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Geometry of the Funk metric on Weil-Petersson spaces

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Abstract

We discuss the Funk function $F(x,y)$ on a Weil-Petersson space $(\mathcal{T}, d)$ introduced in [14], which was originally studied for an open convex subset in a Euclidean space by Funk (cf. [10]). $F(x,y)$ is an asymmetric distance and invariant by the action of the mapping class group. Unlike the original one, $F(x,y)$ is not always convex on y with x fixed (Cor 2.11, Thm 5.1).

For each pseudo-Anosov mapping class g and a point $x \in \mathcal{T}$, there exists E such that for all $n \neq 0$, $\log|n| - E \leq F(x, g^n.x) \leq \log|n| + E$ (Cor 2.10), while $F(x, g^n.x)$ is bounded if g is a Dehn twist (Prop 2.13). The *translation length* is defined by $|g|_F = \inf_{x \in \mathcal{T}} F(x, g.x)$ for a map $g : \mathcal{T} \to \mathcal{T}$. If g is a pseudo-Anosov mapping class, there exists Q such that for all $n \neq 0$, $\log|n| - Q \leq |g^n|_F \leq \log|n| + Q$. For sufficiently large n, $|g^n|_F > 0$ and the infimum is achieved. If g is a Dehn twist, then $|g^n|_F = 0$ for each n (Thm 2.16).

Some geodesics in $(\mathcal{T}, d)$ are geodesics in terms of F as well. We find a decomposition of $\mathcal{T}$ by sets, each of which is foliated by those geodesics (Thm 4.10).

1 Introduction

1.1 Weil-Petersson spaces

Let S be an orientable hyperbolic surface and $(\mathcal{T}(S), d)$ its Weil-Petersson space. $S_{g,n}$ denotes a surface of genus g and n-punctures. We may denote the distance $d(x, y)$ by $|x - y|$. $\mathcal{T}$ is a Riemannian manifold of negative sectional curvature, K, but not complete as a metric space, and its metric completion is obtained as $\overline{\mathcal{T}} = \mathcal{T} \cup (\cup_{\sigma \in \mathcal{C}(S)} \mathcal{T}(\sigma))$, where σ is a simplex in the curve complex $\mathcal{C}(S)$ of S and $\mathcal{T}(\sigma)$ consists of all hyperbolic surfaces with nodes along the vertices in σ. (In this paper, $\sigma \in \mathcal{C}(S)$ means that σ is a simplex.) We denote $\overline{\mathcal{T}} \backslash \mathcal{T}$ by $\partial \overline{\mathcal{T}}$. $\overline{\mathcal{T}}$ is a CAT(0) space, and each $\mathcal{T}(\sigma)$, called a *stratum*, is a convex subset, which is isometric to the Weil-Petersson space of the surface obtained from S with all curves in σ removed.

Let $\overline{\mathcal{T}(\sigma)}$ denote the closure in $\overline{\mathcal{T}}$. The nearest point projection $\pi_{\overline{\mathcal{T}(\sigma)}}$: $\mathcal{T} \to \overline{\mathcal{T}(\sigma)}$ is well defined, and the distance between $x \in \mathcal{T}$ and $\overline{\mathcal{T}(\sigma)}$ is achieved by a unique geodesic, called a *projection geodesic* (or *ray*). We denote the distance by $|x - \overline{\mathcal{T}(\sigma)}|$ or $|x - \mathcal{T}(\sigma)|$.

In this paper we do not regard $\mathcal{T}$ as a stratum, and say $\mathcal{T}(\sigma)$ is *maximal dimensional* if σ is a vertex of $\mathcal{C}(S)$. The mapping class group $MCG(S)$ acts on $\mathcal{T}(S)$ by isometries, mapping x to $g.x$ (or denoted by $g(x)$) for $x \in \mathcal{T}, g \in MCG$. The quotient is the *moduli space*, $\mathcal{M}(S)$. We denote the projection by $\pi_{\mathcal{M}}$. The action by MCG is properly discontinuous in the sense that for each point $x \in \mathcal{T}$, there exists $r > 0$ such that there are at most finitely many $g \in MCG$ with $|x - g.x| \leq r$. The action extends to $\overline{\mathcal{T}}$ such that $g.\mathcal{T}(\sigma) = \mathcal{T}(g.\sigma)$, where $g.\sigma$ is the result of the action of g on $\mathcal{C}(S)$. The action of each mapping class g on $\overline{\mathcal{T}}$ is elliptic, i.e., there is a fixed point; or hyperbolic, i.e., not elliptic and there is an invariant geodesic, called *axis*. For example, pseudo-Anosov maps are hyperbolic ([6]) and Dehn twists are elliptic (cf. [13]). For more information, see [7] for mapping class groups and curve complexes, see [13] for Weil-Petersson spaces, and [2] for CAT(0) spaces.

1.2 Funk function

Yamada [14] introduced a function $F(x, y)$ on $\mathcal{T}$, as a generalization of the Funk metric on a convex subset in a Euclidean space (cf. [10]), defined by for $x, y \in \mathcal{T}$

$$F(x,y) = \sup_{\sigma \in \mathcal{C}(S)} \log \frac{|x - \mathcal{T}(\sigma)|}{|y - \mathcal{T}(\sigma)|}.$$

If the supremum is realized by a stratum $\mathcal{S}$ for x, y, then we say $F(x,y)$ is *realized* by $\mathcal{S}$ for x, y.

The function F satisfies

(1) $F(x,y) \geq 0$. $F(x,y) > 0$ if $x \neq y$ ([9]).

(2) $F(x,y) + F(y,z) \leq F(x,z)$ (the triangle inequality).

In general $F(x,y) \neq F(y,x)$. Therefore F is an asymmetric distance. (2) is straightforward from the definition of F. (1) is not obvious from the definition. If $x \neq y$, let $\gamma(t)$ be a unit speed geodesic starting from x such that $y \in \gamma$. If γ reaches some stratum $\mathcal{S}$ at $p \in \mathcal{S}$, then since $|\gamma(t) - \mathcal{S}|$ is convex on t by CAT(0) geometry, it follows that $F(x,y) \geq \log(|x - \mathcal{S}|/|y - \mathcal{S}|) \geq \log(|x-p|/|y-p|) > 0$. But γ may not hit any stratum. The idea is that in fact one can always find a stratum $\mathcal{S}$ such that $|x - \mathcal{S}| - |y - \mathcal{S}|$ is approximately $|x-y|$, therefore $F(x,y) > 0$ (see Remark 3.5).

Since the action by MCG preserves the set of strata, it is clear from the definition that each element $g \in MCG(S)$, $F(x,y) = F(g.x, g.y)$.

1.3 Main results

We are interested in isometries and geodesics in terms of F, as well as the relationship between $|x-y|$ and $F(x,y)$. We will obtain an estimate of $F(x,y)$ by $|x-y|$ on the axis of a pseudo-Anosov element.

Theorem 2.9. *Let γ be the (WP-)axis of a pseudo-Anosov element. Then there exists $E > 0$ such that for any $x, y \in \gamma$*

1. $\log|x-y| - E \leq F(x,y)$.
2. *If $|x-y| \geq 1$, then $F(x,y) \leq \log|x-y| + E$.*
   If $|x-y| < 1$, then $F(x,y) \leq E$.

One immediate consequence is non-convexity of F.

Corollary 2.11. *Let $\gamma(t)$ be a unit speed axis. Then, $f(t) = F(\gamma(0), \gamma(t))$ is not convex on t.*

For the Funk metric in the Euclidean spaces, $F(x, s(t))$ is always convex on t for a geodesic $s(t)$ (see [10], [14]).

Corollary 2.10. *Let g be a pseudo-Anosov map with the axis γ. There exists $C > 0$ such that for any point $x \in \gamma$ and any $n \neq m$,*

$$\log|n-m| - C \leq F(g^m.x, g^n.x) \leq \log|n-m| + C.$$

In contrast, we prove

Proposition 2.13. *If g is a Dehn-twist, then for a point $x \in \mathcal{T}$, there exists C such that for all n, m, $F(g^n.x, g^m.x) \leq C$.*

Although the orbit $\{g^n(x)\}$ is bounded in terms of F, a Dehn twist does not have a fixed point in $\mathcal{T}$.

For a map $g : \mathcal{T} \to \mathcal{T}$ which preserves $F(x,y)$, define the *translation length* by

$$|g|_F = \inf_{x \in \mathcal{T}} F(x, g.x).$$

Theorem 2.16.

1. *If g is a Dehn twist, then for any n, $|g^n|_F = 0$. The infimum is not achieved (unless $n = 0$).*
2. *If g is a pseudo-Anosov mapping class, there exists Q such that for all $n \neq 0$, $\log|n| - Q \leq |g^n|_F \leq \log|n| + Q$. For sufficiently large n, $|g^n|_F > 0$ and the infimum is achieved by some point $x \in \mathcal{T}$ for each n.*

As for geodesics, we will show some of the WP-geodesics, i.e., geodesics in $(\mathcal{T}, d)$, are geodesics for F as well (see Definitions 4.1 and 3.1). For each stratum $\mathcal{S}$, let $N(\mathcal{S})$ be the union of points $x \in \mathcal{T}$ such that $\mathcal{S}$ is the unique nearest stratum. $\overline{N(\mathcal{S})}$ denotes the closure in $\mathcal{T}$.

Theorem 4.10. *$\mathcal{T}$ is decomposed as*

$$\mathcal{T} = \cup_{\sigma \in \mathcal{C}(S)} \overline{N(\mathcal{T}(\sigma))},$$

where $\sigma \in \mathcal{C}(S)$ is a vertex, such that the decomposition is preserved by MCG. $N(\mathcal{T}(\sigma)) \cup \mathcal{T}(\sigma)$ is an open neighborhood of $\mathcal{T}(\sigma)$ in $\overline{\mathcal{T}}$. $N(\mathcal{T}(\sigma))$ and $N(\mathcal{T}(\tau))$ are disjoint if $\sigma \neq \tau$. $\overline{N(\mathcal{T}(\sigma))}$ is foliated by finite WP-projection rays to $\mathcal{T}(\sigma)$, which are also infinite F-geodesics. In particular, for any point $x \in \mathcal{T}$, there is an infinite F-geodesic emanating from x.

2 Action by mapping classes

2.1 $F(x,y)$ vs. $d(x,y)$

For a subset $X \subset \overline{\mathcal{T}}$ (maybe a point), $N_r(X)$ denotes the closed r-neighborhood of X. The diameter of X is denoted by $\operatorname{diam} X$. We denote the distance between subsets X, Y by $|X - Y|$. Since $\overline{\mathcal{T}}$ is CAT(0) (cf. [2]), if X is a closed convex subset, the nearest point projection is well-defined, and we denote it by π_X. For each point $x \in \overline{\mathcal{T}}$, the distance to X is achieved uniquely at $\pi_X(x)$. The distance $|x - X|$ is convex on $x \in \overline{\mathcal{T}}$. π_X does not increase the distance of two points. For $x, y \in \overline{\mathcal{T}}$, there is a unique geodesic between them, which we denote by $[x, y]$.

For a geodesic $\gamma(t)$, there are two canonical orders, one by t and the other by $-t$. For given two points $x, y \in \gamma$, there is a unique order with $x < y$. We use this order in the following argument.

We start with an upper bound in general. We use $\log(x+1) \le \log x + 1$ if $x \ge 1$.

Proposition 2.1. *Let $g \in MCG$. For each $x \in \mathcal{T}$, there exists E such that for all $n \neq 0$, we have*

$$F(x, g^n.x) \le \log|n| + E.$$

Proof. It suffices to argue for one point x since for other point y, $F(y, g^n.y) \le F(y,x) + F(x, g^n.x) + F(g^n.x, g^n.y) = F(y,x) + F(x, g^n.x) + F(x,y)$.

For a given x, there exists $R > 0$ such that for any stratum $\mathcal{S}$, $|x - \mathcal{S}| \ge R$. Since g preserves the set of strata, for any n, $|g^n.x - \mathcal{S}| \ge R$.

It follows $|x - \mathcal{S}|/|g^n.x - \mathcal{S}| \le (|x - g^n.x| + |g^n.x - \mathcal{S}|)/|g^n.x - \mathcal{S}| \le 1 + |n||g.x - x|/R$ since $|g^n.x - x| \le |n||g.x - x|$ by the triangle inequality.

If $|n||g.x - x|/R \ge 1$, then $\log(|x - \mathcal{S}|/|g^n.x - \mathcal{S}|) \le \log(1 + |n||g.x - x|/R) \le \log(|n||g.x - x|/R) + 1 \le \log|n| + \log(|g.x - x|/R) + 1$. If $|n||g.x - x|/R \le 1$, then $\log(|g^n.x - \mathcal{S}|/|x - \mathcal{S}|) \le \log 2$. Therefore, $F(x, g^n.x) \le \log|n| + \log(|g.x - x|/R) + 1 + \log 2$. Set $E = \log(|g.x - x|/R) + 1 + \log 2$. □

We are only using the property that g preserves the distance on $\mathcal{T}$ in the proof. It is known that such g is an element in the extended mapping class group of S, [8], [4], [11].

Question 2.2. *Is there a map $g : \mathcal{T} \to \mathcal{T}$ which preserves $F(x,y)$ such that $F(x, g^n.x)$ grows faster than $\log n$, or linearly ?*

We now discuss a lower bound of $F(x, g^n.x)$. We state a result which is a consequence of negative curvature along an axis.

Proposition 2.3. *Let γ be the axis of a pseudo-Anosov map. Then*

1. *There exists $D > 0$ such that for any stratum $\mathcal{S}$, $\operatorname{diam} \pi_\gamma(\overline{\mathcal{S}}) \leq D$, i.e., there exist $P, Q \in \gamma$ with $|P - Q| \leq D$ and $\pi_\gamma(\overline{\mathcal{S}}) \subset [P, Q]$.*

2. *There exists $D'' > 0$ such that for any stratum $\mathcal{S}$, $\operatorname{diam} \pi_{\overline{\mathcal{S}}}(\gamma) \leq D''$.*

3. *There exists $D' > 0$ such that for any stratum $\mathcal{S}$, there exist $P', Q' \in \gamma$ with $|P' - Q'| \leq D'$ and $P' < Q'$ such that for any $x \in \gamma$ with $x \leq P'$ (or $Q' \leq x$), the geodesic $[x, \pi_{\overline{\mathcal{S}}}(x)]$ passes through $N_1(P')$ (or $N_1(Q')$).*

After we prove this proposition, we will denote $\max(D, D', D'')$ by D, which depends on γ.

Proof. (1) Since γ maps to a loop in $\mathcal{T}/MCG$, there exists $C > 0$ such that for any stratum $\mathcal{S}$, $|\gamma - \mathcal{S}| \geq 2C$. Let $k > 0$ be a constant such that the sectional curvature K in $N_C(\gamma)$ is at most $-k$. Such k exists since $N_C(\gamma)$ maps to a compact set in $\mathcal{T}/MCG$.

The following argument is very similar to the one in [1, Section 8] (cf. [5, Section 4]). Fix a stratum $\mathcal{S}$ and a point $p \in \mathcal{S}$. Let $q \in \mathcal{S}$ be any point. Let $p' = \pi_\gamma(p), q' = \pi_\gamma(q)$. We construct two ruled triangles from those four points. One, Δ, is obtained by coning off the geodesic $[p, q]$ with the cone point p' by geodesics. The other, Δ', is by coning off $[p', q']$ with the cone point q. The two triangles Δ, Δ' have a Riemannian metric induced from $\mathcal{T}$. The metric is non-degenerate. Let K' denote the curvature of the induced metric on the triangles. At each point, K' is at most the upper bound of the sectional curvature K of the same point (see [1]). All sides of the triangles are geodesics for the induced metric since they are WP-geodesics. See the right part of Figure 1.

Set $A = \Delta \cup \Delta'$. Using the Gauss-Bonnet theorem to each of Δ, Δ', we have

$$\int_A K' dA \geq -2\pi.$$

(In fact we can take $-\pi$ as the lower bound, but it does not matter.) Since $K < 0$ on $\mathcal{T}$ and $K \leq -k$ on $N_C(\gamma) \cap A$, therefore $K' \leq -k$ on A, we obtain $C(|\pi_\gamma(p) - \pi_\gamma(q)| - 2C) \leq 2\pi/k$, since the area of $N_C(\gamma) \cap A$ is at least $C(|\pi_\gamma(p) - \pi_\gamma(q)| - 2C)$. This estimate of the area follows from that $[p, q]$

does not intersect $N_C(\gamma)$. It follows $|\pi_\gamma(p) - \pi_\gamma(q)| \leq 2C + 2\pi/Ck$, therefore $\operatorname{diam} \pi_\gamma(\mathcal{S}) \leq 2(2C + 2\pi/Ck) = D$. Use $\operatorname{diam} \pi_\gamma(\mathcal{S}) = \operatorname{diam} \pi_\gamma(\overline{\mathcal{S}})$. It is clear that desired points P, Q exist.

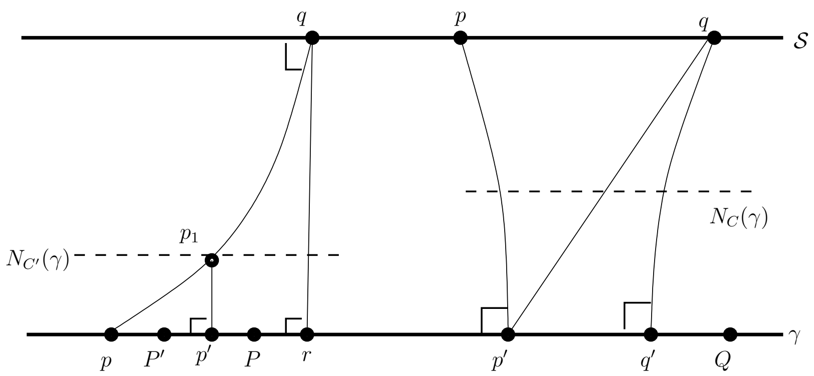

Figure 1: Negative curvature near the axis γ

(3). Set $C' = \min(1, C)$. Let $k > 0$ be such that the sectional curvature $K \leq -k$ on $N_{C'}(\gamma)$. Let $\mathcal{S}$ be any stratum and $p \in \gamma \backslash [P, Q]$ any point, and set $q = \pi_{\overline{\mathcal{S}}}(p), r = \pi_\gamma(q)$. By (1), $r \in [P, Q]$. Assume $p \leq P < Q$ (the other case is $P < Q \leq p$. The argument is similar). Let $p_1 \in [p, q]$ be the point with $|p_1 - \gamma| = C'$, and set $p' = \pi_\gamma(p_1)$. The geodesic $[p_1, q]$ is outside of $N_{C'}(\gamma)$. We claim $|p' - r| \leq 2C' + 2\pi/C'k$. See the left part of Figure 1.

The argument is same as for (1). In this case, we construct ruled triangles Δ, Δ' using p_1, q instead of p, q. The ruled rectangle $A = \Delta \cup \Delta'$ have four vertices p_1, q, p', r, and on $N_{C'}(\gamma) \cap A$, the sectional curvature K' of A is at most $-k$ as before. We omit details.

Let $P' \in \gamma$ be a point such that $|P - P'| = 2C' + 2\pi/C'k$ and $P' \leq P < Q$. Then, from the claim, $P' \leq p'$. Now, if $p \leq P'$, then since $|p_1 - p'| = |p_1 - \gamma| = C' \leq 1$, by CAT(0) geometry, $[p, q]$ passes through $N_1(P')$.

Similarly, choose $Q' \in \gamma$ with $P < Q < Q'$ with $|Q - Q'| = 2C' + 2\pi/C'k$. Now, $|P' - Q'| \leq 2(2C' + 2\pi/C'k) + D = D'$.

(2). By (3), we see $\operatorname{diam} \pi_{\overline{\mathcal{S}}}(\gamma) \leq 2 + |P' - Q'| \leq 2 + D' = D''$. □

Remark 2.4. *(3) says that $\gamma \backslash [P', Q']$ is almost "perpendicular" to $\mathcal{S}$.*

We quote one fact.

Fact 2.5. *It is known (cf. [3]) that for a given surface S, there exists a constant $B > 0$ such that for any point $x \in \mathcal{T}(S)$, there exists a stratum $\mathcal{S}$ with $|x - \mathcal{S}| \leq B$. We may assume $\mathcal{S}$ is maximal dimensional.*

Here is a lower bound on $F(x, y)$ along an axis.

Proposition 2.6. *Let γ be the axis of a pseudo-Anosov map. There exists $E > 0$ such that for any $x, y \in \gamma$, $F(x, y) \geq \log |x - y| - E$.*

Proof. Let D be the constant from the proposition 2.3. Let B be the constant in the fact 2.5. Let $z \in \gamma$ be the point such that $|x-z| = |x-y|+2B+D+3$, where $x < y < z$. Choose a stratum $\mathcal{S}$ with $|z - \mathcal{S}| \leq B$. Apply the proposition 2.3 to $\mathcal{S}$ then we get P', Q' with $P' < Q'$. Then, $y < P'$. Indeed, let $s \in \mathcal{S}$ with $|s - z| \leq B$. (If $s \in \overline{\mathcal{S}} \backslash \mathcal{S}$, then we retake $\mathcal{S}$ such that $s \in \mathcal{S}$.) Set $t = \pi_\gamma(s)$. Then, $|t - s| \leq B$, therefore $|t - z| \leq 2B$. Since $P' \leq t \leq Q'$, and $|P' - Q'| \leq D$, we find $|P' - z| \leq D + 2B$. It follows that $y < P'$ and $|y - P'| \geq 3$.

Notice that $|x - y| \leq |x - \mathcal{S}|$. This is because $[x, \pi_{\overline{\mathcal{S}}}(x)]$ passes through $N_1(y)$ and $N_1(P')$, by the proposition 2.3 (3), and $|y - P'| \geq 3$.

Then $|y-\mathcal{S}| \leq |y-z|+|z-\mathcal{S}| \leq (2B+D+3)+B$. Now, $\log |x-\mathcal{S}|-\log |y-\mathcal{S}| \geq \log |x-\mathcal{S}| - \log(3B+D+3) \geq \log |x-y| - \log(3B+D+3)$. It follows that $F(x, y) \geq \log |x - y| - \log(3B + D + 3)$. Set $E = \log(3B + D + 3)$. □

We state a result which we use later.

Lemma 2.7. *Let γ be the axis of a pseudo-Anosov map, $x \in \gamma$ and $\mathcal{S}$ a stratum. Then there exists F such that for any $y \in \gamma$, we have $|y - \mathcal{S}| \geq |x - y| - F$.*

Proof. The argument is very similar to the one for the proposition 2.6. Choose $P', Q' \in \gamma$ for the given $\mathcal{S}$. We first assume $x = P'$ (or $x = Q'$) and prove the inequality, then add $|x - P'| + D$ to the constant F at the end. We omit details. □

Here is an upper bound on $F(x, y)$ along an axis.

Proposition 2.8. *Let γ be the axis of a pseudo-Anosov map. There exists a constant E' such that for any $x, y \in \gamma$, the following holds.*

1. *If $|x - y| \geq 1$, then for any stratum $\mathcal{S}$,*

$$\log \frac{|x - \mathcal{S}|}{|y - \mathcal{S}|} \leq \log |x - y| + E'.$$

Therefore, $F(x, y) \leq \log |x - y| + E'$.

2. *If* $|x-y| \leq 1$*, then for any stratum* $\mathcal{S}$*,*

$$\log \frac{|x-\mathcal{S}|}{|y-\mathcal{S}|} \leq E'.$$

Proof. The argument is same as Proposition 2.1. Since γ maps to a loop in $\mathcal{M}$, there exists $R > 0$ such that $|\gamma - \mathcal{S}| \geq R$ for any stratum $\mathcal{S}$. Set $E' = \log(1+1/R)$.

(1). $\frac{|x-\mathcal{S}|}{|y-\mathcal{S}|} \leq \frac{|x-y|+|y-\mathcal{S}|}{|y-\mathcal{S}|} \leq 1 + |x-y|/R \leq |x-y|(1+1/R)$. Take log.

(2). In this case, $\frac{|x-\mathcal{S}|}{|y-\mathcal{S}|} \leq 1+1/R$. Take log. □

From now on, we denote $\max(E, E')$ as E. Combining the propositions 2.6 and 2.8, we immediately get

Theorem 2.9. *Let* γ *be the axis of a pseudo-Anosov map. Then there exists* $E > 0$ *such that for any* $x, y \in \gamma$

1. $\log|x-y| - E \leq F(x,y)$.
2. *If* $|x-y| \geq 1$*, then* $F(x,y) \leq \log|x-y| + E$.
   If $|x-y| < 1$*, then* $F(x,y) \leq E$.

Corollary 2.10. *Let* g *be a pseudo-Anosov map with the axis* γ*.*

1. *There exists* $C > 0$ *such that for any point* $x \in \gamma$ *and any* $n \neq 0$*,*

$$\log|n| - C \leq F(x, g^n.x) \leq \log|n| + C.$$

   It follows that if $n \neq m$*,*

$$\log|n-m| - C \leq F(g^m.x, g^n.x) \leq \log|n-m| + C.$$

2. *For any point* $x \in \mathcal{T}$*, there exists* $D > 0$ *such that for any* $n \neq 0$*,*

$$\log|n| - D \leq F(x, g^n.x) \leq \log|n| + D.$$

Proof. (1). Set $|g| = |x - g.x| > 0$. Then, $|x - g^n.x| = |n||g|$. Set

$$C = |\log|g|| + E.$$

By the theorem 2.9, $F(x, g^n.x) \geq \log|x - g^n.x| - E = \log|n| + \log|g| - E \geq \log|n| - C$.

For the other inequality, again by the theorem, if $|x - g^n.x| \geq 1$, then $F(x, g^n.x) \leq \log|n| + \log|g| + E \leq \log|n| + C$. If $|x - g^n.x| < 1$, then $F(x, g^n.x) \leq E \leq C$. (Or, we can use Proposition 2.1 for this direction.)

Now, for $F(g^m.x, g^n.x)$, use $F(g^m.x, g^n.x) = F(x, g^{n-m}.x)$.

(2). By the triangle inequality for F, $F(y, g^n.y) \leq F(y, x) + F(x, g^n.x) + F(g^n.x, g^n.y) = F(y, x) + F(x, g^n.x) + F(x, y) \leq F(x, g^n.x) + P$, where $P = F(x, y) + F(y, x)$. Therefore, $|F(x, g^n.x) - F(y, g^n.y)| \leq P$. Now, set $D = C + P$ and use (1). □

Corollary 2.11. *Let $\gamma(t)$ be the axis for a pseudo-Anosov map with unit speed. Then, $f(t) = F(\gamma(0), \gamma(t))$ is not convex on t.*

Proof. By the lower bound in Theorem 2.9 (1), for sufficiently large t, $f(1) < f(t)$. If f was convex, then there must be $C > 0$ such that for all sufficiently large t, $f(t) \geq Ct$. This is impossible by the upper bound in the same theorem. □

We now discuss a lower bound of F along an orbit of a partial pseudo-Anosov map.

Theorem 2.12. *Let $g \in MCG(S)$ be a reducible element such that some power of g, g^N, preserves some connected, proper subsurface $S_1 \subset S$ and g^N is pseudo-Anosov on S_1. Then, there exists a stratum $\mathcal{S}$ such that for any $y \in \mathcal{T}$, there exist $L > 0, E$ such that for any n, $|\mathcal{S} - g^n.y| \geq L|n| - E$.*

It follows that there exists D such that $\log|n| - D \leq F(y, g^n.y)$ for any $n \neq 0$.

The difference from the pseudo-Anosov case is that we have the lower bound only for particular strata $\mathcal{S}$ (cf. Lemma 2.7).

Proof. For the lower bound on $|\mathcal{S} - g^n.y|$, it suffices to show for g^N instead of g by the triangle inequality on $(\mathcal{T}, d)$. Therefore, in the following, by rewriting g^N by g, we assume that $N = 1$, i.e., g itself is pseudo-Anosov on S_1. The boundary of S_1 gives a simplex $\sigma \in \mathcal{C}(S)$. Let $S_2 = S \backslash S_1$, and write $g = (g_1, g_2)$, where g_1 is pseudo-Anosov on S_1 and g_2 is some mapping class on S_2. If a component of S_2 is a circle, we enlarge it to an annulus. Let $\mathcal{T}_1, \mathcal{T}_2$ be the Weil-Petersson spaces for S_1, S_2. Then $\mathcal{T}(\sigma) = \mathcal{T}_1 \times \mathcal{T}_2$.

For any stratum $\mathcal{S}_1$ of $\mathcal{T}_1$ and any point $x_1 \in \mathcal{T}_1$, there exist $L > 0, E$ such that for any n, $|\mathcal{S}_1 - g_1^n.x_1|_{\mathcal{T}_1} \geq L|n| - E$, where $|p - q|_{\mathcal{T}_1}$ is the distance on $\mathcal{T}_1$. To see this, let $\gamma_1 \subset \mathcal{T}_1$ be the axis for g_1. Assume that $x_1 \in \gamma_1$. Then by the lemma 2.7, there exists E such that for any n, $|\mathcal{S}_1 - g_1^n.x_1|_{\mathcal{T}_1} \geq$

$|x_1 - g_1^n.x_1|_{\mathcal{T}_1} - E = |n||x_1 - g_1.x_1|_{\mathcal{T}_1} - E$. Set $L = |x_1 - g_1.x_1|_{\mathcal{T}_1}$. For a point $x_1 \in \mathcal{T}_1$ in general, use the triangle inequality and modify E appropriately.

Set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{T}_2$. Then $\mathcal{S}$ is a stratum in $\mathcal{T}(\sigma)$. If $\mathcal{S}_1$ is $\mathcal{T}_1(\sigma_1)$, then $\mathcal{S} = \mathcal{T}(\sigma \cup \sigma_1)$. It follows from the previous estimate that for any $x = (x_1, x_2) \in \mathcal{T}(\sigma)$, $|\mathcal{S} - g^n.x|_{\mathcal{T}(\sigma)} \geq |\mathcal{S}_1 - g_1^n.x_1|_{\mathcal{T}_1} \geq L|n| - E$.

Now for given $y \in \mathcal{T}$, let $\pi_{\overline{\mathcal{T}(\sigma)}}(y) = x = (x_1, x_2) \in \mathcal{T}(\sigma)$. (Here, we are using that $x \in \mathcal{T}(\sigma)$, not only in $\overline{\mathcal{T}(\sigma)}$. This holds by a general fact, see Lemma 4.4.) Since $\mathcal{T}(\sigma)$ is invariant by g, $\pi_{\overline{\mathcal{T}(\sigma)}}(g.y) = g.x$. Now, since $\mathcal{T}(\sigma)$ is convex in $\mathcal{T}$ and $\mathcal{S} \subset \overline{\mathcal{T}(\sigma)}$, we have $|\mathcal{S} - g^n.y| \geq |\mathcal{S} - g^n.x|_{\mathcal{T}(\sigma)} \geq L|n| - E$ for any n from the previous discussion.

It follows that $|\mathcal{S} - g^n.y| \geq L|n|/2$ for all but finitely many n. Thus, $|g^n.\mathcal{S} - y| \geq L|n|/2$ for all but finitely many n. Therefore, for all but finitely many $n \neq 0$, $F(y, g^n.y) \geq \log|y - g^n.\mathcal{S}| - \log|g^n.y - g^n.\mathcal{S}| \geq \log|n| + \log(L/2) - \log|y - \mathcal{S}|$. Since $\log(L/2) - \log|y - \mathcal{S}|$ is a constant, there exists a desired D for all $n \neq 0$. □

Proposition 2.13. *If g is a Dehn-twist, then for each $x \in \mathcal{T}$, there exists C such that for all n, m, $F(g^n(x), g^m(x)) \leq C$.*

The same conclusion holds if g (or more generally, g^N for some $N \neq 0$) is a product of powers of commuting Dehn twists.

A product of powers of commuting Dehn twists is called a *multi twist.*

Proof. For a given $x \in \mathcal{T}$, there exists $R > 0$ such that $|x - \mathcal{S}| \geq R$ for all strata $\mathcal{S}$. Therefore, for any n, $|g^n(x) - \mathcal{S}| \geq R$ since g^n preserves the set of strata.

On the other hand since g has a fixed point in $\overline{\mathcal{T}}$, there exists D such that for any n, m, $|g^n(x) - g^m(x)| \leq D$.

Now for any stratum $\mathcal{S}$, $\frac{|g^n.x - \mathcal{S}|}{|g^m.x - \mathcal{S}|} \leq \frac{|g^m.x - \mathcal{S}| + D}{|g^m.x - \mathcal{S}|} = 1 + D/|g^m.x - \mathcal{S}| \leq 1 + D/R$. Therefore $F(g^n.x, g^m.x) \leq \log(1 + D/R) = C$.

If g is a multi twist, then by definition there must be a simplex $\sigma \in \mathcal{C}(S)$ such that g is a product of powers of Dehn twists along curves in σ. Then g fixes each point in the stratum $\mathcal{T}(\sigma)$. The rest of the argument is same. If g^N is a multi twist, use the triangle inequality for F. □

Remark 2.14. *(1). There is a bounded set in $\mathcal{T}$ with respect to d which is not bounded in terms of F. See Proposition 4.9.*
(2). The bound C depends on x, but $\inf_{x \in \mathcal{T}} C = 0$. If g is a Dehn twist, or more generally a multi twist along curves in a simplex $\sigma \in \mathcal{C}(S)$, then each

point in $\mathcal{S} = \mathcal{T}(\sigma)$ is fixed by g. Therefore, if $|x - \mathcal{S}| \leq e$, then $D = 2e$. Now fix $x \in \mathcal{T}$ and choose $x_n \in [x, \pi_{\overline{\mathcal{S}}}(x))$ with $e_n = |x_n - \mathcal{S}| \to 0$ as $n \to \infty$. Then there exists $R > 0$ such that for all n, $|x_n - \mathcal{S}'| \geq R$ for all strata $\mathcal{S}' \neq \mathcal{S}$. It follows $F(g^p.x_n, g^q.x_n) \leq \log(1 + 2e_n/R) \to 0$. Notice that the stratum $\mathcal{S}$ does not count since $|g^p.x_n - \mathcal{S}| = |g^q.x_n - \mathcal{S}|$ for all p, q, n.

Corollary 2.15. *Let g be a mapping class on a surface S. If g is of finite order, or some power g^N is a multi twist, then for any $x \in \mathcal{T}$, $F(x, g^n.x)$ is bounded on n. Otherwise, i.e., g is pseudo-Anosov or some power g^N acts as a pseudo-Anosov map on some proper subsurface of S, for each $x \in \mathcal{T}$, there exists $C > 0$ such that $|F(x, g^n.x) - \log |n|| \leq C$ for any $n \neq 0$.*

Proof. The argument is based on the classification of mapping classes due to Thurston (see for example [7], [13]). Any mapping class g is either:

1. a finite order element.
2. *reducible* in the sense that there exists $\sigma \in \mathcal{C}(S)$ such that $g.\sigma = \sigma$.
3. a pseudo-Anosov element.

1. Clearly, $F(x, g^n.x)$ is bounded on n.

2. If g is reducible, we may assume that for some power g^N, each vertex of σ is fixed by g^N, and moreover, each connected component of $S \backslash \sigma$ is preserved and g^N has infinite order on each component unless it is trivial, where $S \backslash \sigma$ is the subsurface we obtain from S by removing all curves in σ. Then, unless g^N is pseudo-Anosov on at least one component of $S \backslash \sigma$, g^N is a multi twist along curves in σ.

If g^N is a multi-twist, by Proposition 2.13, $F(x, g^n.x)$ is bounded on n (use the triangle inequality for F). Otherwise, by Theorem 2.12 we obtain D such that $\log |n| - D \leq F(x, g^n.x)$. Also, by Proposition 2.1, we obtain E with $F(x, g^n.x) \leq \log |n| + E$. Set $C = \max(|D|, |E|)$.

3. See Corollary 2.10 (1). □

2.2 Translation length

For a map $g : \mathcal{T} \to \mathcal{T}$ which preserves $F(x, y)$, define the *translation length* by

$$|g|_F = \inf_{x \in \mathcal{T}} F(x, g.x).$$

For an element in MCG, we have the following.

Theorem 2.16. *1. If $g \in MCG$ has finite order, then $|g^n|_F = 0$ for each n and the infimum is achieved.*

2. If g is a Dehn twist or a multi twist, then for any n, $|g^n|_F = 0$. The infimum is not achieved (unless $n = 0$).

3. If g is pseudo-Anosov, then for sufficiently large n, $|g^n|_F > 0$ and it is achieved by some point $x \in \mathcal{T}$ for each n.

4. Moreover, if g is pseudo-Anosov, then there exists Q such that for all $n \neq 0$, $\log|n| - Q \leq |g^n|_F \leq \log|n| + Q$.

Question 2.17. *1. How about for a partial pseudo-Anosov map g (for example, g is pseudo Anosov on a connected proper subsurface and trivial on the rest) ?*

2. In (3), what is the set of points where $|g^n|_F$ is achieved ? It is invariant by g.

Proof. (1). Clear since g fixes a point $x \in \mathcal{T}$.
(2). For a Dehn twist, $|g^n|_F = 0$ is already shown in Remark 2.14 (2). g^n ($n \neq 0$) does not fix any point in $\mathcal{T}$, therefore the infimum is not achieved. The same argument applies to a multi twist.
(3). It is natural to expect that a point which achieves $|g^n|_F$ is near the axis of g. The following lemma verifies that. As we mentioned (Fact 2.5), there exists a constant K, which depends only on S such that for each $x \in \mathcal{T}$, there exists a stratum $\mathcal{S}$ with $|x - \mathcal{S}| \leq K$.

Lemma 2.18. *Let γ be the axis for g. Then there exists a constant $L > 0$ with the following property. Let $x \in \mathcal{T}$ and set $X = \pi_\gamma(x)$. Assume n satisfies that $|X - g^n.X| \geq L$. Choose a stratum $\mathcal{S}$ with $|g^n.x - \mathcal{S}| \leq K$. Then $|x - \mathcal{S}| \geq |x - X|/2$. Also, $|x - \mathcal{S}| \geq |X - g^n.X|/2$.*

Proof. For the notational simplicity, we rewrite g^n as g. Then $\pi_\gamma(g.x) = g.X$. Set $y = g.x$, $Y = g.X$. We will show that if $|X - Y| \geq L$, then $|x - \mathcal{S}| \geq |x - X|/2$ and $|x - \mathcal{S}| \geq |X - Y|/2$. We will choose L in the argument.

Roughly speaking, since most part of $[Y, X]$ is perpendicular to $\mathcal{S}$ (see Remark 2.4), the distance to $\mathcal{S}$ increases approximately by $|Y - X|$ on $[Y, X]$. Then, by the convexity of the distance to $\mathcal{S}$, it further increases on $[X, x]$ approximately by $|X - x|$.

The points on γ in the following argument satisfy

$$X < A < C' < C < B < Y,\ C' < P',\ C < Q',$$

and accordingly on $[x,y]$,

$$x < a < c' < c < b < y.$$

Claim 1. For the axis γ, there exist M, N with the following property. Let $x, y \in \mathcal{T}$ be such that $X = \pi_\gamma(x)$ and $Y = \pi_\gamma(y)$ satisfy $|X - Y| \geq M$. Then there exist $X \leq A \leq B \leq Y$ on γ with $|X - A|, |Y - B| \leq N$ such that $|A - [x,y]|, |B - [x,y]| \leq 1$. We will choose such that $2N \leq M$.

This claim says that the geodesic $[x,y]$ and the union $[x,X] \cup [X,Y] \cup [Y,y]$ are close to each other (at most $N+1$ apart). See Figure 3.

This is a consequence of the Gauss-Bonnet theorem and the argument is similar to the one for the proposition 2.3. We use the same notation $C' \leq 1, k$ as in the proof of the proposition. We form a ruled rectangle with the sides $[x,X], [X,Y], [Y,y], [y,x]$. If $[x,y]$ does not intersect $N_{C'}(\gamma)$, then $|X - Y| \leq 2C' + 2\pi/C'k$. So, if $[X,Y]$ is longer than this, then there are $a, b \in [x,y]$ with $|a - \gamma| = |b - \gamma| = C'$. By convexity, $[a,b] = N_{C'}(\gamma) \cap [x,y]$. Then for $A = \pi_\gamma(a), B = \pi_\gamma(b)$, we can show $|X - A|, |Y - B| \leq 2C' + 2\pi/C'k$. This is again by Gauss-Bonnet theorem applied to the ruled rectangles whose vertices are x, a, X, A and y, b, Y, B. We leave the details to the readers. Set $N = 2C' + 2\pi/C'k$ and $M = 2N$.

Claim 2. Let γ, x, y, X, Y be as in the claim 1. Set $L = M + N + D + K + 16$, where D is from the proposition 2.3 for γ, and assume $|X - Y| \geq L$. Let $\mathcal{S}$ be a stratum with $|y - \mathcal{S}| \leq K$. Choose $C \in \gamma$ with $C < B$ and $|C - B| = K + 4$. Then $A \leq C$. Let $P', Q' \in \gamma$ be the points from Proposition 2.3 for γ and $\mathcal{S}$ with $P' < Q'$. Then, $C \leq Q'$.

Indeed, for $z \in \mathcal{T}$, define $f(z) = |z - \mathcal{S}|$. Since $f(z)$ is convex on $[y,x]$ and $f(y) \leq K$, on any segment in $[y,x]$, f can decrease at most by K. (f may decrease only at the beginning, but at most by K since f is positive.) To argue by contradiction, assume $Q' \leq C$. Then $f(B) \geq f(C) + |C - B| - 1$. To see this, evaluate f on $[B, \pi_{\overline{\mathcal{S}}}(B)]$, which must pass through $N_1(C)$ since $Q' \leq C$ (See Figure 2). The inequality follows.

Using the claim 1, choose $b, c \in [y,x]$ with $|b - B|, |c - C| \leq 1$. It follows that $f(b) \geq f(c) + |C - B| - 3 \geq f(c) + K + 1$. But b is closer to y than c is on

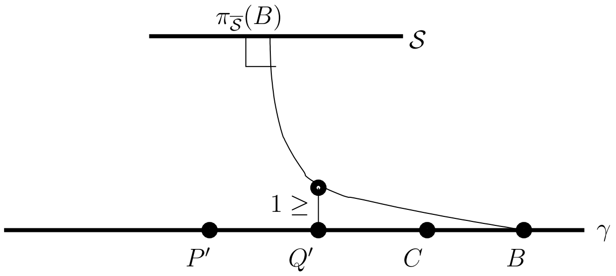

Figure 2: If $Q' \leq C$, the distance to $\mathcal{S}$ decreases approximately by $|B - C|$ from B to C.

$[y, x]$. It means that f decreases at least $K + 1$ on $[b, c] \subset [y, x]$, impossible. The argument for the claim 2 is complete.

We keep the assumption $|X - Y| \geq L$. Choose $C' \in \gamma$ with $C' < C$ and $|C' - C| = D$. Then by the proposition 2.3 and the claim 2, $C' \leq P'$.

Also, $|Y - C''| = N + K + 4 + D$, therefore $|A - C'| \geq |X - Y| - (2N + K + 4 + D) \geq 12 + N$. (We used $2N \leq M$.)

Choose $a, c' \in [x, y]$ with $|a - A| \leq 1, |c' - C'| \leq 1$. Then $|a - c'| \geq 10 + N$. Then we find $f(A) \geq f(C') + |A - C'| - 1$ by evaluating f on $[A, \pi_{\overline{\mathcal{S}}}(A)]$, which passes through $N_1(C')$ as $C' \leq P'$. See Figure 3. Therefore, $f(a) \geq f(c') + |a - c'| - 5$ by the triangle inequality.

It follows that $f(a) - f(c') \geq |a - c'|/2$ as $|a - c'| \geq 10$. Since f is convex, $f(x) - f(c') \geq |x - c'|/2$. Also, $|x - c'| \geq |x - X|$ since $|a - c'| \geq N + 1$ and $|X - a| \leq N + 1$. Now, $|x - \mathcal{S}| = f(x) > f(x) - f(c') \geq |x - c'|/2 \geq |x - X|/2$. We have shown the following.

Claim 3. If $|X - Y| \geq L$, then $|x - \mathcal{S}| \geq |x - X|/2$.

We will show

Claim 4. If $|X - Y| \geq 2L$, then $|x - \mathcal{S}| \geq |X - Y|/2$.

Since $f(a) \geq f(c')$ and f is convex, $f(x) > f(a)$. Therefore, $|x - \mathcal{S}| = f(x) > f(a) > f(a) - f(c') \geq |a - c'| - 5 \geq |A - C'| - 7 \geq |X - Y| - (2N + K + D + 11) > |X - Y| - L \geq |X - Y|/2$.

Now, we retake $2L$ as L. The proof of the lemma is complete. □

We resume the proof of the theorem. If n is sufficiently large, then for any maximal dimensional stratum $\mathcal{S} = \mathcal{T}(\sigma)$, $\overline{\mathcal{S}}$ and $g^n.\overline{\mathcal{S}}$ do not intersect. This is because if n is large enough, then for any vertex $\sigma \in \mathcal{C}(S)$, distance between σ and $g^n.\sigma$ is ≥ 2 in $\mathcal{C}(S)$ (this is due to the fact that g has positive translation length in $\mathcal{C}(S)$, see for example [7]), which means that $\overline{\mathcal{S}}$ and

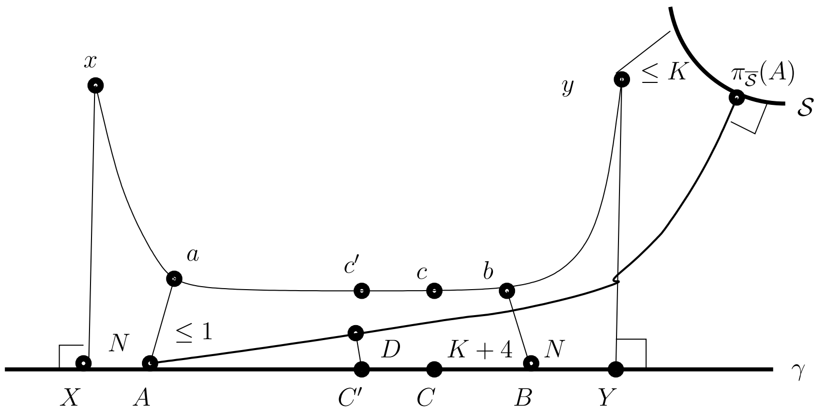

Figure 3: Since $C' \leq P'$, $[A, \pi_{\overline{\mathcal{S}}}(A)]$ passes $N_1(C')$.

$g^n.\overline{\mathcal{S}}$ do not intersect. In the following we assume n is large enough in this sense. It is a fact that for S, there exists $C > 0$ such that if the closure of two strata $\overline{\mathcal{S}}, \overline{\mathcal{S}}'$ do not intersect, then $|\mathcal{S} - \mathcal{S}'| \geq C$ (see Fact 4.2).

Also, we assume that n satisfies the assumption of the lemma 2.18. Then, since $|g^n.x - \mathcal{S}| \leq K$,
(I). $F(x, g^n.x) \geq \log(|x - \mathcal{S}|/|g^n.x - \mathcal{S}|) \geq \log|x - X| - \log(2K) \to \infty$ as $|x - X| = |x - \gamma| \to \infty$, uniformly on $|x - \gamma|$.

On the other hand,
(II) for $x \in \mathcal{T}$, $F(x, g^n.x) \to \infty$ as $|x - \partial\overline{\mathcal{T}}| \to 0$, uniformly on $|x - \partial\overline{\mathcal{T}}|$. Indeed, if $|x - \mathcal{S}| \leq e$ for some $\mathcal{S}$, then $|g^n.x - g^n.\mathcal{S}| \leq e$. But since $\overline{\mathcal{S}}$ and $g^n.\overline{\mathcal{S}}$ do not intersect, $|\mathcal{S} - g^n.\mathcal{S}| \geq C$, therefore $|x - g^n.\mathcal{S}| \geq C - e$. Letting $e \to 0$, $F(x, g^n.x) \geq |x - g^n.\mathcal{S}|/|g^n.x - g^n.\mathcal{S}| \geq (C - e)/e \to \infty$.

It now follows from (I) and (II) that there exist $M > 0$ and $e > 0$ such that $\inf_{x \in \mathcal{T}} F(x, g^n.x) = \inf_{x \in A} F(x, g^n.x)$, where

$$A = \{x \in \mathcal{T} \,|\, |x - \gamma| \leq M, |x - \partial\overline{\mathcal{T}}| \geq e\}.$$

The set A is invariant by g and the quotient of A by the action of $< g^n >$, $A/ < g^n >$, is compact. Choose a compact set $A' \subset A$ such that $< g^n > .A' = A$.

Let $\{x_m\} \subset \mathcal{T}$ be a sequence of points such that $\lim_m F(x_m, g^n.x_m) = |g^n|_F$. By the above discussion, we may assume that $x_m \in A$. Moreover, since $F(x, g^n.x) = F(g^p.x, g^n.(g^p.x))$ for any p, we may assume that $x_m \in A'$. Since A' is compact, we may assume that there exists $x \in A'$ such that

$\lim_m x_m = x$, i.e., $\lim_m |x_m - x| = 0$. Now since $|x - \partial\overline{\mathcal{T}}| \geq e$, it follows that $\lim_m F(x_m, g^n.x_m) = F(x, g^n.x)$. Therefore $|g^n|_F = F(x, g^n.x) > 0$.

(4). We keep the notation. Suppose n satisfies $|X - g^n.X| \geq L$ to apply the lemma 2.18. We will show that there exists P such that for such n, $|g^n|_F \geq \log|n| + P$. (Notice that, then by retaking P properly, we have this inequality for all $n \neq 0$.) Indeed, using $|x - \mathcal{S}| \geq |X - Y|/2$ for any $x \in \mathcal{T}$, since $Y = g^n.X$, we have $F(x, g^n.x) \geq \log(|x - \mathcal{S}|/|g^n.x - \mathcal{S}|) \geq \log(|X - Y|) - \log(2K) = \log|n| + \log|X - g.X| - \log(2K)$. Set $P = \log|X - g.X| - \log(2K)$. Notice that $|X - g.X|$ does not depend on $X \in \gamma$ since γ is the axis for g. It now follows that (after retaking P if needed as we explained) $|g^n|_F \geq \log|n| + P$.

On the other hand, $|g^n|_F \leq F(X, g^n.X) \leq \log|n| + C$ by Corollary 2.10 (1) for $n \neq 0$. Set $Q = \max(|P|, |C|)$. We have shown (4). The proof of the theorem is complete. □

3 Density of finite projection rays

This section is independent from other sections.

Definition 3.1. *A geodesic from some point $x \in \mathcal{T}$ to a point in a stratum, $y \in \mathcal{S}$, is called a finite ray from x. If this realizes the distance $|x - \mathcal{S}|$, we call it a (finite) projection ray, since $y = \pi_{\overline{\mathcal{S}}}(x)$.*

The visual sphere at x, $V_{\mathcal{T}}(x)$, is the collection of all finite rays from x and all infinite geodesics emanating from x. It is homeomorphic to the unit tangent sphere at x, $UT_x\mathcal{T}$, by assigning the unit tangent vector at x to each geodesic.

Using the fact 2.5, Brock [3] proved the following:

Theorem 3.2. *[3, Theorem 1.5] For each $x \in \mathcal{T}$, the finite rays from x are dense in the visual sphere $V_{\mathcal{T}}(x)$.*

We quote another density theorem due to Brock-Masur-Minsky:

Theorem 3.3. *[5, Theorem 1.8 (Closed Orbits Dense)] Closed orbits of the geodesic flow are dense in $UT\mathcal{M}$, the unit tangent bundle of $\mathcal{M}$.*

In other words, any vector $V \in UT_x\mathcal{M}$ at each $x \in \mathcal{M}$ is approximated by vectors in $UT\mathcal{M}$ which are tangent to the image by $\pi_{\mathcal{M}}$ of the axes in $\mathcal{T}$ of pseudo-Anosov elements. It follows that for given $x, y \in \mathcal{T}$ and for any $e > 0$, there is a pseudo-Anosov element $g \in MCG$ such that its axis γ passes through the e-neighborhood of x and the e-neighborhood of y (let V be the unit tangent vector to $[x, y]$ at x and apply the theorem to the image of V in $UT\mathcal{M}$).

From the two density theorems, we have the following corollary. In fact, as it is pointed out in [3], for the density result, we do not need all finite rays in Theorem 3.2. For example, finite rays to strata of least dimension, i.e., points, are already dense. In view of that, this result is not surprising.

Corollary 3.4. [1] *Let $\mathcal{S}$ be a stratum. Let $p.A.(\mathcal{S})$ denote the collection of strata of the form $g.\mathcal{S}$ such that $g \in MCG$ are pseudo-Anosov elements.*

Then, for each $x \in \mathcal{T}$, the collection of finite projection rays from x to the closure of strata in $p.A.(\mathcal{S})$ is dense in the visual sphere $V_{\mathcal{T}}(x)$.

Proof. Fix $y \in \mathcal{T}$ with $y \neq x$. We first show that a sequence of finite projection rays converges to $[x, y]$ in $V_{\mathcal{T}}(x)$. For a given $e > 0$, by the theorem 3.3, let γ be the axis of some pseudo-Anosov map g such that x and y are in the e-neighborhood of γ. Let $x', y' \in \gamma$ be with $|x - x'|, |y - y'| \leq e$. Choose n such that the projection ray from x' to $g^n.\overline{\mathcal{S}} = \overline{g^n.\mathcal{S}}$ passes through the e-neighborhood of y'. Such n exists by Proposition 2.3 (3).

Indeed, let $P', Q' \in \gamma$ be the points for $\mathcal{S}$ from the proposition. We may assume that $x' < y'$ and $P' < Q'$ on γ (swap P' and Q' if necessary). We may also assume $x' < g.x'$ (otherwise, consider g^{-1}). If n is large enough, then $y' < g^n.P'$. Then the projection ray from x' to $g^n.\overline{\mathcal{S}}$ passes through $N_1(g^n.P')$. By CAT(0) geometry, the distance between the projection ray and y' converges to 0 as n tends to ∞ since $|x' - g^n.P'|$ tends to ∞.

Choose n such that the distance between y' and the projection ray from x' to $g^n.\overline{\mathcal{S}}$ is $\leq e$. Let y'' be a point on the projection ray with $|y' - y''| \leq e$.

Let π be the projection of $\mathcal{T}$ to $g^n.\overline{\mathcal{S}}$. Set $p = \pi(x), p' = \pi(x')$. Then $|p - p'| \leq |x - x'| \leq e$. Choose a point $y''' \in [x, p]$ with $|y''' - y''| \leq e$ (such point exists by CAT(0) geometry), therefore, $|y - y'''| \leq 3e$. Letting $e \to 0$, $[x, y''']$ converges to $[x, y]$. See Figure 4.

Now let $[x, z]$ be any finite ray. Fix a point $y \in (x, z)$, and we approximate $[x, y]$ by finite projection rays by the previous discussion. Now, use the theorem 3.2 to approximate any element in $V_{\mathcal{T}}(x)$. □

[1] After the first draft has been completed, the author was informed that Miyachi-Ohshika-Yamada have been aware of a result similar to Corollary 3.4.

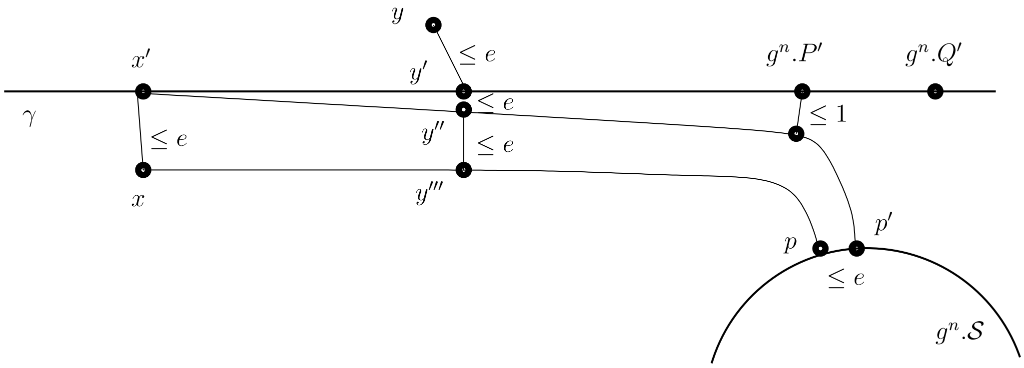

Figure 4: Projection rays are dense

Remark 3.5. *For any $x \neq y \in \mathcal{T}$, the above argument says that for any $e > 0$, there exists a stratum $\mathcal{S}' = g^n.\mathcal{S}$ such that $|x - \mathcal{S}'| - |y - \mathcal{S}'| \geq |x - y| - 4e$. Taking e sufficiently small, we see $|x - \mathcal{S}'| > |y - \mathcal{S}'|$, therefore $F(x, y) > 0$.*

4 WP-geodesics and F-geodesics

In this section we look for geodesics in the following sense:

Definition 4.1. *An (F-)geodesic is a map $f : [a, b] \to \mathcal{T}$ such that for any $a \leq t \leq s \leq u \leq b$,*

$$F(f(t), f(s)) + F(f(s), f(u)) = F(f(t), f(u)).$$

We quote one fact on the geometry of $\overline{\mathcal{T}}$ ([11, Corollary 22]).

Fact 4.2. *For a surface S, there exists $C > 0$ such that if the closure of two strata $\overline{\mathcal{S}}, \overline{\mathcal{S}}'$ do not intersect, then $|\mathcal{S} - \mathcal{S}'| \geq C$.*

We quote another result by Wolpert which describes the geometry of tangent cones of $\overline{\mathcal{T}}$, [12, Section 4]. By a general theory of CAT(0) spaces (cf. [2]), the *Alexandrov tangent cone* of $\overline{\mathcal{T}}$ at $p \in \overline{\mathcal{T}}$, $AC_p\overline{\mathcal{T}}$ is defined as the equivalence classes of constant speed geodesics emanating from x. Two of them are equivalent if they have same speed and the angle between them is 0. The Alexandrov tangent cone has a structure of a cone over an inner product space. If $p \in \mathcal{T}$, then $AC_p\overline{\mathcal{T}}$ is the usual tangent space since $\mathcal{T}$ is a Riemannian manifold.

Suppose $p \in \mathcal{S} = \mathcal{T}(\sigma)$. Let $\{\alpha\}$ be the collection of vertices in σ. We write $\alpha \in \sigma$. *A relative length basis* [12, Definition 4.1] is a collection τ of vertices in $\mathcal{C}(S)$ which are disjoint (when viewed as curves on S) from any vertex $\alpha \in \sigma$, where $|\tau| = \dim \mathcal{S}$ and the tuple $(\ell_\beta(x))_{\beta\in\tau}$ is a coordinate in a neighborhood of p in the manifold $\mathcal{S}$.

In a neighborhood of p in $\overline{\mathcal{T}}$, define

$$\mathcal{L}(x) = (\ell_\alpha^{1/2}(x), \ell_\beta^{1/2}(x))_{\alpha\in\sigma,\beta\in\tau} \in \mathbb{R}^{|\sigma|}_{\geq 0} \times \mathbb{R}^{|\tau|}_{>0}.$$

Let $\gamma(t)$ be a (short) geodesic from p. Each $\ell_\alpha(\gamma(t))$ and $\ell_\beta(\gamma(t))$ is convex and we can differentiate. Define

$$\Lambda(\gamma) = (2\pi)^{1/2} \frac{d\mathcal{L}(\gamma(t))}{dt}(0) \in \mathbb{R}^{|\sigma|}_{\geq 0} \times \mathbb{R}^{|\tau|},$$

where $\mathbb{R}^{|\tau|}$ is isometric to $T_p\mathcal{T}(\sigma)$.

Theorem 4.3. *[12, Thm 4.18] Let $\mathcal{S} = \mathcal{T}(\sigma)$ be a stratum, and $p \in \mathcal{S}$. The Alexandrov tangent cone of $\overline{\mathcal{T}}$ at p, $AC_p\overline{\mathcal{T}}$, is isometric to $\mathbb{R}^{|\sigma|}_{\geq 0} \times T_p\mathcal{T}(\sigma)$ by the map Λ. If $\frac{d\ell_\alpha^{1/2}}{dt}(0) = 0$ (i.e., the α-coordinate of $\Lambda(\gamma) = 0$), then γ is contained in the stratum $\mathcal{T}(\alpha)$, where $\mathcal{T}(\sigma) \subset \overline{\mathcal{T}(\alpha)}$.*

Here, $\mathbb{R}^{|\sigma|}_{\geq 0} \times T_p\mathcal{T}(\sigma)$ is a subset of $\mathbb{R}^{|\sigma|} \times T_p\mathcal{T}(\sigma)$ with the Euclidean inner product on $\mathbb{R}^{|\sigma|}$ and the one from the Riemannian metric on $T_p\mathcal{T}(\sigma)$.

The following consequence is implicitly contained in the discussion after Theorem 4.18 in [12]. This statement appears in [14] as well.

Lemma 4.4. *Let $\mathcal{S} = \mathcal{T}(\sigma), \mathcal{S}' = \mathcal{T}(\sigma')$ be strata with $\mathcal{S}' \neq \mathcal{S}$ and $\mathcal{S}' \in \overline{\mathcal{S}}$. Then for any point $x \in \mathcal{T}$, $|x - \mathcal{S}| < |x - \mathcal{S}'|$. In particular, $p = \pi_{\overline{\mathcal{S}}}(x) \in \mathcal{S}$.*

Proof. By the assumption, $\sigma \subsetneq \sigma'$. In the following, we only discuss the case such that $|\sigma| = |\sigma'| - 1$, since we can then argue inductively on $|\sigma'| - |\sigma|$. Let α be the vertex for $\sigma' \backslash \sigma$.

Let $p \in \mathcal{S}'$ and $\gamma = [p, x]$. By the theorem 4.3 we have $\Lambda(\gamma) \in \mathbb{R}^{|\sigma'|}_{\geq 0} \times T_p\mathcal{S}'$. In fact, $\Lambda(\gamma) \in \mathbb{R}^{|\sigma'|}_{>0} \times T_p\mathcal{S}'$, since otherwise, γ must be contained in some stratum $\mathcal{T}(\beta)$, impossible since $x \in \mathcal{T}$. It follows that if we move p into $\mathcal{S}$, then the distance from x decreases, therefore $|x - \mathcal{S}'| > |x - \mathcal{S}|$.

Here are some details. Let $\delta(t)$ be a unit speed (short) geodesic in $\overline{\mathcal{S}}$ from p such that all coordinates of $\Lambda(\delta)$ is 0 except for the one for α, which is 1. Then δ is contained in $\mathcal{S}$ except for $p = \delta(0)$. By the first variation formula for the distance from x, $-\frac{d(|x-\delta(t)|)}{dt}(0)$ is equal to the inner product of $\Lambda(\gamma)$ and $\Lambda(\delta)$, which is the α-coordinate of $\Lambda(\gamma)$, therefore positive. It means that $|x - \delta(t)|$ decreases when we increase t from 0.

Now, $p = \pi_{\overline{\mathcal{S}}}(x) \in \mathcal{S}$ is obvious, since otherwise $p \in \mathcal{S}' \subset \overline{\mathcal{S}}$ for some $\mathcal{S}' \neq \mathcal{S}$, impossible. □

For a point $x \in \mathcal{T}$, define

$$r(x) = \inf_{\sigma \in \mathcal{C}(S)} |x - \mathcal{T}(\sigma)|.$$

Clearly, $r(x) > 0$ since $\mathcal{T}$ is a Riemannian manifold.

Lemma 4.5. *For each x, the infimum in the definition of $r(x)$ is achieved by finitely many $\mathcal{T}(\sigma)$. Those strata are maximal dimensional, and $\pi_{\overline{\mathcal{T}(\sigma)}}(x) \in \mathcal{T}(\sigma)$.*

Moreover, $r(x)$ is an isolated point of the set $\{|x - \mathcal{T}(\sigma)|| \sigma \in \mathcal{C}(S)\} \subset \mathbb{R}$.

Proof. Suppose that the infimum is not achieved for x. Let $\{\mathcal{S}_i\}$ be a sequence of strata such that $|x - \mathcal{S}_i| \to r(x)$. Without loss of generality, we may assume that those strata are maximal dimensional, since otherwise, for each i we can find a stratum $\mathcal{S}_i'$ with $\mathcal{S}_i \subset \overline{\mathcal{S}}_i'$ whose dimension is larger than $\mathcal{S}_i$'s such that $|x - \mathcal{S}_i'| < |x - \mathcal{S}_i|$ (Lemma 4.4). Then we replace $\mathcal{S}_i$ with $\mathcal{S}_i'$. Let $[x, p_i](t)$ be the projection ray to $\overline{\mathcal{S}}_i$ with unit speed. We know $p_i \in \mathcal{S}_i$ (Lemma 4.4). The initial unit tangent vectors are subconvergent in $T_x\mathcal{T}$. After passing to a convergent subsequence, let's assume that it converges to $V \in T_x\mathcal{T}$. Let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = x$ and $\gamma'(0) = V$, where $\gamma'(t)$ is the tangent vector. By the definition of $r(x)$, $\gamma(t)$ is defined for $0 \leq t < r(x)$, and in fact at $t = r(x)$ as well by our assumption (otherwise γ reaches a stratum at $t = r(x)$, contradiction). Therefore $\gamma(r(x)) \in \mathcal{T}$, and its small open neighborhood is also contained in $\mathcal{T}$. On the other hand, p_i converges to $\gamma(r(x))$, which implies that $p_i \in \mathcal{T}$ for sufficiently large i. This is impossible since $p_i \notin \mathcal{T}$.

If a stratum $\mathcal{S}$ with $|x - \mathcal{S}| = r(x)$ is not maximal dimensional, then one can find $\mathcal{S}'$ with $|x - \mathcal{S}'| < |x - \mathcal{S}|$ by Lemma 4.4, impossible. Also by Lemma 4.4, $\pi_{\overline{\mathcal{S}}}(x) \in \mathcal{S}$.

Suppose there are infinitely many strata $\mathcal{S}_i$ which achieve $r(x)$ for x, at $p_i \in \mathcal{S}_i$. As in the previous discussion, one may assume that the initial tangent vectors of $[x, p_i]$ converges to $V \in T_x\mathcal{T}$. Then, the sequence p_i is Cauchy. This is impossible since two distinct strata are C-apart by the fact 4.2.

For the moreover part, to argue by contradiction, let $\mathcal{S}_i$ be a sequence of strata such that $r(x) < |x - \mathcal{S}_i| \to r(x)$, and $\pi_{\overline{\mathcal{S}}_i}(x) = p_i$. Passing to a subsequence, we may assume that the initial tangent vectors of $[x, p_i]$ are convergent, to a unit vector $V \in T_x\mathcal{T}$. V must be the initial vector for the projection ray $[x, p]$ for some maximal dimensional stratum $\mathcal{S}$ (only finitely many possibilities for $\mathcal{S}$). Then $[x, p_i](r(x))$ converge to $p \in \overline{\mathcal{T}}$, therefore $\lim_i p_i = p$. This is impossible since there exists $C > 0$ such that $|p - p_i| \geq C$ by the fact 4.2. □

Question 4.6. *What is the structure of the set* $\{|x - \mathcal{T}(\sigma)||\sigma \in \mathcal{C}(S)\} \subset \mathbb{R}$ *? Is it discrete, well-ordered ?*

For each stratum $\mathcal{S}$, let $N(\mathcal{S})$ be the union of points $x \in \mathcal{T}$ such that $\mathcal{S}$ is the unique nearest stratum.

Lemma 4.7. *If* $\mathcal{S}$ *is maximal dimensional, then* $N(\mathcal{S})$ *is not empty, open in* $\mathcal{T}$*, and* $N(\mathcal{S}) \cup \mathcal{S}$ *is an open neighborhood of* $\mathcal{S}$ *in* $\overline{\mathcal{T}}$*.*

If $\mathcal{S}$ *is not maximal dimensional, then* $N(\mathcal{S})$ *is empty.*

Proof. Let $\mathcal{S}$ be a maximal dimensional stratum. Choose a point $x \in \mathcal{S}$ and let $R > 0$ be such that for any stratum $\mathcal{S}' \subset \overline{\mathcal{S}}$ with $\mathcal{S}' \neq \mathcal{S}$, $|x - \mathcal{S}'| > R$. Without loss of generality, we may assume $R \leq C$, where C is the constant from the fact 4.2. Then for any stratum $\mathcal{S}' \neq \mathcal{S}$, $|x - \mathcal{S}'| \geq R$. Now, let $y \in \mathcal{T}$ be such that $|x - y| \leq R/3$. Then, by the triangle inequality, for any stratum $\mathcal{S}' \neq \mathcal{S}$, $|y - \mathcal{S}'| \geq 2R/3$. This implies that $y \in N(\mathcal{S})$. In particular, $N(\mathcal{S})$ is not empty.

Let $x \in N(\mathcal{S})$, and $[x, p]$ the projection ray to $\mathcal{S}$. Then $r(x) = |x - p|$. Since $r(x)$ is isolated, there exists $e > 0$ such that $|x - \mathcal{S}'| \geq r(x) + e$ for any stratum $\mathcal{S}'$ except for $\mathcal{S}$. This means that the ball of radius $e/3$ at x is contained in $N(\mathcal{S})$ by the triangle inequality. Therefore $N(\mathcal{S})$ is open. Combined with the previous paragraph, we have shown that $\mathcal{S} \cup N(\mathcal{S})$ is an open neighborhood of $\mathcal{S}$ in $\overline{\mathcal{T}}$.

Let $\mathcal{S} = \mathcal{T}(\sigma)$ be a stratum which is not maximal dimensional. Then choose a stratum $\mathcal{S}'$ such that $\mathcal{S} \subset \overline{\mathcal{S}}'$ and $\mathcal{S} \neq \mathcal{S}'$, then by the lemma 4.4, $|x - \mathcal{S}'| < |x - \mathcal{S}|$ for any $x \in \mathcal{T}$. Therefore $N(\mathcal{S})$ is empty. □

Lemma 4.8. *Let* $\mathcal{S}$ *be a maximal dimensional stratum. Then* $N(\mathcal{S})$ *is foliated by finite projection rays to* $\mathcal{S}$*.*

Proof. $N(\mathcal{S})$ is not empty by Lemma 4.7. If $x \in N(\mathcal{S})$, then $y = \pi_{\overline{\mathcal{S}}}(x) \in \mathcal{S}$, and the projection ray $[x, y)$ is contained in $N(\mathcal{S})$. Indeed, let $p \in [x, y)$

and $\mathcal{S}'$ be a nearest stratum for p. Then, $|x - \mathcal{S}| = |x - p| + |p - \mathcal{S}| \geq |x - p| + |p - \mathcal{S}'| \geq |x - \mathcal{S}'|$. This implies $\mathcal{S} = \mathcal{S}'$ by the definition of $N(\mathcal{S})$.

For $x, x' \in N(\mathcal{S})$, the finite projection rays to $\mathcal{S}$, $[x, y), [x', y')$ are disjoint, or one contains the other. Therefore $N(\mathcal{S})$ is foliated by finite projection rays to $\mathcal{S}$. □

Proposition 4.9. *Let $\mathcal{S}$ be a maximal dimensional stratum.*

1. *For each $x \in N(\mathcal{S})$, the projection ray $[x, \pi_{\overline{\mathcal{S}}}(x))$ is an infinite F-geodesic.*

2. *For any $x, x' \in N(\mathcal{S})$, there exists Q such that for any $p \in [x, \pi_{\overline{\mathcal{S}}}(x))$, there exists $p' \in [x', \pi_{\overline{\mathcal{S}}}(x'))$ with $F(p, p') \leq Q$.*

By this proposition, $N(\mathcal{S})$ is foliated by infinite F-geodesics such that any two of them are in finite Hausdorff-distance from each other in terms of F.

Proof. (1). Set $y = \pi_{\overline{\mathcal{S}}}(x)$. Suppose $|x - y| = e^{-t_0}$. For $t \in [t_0, \infty)$, define a map $[x, y]$ by letting $[x, y](t)$ be the point p on the Weil-Petersson geodesic $[x, y]$ with $|p - y| = e^{-t}$.

We will show that for $t_0 \leq t < s$, $F([x, y](t), [x, y](s))$ is realized by $\mathcal{S}$. Then it follows that $F([x, y](t), [x, y](s)) = \log(e^{-t}/e^{-s}) = s - t$. This shows that $[x, \pi_{\overline{\mathcal{S}}}(x))$ is an F-geodesic.

Set $p = [x, y](t), q = [x, y](s)$. Then $|p - q| = e^{-t} - e^{-s}$ and $e^{-t} = |p - \mathcal{S}| \geq |q - \mathcal{S}| = e^{-s}$. Let $\mathcal{S}'$ be a stratum. Then $|q - \mathcal{S}'| \geq |q - \mathcal{S}|$ (otherwise, by the triangle inequality, $|x - \mathcal{S}'| \leq |x - q| + |q - \mathcal{S}'| < |x - q| + |q - \mathcal{S}| = |x - \mathcal{S}|$, a contradiction). Now, $|p - \mathcal{S}'|/|q - \mathcal{S}'| \leq (|p - q| + |q - \mathcal{S}'|)/|q - \mathcal{S}'| = 1 + (e^{-t} - e^{-s})/|q - \mathcal{S}'| \leq 1 + (e^{-t} - e^{-s})/|q - \mathcal{S}| = (|q - \mathcal{S}| + (|p - \mathcal{S}| - |q - \mathcal{S}|))/|q - \mathcal{S}| = |p - \mathcal{S}|/|q - \mathcal{S}|$. This shows that $F(p, q)$ is realized by $\mathcal{S}$.

(2). By the triangle inequality for F, it suffices to argue in the case that $|x - \mathcal{S}| = |x' - \mathcal{S}|$. Also, for the same reason, we may assume that $|x - \mathcal{S}| \leq C/2$, where C is the constant from the fact 4.2. Set $y' = \pi_{\overline{\mathcal{S}}}(x')$, and $L = |x - x'|$. For $p = [x, y)(t)$, we take $p' = [x', y')(t)$. Then $|p - \mathcal{S}| = |p' - \mathcal{S}| = e^{-t}$, therefore $\mathcal{S}$ does not count for $F(p, p')$.

On the other hand, there exists $K > 0$ such that for any stratum $\mathcal{S}' \neq \mathcal{S}$, we have $|p - \mathcal{S}'| \geq K$ and $|p' - \mathcal{S}'| \geq K$. Indeed, let $R > 0$ be a constant such that for any stratum $\mathcal{S}' \neq \mathcal{S}$ with $\mathcal{S}' \subset \overline{\mathcal{S}}$, $|y - \mathcal{S}'|, |y' - \mathcal{S}'| \geq R$ (cf. the definition of $r(x)$ after this proof). Since $|p - \mathcal{S}'| > |y - \mathcal{S}'|, |p' - \mathcal{S}'| > |y' - \mathcal{S}'|$, we have $|p - \mathcal{S}'|, |p' - \mathcal{S}'| \geq R$. Also, for a maximal dimensional stratum $\mathcal{S}' \neq \mathcal{S}$, we have $|p - \mathcal{S}'|, |p' - \mathcal{S}'| \geq C/2$. Now, set $K = \min(C/2, R)$.

By convexity, $|p - p'| \leq |x - x'| = L$. It follows that $|p - \mathcal{S}'|/|p' - \mathcal{S}'| \leq (K + L)/K$, therefore $F(p, p') \leq \log(K + L) - \log K$ for any $p \in [x, y)$. Set $Q = \log(K + L) - \log K$. □

Let $\overline{N(\mathcal{S})}$ be the closure in $\mathcal{T}$. A point x in $\overline{N(\mathcal{S})} \setminus N(\mathcal{S})$ has another stratum $\mathcal{S}'$ than $\mathcal{S}$ such that $|x - \mathcal{S}'| = r(x)$. Since for each such x, there is a unique projection ray to $\mathcal{S}$, the foliation in $N(\mathcal{S})$ in Lemma 4.8 extends to $\overline{N(\mathcal{S})}$.

We summarize the discussion as follows. It is clear that the decomposition is preserved by MCG since $g \in MCG$ preserves $|x - y|$ and also maps a stratum to a stratum.

Theorem 4.10. *$\mathcal{T}$ is decomposed as $\mathcal{T} = \cup_{\sigma \in \mathcal{C}(S)} \overline{N(\mathcal{T}(\sigma))}$ such that*

1. *the decomposition is preserved by MCG.*
2. *$N(\mathcal{T}(\sigma))$ is non-empty if and only if $\mathcal{T}(\sigma)$ is maximal dimensional (i.e. σ is a vertex of $\mathcal{C}(S)$). If $\mathcal{T}(\sigma)$ is maximal dimensional, then $N(\mathcal{T}(\sigma))$ is an open submanifold in $\mathcal{T}$, and $N(\mathcal{T}(\sigma)) \cup \mathcal{T}(\sigma)$ is an open neighborhood of $\mathcal{T}(\sigma)$ in $\overline{\mathcal{T}}$.*
3. *If $\sigma \neq \tau$, then $N(\mathcal{T}(\sigma)) \cap N(\mathcal{T}(\tau)) = \emptyset$, and a point $x \in \overline{N(\mathcal{T}(\sigma))} \cap \overline{N(\mathcal{T}(\tau))}$, if it exists, is in the boundary of each of them. For each point $x \in \mathcal{T}$, there are at most finitely many $\overline{N(\mathcal{T}(\sigma))}$ which contains x.*
4. *Each $\overline{N(\mathcal{T}(\sigma))}$ is foliated by finite WP-projection rays to $\mathcal{T}(\sigma)$, which are also infinite F-geodesics, such that the Hausdorff-distance between any two such rays is finite. In particular, for any point $x \in \mathcal{T}$, there is an infinite F-geodesic emanating from x.*

Proof. 1. We already discussed.
2. By Lemma 4.7.
3. By the definition of $N(\mathcal{S})$ and Lemma 4.5.
4. By Proposition 4.9. We already mentioned that the foliation in $N(\mathcal{S})$ extends to $\overline{N(\mathcal{S})}$. □

5 Non-convexity of $F(x, y)$

We briefly discuss convexity of F. For the original Funk metric for the Euclidean spaces, $F(x, s(t))$ is convex on t for a geodesic $s(t)$ (see [10], [14]). By Theorem 2.11, we already know that the same convexity does not hold on $\mathcal{T}$ along an axis. Convexity does not hold near a stratum as well.

Theorem 5.1. *Let $\sigma \in \mathcal{C}(S)$ be a vertex and g the Dehn twist with respect to σ. For $x \in \mathcal{T}$, $F(x, g^n.x)$ is bounded for $n \geq 1$. But, if x is sufficiently close to $\mathcal{S} = \mathcal{T}(\sigma)$, then $F(x, y)$ is unbounded on $y \in \cup_{n \geq 2}[g.x, g^n.x]$.*

In particular, there is a point $x \in \mathcal{T}$ and a (WP-)geodesic segment $\gamma(t)$ such that $F(x, \gamma(t))$ is not convex on t.

Proof. We already know that $F(x, g^n.x)$ is bounded by Proposition 2.13. Let C be the constant from the fact 4.2 and let $x \in \mathcal{T}$ such that $|x - \mathcal{S}| \leq C/5$. Let $\gamma_n = [g.x, g^n.x]$. Then, for any stratum $\mathcal{S}' \neq \mathcal{S}$, $|\gamma_n - \mathcal{S}'| \geq 4C/5$. We claim that there is no $e > 0$ such that for all $n > 0$, $|\gamma_n - \mathcal{S}| \geq e$. This is because otherwise, γ_n must subconverge to some geodesic $[x, y]$ in $\mathcal{T}$. It follows that $g^n.x$ subconverges to y. This is impossible since the action of MCG on $\mathcal{T}$ is properly discontinuous, there exists $K > 0$ such that $|g^n.x - g^m.x| \geq K$ if $n \neq m$.

Now let γ_{m_n} be a subsequence such that $|\gamma_{m_n} - \mathcal{S}| \to 0$, and let $p_{m_n} \in \gamma_{m_n}$ such that $|p_{m_n} - \mathcal{S}| \to 0$. (It is not difficult to show that we can take the mid points of γ_{m_n} as p_{m_n}.) Then, $F(x, p_{m_n}) \geq \log(|x - \mathcal{S}|/|p_{m_n} - \mathcal{S}|) \to \infty$.

Let L be a bound on $F(x, g^n.x)$, $n > 0$. Choose γ_n such that $\max_{y \in \gamma_n} F(x, y) > L$. Clearly, $F(x, y)$ is not convex on $y \in \gamma_n$. □

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