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The deficiencies of Kähler groups

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THE DEFICIENCIES OF KÄHLER GROUPS

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ABSTRACT. Generalizing the theorem of Green–Lazarsfeld and Gromov, we classify Kähler groups of deficiency at least two. As a consequence we see that there are no Kähler groups of even and strictly positive deficiency. With the same arguments we prove that Kähler groups that are non-Abelian and are limit groups in the sense of Sela are surface groups.

It is only fair to say that our knowledge of deficiency is quite deficient.[1]

1. INTRODUCTION

The deficiency $\mathrm{def}(\Gamma)$ of a finitely presentable group Γ is the maximum over all presentations of the difference of the number of generators and the number of relators. This invariant, which arises naturally in combinatorial group theory, is often difficult to compute. Its investigation has, almost from the beginning, had close connections to low-dimensional topology. For example, it is well-known that knot groups have deficiency one, and that the fundamental groups of closed three-manifolds have non-negative deficiency; cf. Epstein [19].

In Kähler geometry, the deficiency has also long been known to play a rôle because of the following:

Theorem 1. (Green–Lazarsfeld [22], Gromov [24]) *If the fundamental group of a compact Kähler manifold X has deficiency ≥ 2, then the Albanese image of X is a curve.*

In fact, Gromov proved that the fundamental group is commensurable with a surface group, and so one might hope for a more detailed classification. For example, it is natural to wonder whether the kernel of the homomorphism on fundamental groups induced by the Albanese is finite. However, in the more than twenty years since Theorem 1 was proved, there have been no improvements on this result. The main purpose of this paper is to prove the following definitive version of Theorem 1:

Theorem 2. *A group of deficiency ≥ 2 is the fundamental group of a compact Kähler manifold X if and only if it is isomorphic to the orbifold fundamental group of a curve of genus $g \geq 2$. The curve is the Albanese image of X and the isomorphism is induced by the Albanese map of X.*

The orbifold structure on the target is determined by the multiplicities of the singular fibers of the Albanese. Since the kernel of the map from the orbifold fundamental group to the ordinary fundamental group is huge (not even finitely generated [23]) whenever the orbifold structure is non-trivial, we see that the Albanese map usually does not induce a homomorphism with finite kernel on the ordinary fundamental group. It is straightforward to see that all such orbifold fundamental groups are in fact fundamental groups of smooth complex projective varieties, and that the deficiency equals $2g-1$; cf. Section 2 below. The latter statement gives the following result:

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[1] F. R. Beyl and J. Tappe [6, p. 191]

Corollary 1. *There are no Kähler groups with positive even deficiency.*

Theorem 2 cannot be obtained from the arguments of Green–Lazarsfeld [22] or Gromov [24], and a new ingredient is required. Our proof, like Gromov's, uses ℓ^2-cohomology, starting from the well known fact that groups of deficiency at least two have positive first ℓ^2-Betti number. However, unlike in Gromov's approach, we do not use any ℓ^2-Hodge theory or complex analysis, but instead shift the focus onto the formal properties of the first ℓ^2-Betti number, replacing analytic techniques by algebraic ones. A crucial tool in our argument is the vanishing theorem for the first ℓ^2-Betti numbers of groups admitting finitely generated infinite normal subgroups of infinite index due to Lück [29] and Gaboriau [21].

The question of classifying one-relator Kähler groups, which appeared in [1, 3] not long after the works of Green–Lazarsfeld [22] and Gromov [24], is settled by the following result, to which a completely different route was taken recently by Biswas and Mj [9].

Theorem 3. *An infinite one-relator group is the fundamental group of a compact Kähler manifold X if and only if it is isomorphic to the orbifold fundamental group of an orbifold of genus $g \geq 1$ with at most one point with multiplicity > 1. Moreover, the isomorphism is induced by the Albanese map of X.*

If the number of generators of a one-relator group is at least 3, then this follows from Theorem 2. However, instead of deducing most of Theorem 3 from Theorem 2, we will give a direct argument for Theorem 3 in Section 4 as a warmup for the proof of Theorem 2. On the one hand, some of the details are actually simpler in this special case, and, on the other hand, this proof will also handle most one-relator groups with two generators, to which Theorem 2 does not apply because their deficiency is one.

In Section 5.1 we state and prove a more detailed version of Theorem 2, and we discuss a possible extension to groups of deficiency one. In Section 5.2 we apply some of the arguments from the proof of Theorem 2 to show that a non-Abelian limit group in the sense of Sela [34] is a Kähler group if and only if it is a surface group of genus at least two. In Section 7 we show that our results are valid not only for Kähler manifolds, but also for non-Kähler compact complex surfaces and for so-called Vaisman manifolds. The latter are a subclass of the locally conformally Kähler manifolds, cf. [18].

2. Fibered Kähler manifolds and orbifolds

Suppose we have a surjective holomorphic map $f\colon X \longrightarrow C_g$ with connected fibers from a compact complex manifold to a curve of genus $g \geq 1$. Then the induced map on fundamental groups is surjective. A subtle problem, which is easy to overlook, is that the kernel of this induced map may not be finitely generated. Indeed, if there are multiple fibers, it is not. For me, this point was clarified by Catanese's paper [15], although it seems that Simpson [35] and perhaps others understood it much earlier. In order to have a finitely generated kernel, one has to replace the usual fundamental group of C_g by its orbifold fundamental group that takes into account the multiplicities of the multiple fibers.

Lemma 1. ([15]) *Let $f\colon X \longrightarrow C_g$ be a surjective holomorphic map with connected fibers from a compact complex manifold to a curve of genus $g \geq 1$. By marking the critical values $p_1, \ldots, p_k$ of f with suitable integral multiplicities $m_i \geq 1$, one can define the orbifold fundamental group $\pi_1^{orb}(C_g)$ of C_g with respect to these multiplicities, so that one obtains a short exact sequence*

$$1 \longrightarrow K \longrightarrow \pi_1(X) \longrightarrow \pi_1^{orb}(C_g) \longrightarrow 1 \qquad (1)$$

in which the kernel K is finitely generated, since it is a quotient of the fundamental group of a regular fiber of f.

All these orbifold fundamental groups are also fundamental groups of compact Kähler manifolds, and even of smooth complex projective varieties. They are real surface groups only when the orbifold structure is trivial, but in all cases they can be realized by projective complex elliptic surfaces with appropriate multiple fibers with smooth reduction, see [20, Section 2.2.1]. Alternatively, and less explicitly, one can use the fact that each such orbifold fundamental group contains a real surface group as a subgroup of finite index [12], and then apply [2, Lemma 1.15].

The orbifold fundamental groups are of the form

$$\Gamma = \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_k \mid [x_1, y_1] \ldots [x_g, y_g] \cdot z_1 \ldots z_k = z_1^{m_1} = \ldots = z_k^{m_k} = 1 \rangle \tag{2}$$

for some $g \geq 1$ and $k \geq 0$, with $m_i \geq 2$ for all i. (Singular fibers with multiplicity 1 do not contribute.)

Lemma 2. *The deficiency of an orbifold group as in* (2) *is* $\mathrm{def}(\Gamma) = 2g - 1$.

Proof. The given presentation shows that the deficiency cannot be smaller than $2g - 1$. It cannot be larger either because of the Morse inequality[2]

$$\mathrm{def}(\Gamma) \leq b_1(\Gamma) - b_2(\Gamma) \; . \tag{3}$$

in which the right-hand-side is $2g - 1$ in this case. □

If the number k of orbifold points is positive, then one can use the first relation in (2) to eliminate one of the generators z_i from the presentation. Therefore, for $k \leq 1$ the orbifold fundamental group is a one-relator group of the form

$$\Gamma = \langle x_1, y_1, \ldots, x_g, y_g \mid ([x_1, y_1] \cdot \ldots \cdot [x_g, y_g])^m = 1 \rangle \tag{4}$$

for some g and m, both ≥ 1. These are the groups appearing in Theorem 3.

3. Background on ℓ^2-Betti numbers

The ℓ^2-Betti numbers are defined for all countable groups. In this section we collect those results about them that will be needed in the proofs of our theorems. For details we refer to Lück's monograph [30].

For our purposes, the starting point is the relationship between the deficiency and the first ℓ^2-Betti number β_1.

Proposition 1. ([25]) *For every finitely presented group one has*

$$\beta_1(\Gamma) \geq \mathrm{def}(\Gamma) - 1 \; . \tag{5}$$

In the case of equality the presentation complex of a presentation realizing the deficiency is aspherical.

The inequality (5) has been part of the folklore for a very long time, and appears already in [24]. It is a special case of the Morse inequalities in the ℓ^2 setting. For the case of equality we refer to the paper by Hillman [25].

Proposition 1 shows that groups of deficiency ≥ 2 have positive first ℓ^2-Betti number. For one-relator groups we have the following more precise statement, sharpening the predictions in [24, 30]:

Proposition 2. (Dicks–Linnell [17]) *Let* Γ *be an infinite group given by* $a \geq 2$ *generators and one relation* r*. One may assume that* r *is cyclically reduced and of the form* $r = s^n$*, with* $n \geq 1$ *and* s *not a proper power. Then*

$$\beta_1(\Gamma) = -\chi(\Gamma) = a - 1 - \frac{1}{n} \; .$$

Here χ denotes the rational Euler characteristic, which is known to equal the ℓ^2-Euler characteristic. This proposition contains as special cases the computation of the first ℓ^2-Betti numbers for surface groups and for the fundamental groups of two-dimensional orbifolds with exactly one point with multiplicity > 1. We can easily extend to orbifolds with an arbitrary number of points with multiplicity, although this is not needed for the proofs of our main theorems.

[2] This instance of the "Morse inequality" is often called "Philip Hall's inequality" by algebraists, because Epstein [19] wrote it was "essentially due" to Hall.

Proposition 3. *Let $\pi_1^{orb}(C_g)$ be the fundamental group of an orbifold of genus $g \geq 1$ as in* (2). *Then*

$$\beta_1(\pi_1^{orb}(C_g)) = -\chi(\pi_1^{orb}(C_g)) = 2g - 2 + k - \sum_{i=1}^{k} \frac{1}{m_i} \ .$$

Proof. By the solution of the Fenchel conjecture due to Bundgaard and Nielsen [12], $\pi_1^{orb}(C_g)$ has a surface group as a finite index subgroup. Since both β_1 and χ are multiplicative under passage to finite index subgroups, the equality $\beta_1 = -\chi$ holds for $\pi_1^{orb}(C_g)$ because it holds for surface groups. □

The deepest theorem about ℓ^2-Betti numbers that we use is the following vanishing result:

Theorem 4. (Lück [29], Gaboriau [21]) *If a finitely presentable Γ fits into an exact sequence*

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

with K and Q infinite, and K finitely generated, then $\beta_1(\Gamma) = 0$.

This was originally proved by Lück [29] under the assumption that Q is not a torsion group, and then by Gaboriau [21] in the general case. We apply Theorem 4 to the exact sequence (1), where Lück's technical assumption is satisfied, so that the full strength of Gaboriau's result is not required for our proofs. A quick proof of Lück's result is given in [11].

We will use Theorem 4 to show that in certain situations the kernel K in (1) is finite. It is then interesting to bound the order of K, which one can do with the following consequence of the Lyndon–Hochschild–Serre spectral sequence in the ℓ^2 setting:

Lemma 3. *Suppose the group Γ fits into an exact sequence*

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

with K finite. Then $\beta_1(\Gamma) = \frac{1}{|K|}\beta_1(Q)$.

4. A DIRECT APPROACH TO THEOREM 3

As remarked in Section 2, all the one-relator orbifold groups of the form (4) are fundamental groups of smooth complex projective varieties. So one only has to prove the converse.

Let Γ be an infinite one-relator group, and X a compact Kähler manifold with $\pi_1(X) = \Gamma$. Since the first Betti number of Γ is positive, X has a non-constant Albanese map α_X. The dimension of the Albanese image is bounded above by the cup-length of degree one cohomology classes on X, see Catanese [13] and [2, Ch. 2]. By the Hard Lefschetz theorem, this cup-length is at least two, and since $b_2(\Gamma) \leq 1$ because there is only one relation, we conclude that the cup-length is exactly two. Therefore, the Albanese image of X has real dimension equal to 2. It is a complex curve C_g, necessarily of genus $g \geq 1$. It is well known, and easy to see, that a one-dimensional Albanese image must be smooth, and that the fibers of the Albanese map are connected in this case. Thus

$$\alpha_X : X \longrightarrow C_g$$

is a surjective holomorphic map with connected fibers onto a smooth curve, which induces an isomorphism on $H^1(-;\mathbb{Z})$ and an injection on $H^2(-;\mathbb{Z})$. Since the target of the Albanese map is aspherical, α_X factors through the classifying map

$$c_X : X \longrightarrow B\Gamma$$

of the universal covering of X, showing that the rank of $H^2(\Gamma;\mathbb{Z})$ is positive. This means that the defining relation of Γ is in the commutator subgroup of the free group on the generators. Thus there are precisely $2g$ generators.

Now we apply Lemma 1 to α_X. Let k be the number of critical values with multiplicity > 1. If $k > 0$, then the minimal number of generators of $\pi_1^{orb}(C_g)$ is $2g + k - 1$, see [32]. But $\pi_1^{orb}(C_g)$ is a homomorphic image of Γ, which is generated by $2g$ elements. Therefore we conclude that there is at most one critical value p with a multiplicity $m > 1$ and the proof of Theorem 3 will be complete once one shows that the kernel K in Lemma 1 is trivial in this case.

By Proposition 2 the first ℓ^2-Betti number of a one-relator group Γ on $2g$ generators is

$$\beta_1(\Gamma) = 2g - 1 - \frac{1}{n}\ , \tag{6}$$

where n is the maximal positive integer such that the defining relator can be written as an n^{th} power. Thus $\beta_1(\Gamma) > 0$ as soon as $g \geq 2$, or $g = 1$ and $n > 1$. We can then apply Theorem 4 to conclude that K in (1) is finite. Here it is crucial to have a finitely generated K.

We can estimate the order of the finite group K using Lemma 3 and (6):

$$|K| = \frac{\beta_1(\pi_1^{orb}(C_g))}{\beta_1(\Gamma)} = \frac{2g-1-\frac{1}{m}}{2g-1-\frac{1}{n}} < \frac{2g-1}{2g-1-\frac{1}{n}} = 1 + \frac{1}{(2g-1)n-1} \leq 2 \tag{7}$$

if $g \geq 2$, and if $g = 1$ and $n > 1$. Therefore, in all these cases, the proof is complete.

Finally, consider the case $g = n = 1$. The condition $n = 1$ implies that Γ is torsion-free, so that K would have to be infinite if it were non-trivial. The condition $g = 1$ means that in this case the Albanese map is onto an elliptic curve. If there are no multiple fibers, then $\pi_1^{orb}(C_1) = \mathbb{Z}^2$, and the map $\Gamma \longrightarrow \mathbb{Z}^2$ induced by the Albanese is the Abelianization. Then the kernel K is the commutator subgroup of Γ, and if it is finitely generated, then it is trivial by a result of Bieri [8, Corollary B(b)].

5. The main results

5.1. **Proof of Theorem 2.** We now prove the following precise version of Theorem 2.

Theorem 5. *A group Γ of deficiency ≥ 2 is the fundamental group of a compact Kähler manifold X if and only if it is isomorphic to $\pi_1^{orb}(C_g)$ for some $g \geq 2$.*

The curve C_g is the Albanese image of X and the isomorphism is induced by the Albanese map. The number k of multiple fibers of the Albanese map, equivalently the number of orbifold points with multiplicity > 1 in C_g, is 0 or 1 if Γ is a one-relator group, and equals the minimal number of relations in any finite presentation of Γ in all other cases.

Proof. We noted in Section 2 that all orbifold fundamental groups $\pi_1^{orb}(C_g)$ are in fact fundamental groups of smooth complex projective varieties, and that they have deficiency $\mathrm{def}(\pi_1^{orb}(C_g)) = 2g - 1 \geq 2$ if and only if $g \geq 2$.

For the non-trivial direction of the theorem, let X be a compact Kähler manifold with $\pi_1(X) = \Gamma$ of deficiency ≥ 2. The first step is to prove that the Albanese image is indeed a curve. Unlike for one-relator groups, this is not obvious, and was the content of the original result of Green–Lazarsfeld and Gromov, cf. Theorem 1. For the sake of completeness we give a quick proof following Catanese [14]. Using $\mathrm{def}(\Gamma) \geq 2$, the Morse inequality (3) shows that every element of $H^1(\Gamma, \mathbb{R})$ is contained in an isotropic subspace of dimension ≥ 2 for the cup product $H^1(X, \mathbb{R}) \times H^1(\Gamma, \mathbb{R}) \longrightarrow H^2(\Gamma, \mathbb{R})$. Therefore, by Catanese's proof of the Siu–Beauville theorem, see [13] and [2, Ch. 2], every element of $H^1(\Gamma, \mathbb{R})$ is in the image of a pullback f^*, where $f\colon X \longrightarrow C$ is a holomorphic map with connected fibers to a curve of genus ≥ 2. Since each of these fibrations is uniquely determined by the map it induces on integral cohomology, there are at most countably many of them. But $H^1(\Gamma, \mathbb{R})$ cannot equal a union of countably many proper subspaces, and so at least one of these subspaces equals $H^1(\Gamma, \mathbb{R})$. Therefore, there is a holomorphic fibration of X over a curve which induces an isomorphism on H^1. It follows by the universal property of the Albanese map that it factors through this fibration.

Since the Albanese map of X has one-dimensional image, by Lemma 1 it induces an exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow \pi_1^{orb}(C_g) \longrightarrow 1\ , \tag{8}$$

with C of genus $g = \frac{1}{2}b_1(\Gamma)$, and K finitely generated. By Proposition 1, $\beta_1(\Gamma) \geq \text{def}(\Gamma) - 1 \geq 1$, and so Theorem 4 implies that K is finite. It is this application of Theorem 4 that requires the a priori knowledge that K is finitely generated, which we arranged by considering the Albanese image as an orbifold.

If k is the number of multiple fibers, then by [32] the orbifold fundamental group $\pi_1^{orb}(C_g)$ is generated by no fewer than $2g + k - 1$ elements. As Γ surjects to $\pi_1^{orb}(C_g)$, we conclude that

$$k \leq a - b_1(\Gamma) + 1 \leq b \tag{9}$$

if Γ has a presentation with a generators and b relations. The second inequality follows from (3) since $b_2(\Gamma) \geq 1$. One can use (9) to bound the order of the finite group K, but, unlike in (7), this bound is not good enough to show that K is trivial, because we do not have enough control over $\beta_1(\Gamma)$. We therefore resort to an argument using group cohomology with mod p coefficients to bound the deficiency of a group in order to rule out a non-trivial finite K in (8).

Suppose that $\bar{\Gamma} \subset \Gamma$ is a subgroup of finite index d. If Γ has a presentation with a generators and b relations, then, by the Reidemeister–Schreier process, $\bar{\Gamma}$ has a presentation with $\bar{a} = (a-1)d+1$ generators and $\bar{b} = bd$ relations. Therefore

$$\text{def}(\bar{\Gamma}) \geq \bar{a} - \bar{b} = (a - b - 1)d + 1 \ .$$

This shows that the properties of having deficiency ≥ 2, or of having positive deficiency, are preserved under passage to finite index subgroups. Therefore, the triviality of the kernel K in (8) follows from the following:

Lemma 4. (cf. [26, Thm. 1]) *If the finite subgroup K in* (8) *is non-trivial, then Γ has a finite index subgroup of negative deficiency.*

Proof. Suppose K is non-trivial. Let $a \in K$ be an element of prime order p, and $C \subset K$ the cyclic subgroup generated by a. The centralizer $\bar{\Gamma} = C_\Gamma(a)$ has finite index in Γ, and fits into a central extension

$$1 \longrightarrow C \longrightarrow \bar{\Gamma} \longrightarrow \Delta \longrightarrow 1 \ ,$$

where Δ is a finite index subgroup of $\pi_1^{orb}(C_g)$. (For simplicity we may arrange that Δ is an ordinary surface group of positive genus [12].) This central extension corresponds to an Euler class $e \in H^2(\Delta; \mathbb{Z}_p) = \mathbb{Z}_p$. If we pull back to a subgroup $\bar{\Delta} \subset \Delta$ of index p, then e is multiplied by p, and so becomes zero, so that the pulled-back extension splits. So $\bar{\Gamma}$ has a finite index subgroup isomorphic to $C \times \bar{\Delta}$, with $\bar{\Delta}$ a surface group. Now by the Morse inequality for cohomology with $\mathbb{Z}_p$-coefficients and the Künneth formula, we have

$$\text{def}(C \times \bar{\Delta}) \leq \dim_{\mathbb{Z}_p} H^1(C \times \bar{\Delta}; \mathbb{Z}_p) - \dim_{\mathbb{Z}_p} H^2(C \times \bar{\Delta}; \mathbb{Z}_p) = -1 \ .$$

This completes the proof of the Lemma. ☐

We have now proved that Γ is isomorphic to $\pi_1^{orb}(C_g)$ via the homomorphism induced by the Albanese map. Moreover, by (9), the minimal number of relations in any finite presentation of Γ is bounded from below by the number of multiple fibers of the Albanese map. If Γ has a presentation with one relation, then there is at most one multiple fiber (but there may be none). For a two-relator group Γ which is not isomorphic to a one-relator group we have at most two multiple fibers. If there were strictly fewer, then Γ would in fact have a presentation with one relation only, compare Section 2. Therefore, there are exactly two multiple fibers in this case. Proceeding by induction, we see that for all larger values of the minimal number of relations in a presentation of Γ, this number equals the number of multiple fibers of the Albanese. This completes the proof of Theorem 5. ☐

Remark 1. It would be interesting to extend Theorem 5 to groups with $\text{def}(\Gamma) = 1$. The first step is to see that a Kähler manifold X with a fundamental group Γ of deficiency one fibers over a curve of positive genus. The arguments used for Theorems 3 and 5 do not immediately give this. However $\text{def}(\Gamma) = 1$ still gives $b_1(\Gamma) > 0$, and if $b_1(\Gamma) = 2$, then the Albanese map of X is surjective onto an elliptic curve. If $b_1(\Gamma) > 2$, then a result of Amorós [1, Cor. 5.7] shows that X fibers; see also [2, Cor. 3.45].

By looking at a fibration of X over a curve of maximal genus, and considering the curve as an orbifold, we obtain the exact sequence (1) with a finitely generated kernel K. Using $\mathrm{def}(\Gamma) > 0$, we can rule out a finite K as in the proof of Theorem 5 using Lemma 4.

Finally, if $\beta_1(\Gamma) > 0$, then Theorem 4 rules out an infinite K, giving the desired result. However, it is now possible that $\beta_1(\Gamma) = \mathrm{def}(\Gamma) - 1 = 0$. This is the case of equality in Proposition 1, and so Γ would have an aspherical presentation complex. This shows that Γ would be torsion-free of cohomological dimension two, and the subgroup K would be of cohomological dimension ≤ 2. One can actually rule out a K of cohomological dimension one, so that in this case the desired generalization of Theorem 5 follows.

Unfortunately, I do not know how to obtain the conclusion that K is of cohomological dimension one in all cases. It can be obtained from a result of Bieri [7, Thm. 5.5] if one knows that K is finitely presentable, or at least of type FP_2. Therefore, adding, for example, the assumption that the fundamental group is coherent, one can circumvent this problem. There are other assumptions one can add, which also produce the desired conclusion, compare the very recent preprint by Biswas and Mj [10]. However, I believe that none of these additional assumptions are necessary.

5.2. **Limit groups.** The class of limit groups was introduced by Sela [34], and has turned out to be the same as the class of fully residually free groups, see [16, Cor. 3.10]. A group Γ is fully residually free if for every finite set $S \subset \Gamma$ there is a homomorphism $\varphi\colon \Gamma \longrightarrow F_n$ to a free group which is injective on S. As soon as Γ is non-Abelian, this means that Γ surjects onto a non-Abelian free group.

It was shown by Pichot [33] that non-Abelian limit groups Γ have $\beta_1(\Gamma) \geq 1$. This allows us to use the same arguments as in the proof of Theorem 2 to conclude the following:

Theorem 6. *A non-Abelian limit group Γ is a Kähler group if and only if it is isomorphic to a surface group of genus ≥ 2.*

Proof. A non-Abelian limit group Γ admits a surjective homomorphism to a non-Abelian free group. Therefore, if such a Γ is the fundamental group of a compact Kähler manifold X, then by the Siu–Beauville theorem, cf. [2, Ch. 2], X admits a surjective holomorphic map $f\colon X \longrightarrow C_g$ with connected fibers to a curve of genus $g \geq 2$. We now apply Lemma 1 to f. Since Γ has positive first ℓ^2-Betti number [33], the kernel K must be finite by Theorem 4. However, limit groups are torsion-free [16, Prop. 3.1], and so the finite kernel K is actually trivial. Thus $\Gamma = \pi_1^{orb}(C_g)$, and the torsion-freeness implies that there are no points with multiplicity > 1 in C_g, equivalently f has no multiple fibers. Thus Γ is an ordinary surface group.

Conversely, surface groups are indeed limit groups, cf. [16, p. 27]. □

6. The possible deficiencies of Kähler groups

We have seen in Corollary 1 that there are no Kähler groups of positive even deficiency. This raises the question whether there are other constraints on the deficiencies of Kähler groups. In the case of positive deficiency there are no further constraints, since all odd positive integers are indeed deficiencies of Kähler groups because they are deficiencies of surface groups by Lemma 2.

For negative deficiencies we have the following result.

Proposition 4. *Let Σ_g be a closed oriented surface of genus $g \geq 1$, and $p > 0$ a prime number. We have*

$$\begin{aligned}
\mathrm{def}(\pi_1(\Sigma_g) \times \mathbb{Z}^2) &= -2g \ , \\
\mathrm{def}(\pi_1(\Sigma_g) \times \mathbb{Z}_p) &= -1 \ , \\
\mathrm{def}(\pi_1(\Sigma_g) \times \mathbb{Z}^4) &= -3 - 6g \ , \\
\mathrm{def}(\pi_1(\Sigma_g) \times \mathbb{Z}^2 \times (\mathbb{Z}_p)^2) &= -5 - 6g \ , \\
\mathrm{def}(\pi_1(\Sigma_g) \times (\mathbb{Z}_p)^4) &= -7 - 6g \ .
\end{aligned}$$

Furthermore, all these groups are Kähler.

Proof. It is clear that all the groups are Kähler, because they are direct products of fundamental groups of curves and Abelian surfaces, and of finite groups. (All finite groups are Kähler by a general result of Serre, but for $\mathbb{Z}_p$ one does not need this general result.)

The calculation of the deficiency proceeds in the same way for all the examples. On the one hand, there is an obvious presentation in each case, which shows that the deficiency cannot be smaller than the value claimed. On the other hand, the deficiency cannot be larger either, because of the Morse inequality (3). In the cases where finite factors appear, the Morse inequality for cohomology with $\mathbb{Z}_p$ coefficients is used. □

The proposition shows that almost all negative integers are deficiencies of Kähler groups. The only values not covered are -3, -5 and -7. Of these, the first is taken care of by $\mathrm{def}((\mathbb{Z}_p)^3) = -3$. Finally, there are many Kähler groups of deficiency zero, so that -5 and -7 are the only integers for which we have not decided the question whether they are deficiencies of Kähler groups. It is very likely that these two values can be realized, for example by finite groups.

7. Relaxing the Kähler condition

It was proved by Taubes [36] that every finitely presentable group is the fundamental group of a compact complex three-fold. Therefore, it is interesting to look at the fundamental groups of complex manifolds that are not necessarily Kähler, but have Hermitian metrics satisfying geometrically interesting conditions weaker than the Kähler condition, in order to see where the strong restrictions on Kähler groups break down.

Kokarev [27] introduced the class of so-called pluri-Kähler–Weyl manifolds, and in [28] it was proved that non-Abelian free groups cannot be fundamental groups of such manifolds. We now generalize this result by looking at arbitrary groups with deficiency ≥ 2.

It was shown by Ornea–Verbitsky [31] that non-Kähler pluri-Kähler–Weyl manifolds of complex dimension ≥ 3 also have non-Kähler Vaisman structures, meaning that there is a complex structure with a Hermitian metric whose fundamental two-form satisfies $d\omega = \omega \wedge \theta$ for a parallel non-zero one-form θ, cf. [18]. Therefore [27, 28], the class of pluri-Kähler–Weyl manifolds in the sense of [27] consists of the union of the following three classes: Kähler manifolds, arbitrary compact complex surfaces, and manifolds that are, up to diffeomorphism, non-Kähler Vaisman of complex dimension ≥ 3. We now show that for these manifolds there are no additional deficiency ≥ 2 fundamental groups other than those occurring in Theorem 2.

Lemma 5. *For surfaces S with $b_1(S) = 1$ one has $\mathrm{def}(\pi_1(S)) \leq 1$. This bound is sharp.*

Proof. The implication is clear, cf. (3). The bound is sharp because $\mathbb{Z}$ occurs as the fundamental group of certain Hopf surfaces of class VII. □

Proposition 5. *Let S be a non-Kähler elliptic surface with $b_1(S) \geq 3$. Then $\beta_1(\pi_1(S)) = 0$ and $\mathrm{def}(\pi_1(S)) \leq 0$.*

Proof. The elliptic fibration induces an exact sequence of the form

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \pi_1(S) \longrightarrow \pi_1^{orb}(C_g) \longrightarrow 1 ,$$

with $g \geq 1$; cf. [20, Section 2.7.2]. Therefore, by Theorem 4, one has $\beta_1(\pi_1(S)) = 0$, implying $\mathrm{def}(\pi_1(S)) \leq 1$ via (5). In the case of equality in (5), the group $\pi_1(S)$ would have to be of cohomological dimension at most 2 by Proposition 1, which is not possible. In fact, the cohomological dimension is 4, see [2, Cor. 1.41]. □

Proposition 6. *For a non-Kähler Vaisman manifold V one has $\beta_1(\pi_1(V)) = 0$ and $\mathrm{def}(\pi_1(V)) \leq 1$. This bound is sharp.*

Proof. By definition, a non-Kähler Vaisman manifold V carries a parallel non-zero one-form θ, and therefore is the total space of a smooth fiber bundle over the circle. Thus, its fundamental group fits into an extension of the form

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(V) \longrightarrow \mathbb{Z} \longrightarrow 1 ,$$

with F the fiber of the fibration over S^1. If $\pi_1(F)$ is finite, then $\beta_1(\pi_1(V)) = 0$ follows from $\beta_1(\mathbb{Z}) = 0$ by Lemma 3. If $\pi_1(F)$ is infinite, then $\beta_1(\pi_1(V)) = 0$ follows from Theorem 4. Now the vanishing of $\beta_1(\pi_1(V))$ implies $\operatorname{def}(\pi_1(V)) \leq 1$ by (5). This bound is sharp because $\mathbb{Z}$ occurs as the fundamental group of certain Vaisman manifolds of Hopf type. □

Putting together these observations, we see that Theorems 2 and 6 extend in the following way:

Corollary 2. *If Γ is either a group with $\operatorname{def}(\Gamma) \geq 2$ or a non-Abelian limit group and Γ is the fundamental group of a compact pluri-Kähler–Weyl manifold X, then X is Kähler.*

Proof. A compact complex surface is Kähler if and only if its first Betti number is even [5]. Thus we have to consider surfaces with odd b_1. By the Kodaira classification surfaces S with odd first Betti number ≥ 3 are elliptic, see [2, 5].

The vanishing of their first ℓ^2-Betti number shows that fundamental groups of elliptic surfaces with $b_1 \geq 3$ or of non-Kähler Vaisman manifolds cannot have deficiency ≥ 2, or be non-Abelian limit groups.

Groups with $b_1 = 1$ have deficiency at most one, and they cannot surject to non-Abelian free groups. □

Finally, we note that Proposition 6 has the following consequence:

Corollary 3. *The fundamental group of a Vaisman manifold has finitely many ends.*

Proof. As is now well-known, groups with infinitely many ends have positive first ℓ^2-Betti number, cf. [2, 4]. Thus a Vaisman manifold with a fundamental group with infinitely many ends would have to be Kähler by Proposition 6. However, in the Kähler case the number of ends is at most one by Gromov's result [24]; cf. also [2, 4]. □

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