fibration and to the horizontal foliation associated to the flat connexion. Hence, we have $\sigma^2 = \chi$, where $\chi = 2 - 2g$ is the Euler characteristic of Σ , see [39]. It is a classical fact that a \mathbb{P}^1 -bundle having a section of even self-intersection is indeed diffeomorphic to the product bundle. Hence, we have $H^2(X, \mathbb{Z}) = \mathbb{Z}[\sigma] + \mathbb{Z}[\varphi]$ where φ is a fiber of the \mathbb{P}^1 -fibration. Adjunction formula then yields $[K_X] = -2[\sigma]$. In order to perform the ramified covering construction, we need the following lemma.

Lemma 2.3. *Let $E' := 6\sigma + \sum_{i=1}^{4(1-2\chi)} \varphi_i$, where the φ_i 's are disjoint fibers of X . Then there exists a line bundle E such that $2E = E'$. Moreover, E' is very ample, and $E + K_X$ is ample.*

Let s^0 be a non zero section of E' vanishing on $6\sigma + \sum_i \varphi_i$. Because E' is very ample, there exists a small deformation s of s^0 which is smooth. This section intersects the Levi-flat M in a link isotopic to a family consisting of $4(-2\chi + 1)$ distinct fibers φ_i of the fibration. We denote by $c : S \rightarrow X$ the double covering defined by s as indicated in subsection 2.1.

We now claim that $M' = c^{-1}(M)$ is a circle bundle which is the pull-back (as a circle bundle) of M by a double ramified covering $\Sigma' \rightarrow \Sigma$. To verify this, we begin by considering an isotopy which sends the $\mathbb{P}^1(\mathbb{R})$ -fibration of M to a fibration F having the link $(s)_0 \cap M$ as sections. Then, the pull-back of this fibration by c is a Seifert fibration in $M' = c^{-1}(M)$. Since the fibers of F which are far away from $(s)_0 \cap M$ bound discs in $X \setminus (s)_0$ (this is true for $6\sigma + \sum_i \varphi_i$), this Seifert fibration is indeed a genuine fibration, and our claim follows.

Finally, the Euler class of the bundle structure on M' is twice the Euler class of the base, hence $eu' = 2eu = 2\chi$. Moreover the Euler characteristic of Σ' is $2\chi + 4(2\chi - 1) = 10\chi - 4$ by Hurwitz formula. That completes the proof of proposition 2.2. \square

Proof. (of lemma 2.3) If φ is a fixed fiber, the line bundle $E' - (6\sigma + 4(1 - 2\chi)\varphi)$ has a vanishing Chern class. Hence it belongs to the Jacobian of X ,

and consequently has a square root E'' , namely $E' - (6\sigma + 4(1 - 2\chi)\varphi) = 2E''$. We then set $E = E'' + 3\sigma + 2(1 - 2\chi)\varphi$.

Let us prove that $E + K_X$ is ample. Assuming that σ^2 is negative, a classical fact is that a line bundle L on X with cohomology class $[L] = a[\sigma] + b[\varphi]$ is ample iff $a > 0$ and $\chi a + b > 0$. That relies on Nakai-Moishezon criterion, stating that a line bundle is ample iff its intersection with every algebraic curve is positive, and its self-intersection is positive. Since $[E + K_X] = [\sigma] + 2(1 - 2\chi)[\varphi]$, we infer that $E + K_X$ is ample.

To prove the very ampleness of $2E$, we use a criterion of Reider [49], namely that if a line bundle L is ample, then $4L + K_X$ is very ample. So we look for L such that $2E = 4L + K_X$. The existence of such a line bundle L relies on the fact that the line bundle $2E - K_X - 4(2\sigma + (1 - 2\chi)\varphi)$ has a vanishing cohomology class. Hence it belongs to the Jacobian, and consequently has a fourth root, namely there exists F such that $4F = 2E - K_X - 4(2\sigma + (1 - 2\chi)\varphi)$. We just set $L = F + 2\sigma + (1 - 2\chi)\varphi$ to get the desired line bundle. It remains to see that L is ample, this can be done using the previous criterion. \square

3. HARMONIC CURRENTS

In this section, M is either a minimal set or a C^1 Levi-flat hypersurface in a compact complex surface S . We introduce harmonic currents on M with respect to the foliation. These currents generalize the notion of foliated cycles introduced by Sullivan [51] and permit to interpret M as a curve [25]. Assuming that S is Kähler, any harmonic current has a cohomology class in the Dolbeault cohomology group $H^{1,1}(S, \mathbb{C})$ which allows to use intersection theory, see [13, 18, 19]. We stress that all the arguments also hold for immersed Levi-flats, we work in the embedded case for simplicity.

3.1. Leaves and charts. Any leaf of M is an immersed Riemann surface. We call it *elliptic*, *parabolic* or *hyperbolic* depending on its universal covering \mathbb{P}^1 , \mathbb{C} or \mathbb{D} . The foliation \mathcal{F} is *hyperbolic* if every leaf is hyperbolic. M is covered by finitely many foliated charts $U_j \simeq \mathbb{D} \times F_i$, with $F_i \subset \mathbb{D}$, and overlapping as

$$(1) \quad (z, t) = (\alpha_{jj'}(z', t'), \beta_{jj'}(t')).$$

The functions α, β are continuous and $\alpha(\cdot, t')$ is holomorphic for every t' . For C^k Levi-flat we have $F_i =]0, 1[$, the functions α, β are of class C^k by [3], and $\alpha(\cdot, t')$ depends continuously on t' in the C^∞ topology.

3.2. Calculus. Let $\mathcal{O}_{\mathcal{F}}$ (resp. $C_{\mathcal{F}}^\infty$) denote the sheaf of functions which are leafwise holomorphic (resp. smooth). Let $A_{\mathcal{F}}^p$ (resp. $A_{\mathcal{F}}^{p,q}$) denote the space of leafwise p -forms (resp. (p, q) forms). A form $\eta \in A_{\mathcal{F}}^{1,1}$ is *positive* if $\eta = c(z, t) i dz \wedge d\bar{z}$ locally, where $c \in C_{\mathcal{F}}^\infty$ is positive. For $f \in A_{\mathcal{F}}^0$ we define $\partial f = \partial_z f \cdot dz$ and $\bar{\partial} f = \partial_{\bar{z}} f \cdot d\bar{z}$, where $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. The leafwise differential operator $d : A_{\mathcal{F}}^p \rightarrow A_{\mathcal{F}}^{p+1}$ is then equal to $\partial + \bar{\partial}$ and $i\partial\bar{\partial} : A_{\mathcal{F}}^0 \rightarrow A_{\mathcal{F}}^{1,1}$ is a real operator.

Given a metric m on the tangent bundle $T_{\mathcal{F}}$ (see below), we denote by vol_m the leafwise volume form and by $\Delta_m = \text{div grad}$ the leafwise Laplacian. They satisfy $2i\partial\bar{\partial} = \Delta_m \cdot vol_m$.

3.3. Foliated cycles. A *foliated cycle* is a continuous linear form $T : A_{\mathcal{F}}^{1,1} \rightarrow \mathbb{R}$ such that

- $T(d\varphi) = 0$ for any $\varphi \in A_{\mathcal{F}}^1$,
- $T(\eta) > 0$ for any positive form $\eta \in A_{\mathcal{F}}^{1,1}$.

The existence of a foliated cycle is a strong condition. There is a correspondence between foliated cycles and transverse invariant measures [51]. Any compact leaf yields a foliated cycle *via* its current of integration. Any parabolic leaf produces foliated cycles as follows. Let $f : \mathbb{C} \rightarrow L$ be a uniformization. A metric on the tangent bundle $T_{\mathcal{F}}$ being fixed, let a_r denote the area of $D_r := f(\mathbb{D}_r)$ and l_r denote the length of ∂D_r . By an argument due to Ahlfors, there exists a sequence $r_n \rightarrow \infty$ such that $l_{r_n}/a_{r_n} \rightarrow 0$. The normalized current of integration $A_n := \frac{1}{a_{r_n}}[D_{r_n}]$ then converges to a closed positive current A , called an *Ahlfors current*. Those considerations show that \mathcal{F} is hyperbolic if M has no transverse invariant measure. The converse is false, see subsection 4.1.

3.4. Harmonic currents. A *harmonic current* is a continuous linear form $T : A_{\mathcal{F}}^{1,1} \rightarrow \mathbb{R}$ such that

- $T(\partial\bar{\partial}\varphi) = 0$ for any $\varphi \in A_{\mathcal{F}}^0$,
- $T(\eta) > 0$ for any positive form $\eta \in A_{\mathcal{F}}^{1,1}$.

Hahn-Banach theorem allows to prove the existence of harmonic currents, see [25]. They can also be constructed from a hyperbolic leaf by using a weighted Ahlfors procedure, see [18]. A harmonic current has the following local expression in $U_j \simeq \mathbb{D} \times F_i$:

$$T = [H_j(z, t) i dz \wedge d\bar{z}] \otimes \nu_j,$$

where $H_j(\cdot, t)$ are positive harmonic functions and ν_j is a positive measure on F_i . This is a foliated cycle if and only if $H_j(\cdot, t)$ is constant.

The harmonic current is unique for laminations and singular foliations of \mathbb{P}^2 [18], [19], it is also unique for conformal foliations which do not admit any transverse invariant measure [15]. Let us see how a harmonic current defines a cohomology class $[T]$ in the Dolbeault cohomology group $H^{1,1}(S, \mathbb{C})$. The Bott-Chern cohomology group $H_{BC}^{1,1}(S, \mathbb{C})$ is the quotient of the space of smooth closed $(1, 1)$ -forms by the space of $\partial\bar{\partial}$ -exact ones (see [12], section VI.8). When S is a Kähler surface, the natural morphism $H_{BC}^{1,1}(X, \mathbb{C}) \rightarrow H^{1,1}(X, \mathbb{C})$ is an isomorphism by the classical $\partial\bar{\partial}$ -lemma. In particular, since a harmonic current belongs to the dual of $H_{BC}^{1,1}(S, \mathbb{C})$, it also belongs to the dual of $H^{1,1}(S, \mathbb{C})$, which is isomorphic to $H^{1,1}(S, \mathbb{C})$ by duality. That allows to define $[T] \in H^{1,1}(S, \mathbb{C})$ as desired.

3.5. Line bundles. These are the elements of $H^1(M, \mathcal{O}_{\mathcal{F}}^*)$. They can be defined using the charts by identifying $(z, t, \xi) \in U_j \times \mathbb{C}$ with $(z', t', \xi') \in U_{j'} \times \mathbb{C}$ using (1) and the rule $\xi = \gamma_{jj'}(z', t') \cdot \xi'$ for some $\gamma_{jj'} \in \mathcal{O}_{\mathcal{F}}^*$. We shall use the following line bundles on M

- *Tangent line bundle* $T_{\mathcal{F}}$: $\gamma_{jj'} := \frac{\partial\alpha_{jj'}}{\partial z'}$.
- *Canonical line bundle* $K_{\mathcal{F}}$: $\gamma_{jj'} := \left(\frac{\partial\alpha_{jj'}}{\partial z'}\right)^{-1}$ (dual of $T_{\mathcal{F}}$)
- *Normal line bundle* $N_{\mathcal{F}}$: $\gamma_{jj'} := \frac{d\beta_{jj'}}{dt'}$.

The definition of $N_{\mathcal{F}}$ concerns Levi-flat hypersurfaces, β is in that case (at least) of class C^1 . The definition of $N_{\mathcal{F}}$ for minimal sets uses the ambient foliation. In any case the transition functions of $N_{\mathcal{F}}$ are leafwise constant. We have the following adjunction formula (see [5], section 1), which also holds for immersed Levi-flats (see [13], section 3).

Proposition 3.1. $K_{\mathcal{F}} = K_{S|_{\mathcal{F}}} + N_{\mathcal{F}}$.

A metric h on a line bundle L is locally defined by $h_j(z, t)|\xi|$, where $h_j \in C_{\mathcal{F}}^{\infty}$ is positive. If L is defined by $(\gamma_{jj'})$ then

$$h_{j'}(z', t') = h_j(z, t)|\gamma_{jj'}(z', t')|.$$

Metrics always exist, they can be constructed by using partitions of unity. If h' is another metric on L , then $h' = he^{\tau}$ for some function $\tau \in C_{\mathcal{F}}^{\infty}$. The Chern curvature of h is the form $\omega_h \in \mathcal{A}_{\mathcal{F}}^{1,1}$ locally defined by

$$\omega_h := \frac{1}{2i\pi} \partial \bar{\partial} \log h_j(z, t)^2.$$

If $h' = he^{\tau}$ is another metric, then $\omega_{h'} = \omega_h + \frac{1}{i\pi} \partial \bar{\partial} \tau$. If T is a harmonic current, we define the intersection

$$T \cdot L := [T] \cdot [L] = T(\omega_h),$$

the last equality being independent of the metric h on L .

3.6. Some intersections. The Euler-Poincaré characteristic of a harmonic current is $\chi(T) := T \cdot T_{\mathcal{F}}$. That definition extends Gauss-Bonnet formula for compact smooth curves. We have the following result, see [9].

Proposition 3.2 (Candel). *If \mathcal{F} has a parabolic leaf L , then there exists an Ahlfors current such that $\chi(A) = 0$.*

The arguments use Gauss-Bonnet formula for open Riemann surfaces: the idea is to perturb the boundary of A_n (see subsection 3.3 for the definition) in order to control its geodesic curvature. The following lemma is crucial, the proof uses the Bott connexion on $N_{\mathcal{F}}$ (see [10], chapter 6), which is flat when restricted to \mathcal{F} . The intersection $T \cdot N_{\mathcal{F}}$ will be interpreted later as a Lyapunov exponent.

Lemma 3.3.

- (1) $T \cdot N_{\mathcal{F}} = 0$ for every foliated cycle T ,
- (2) $C \cdot C = 0$ for every compact leaf C .

Proof. The first item (see [8], theorem 2 for a related result) follows from the fact that if T is a foliated cycle and L is a line bundle, then $T \cdot L = T(\omega)$ where ω is the curvature of any smooth connexion on L . Indeed the curvatures of two connexions differ by a closed 2-form. The second item is a consequence of the first one and the fact that since C is embedded, then $C \cdot C = T \cdot N_C$ where T is the current of integration on C and N_C is the normal bundle. \square

We shall need the following proposition. It was proved in [25], lemma 4.5 for foliated cycles.

Proposition 3.4. *Let L be a line bundle on M . Assume that it has a holomorphic section s which does not vanish identically on any leaf. Then $T \cdot L \geq 0$ for any harmonic current T .*

Proof. It suffices to construct a smooth metric $|\cdot|$ on L whose curvature is non negative. To do so we construct a smooth leafwise surharmonic function $\varphi : M \rightarrow \mathbb{R}$ with logarithmic singularities along $D := \{s = 0\}$ corresponding exactly with the multiplicities defined by s . We then define the metric by $|s| := e^\varphi$.

Let $\{U_j\}$ be a finite covering of M such that the line bundle and the foliation are trivial on U_j . We fix diffeomorphisms $U_j \simeq \mathbb{D} \times (0, 1)$ such that the foliation is horizontal (for a minimal set replace the interval by F_j), let also $1_j : U_j \rightarrow L$ be a non vanishing holomorphic section. Hence $s = s_j 1_j$ for some holomorphic function $s_j : U_j \rightarrow \mathbb{C}$. Shrinking the open sets U_j and multiplying the sections 1_j by small constants, we can suppose that there exists $0 < r < 1$ such that $|s_j(z, t)| > 1$ if $|z| \geq r$. In particular, $\varphi_j(z, t) := \log^- |s_j|$ is leafwise surharmonic on U_j . Let us regularize that function. Let $\varepsilon := \min\{r/2, (1-r)/2\}$ and $K : [0, 1] \rightarrow \mathbb{R}^+$ be a smooth function with support in $[0, \varepsilon]$ satisfying $\int_0^1 K(t) dt = 1$. Let us define $\varphi_j^{reg} : U_j \rightarrow \mathbb{R}$ by $\varphi_j^{reg}(z, t) := \varphi_j(z, t)$ if $|z| < r/2$ and

$$\varphi_j^{reg}(z, t) := \int K(|z - (x + iy)|) \varphi_j(x + iy) dx dy \quad \text{if } |z| \geq r/2.$$

That function is leafwise surharmonic. (Observe that by construction φ_j^{reg} is 0 if U_j does not intersect D .)

Let $0 < \eta < 1/2$ such that $\hat{U}_j \simeq \mathbb{D} \times (\eta, 1 - \eta) \subset U_j$ is still a covering of M . Let $\hat{\psi}_j : U_j \rightarrow [0, 1]$ be a smooth function of the form $\hat{\psi}_j(z, t) = \hat{\psi}_j(t)$, which is equal to 1 on \hat{U}_j and vanishes on $\mathbb{D} \times ([0, \eta/2] \cup [1 - \eta/2, 1])$. Then $\psi_j = \hat{\psi}_j / \sum_j \hat{\psi}_j$ is smooth on U_j and we have $\sum_j \psi_j = 1$ on M . The function $\psi_j \varphi_j^{reg} : U_j \rightarrow \mathbb{R}$ is smooth, leafwise surharmonic and vanishes on a neighborhood of ∂U_j . We extend it by 0 outside U_j , so that it becomes a smooth and leafwise surharmonic function $\psi_j \varphi_j^{reg}$ on M . Let us verify that $\varphi = \sum_j \psi_j \varphi_j^{reg}$ is convenient. Let U_k be a flow box intersecting D . In some neighborhood of D in U_k , the function $\varphi - \log |s_k|$ is equal to

$$\sum_j \psi_j (\varphi_j^{reg} - \log |s_k|) = \sum_j \psi_j (\log |s_j| - \log |s_k|) = \sum_j \psi_j \log |1_k / 1_j|.$$

Hence it extends as a smooth leafwise surharmonic function on U_k . Since $\log |1_k| = \varphi - \log |s_k|$ is smooth, the metric $|\cdot|$ on L can be defined by $|s| = e^\varphi$ as desired. \square

4. HYPERBOLICITY

In this section, we prove that Levi-flats and minimal sets in surfaces of general type are hyperbolic, and deduce consequences for their topology. We remark that the results are also valid for immersed Levi-flats, but as in section 3, for simplicity we assume M is embedded.

4.1. Leaves are hyperbolic.

Proposition 4.1. *Let $M \subset S$ be a minimal set or a Levi-flat hypersurface of class C^1 in a surface of general type. Then \mathcal{F} is hyperbolic. In particular, the genus of any compact leaf (if it exists) satisfies $g \geq 2$.*

Before giving the proof of theorem 4.1 let us recall classical facts about minimal surfaces and surfaces of general type, see [4], [21], [33] and subsection 2.1. A $(-k)$ -curve is a curve biholomorphic to \mathbb{P}^1 with self-intersection $-k$. A surface S is *minimal* if it does not have any (-1) -curve. Every non minimal surface can be obtained from a minimal one by performing finitely many blowups. A minimal surface of general type has finitely many (-2) -curves. Let \mathcal{C} denote the union of these curves. For n large enough, the pluricanonical map $S \rightarrow \mathbb{P}^N$ associated to $|nK_S|$ contracts each curve of \mathcal{C} to singular points and is biholomorphic elsewhere. In particular, by considering the pullback of the Fubini-Study metric, K_S supports a metric with positive curvature outside \mathcal{C} .

Proof. Assume that there exists a compact leaf C isomorphic to \mathbb{P}^1 (elliptic leaf). The classical adjunction formula $-2 = K_S \cdot C + C^2$ then contradicts $K_S \geq 0$ and $C^2 = 0$ (see lemma 3.3). Assume now that there exists a parabolic leaf L and let A be an Ahlfors current such that $\chi(A) = 0$ (see proposition 3.2). Using the leafwise adjunction formula

$$-\chi(A) = A \cdot K_S|_{\mathcal{F}} + A \cdot N$$

and the fact that $A \cdot N = 0$ (see lemma 3.3), we deduce $A \cdot K_S|_{\mathcal{F}} = 0$. If S is a minimal surface, then K_S has a metric of positive curvature outside \mathcal{C} . Since the curves of \mathcal{C} are not leaves of \mathcal{F} (they are elliptic and use the argument above, or their self intersection is -2 and use lemma 3.3), the local structure of foliated cycle shows that $A \cdot K_S|_{\mathcal{F}} > 0$, contradiction.

Now we focus on the non minimal case. It suffices to consider a blowup $\pi : S' \rightarrow S$ of a minimal surface of general type. By restricting the formula $K_{S'} = \pi^*K_S + E$ (see [21], section 3) to the foliation \mathcal{F} , and intersecting with A , we obtain

$$(2) \quad A \cdot K_{S'}|_{\mathcal{F}} = A \cdot (\pi^*K_S)|_{\mathcal{F}} + A \cdot E|_{\mathcal{F}}.$$

Now π^*K_S has a metric of positive curvature outside E and the proper transforms (of the curves) of \mathcal{C} . Moreover, by lemma 3.3, these curves are not leaves of \mathcal{F} : we have $E^2 = -1$ for the exceptional divisor, and a blow up does not increase the self-intersection of the proper transforms of the curves of \mathcal{C} . Hence the first term of the right side of (2) is positive from the local structure of foliated cycles. The second term is non negative by [25], lemma 4.5 (or by proposition 3.4 for foliated cycles). \square

Observe that even if their CR foliation is hyperbolic, there exist Levi-flats in surface of general type carrying a transverse invariant measure. Indeed, one can consider pull-backs by ramified coverings of a flat \mathbb{P}^1 -bundle whose representation $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ has a dense image in $\mathrm{PSO}(2, \mathbb{R})$, see section 2.

4.2. Poincaré metric. We consider the Poincaré metric in the unit disc given by

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

It is complete of gaussian curvature² -1 . This metric being invariant by the group of biholomorphisms of the unit disc, it induces a metric on any hyperbolic Riemann surface. Hence, on a minimal set or a Levi-flat in a surface of general type, the Poincaré metric on each leaf defines a metric on the tangent bundle to the foliation. The following continuity/compactness result is due to Verjovsky and Candel. It will be used in proposition 6.7:

Theorem 4.2. [53, 9] *Assume that \mathcal{F} is hyperbolic. Then the Poincaré metric is continuous. Equivalently, the set $U(\mathcal{F})$ of uniformization mappings $\pi : \mathbb{D} \rightarrow M$ of some leaf, endowed with the topology of uniform convergence, is compact.*

4.3. Exponential growth. In this paragraph we first observe that Levi-flats in Kähler surfaces have trivial second fundamental group. We then prove that the fundamental group of Levi flats in surfaces of general type have exponential growth. In particular they do not carry the geometries of \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{R}$, \mathbb{R}^3 , nor Nil. Analogous results were obtained in [29] for Levi-flats in the complex projective plane. We follow their argument, using in our context the hyperbolicity of the foliation (see proposition 4.1) and a particular – and presumably well-known – case of Plante’s theorem for codimension one foliations without transverse invariant measure (see [48], corollary 6.4).

Novikov’s theorems are central, let us recall the statements (see [45] and [10], chapter 9). Let M be a compact orientable 3-manifold endowed with a transversely orientable 2-dimensional foliation \mathcal{F} of class C^2 . If one of the conditions is satisfied

- (a) there exists a leaf L such that the inclusion map $\pi_1(L) \rightarrow \pi_1(M)$ has a non-trivial kernel,
- (b) the second fundamental group $\pi_2(M)$ is not trivial

then either the foliation \mathcal{F} has a Reeb component $\mathbb{D} \times \mathbb{S}^1$ or the 3-manifold M is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and \mathcal{F} is the product foliation. Let us notice that (a) is equivalent to the existence of a Reeb component and to the existence of a vanishing cycle.

In the context of Kähler surfaces, the CR foliation of a Levi-flat is taut (the Kähler form is positive on complex directions, see [51]), in particular it has no Reeb component. The proof of this last fact is actually very simple: the homology class of any compact holomorphic curve can not be trivial by Stokes formula. We deduce that if M is a C^2 Levi-flat in a Kähler surface, then either M has trivial second fundamental group or M is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. In the latter case the leaves have zero self-intersection, that implies that S is a rational fibration (up to a modification, see [4], section V.4) and that M is tangent to the fibration.

²The gaussian curvature of $ds^2 = \rho^2|dz|^2$ is $-\Delta_m \log \rho = -\frac{1}{\rho^2} \Delta \log \rho$.

Let us now focus on the exponential growth of the fundamental group. The following result shows in particular that Levi-flats in surfaces of general type are not diffeomorphic to the unitary tangent bundle of \mathbb{P}^1 or an elliptic curve. That allows to assume $g \geq 2$ in the rigidity theorem 7.1.

Proposition 4.3. *Let M be a C^2 Levi-flat hypersurface in a compact complex surface of general type. Then the fundamental group of M has exponential growth.*

Proof. By proposition 4.1, all leaves are hyperbolic Riemann surfaces. Because the Poincaré metric is continuous, the universal coverings of the leaves have exponential growth. Assume by contradiction that the fundamental group of M has sub-exponential growth. By Novikov, the map $\pi_1(L) \rightarrow \pi_1(M)$ is injective for every leaf. Hence the pull-back foliation $\tilde{\mathcal{F}}$ on the universal covering \tilde{M} has simply connected leaves. Let us consider a finite covering $\{U_i\}_{i \in I}$ of M by foliated charts. Let U_i^0 be some lift of U_i in \tilde{M} . Observe that $\{gU_i^0\}$ for $g \in \pi_1(M)$ and $i \in I$ is a covering of \tilde{M} by foliated charts of $\tilde{\mathcal{F}}$. Moreover, because $\pi_1(M)$ has subexponential growth δ , the number of charts of this cover in $B_{\tilde{M}}(x, R)$ is subexponential. By the pigeon hole principle, there exists gU_i^0 whose intersection with L contains at least two different plaques. Hence there exists a simple closed loop in \tilde{M} which is transverse to $\tilde{\mathcal{F}}$ (Novikov). Its projection in M is a loop transverse to \mathcal{F} which is homotopically trivial. But that must produce a Reeb component. \square

5. LEAFWISE BROWNIAN MOTION AND LYAPUNOV EXPONENT

As in the preceding sections the arguments work for immersed Levi-flats and minimal sets, for simplicity we assume that they are embedded.

5.1. Heat kernel and Brownian motion. Let M be a minimal set or a Levi-flat hypersurface in a compact complex surface. Given a metric m on the tangent bundle $T_{\mathcal{F}}$, we denote by vol_m the leafwise volume form and by Δ_m the leafwise Laplace-Beltrami operator. We have $2i\partial\bar{\partial} = \Delta_m \cdot vol_m$ and $2i\partial\bar{\partial} = \Delta \cdot dx \wedge dy$ using usual laplacian. Let $p_m(t, x, y)$ be the leafwise heat kernel: this is the smallest positive solution of the heat equation

$$\frac{\partial}{\partial t} = \Delta_m \quad , \quad \lim_{t \rightarrow 0} p_m(t, x, y) = \delta_x(y) \quad ,$$

where δ_x denotes the Dirac mass at x , the limit being in the sense of distributions (see [11]). We consider the Brownian motion on L_x with transition probability $p_m(t, x, y)$. This is a diffusion process with continuous sample paths. Since the leaves are complete and of bounded geometry, the latter are defined for every $t \in \mathbb{R}^+$. Let $\Gamma_x := \{\gamma : \mathbb{R}^+ \rightarrow L_x, \gamma(0) = x\}$ be the leafwise continuous paths, and W_x be the Wiener probability measure on Γ_x given by Brownian motion.

5.2. Harmonic measure and Garnett's theory. We refer to [22]. A *harmonic measure* is a probability measure μ on M satisfying

$$\forall \psi \in A_{\mathcal{F}}^0 \quad , \quad \int \Delta_m \psi \, d\mu = 0.$$

There always exist harmonic measures, this can be proved using Hahn-Banach theorem or Markov-Kakutani fixed point theorem. The set of harmonic measures is convex, its extremal points are called *ergodic*. Harmonic measures allows to introduce ergodic theory and Lyapunov exponents. Let us specify the associated dynamical system $(\Gamma, (\sigma_t)_{t \geq 0}, \bar{\mu})$. The phase space is $\Gamma := \cup_{x \in M} \Gamma_x$ and $(\sigma_t)_{t \geq 0}$ is the shift semi-group acting on Γ by

$$\sigma_t(\gamma)(u) := \gamma(t + u).$$

The invariant probability measure $\bar{\mu}$ on Γ is defined by

$$\bar{\mu} := \int_M W_x d\mu(x).$$

5.3. Lyapunov exponent. For any $x \in M$ and $\gamma \in \Gamma_x$, let

$$h_{\gamma,t} : \tau_x \rightarrow \tau_{\gamma(t)}$$

denote the holonomy map over the path $\gamma : [0, t] \rightarrow L_x$. Here $(\tau_x)_{x \in M}$ denotes a family of discs (or intervals if we deal with Levi-flats) transversal to \mathcal{F} and centered at x , it can be constructed from transversal discs in the foliated charts. To simplify the exposition, we denote $h'_{\gamma,t}$ for $h'_{\gamma,t}(0)$. Let $|\cdot|$ be a metric on $N_{\mathcal{F}}$.

Theorem 5.1. *Let μ be an ergodic harmonic measure on M . There exists $\lambda \in \mathbb{R}$ such that for μ -almost every $x \in M$ and W_x -almost every $\gamma \in \Gamma_x$*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}| = \lambda.$$

The number λ is called the Lyapunov exponent of μ .

The proof uses Birkhoff's ergodic theorem for $(\Gamma, (\sigma_t)_{t \geq 0}, \bar{\mu})$. The compactness of M ensures that the exponent is finite and does not depend on $|\cdot|$. However, it depends on m via the Wiener measure W_x on the leaves.

5.4. Cohomological expression of the Lyapunov exponent. The following observation gives the relation between harmonic measures and harmonic currents, see [25] for more details.

Remark 5.2. *Given a metric m on the tangent bundle $T_{\mathcal{F}}$, there is a bijection between the projective classes of harmonic currents on M and the harmonic measures supported on M given by $\mu \mapsto T$, where T is the unique current such that $\mu = T \wedge \text{vol}_m$.*

Now we can state a fundamental formula.

Proposition 5.3. *Let M be a minimal set or a Levi-flat hypersurface in a compact complex surface S . Let μ be an ergodic harmonic measure and T be the unique harmonic current such that $\mu = T \wedge \text{vol}_m$. Then the Lyapunov exponent of μ for the metric m is equal to*

$$\lambda = -2\pi T \cdot N_{\mathcal{F}}.$$

The proof is based on the formula

$$\lambda = \int_M \Delta_m \log |\cdot| d\mu,$$

where $|\cdot|$ is any metric on $N_{\mathcal{F}}$ (see [15], theorem 2.10). That expression comes from the heat equation $\Delta_m = \frac{\partial}{\partial t}$ and the ergodicity of the generator Δ_m . The conclusion follows from the relation $2i\partial\bar{\partial} = \Delta_m \cdot \text{vol}_m$.

5.5. Application.

Proposition 5.4. *Let M be a minimal set or a Levi-flat hypersurface of class C^1 in a surface of general type S , endowed with the Poincaré metric on the leaves. Then the Lyapunov exponent of any ergodic harmonic measure μ for the Poincaré metric satisfies $\lambda > -1$.*

Proof. Let T be the unique harmonic current such that $\mu = T \wedge \text{vol}_m$. Then, by proposition 5.3 and adjunction formula (proposition 3.1), we have

$$\lambda = -2\pi T \cdot T_{\mathcal{F}} + 2\pi T \cdot K_S|_{\mathcal{F}} = -1 + 2\pi T \cdot K_S|_{\mathcal{F}}$$

If $ds^2 = \rho^2|dz|^2$ denote the Poincaré metric, then

$$T \cdot T_{\mathcal{F}} = T\left(\frac{1}{2i\pi}\partial\bar{\partial}\log\rho^2\right) = T\left(-\frac{1}{2\pi}\Delta_m\log\rho \cdot \text{vol}_m\right) = -\frac{1}{2\pi},$$

since the gaussian curvature of the metric is equal to -1 . It remains to prove that $T \cdot K_S|_{\mathcal{F}}$ is positive. If S is a minimal surface, then K_S has a metric of positive curvature outside \mathcal{C} , and these curves are not leaves of \mathcal{F} . That implies $T \cdot K_S|_{\mathcal{F}} > 0$ from the local structure of harmonic currents. For the non minimal case, we follow the arguments of theorem 4.1, replacing foliated cycles by harmonic currents. If $\pi : S' \rightarrow S$ is a blowup of a minimal surface of general type, then

$$(3) \quad T \cdot K_{S'}|_{\mathcal{F}} = T \cdot (\pi^*K_S)|_{\mathcal{F}} + T \cdot E|_{\mathcal{F}}.$$

As before, π^*K_S has a metric of positive curvature outside E and the proper transforms of \mathcal{C} , which are not leaves of \mathcal{F} . The first term of the right side of (3) is positive from the local structure of harmonic currents. The second term is non negative by proposition 3.4. \square

5.6. Remark. To end this section, we recall a formula for the Lyapunov exponent on exceptional minimal sets in \mathbb{P}^2 , see [15]. They are hyperbolic by [8], we endow the leaves with the Poincaré metric.

Proposition 5.5. *Let \mathcal{F} be a singular holomorphic foliation of \mathbb{P}^2 of degree $d \geq 2$. Let M be a hypothetical minimal set. Then the Lyapunov exponent of its harmonic measure for the Poincaré metric is equal to*

$$\lambda = -\frac{d+2}{d-1}.$$

The proof relies on $N_{\mathcal{F}} = \mathcal{O}(d+2)$ and $K_{\mathcal{F}} = \mathcal{O}(d-1)$, see [6]. Then

$$\lambda = -2\pi T \cdot N_{\mathcal{F}} = 2\pi \frac{d+2}{d-1} T \cdot T_{\mathcal{F}}$$

and the expected value follows. Observe that the exponent of a holomorphic foliation of \mathbb{P}^2 is fixed by its degree. The situation is different for rational maps on \mathbb{P}^1 , where the exponent of the maximal entropy measure depends on the map.

6. ANOSOV LEVI-FLAT

We call a Levi-flat *Anosov* if its CR foliation is topologically conjugated to the weak stable foliation of an Anosov flow on a compact manifold.

Theorem 6.1. *Let M be an Anosov Levi-flat. Then \mathcal{F} has no transverse invariant measure, in particular its CR foliation is hyperbolic. If we endow the leaves with the Poincaré metric then the Lyapunov exponent of any ergodic harmonic measure μ satisfies $\lambda \leq -1$.*

The absence of transverse invariant measure is based on the Anosov property. For the Lyapunov exponent, we prove that we can stretch the trajectories of the Anosov flow in the uniformizations, that allows to construct a continuous flow whose orbits are leafwise geodesics for the Poincaré metric. The resulting flow is called a *point at infinity*. The bound on λ is proved in that more general context, it relies on volume estimates in the spirit of Margulis-Ruelle's inequality.

Corollary 6.2. *A C^1 Levi-flat in a surface of general type is not Anosov.*

Proof. We use Lyapunov exponents. Suppose that there exists such a Levi-flat, let μ be an ergodic harmonic measure and λ be the Lyapunov exponent of μ . On one hand, theorem 5.4 states that $\lambda > -1$ because the surface is of general type. On the other hand, theorem 6.1 asserts that $\lambda \leq -1$. \square

We apply this result in section 7 to the weak stable foliation of the geodesic flow on the unit tangent bundle of negatively curved surfaces and to the weak stable foliation of the suspension of Anosov diffeomorphisms of the 2-torus (hyperbolic torus bundles). Theorem 6.1 is of independant interest since a lot of Anosov flows were constructed on 3-manifolds, including hyperbolic or graph manifolds [3, 20, 27, 28]. We also point out that there is no requirement on the CR structure of the foliation, this is important since there is an infinite number of moduli of CR structures [14].

6.1. Proof of theorem 6.1. We say that a Levi-flat M has a *point at infinity* if it is hyperbolic and if it supports a continuous flow $\psi : M \times \mathbb{R} \rightarrow M$ tangent to the CR foliation which lifts (on the universal covering $\mathbb{D} \rightarrow L$ of every leaf) to a flow whose trajectories are hyperbolic geodesics (parametrized by arc length) tending to the same point at infinity in $\partial\mathbb{D}$.

Lemma 6.3 (Stretching). *An Anosov Levi-flat has a point at infinity.*

The proof is postponed to subsection 6.2. The arguments are based on the fact that trajectories of Anosov flows are quasi-geodesics, hence they have a well-defined limit in $\partial\mathbb{D}$ when lifted to the universal cover. By the Anosov property, this limit does not depend on the trajectory in the leaves. We then consider the flow ψ defined by the geodesics pointing towards this point at infinity. Continuity of the limit point of quasi-geodesics for the compact open topology ensures the continuity of that flow.

Lemma 6.4 (Volume estimates). *Let M be a Levi-flat having a point at infinity. We endow the CR foliation with the Poincaré metric. Then the Lyapunov exponent of any ergodic harmonic measure satisfies $\lambda \leq -1$.*

The proof is given in subsection 6.3. The flow ψ induced by the point at infinity is absolutely continuous, and we relate its Jacobian with the Lyapunov exponent by shadowing geodesics with Brownian trajectories. The bound on the exponent follows from volume estimates in the spirit of Margulis-Ruelle inequality.

6.2. Stretching trajectories of an Anosov flow. By assumption the foliation is topologically conjugated to the weak stable foliation of an Anosov flow. It is classical that the conjugation can be chosen smooth along the leaves. Hence, with no loss of generality, we can suppose until the end of subsection 6.2 that \mathcal{F} is the weak stable foliation \mathcal{F}^s of an Anosov flow $A : M \times \mathbb{R} \rightarrow M$. We shall write $t \mapsto A_x(t)$ or $x \mapsto A^t(x)$ when x and t are respectively fixed.

Let us first observe that \mathcal{F} has no transverse invariant measure. Indeed, a leafwise volume form on \mathcal{F} is uniformly exponentially contracted by the Anosov flow. Hence, if there exists a transverse invariant measure on \mathcal{F} , its product with a leafwise volume form is a finite measure which is uniformly and exponentially contracted by the Anosov flow. This contradicts that the total volume is preserved. As a consequence, the weak stable foliation has hyperbolic leaves.

Now we stretch the trajectories of the Anosov flow. Let us endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric. Let $x \in M$ and $\pi : \mathbb{D} \rightarrow L_x$ be a uniformization satisfying $\pi(0) = x$. We define

$$\forall t \in \mathbb{R}, \alpha(t) := \pi^{-1}A_x(t),$$

which is well defined by the covering property of π .

Proposition 6.5. *There exists a constant $\rho > 0$ (independent of α) satisfying for every $(t, t') \in \mathbb{R}^2$:*

$$\rho^{-1}|t - t'| \leq d_P(\alpha(t), \alpha(t')) \leq \rho|t - t'|.$$

Proof. The upper bound is just the fact that the vector field X is bounded for the Poincaré metric: one can take $\rho = \max_M |X|$. Let us verify the lower bound. Observe that the normal bundle to the weak stable foliation is identified with the strong stable foliation E^{uu} . The Anosov property yields a continuous (adapted) metric on this bundle and $\xi > 0$ such that

$$\forall t \geq 0, \forall v_u \in E^{uu}, |dA^t(v_u)| \geq e^{\xi t}|v_u|.$$

In particular, if h denotes the holonomy of the weak stable foliation, then

$$|h'_{\pi \circ \alpha|_{[t, t']}}| \geq e^{\xi(t' - t)}.$$

On the other hand, there exists $\xi' > 0$ such that the derivative of a holonomy map along a path γ is always $\leq e^{\xi' \text{length}(\gamma)}$ (use local coordinates). Hence $d_P(\alpha(t), \alpha(t')) \geq (\xi - \xi')(t' - t)$ as desired. \square

Proposition 6.5 implies that the α 's are quasi-geodesics. Such a property implies that there exists $\delta := \lim_{t \rightarrow \infty} \alpha(t) \in \partial \mathbb{D}$. We define a vector field on M by setting $Y(x) := d_0 \pi(\delta)$, it does not depend on π (whereas δ does). In other words, $Y(x)$ is the unitary direction at x which looks at the limit of the A -trajectory starting from x .

Let $\psi(x, t)$ be the flow on M associated with Y . By definition, the path

$$\forall t \in \mathbb{R}, \beta(t) := \pi^{-1}\psi_x(t)$$

coincides with the geodesic radius $[0, \delta]$. Observe that because of the Anosov property, if α' is the lift of another trajectory contained in the leaf through x , then the limit $\lim_{t \rightarrow \infty} \alpha'(t)$ is also δ . Hence, the flow ψ lifts to the universal cover of the leaves to a flow whose trajectories are geodesics tending to the same point at infinity. To finish the proof, we need to prove that ψ is continuous. We use the following classical proposition, its proof uses hyperbolic geometry in \mathbb{D} .

Proposition 6.6. *Let $\alpha_n : \mathbb{R}^+ \rightarrow \mathbb{D}$ be a sequence of ρ -quasigeodesics starting at $\alpha_n(0) = 0$ and converging uniformly on compact sets to a ρ -quasigeodesic α . If $\delta_n := \lim_{t \rightarrow +\infty} \alpha_n(t)$ and $\delta := \lim_{t \rightarrow +\infty} \alpha(t)$, then $\lim_n \delta_n = \delta$.*

Let us deduce the continuity of ψ . It suffices to prove that the vector fields Y is continuous on M , hence we want to show $Y(x_n) \rightarrow Y(x)$ for every $x_n \rightarrow x$. By compactness, we can assume $Y(x_n) \rightarrow Y$ up to take a subsequence. Let $\pi_n : \mathbb{D} \rightarrow L_{x_n}$ be uniformizations. Taking again a subsequence, π_n converges uniformly on compact sets to a uniformization $\pi : \mathbb{D} \rightarrow L_x$ (see theorem 4.2). The lift $\alpha_n := \pi_n^{-1}A_{x_n}$ also converges to the lift $\alpha := \pi^{-1}A_x$. Since these paths are ρ -quasigeodesics, there exist $\delta_n := \lim \alpha_n, \delta := \lim \alpha$, and we have $\delta_n \rightarrow \delta$ by proposition 6.6. Composing by $d_0\pi_n$ and $d_0\pi$, we obtain $Y = Y(x)$. Proof of lemma 6.3 is complete.

6.3. Volume estimates. Let $\psi : M \times \mathbb{R} \rightarrow M$ be the continuous flow coming from the point at infinity. The next proposition shows that ψ is absolutely continuous and it computes its Jacobian. Let $\nu := \theta \otimes \text{vol}_P$, where θ is the volume measure of some fixed metric on $N_{\mathcal{F}}$ and vol_P is the leafwise Poincaré volume.

Proposition 6.7. *ψ is absolutely continuous with respect to ν , and*

$$\forall t \geq 0, \forall x \in M, \frac{\psi_*^t \nu}{\nu}(x) = e^t \cdot |h'_{(\psi^{-s}(x))_{0 \leq s \leq t}}|.$$

Proof. Let us compute the jacobian of ψ^t . It suffices to consider t small. Every point is then moved in the same flow box, and the homeomorphism ψ^t takes the form

$$\psi^t(z, u) = (\varphi^t(z, u), u),$$

where $\varphi^t(\cdot, u)$ is smooth and depends continuously on u in the smooth topology. Writing $\theta = f(z, u)du$, elementary considerations show that ψ^t is absolutely continuous and that

$$\frac{\psi_*^t \nu}{\nu} = \frac{(\varphi^{-t})^* \text{vol}_P}{\text{vol}_P} \cdot \frac{f \circ \varphi^{-t}}{f} = \frac{(\varphi^{-t})^* \text{vol}_P}{\text{vol}_P} \cdot |h'_{(\psi^{-s}(x))_{0 \leq s \leq t}}|.$$

We can uniformize any leaf by the upper half plane, in such a way that $\varphi^t(\cdot, u)$ acts as $x + iy \mapsto x + ie^t y$. Since $\text{vol}_P = dx \wedge dy/y^2$ in those coordinates, we have $(\varphi^{-t})^* \text{vol}_P = e^t \text{vol}_P$ as desired. \square

We now relate the Jacobian of ψ to the Lyapunov exponent λ of any ergodic harmonic measure μ (with respect to Poincaré metric).

Proposition 6.8. *There exists a Borel set satisfying $\nu(B_\mu) > 0$ and*

$$\forall x \in B_\mu, \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{(\psi^{-s}(x))_{0 \leq s \leq t}}| = \lambda.$$

We shall need the three results below. Let W denote the Wiener measure on the set of continuous paths Γ in \mathbb{D} starting at the origin.

Proposition 6.9. *For W -almost every $\gamma \in \Gamma$, we have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} d_P(0, \gamma(t)) = 1.$$

This result is classical (see e.g. [47], section 9.6), we provide a proof in an appendix (see section 9). We shall use the well-known shadowing property of geodesics by Brownian paths (see [1], théorème 7.3). Recall that W -a.e. γ has a limit $\delta = \lim_{t \rightarrow \infty} \gamma(t) \in \partial\mathbb{D}$.

Theorem 6.10 (Ancona). *For W -a.e. $\gamma \in \Gamma$, there exists $c > 0$ such that*

$$d_P(\gamma(t), [0, \delta]) \leq c \log t$$

for t large enough.

Finally let us recall the following proposition from [15].

Proposition 6.11. *There exists an open set $U \subset M$ satisfying the following properties. For every $x \in U$, there exists $\Lambda_x \subset \Gamma_x$ such that*

- (1) $W_x(\Lambda_x) > 0$,
- (2) $\forall \gamma \in \Lambda_x, \lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}| = \lambda$.

This result needs the Hölder regularity of the metric, counterexamples are given in [16]. Here we can use the reinforcement [17] of Verjovsky's theorem showing that Poincaré metric is indeed Hölder. An alternative is to apply [15] to a smooth hermitian metric on $T_{\mathcal{F}}$, and to use the conformal invariance of Brownian motion in dimension 2.

We deduce the next lemma, which gives proposition 6.8. Let $x \in U$ and $\pi : \mathbb{D} \rightarrow L_x$ be a uniformization such that $\pi(0) = x$. We identify Γ with Γ_x via these coordinates. Let $A \subset \Gamma \simeq \Gamma_x$ be the set of paths satisfying Ancona's theorem and $\delta_x := \delta(A \cap \Lambda_x)$.

Lemma 6.12. *Let $\delta \in \delta_x$ and $\beta : \mathbb{R} \rightarrow \mathbb{D}$ be a geodesic tending to δ at infinity. Then*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta,t}| = \lambda.$$

That yields for every $x \in U$ a subset of ψ -trajectories in L_x of positive (Lebesgue) measure satisfying the limit. By Fubini's theorem, that limit is satisfied on a set of positive (Lebesgue) measure, and proposition 6.8 follows.

Proof. (lemma 6.12) Let $\epsilon > 0$. Using the three results above we get for every $\gamma \in A \cap \Lambda_x$ and $t \geq t(\gamma, \epsilon)$ a time $\tau_t \geq 0$ satisfying

$$d_P(\gamma(t), \beta(\tau_t)) \leq c \log t \quad \text{and} \quad |\tau_t - t| \leq \epsilon t + c \log t$$

and

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}| = \lambda.$$

Observe that we can replace $[0, \delta]$ by β in Ancona's estimates, since these geodesics tend to the same point at infinity: they are exponentially asymptotic. Let $\tilde{\gamma}$ be the geodesic joining $\gamma(t), \beta(\tau_t)$ and let $\hat{\gamma}$ be the geodesic joining $\beta(\tau_t), \beta(t)$. Let l_t be the length of $\tilde{\gamma}$ (the length of $\hat{\gamma}$ is $|\tau_t - t|$). Using the fact that $\gamma + \tilde{\gamma} + \hat{\gamma}$ is homotopic to β , we have the chain rule

$$(4) \quad h'_{\beta,t} = h'_{\tilde{\gamma},|\tau_t-t|} \cdot h'_{\hat{\gamma},l_t} \cdot h'_{\gamma,t}.$$

Using similar arguments as in the proof of proposition 6.5, we obtain

$$|\log |h'_{\tilde{\gamma},|\tau_t-t|}| | \leq C \cdot l_t \leq C \cdot c \log t,$$

$$|\log |h'_{\hat{\gamma},l_t}| | \leq C \cdot |\tau_t - t| \leq C \cdot (\epsilon t + c \log t).$$

Taking logarithm and dividing by t the chain rule (4), we obtain

$$\lambda - C\epsilon \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta,t}| \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta,t}| \leq \lambda + C\epsilon.$$

The conclusion follows by letting ϵ to zero. □

7. LEVI-FLATS CARRYING THE GEOMETRY OF Sol OR $\mathrm{PSL}(2, \mathbb{R})$

This section is devoted to the following theorem.

Theorem 7.1. *Let M be one of the following 3-manifolds:*

- (1) *a hyperbolic torus bundle,*
- (2) *the unitary tangent bundle $T^1\Sigma$ of a compact surface Σ .*

Then M is not diffeomorphic to a C^2 Levi-flat in a surface of general type.

This is a consequence of corollary 6.2 (dynamical part) and proposition 7.2 (topological part). The proof of 7.2 relies on deep rigidity theorems by Thurston [52], Ghys-Sergiescu [26] and Matsumoto [42]. We can assume for the case (2) that the genus of Σ is ≥ 2 by proposition 4.3.

Proposition 7.2. *Let M be a C^2 Levi-flat in a surface of general type. Suppose that it is diffeomorphic to either (1) or (2)-with Σ of genus ≥ 2 . Then M is an Anosov Levi-flat.*

Our arguments actually prove that M can not be immersed as a C^2 Levi-flat in a surface of general type. Hence M can not be a finite covering of such a Levi-flat, proving that the geometries Sol and $\mathrm{PSL}(2, \mathbb{R})$ do not appear.

7.1. Hyperbolic torus bundle. Let $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ be a torus. For any $A \in \mathrm{GL}_2(\mathbb{Z})$ let T_A be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the relation $(m, t) \sim (Am, t+1)$. That manifold is a *torus bundle* over the circle. It is called *hyperbolic* if A is hyperbolic, namely when the modulus of its trace is > 2 . In that case, A has two distinct real eigenvectors with irrational slopes, these directions define the *stable and unstable foliations* of A on \mathbb{T}^2 . In that case T_A supports the geometry of Sol. The *suspensions of the stable and unstable foliations* are the foliations on T_A respectively defined as the quotients of the product foliations on $\mathbb{T}^2 \times \mathbb{R}$. The family of maps $(m, t) \rightarrow (m, t + t')$ defines an Anosov flow on T_A . We use the following.

Theorem 7.3 (Ghys-Sergiescu, [26]). *Let M be an orientable hyperbolic torus bundle T_A endowed with a transversally orientable C^{k+2} foliation \mathcal{F} ($0 \leq k \leq \infty$). If that foliation does not have any compact leaf, then it is C^k -conjugated to the suspension of the stable foliation on T_A .*

Let us deduce case (1) of proposition 7.2. Let M be a hyperbolic torus bundle T_A and assume that it is diffeomorphic to a C^2 Levi-flat in a surface of general type. Its CR foliation \mathcal{F} is of class C^2 by [3]. Up to take a double covering, we can assume that M is orientable and \mathcal{F} is transversally orientable. Then, in view of theorem 7.3, it suffices to prove that \mathcal{F} has no compact leaf. Suppose to the contrary that \mathcal{F} has a compact leaf, denoted C . It has genus $g \geq 2$ by the general type property (see theorem 4.1). Then the natural map $\pi_1(C) \rightarrow \pi_1(M)$ is not injective, because $g \geq 2$ and the topology of a torus bundle T_A is not rich enough (solvability of the Lie group). Novikov's theorem (see section 4.3) then implies that \mathcal{F} has a compact leaf of genus 0 or 1, which is excluded by theorem 4.1.

7.2. Circle fibrations. Let Σ be a compact oriented surface of genus $g \geq 2$. Let $\pi_1(\Sigma)$ denote its fundamental group and $\text{Diff}_+^r(\mathbb{S}^1)$ be the group of orientation preserving C^r diffeomorphisms of the circle ($0 \leq r \leq \infty$). Let $\rho : \pi_1(\Sigma) \rightarrow \text{Diff}_+^r(\mathbb{S}^1)$ be a representation and $eu(\rho)$ denote its Euler number (see [23, 50]). If X denotes the quotient of $\mathbb{H} \times \mathbb{S}^1$ by the action

$$\gamma(p, \xi) = (\gamma p, \rho(\gamma)\xi) \quad , \quad \gamma \in \pi_1(\Sigma)$$

then $eu(\rho)$ gives the amount of twisting of the \mathbb{S}^1 -bundle X . Let \mathcal{F} be the quotient of the horizontal foliation of $\mathbb{H} \times \mathbb{S}^1$. The Milnor-Wood inequality states that $|eu(\rho)| \leq -\chi(\Sigma) = 2g - 2$. The maximal value is obtained when ρ embeds $\pi_1(\Sigma)$ as a discrete subgroup of $\text{PSL}(2, \mathbb{R})$, i.e. when X is the unitary tangent bundle $T^1\Sigma$. A converse is true in class C^2 , see [42]:

Theorem 7.4 (Matsumoto). *Let $\rho, \rho' : \pi_1(\Sigma) \rightarrow \text{Diff}_+^2(\mathbb{S}^1)$ be representations satisfying $eu(\rho) = eu(\rho') \in \{-\chi(\Sigma), \chi(\Sigma)\}$. Then ρ and ρ' are topologically conjugated.*

In other words, if $|eu(\rho)|$ is maximal, then X is homeomorphic to $T^1\Sigma$ and \mathcal{F} can be topologically conjugated to the weak stable foliation of the geodesic flow for *any* metric of negative curvature on Σ (in particular the Poincaré metric).

Let us prove case (2) of proposition 7.2. Let $M = T^1\Sigma$ be the unitary tangent bundle of a compact surface of genus $g \geq 2$. Assume it is diffeomorphic to a C^2 Levi-flat in a surface of general type, let \mathcal{F} be its CR foliation. Theorem 4.1 ensures that \mathcal{F} has no compact leaf of genus 1. Moreover, it has no compact leaf of genus ≥ 2 . Indeed, the unitary tangent bundle to a hyperbolic compact closed surface do not contain incompressible surfaces of genus ≥ 2 , and by Novikov's theorem, a compact leaf is incompressible, since the CR foliation is taut. Hence, by Thurston's argument (see [52], [37]), there exists an isotopy between \mathcal{F} and a foliation transverse to the circle fibration over Σ . In particular, the foliation \mathcal{F} is a suspension given by a morphism $\rho : \pi_1(\Sigma) \rightarrow \text{Diff}_+^2(\mathbb{S}^1)$. Moreover, the Euler number $eu(\rho)$ is maximal, equal to $\chi(\Sigma)$. Theorem 7.4 then implies that \mathcal{F} is topologically

conjugated to the weak stable foliation of the geodesic flow for the Poincaré metric on Σ , as desired.

7.3. Remark. Let M be a (hypothetical) smooth Levi-flat in \mathbb{P}^2 . By the classical results of section 4.3, M supports neither the geometry of \mathbb{S}^3 and $\mathbb{S}^2 \times \mathbb{S}^1$ (Kähler surface) nor of \mathbb{R}^3 and Nil (M does not have any transverse invariant measure, see [29]). We claim that an analytic Levi-flat in \mathbb{P}^2 can not support the geometry of Sol or $\mathrm{PSL}(2, \mathbb{R})$. First, by using [23, 26] and the same arguments as before, the CR foliation of M has a transverse projective structure. Now, if M is analytic, then every connected component of $\mathbb{P}^2 \setminus M$ is Stein, and its CR foliation extends to a singular holomorphic foliation of \mathbb{P}^2 [38]. In our case it extends to a transversally projective singular holomorphic foliation on \mathbb{P}^2 . But such a foliation has an invariant algebraic curve, which is incompatible with the existence of a minimal set [8].

8. HAUSDORFF DIMENSION OF MINIMAL SETS

Let M be a minimal set in a complex compact surface S . Its holonomy pseudo-group is *not discrete* if there exist transversal discs $\tau' \subset \subset \tau$ and a sequence of holonomy maps $h_n : \tau' \rightarrow \tau$ different from identity which converges uniformly to identity. Otherwise, it is called *discrete*. We say that M is discrete or not. That notion of discreteness was introduced in [24] and [43]. Let us recall that if \mathcal{F} is minimal, then either there exists a transverse invariant measure or there exists a unique harmonic measure and its Lyapunov exponent is negative [15].

Theorem 8.1. *Let M be a minimal set in a complex compact surface S . Assume that M is discrete and does not support any transverse invariant measure. Let m be a metric on the tangent bundle $T_{\mathcal{F}}$ and μ be the unique harmonic measure. Then*

$$\dim_H M \geq \frac{h}{|\lambda|},$$

where h denotes the Kaimanovich entropy of m , and λ denotes the Lyapunov exponent of μ .

In the context of random walks on linear groups, the dimension of the harmonic measure is equal to entropy/exponent [34, 35]. Such a ratio also appears for the dimension of invariant measures for dynamical systems, see e.g. [41] for rational maps on \mathbb{P}^1 . It is possible to establish an analogous formula in our context, however we only need a lower estimate for the Hausdorff dimension.

8.1. Kaimanovich entropy. We refer to [31], [32], [36]. Let M be a minimal set and m be a metric on $T_{\mathcal{F}}$. The *Kaimanovich entropy* is defined by the following theorem. Recall that $p_m(x, y, t)$ denotes the heat kernel on the leaf L_x .

Theorem 8.2. *There exists $h_m \geq 0$ such that for μ -a.e. $x \in M$,*

$$(5) \quad h_m = \lim_{t \rightarrow \infty} -\frac{1}{t} \int_{L_x} p_m(x, y, t) \log p_m(x, y, t) \mathrm{vol}_m(dy).$$

Moreover, if M has no transverse invariant measure, then $h_m > 0$.

If μ -almost every leaf is the unit disc and m is the leafwise Poincaré metric of curvature -1 , then $h_m = 1$. We shall use the following geometrical interpretation of the entropy: h_m is the exponential growth of separated positions for Brownian motion. To be more precise, introduce the following definition. A subset $Z \subset L_x$ is a (C, D) -lattice if its points are C -separated and if any point of L_x lies at a distance $\leq D$ from Z . We define below the sequence of measures $(M_n)_n$ on the lattice Z .

Proposition 8.3. *Let $x \in M$ generic for μ . Let Z be a lattice in L_x and $(Z_n)_n$ be subsets of Z such that $M_n(Z_n) \geq 1/2$. Then*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Z_n| \geq h.$$

To define M_n we introduce $T_z := \cap_{z' \in Z} \{d(y, z) \leq d(y, z')\}$, these are the tiles of the Voronoi tessellation associated to the lattice. Let π be the projection $L_x \rightarrow Z$ (defined a.e.) and B_n be the distribution of Brownian motion starting at x at the time n (with density $p_m(x, y, n) \text{vol}_m(dy)$). Then M_n stands for the probability measure $\pi_* B_n$. As announced, proposition 8.3 can be made more precise: for any lattice Z in a generic leaf, there exists $(Z_n)_n$ such that $M_n(Z_n) \geq 1/2$ and $\frac{1}{n} \log |Z_n| \rightarrow h$.

8.2. Proof of theorem 8.1. Let M be a discrete minimal set with no transverse invariant measure. We fix $\varepsilon > 0$, $x \in M$ and τ_x a transversal to \mathcal{F} at x . It suffices to prove that $\dim_H M \cap \tau_x \geq \frac{h-3\varepsilon}{|\lambda|+3\varepsilon}$. Let $\mathbb{D}(r) \subset \tau_x$ denote the disc centered at zero of radius r , we put a subscript to indicate a center. By Koebe distortion theorem, there exists $\kappa > 0$ such that

$$\mathbb{D}_{h(0)}(e^{-\kappa}|h'(0)|\rho) \subset h(\mathbb{D}(\rho)) \subset \mathbb{D}_{h(0)}(e^\kappa|h'(0)|\rho)$$

for any holomorphic injective function defined on $\mathbb{D}(2\rho)$. We need to control the domain of definition of holonomy maps. Proposition 2.1 of [15] asserts that for μ -a.e. $x \in M$ and W_x -a.e. $\gamma \in \Gamma_x$, there exist $r_\gamma(\varepsilon) > 0$ and $t_\gamma(\varepsilon) \geq 0$ such that for every $r \leq r_\gamma(\varepsilon)$ and $t \geq t_\gamma(\varepsilon)$:

- $h_{\gamma,t}$ is well defined from $\mathbb{D}(r) \subset \tau_x$ to $\tau_{\gamma(t)}$,
- $h_{\gamma,t}$ satisfies $e^{t(\lambda-\varepsilon)} \leq |h'_{\gamma,t}(0)| \leq e^{t(\lambda+\varepsilon)}$.

Let us fix x generic for μ , r_0 small and t_0 large such that

$$\Gamma_\varepsilon := \{\gamma \in \Gamma_x, r_\gamma(\varepsilon) \geq 2r_0, t_\gamma(\varepsilon) \leq t_0\}$$

satisfies $W_x(\Gamma_\varepsilon) \geq 1/2$. Let $r := r_0 e^{-2\kappa}/8$ (κ is the distortion constant). The next proposition is proved in subsection 8.3.

Proposition 8.4. *For n large enough there exist*

- (i) a set $I'_n \subset L_x \cap \tau_x$ of cardinality larger than $e^{n(h-3\varepsilon)}$,
- (ii) a negative number $\lambda_n \in [n(\lambda - 2\varepsilon), n(\lambda + 2\varepsilon)]$

such that for every $z \in I'_n$,

- (1) $\mathbb{D}_z(re^{\lambda_n} e^{-\kappa}) \subset h_{x,z}(\mathbb{D}(r)) \subset \mathbb{D}_z(re^{\lambda_n + \log 2} e^\kappa)$,
- (2) $\{h_{x,z}(\mathbb{D}(r))\}_{z \in I'_n} \subset \tau_x$ are pairwise disjoint.

Let us see how theorem 8.1 follows. Let $m \geq 1$ and

$$\mathcal{H}_m := \{h_{x,z_1} \circ \dots \circ h_{x,z_m}, z_i \in I'_n\}.$$

Next we introduce

$$K_m := \bigcup_{z \in \{h(0), h \in \mathcal{H}_m\}} \mathbb{D}_z \left(r(e^{\lambda_n} e^{-2\kappa})^m \right),$$

these discs are pairwise disjoint and $K_m \subset \bigcup_{h \in \mathcal{H}_m} h(\mathbb{D}(r))$ by distortion. The family \mathcal{H}_m forms an iterated function system. The limit set $K := \bigcap_m K_m$ satisfies

$$\dim_H K \geq \frac{n(h - 3\varepsilon)}{-\lambda_n + 2\kappa} \geq \frac{h - 3\varepsilon}{-\lambda + 3\varepsilon},$$

where the second inequality relies on $\lambda_n \geq n(\lambda - 2\varepsilon)$ and $n\varepsilon \geq 2\kappa$. That estimate holds for the Hausdorff dimension of $M \cap \tau_x$ since it contains K .

8.3. Proof of proposition 8.4. We work for n large enough. The definition of r is given before proposition 8.4. First we show that there exist a set I_n of cardinality larger than $e^{n(h-2\varepsilon)}$ and a negative number λ_n satisfying (ii) such that for any $z \in I_n$:

- (a) $h_{x,z}$ is defined from $\mathbb{D}(2r_0) \subset \tau_x$ to τ_x ,
- (b) $\lambda_n \leq \log|h'_{x,z}(0)| \leq \lambda_n + \log 2$,
- (c) $\mathbb{D}_z(re^{\lambda_n} e^{-\kappa}) \subset h_{x,z}(\mathbb{D}(r)) \subset \mathbb{D}_z(re^{\lambda_n + \log 2} e^{\kappa})$ (which is (1)),
- (d) $h_{x,z}(\mathbb{D}(r)) \subset \mathbb{D}(r)$.

Let $Z := L_x \cap \mathbb{D}(r/2)$. By compactness and minimality of M , the subset Z is a (C, D) -lattice. Recall that π is the projection $L_x \rightarrow Z$ and $M_t = \pi_* B_t$, where B_t is the distribution of Brownian motion at the time t . We set $\Gamma_\varepsilon(n) := \{\gamma(n), \gamma \in \Gamma_\varepsilon\}$ and $J_n := \pi(\Gamma_\varepsilon(n))$. Since $M_n(J_n) \geq 1/2$ by definition of Γ_ε , proposition 8.3 yields $|J_n| \geq e^{n(h-\varepsilon)}$.

By compactness of M , there exist $E > 0$ and $r_1 > 0$ such that if $\gamma \subset L_y$ is a compact smooth path starting at y of length bounded by D , then the holonomy map h_γ is defined on $\mathbb{D}(r_1)$ and satisfies $-E \leq \log|h'_\gamma(y)| \leq E$.

By construction, every point $z \in J_n$ is D -distant from a point $y \in \Gamma_\varepsilon(n)$. By definition of Γ_ε , $r_\gamma(\varepsilon)$, $t_\gamma(\varepsilon)$ and distortion theorem, $h_{x,y}$ is well-defined on $\mathbb{D}(2r_0)$ and its image is contained in a disc of radius $2r_0 e^{n(\lambda+\varepsilon)} e^\kappa$. Since that radius is smaller than r_1 , the map $h_{x,z} = h_{y,z} \circ h_{x,y}$ is well-defined on $\mathbb{D}(2r_0)$ and satisfies

$$-E + n(\lambda - \varepsilon) \leq \log|h'_{x,z}(0)| \leq E + n(\lambda + \varepsilon).$$

That implies (a). That moreover implies that there exist λ_n satisfying

$$-E + n(\lambda - \varepsilon) \leq \lambda_n \leq E + n(\lambda + \varepsilon)$$

and a subset $I_n \subset J_n$ of cardinality

$$|I_n| \geq \frac{|J_n| \log 2}{2(E + \varepsilon n)}$$

such that (b) is satisfied for every $z \in I_n$. The expected bounds $|I_n| \geq e^{n(h-2\varepsilon)}$ and (ii) for λ_n follow by taking n large. Koebe theorem ensures (c) (which is (1) of proposition 8.4), and we get (d) from $h_{x,z}(0) \in \mathbb{D}(r/2)$ and $\text{diam } h_{x,z}(\mathbb{D}(r)) \leq r e^{\lambda_n + \log 2} e^\kappa$.

It remains to build $I'_n \subset I_n$ satisfying (i) and (2). Let $\kappa' := 2\kappa + \log 2$ and

$$\mathcal{H} := \{h : \mathbb{D}(r) \rightarrow \tau_x \text{ holonomy map}, -\kappa' \leq \log|h'(0)| \leq \kappa'\}.$$

That set is finite because the foliation is discrete. Let $r_n := 2re^{\lambda_n + \log 2} e^\kappa$ and $(z, z') \in I_n \times I_n$ satisfying $|z - z'| \leq r_n$. Koebe theorem yields

$$h_{x,z}(\mathbb{D}(2r)) \subset \mathbb{D}_z(r_n),$$

$$h_{x,z'}(\mathbb{D}(r_0)) \supset \mathbb{D}_{z'}(r_0 e^{\lambda_n} e^{-\kappa}) = \mathbb{D}_{z'}(2r_n) \supset \mathbb{D}_z(r_n).$$

We deduce that $h := h_{x,z'}^{-1} \circ h_{x,z} : \mathbb{D}(2r) \rightarrow \mathbb{D}(r_0)$ is well defined. One easily verifies that $-\kappa' \leq \log |h'(0)| \leq \kappa'$, hence the restriction $h : \mathbb{D}(r) \rightarrow \mathbb{D}(r_0)$ belongs to \mathcal{H} . Let $I'_n \subset I_n$ such that $\{h_{x,z}\}_{z \in I'_n}$ represents the classes of I_n modulo \mathcal{H} by right composition. That set has cardinality

$$|I'_n| \geq |I_n|/|\mathcal{H}| \geq e^{n(h-2\varepsilon)}/|\mathcal{H}| \geq e^{n(h-3\varepsilon)}.$$

The fact that $\{h_{x,z}(\mathbb{D}(r))\}_{z \in I'_n}$ are disjoint follows from $\text{diam } h_{x,z}(\mathbb{D}(r)) \leq r_n/2$ for $z \in I_n$ and from $|z - z'| > r_n$ for points of I'_n .

8.4. Applications of theorem 8.1.

8.4.1. *Proof of theorem 1.1.* We use the following lemmas.

Lemma 8.5. *Let S be a foliated complex surface and M a minimal set with no transverse invariant measure. If M is not discrete then it is an analytic Levi-flat.*

Proof. Assuming that M has no transverse invariant measure and is not discrete, there exists a hyperbolic fixed point in M [15, corollary 1.3]. That implies by [40, section 4] that there are local holomorphic flows in the closure of the pseudo-group. The minimal set is then analytic. \square

In surfaces of general type, theorem 8.1 allows to prove a converse (see lemma 8.7 below) when a leaf is simply connected. We begin with the

Lemma 8.6. *Let S be Kähler surface, \mathcal{F} a holomorphic foliation and M a minimal set. If M has a simply connected leaf, then every leaf is simply connected except for a countable number.*

Proof. First observe that the Kähler property implies that there is no vanishing cycle [30]. Then, by minimality and simply connected assumption, the holonomy of a leafwise (homotopically) non trivial loop is not the identity. Indeed, any deformation of that loop in a close leaf remains non trivial, and by minimality the simply connected leaf must contain such a deformation.

Now suppose that there are uncountably many leaves which are not simply connected. Consider a finite covering of M by flow boxes. Let us fix in each non simply connected leaf a non trivial loop. Each of them induces a cyclic sequence of crossed flow boxes. By cardinality, there exists a cyclic sequence attained by uncountably many loops. Since the domain of definition of the corresponding holonomy map is open, one of its connected component has to intersect uncountably many loops. The holonomy map is therefore equal to identity on the component, which has been excluded. \square

Lemma 8.7. *Let S be a surface of general type, \mathcal{F} a holomorphic foliation and M a minimal set with no transverse invariant measure. If M has a simply connected leaf and is discrete then $\dim_H M > 3$.*

Proof. We endow the leaves with the Poincaré metric m . Let μ be the unique harmonic measure and λ its Lyapunov exponent. The measure μ has no atom since it is not transversely invariant. By lemma 8.6, μ -a.e. leaf is isomorphic to the disc, hence the Kaimanovich entropy is then equal to 1. Proposition 5.4 and theorem 8.1 finally imply that the transverse Hausdorff dimension of M is larger than $h_m/|\lambda| > 1$. \square

8.4.2. *Margulis-Ruelle type inequalities.* From $\dim_H M \cap T \leq 2$ for minimal sets and $\dim_H M \cap T = 1$ for Levi-flats, we obtain:

Corollary 8.8. *We take the assumptions of theorem 8.1. The Lyapunov exponent of the harmonic measure satisfies $\lambda \leq -1/2$. If M is Levi-flat, then $\lambda \leq -1$.*

That corollary provides an alternative proof of theorem 6.1 when M is the unitary tangent bundle of a compact surface of genus $g \geq 2$. However, such a method does not work for a hyperbolic torus bundle, because its holonomy pseudo-group is not discrete: it contains an irrational rotation.

8.4.3. *Dimension of minimal sets in \mathbb{P}^2 .* Theorem 8.1 implies the following lower bound for the transverse dimension if there is a simply connected leaf.

Proposition 8.9. *Let M be a minimal set of a holomorphic foliation of \mathbb{P}^2 of degree $d \geq 2$. If M has a simply connected leaf, then*

$$\dim_H M \cap T \geq \frac{d-1}{d+2}.$$

Indeed, if M is not discrete, its transverse dimension is 1 by lemma 8.5. If M is discrete, its transverse dimension is larger than $1/|\lambda|$, and we use proposition 5.5. Note that the conformal dimension of any conformal harmonic current is bounded above by the same constant [7]. Note also that these sets are not too small in the sense of potential theory: their harmonic current has finite energy [18].

9. APPENDIX

For reader's convenience we prove the classical propositions 6.9 (drift of Brownian motion on \mathbb{D}). We propose arguments adapted to the unit disc, in particular we do not use the law of iterated logarithm, see [47]. Recall that \mathbb{D} is endowed with the Poincaré metric of gaussian curvature -1 and that we work with the heat equation $\frac{\partial}{\partial t} = \Delta_m$. Let W denote the Wiener measure on the set of continuous paths starting at the origin. By applying Kingman's theorem to $(\Gamma, (\sigma_t)_{t \geq 0}, W)$ and $H_t(\gamma) := d_P(\gamma(0), \gamma(t))$, there exists $\delta \in \mathbb{R}$ such that for W -almost every $\gamma \in \Gamma$:

$$\lim_{t \rightarrow \infty} \frac{1}{t} d_P(0, \gamma(t)) = \delta.$$

We want to show that $\delta = 1$. Let $\varphi : \mathbb{D} \rightarrow \mathbb{R}^+$ defined by $\varphi(z) = \frac{1-|z|^2}{|1-z|^2}$. This is a harmonic function (it corresponds to the imaginary part in the hyperbolic plane) and the laplacian of $\log \varphi$ with respect to the Poincaré metric is identically -1 . Dynkin's formula asserts that

$$\forall t \geq 0, \mathbb{E}(\log \varphi(\gamma(t))) = \log \varphi(0) + \mathbb{E} \left(\int_0^t \Delta_m \log \varphi(\gamma(s)) ds \right),$$

where the expectation \mathbb{E} holds on Γ with respect to W . We deduce

$$\forall t > 0, \quad \frac{1}{t} \mathbb{E}(\log \varphi(\gamma(t))) = -1.$$

To complete the proof it suffices to show

$$(6) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}(\log \varphi(\gamma(t))) = -\delta.$$

Given a circle centered at the origin, its hyperbolic radius $R \in \mathbb{R}^+$ is related to its euclidian radius $r \in [0, 1[$ by $r(R) = \frac{e^R - 1}{e^R + 1}$. For every $t \geq 0$, let ρ_t be the pushforward measure of W on $[0, 1[$ by the map $\gamma \mapsto r \circ d_P(0, \gamma(t))$. Then $\mathbb{E}(\log \varphi(\gamma(t)))$ is equal to

$$(7) \quad \int_{[0,1[} \frac{1}{2\pi} \int_0^{2\pi} \log \varphi(re^{i\theta}) d\theta d\rho_t(r) =: \int_{[0,1[} M(r) d\rho_t(r).$$

Using $r : \mathbb{R}^+ \rightarrow [0, 1[$, we define

$$(8) \quad I_{t,\epsilon} := [r(t(\delta - \epsilon)), r(t(\delta + \epsilon))].$$

We have $\rho_t(I_{t,\epsilon}) \geq 1 - \epsilon$ for t large enough, since $\lim_{t \rightarrow \infty} \frac{1}{t} d_P(0, \gamma(t)) = \delta$ for almost every γ . We shall decompose the integral of (7) (right side) over $I_{t,\epsilon}$ and $[0, 1[\setminus I_{t,\epsilon}$. For that purpose observe that $M(r) = \log(1 - r^2)$ from the classical formula $\frac{1}{2\pi} \int_0^{2\pi} \log |1 - re^{i\theta}|^2 d\theta = 0$. Now, from

$$M(r) = \log(1 - r(R)^2) = \log \cosh^{-2}(R/2) \in [-R, -R + \log 4]$$

and the definition (8) of $I_{t,\epsilon}$, we get

$$-t(\delta + \epsilon) \leq \int_{I_{t,\epsilon}} M(r) d\rho_t(r) \leq -t(\delta - \epsilon) + \log 4.$$

Since ρ_t is absolutely continuous with respect to Lebesgue measure on $[0, 1[$ (from the heat kernel) and $M \in L^2[0, 1[$, Cauchy-Schwarz inequality yields

$$\int_{[0,1[\setminus I_{t,\epsilon}} M(r) d\rho_t(r) \leq \epsilon \int_{[0,1[} M^2(r) d\rho_t(r) =: c\epsilon.$$

That completes the proof of (6), and $\delta = 1$ as desired.

REFERENCES

- [1] A. ANCONA. Théorie du potentiel sur les graphes et les variétés. *Lecture Notes in Math.* **1427** (1990).
- [2] T. BARBOT, S. FENLEY. Pseudo-Anosov flows in toroidal manifolds. arXiv 1007.0578.
- [3] D. E. BARRETT, J. E. FORNAESS. On the smoothness of Levi foliations. *Publ. Math.* **32** (1988), no. 2, 171-177.
- [4] W. BARTH, C. PETERS, A. VAN DE VEN. Compact complex surfaces. Springer (1984).
- [5] M. BRUNELLA. Feuilletages holomorphes sur les surfaces complexes compactes. *Ann. Sci. École Norm. Sup.*, (4) **30** (1997), 569-594.
- [6] M. BRUNELLA. Birational geometry of foliations. IMPA (2000).
- [7] M. BRUNELLA. Mesures harmoniques conformes et feuilletages du plan projectif complexe. *Bull. Braz. Math. Soc.* **38** (2007), no.4, 517-524.
- [8] C. CAMACHO, A. LINS NETO, P. SAD. Minimal sets of foliations on complex projective spaces. *Inst. Hautes Études Sci. Publ. Math.* **68** (1988), 187-203.

- [9] A. CANDEL. Uniformization of surface laminations. *Ann. Sci. École Norm. Sup.* (4) **26** (1993), no. 4, 489-516.
- [10] A. CANDEL, L. CONLON. *Foliations II*. AMS (2000).
- [11] I. CHAVEL. *Eigenvalues in riemannian geometry*, Academic Press (1984).
- [12] J.-P. DEMAILLY. *Complex analytic and differential geometry* (2009). Available on J.-P. Demailly's webpage.
- [13] B. DEROIN. Hypersurfaces Levi-plates immergées dans les surfaces complexes de courbure positive. *Ann. Sci. École Norm. Sup.* (4) **38** (2005), no. 1, 57-75.
- [14] B. DEROIN. Non rigidity of Riemann surface laminations. *Proc. Amer. Math. Soc.* **135** (2007), 873-881.
- [15] B. DEROIN, V. KLEPTSYN. Random conformal dynamical systems. *Geom. Funct. Anal.* **17** (2007) no. 4, 1043-1105.
- [16] B. DEROIN, C. VERNICOS. Feuilletage de Hirsch, mesure harmonique, et g -mesure. *Proceedings of the IFUM Colloquium* (2009).
- [17] T.-C. DINH, V.A. NGUYEN, N. SIBONY. Entropy for hyperbolic Riemann surface laminations I. arXiv 1105.2307.
- [18] J.E. FORNAESS, N. SIBONY. Harmonic currents of finite energy and laminations. *Geom. Funct. Anal.* **15** (2005), no. 5, 962-1003.
- [19] J.E. FORNAESS, N. SIBONY. Unique ergodicity of harmonic currents on singular foliations of \mathbb{P}^2 . *Geom. Funct. Anal.* **19** (2010), no. 5, 1334-1377.
- [20] J. FRANKS, B. WILLIAMS. Anomalous Anosov flows. *Lecture Notes in Math.*, **819** (1980), 158-174.
- [21] R. FRIEDMAN. *Algebraic surfaces and holomorphic vector bundles*. Springer, 1998.
- [22] L. GARNETT. Foliations, the ergodic theorem and Brownian motion. *J. Funct. Anal.* **51** (1983), no. 3, 285-311.
- [23] É. GHYS. Rigidité différentiable des groupes fuchsien. *Inst. Hautes Études Sci. Publ. Math.*, **78** (1993), 163-185.
- [24] É. GHYS. Sur les groupes engendrés par des difféomorphismes proches de l'identité. *Bol. Soc. Brasil. Mat.* (N.S.) **24** (1993), no. 2, 137-17.
- [25] É. GHYS. Laminations par surfaces de Riemann. *Panorama & Synthèses.* (1999), 49-95.
- [26] É. GHYS, V. SERGIESCU. Stabilité et conjugaison différentiable pour certains feuilletages. *Topology* **19** (1980), no. 2, 179-197.
- [27] S. GOODMAN. Dehn surgery on Anosov flows. *Lecture Notes in Math.*, **1007** (1983), 300-307.
- [28] M. HANDEL, W. THURSTON. Anosov flows on new three manifolds. *Invent. Math.* **59** (1980), no. 2, 95-103.
- [29] T. INABA, M. A. MISHCHENKO. On real submanifolds of Kähler manifolds foliated by complex submanifolds. *Proc. Japan Acad.* **70** Ser. A (1994)
- [30] S. IVASHKOVICH. Vanishing cycles in holomorphic foliations by curves and foliated shells. *Geom. Funct. Anal.* **21** (2011), no. 1, 70-140.
- [31] V.A. KAIMANOVICH. Brownian motion and harmonic functions on covering manifolds. An entropic approach. *Soviet Math. Dokl.*, **33** (1986), 812-816.
- [32] V.A. KAIMANOVICH. Brownian motion on foliations: entropy, invariant measures, mixing. *Funct. Anal. Appl.* **22** (1988), no. 4, 326-328.
- [33] S. KOBAYASHI. *Hyperbolic complex spaces*. Springer (1998).
- [34] F. LEDRAPPIER. Quelques propriétés des exposants caractéristiques. *Lecture Notes in Math.*, **1097** (1984), 305-396.
- [35] F. LEDRAPPIER. Poisson boundaries of discrete groups of matrices. *Israel J. Math.*, **50** (1985), no. 4, 319-336.
- [36] F. LEDRAPPIER. Profil d'entropie dans le cas continu. *Astérisque*, **236** (1996), 189-198.
- [37] G. LEVITT. Feuilletages des variétés de dimension 3 qui sont des fibrés en cercles. *Comment. Math. Helvetici*, **53** (1978), 572-594.
- [38] A. LINS NETO. A note on projective Levi flats and minimal sets of algebraic foliations. *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 4, 1369-1385.

- [39] F. LORAY, D. MARÍN PÉREZ. Projective structures and projective bundles over compact Riemann surfaces. *Astérisque* **323** (200), 223-252.
- [40] F. LORAY, J. REBELO. Minimal, rigid foliations by curves on $\mathbb{C}P^n$. *J. Eur. Math. Soc.* **5** (2003), 147-201.
- [41] R. MAÑÉ. The Hausdorff dimension of invariant probabilities of rational maps. *Lecture Notes in Math.*, **1331** (1988).
- [42] S. MATSUMOTO. Some remarks on foliated S^1 -bundles. *Invent. Math.* **90** (1987), 343-358.
- [43] I. NAKAI. Separatrices for nonsolvable dynamics on $(\mathbb{C}, 0)$. *Ann. Inst. Fourier* **44** (1994), no. 2, 569-599.
- [44] S. YU. NEMIROVSKII. Stein domains with Levi-plane boundaries on compact complex surfaces. *Math. Notes* **66** (2000), no. 3-4, 522-525.
- [45] S.P. NOVIKOV. Topology of foliations. *Trans. Moscow Math. Soc.* **14** (1965), 268-305.
- [46] T. OHSAWA. A Levi-flat in a Kummer surface whose complement is strongly pseudoconvex. *Osaka J. Math.* **43** (2006), no. 4, 747-750.
- [47] R.G. PINSKY. Positive Harmonic Functions and Diffusion. Cambridge, 1995.
- [48] J.F. PLANTE. Foliations with measure preserving holonomy. *Ann. of Math.* (2) **102** (1975), no. 2, 327-361.
- [49] I. REIDER. Vector bundles of rank 2 and linear systems on algebraic surfaces. *Ann. of Math.* (2) **127** (1988), no. 2, 309-316.
- [50] P. SCOTT. The geometries of 3-manifolds. *Bull. London Math. Soc.* **15** (1983), no. 5, 401-487.
- [51] D. SULLIVAN. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.* **36** (1976), 225-255.
- [52] W. THURSTON. Foliations on 3-manifolds which are circle bundles. PhD Thesis, Berkeley, 1972.
- [53] A. VERJOVSKY. A uniformization theorem for holomorphic foliations. *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, p. 233-253, *Contemp. Math.*, 58, III, AMS, Providence, RI, 1987.

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