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COHOMOLOGICAL FINITENESS CONDITIONS FOR A CLASS OF METABELIAN GROUPS

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We consider a class of finitely presented metabelian groups and the cohomological finiteness conditions they satisfy. It was the work of Baumslag and Remeslennikov in the 1970s which suggested that the theory of finitely presented soluble groups might be richer than once thought. For example, it was demonstrated in [2] that there exists a finitely presented metabelian group with a free abelian derived group of infinite rank. Remeslennikov [22] found the same example independently. This could be compared with the fact, proved in [3], that every finitely generated metabelian group can be embedded in a finitely presented metabelian group. Recall that a group G is said to be metabelian if there exists a short exact sequence

$$A \hookrightarrow G \twoheadrightarrow Q \tag{1}$$

of groups with A and Q abelian. Hence metabelian groups are precisely the soluble groups of derived length two. We think of A as a $\mathbb{Z}Q$-module via conjugation; i.e. we define $a \circ q = g^{-1}ag$ where $g \mapsto q$ under the projection map $\pi : G \twoheadrightarrow Q$. Since A is abelian, this is well-defined. Then G is finitely generated if and only if Q is a finitely generated group and A is finitely generated as a $\mathbb{Z}Q$-module. From now on we assume that G is finitely generated.

In 1980 Bieri and Strebel [13] characterized finitely presented metabelian groups in terms of the invariant Σ_A. This is defined in the following way. A *character* of Q is a non-zero group homomorphism $v : Q \to \mathbb{R}$; two characters are equivalent if they are positive real multiples of each other. The set $S(Q)$ of all equivalence classes $[v]$ of characters of Q can be identified with the unit sphere $S^{n-1} \subset \mathbb{R}^n \cong \mathrm{Hom}(Q, \mathbb{R})$, where n is the $\mathbb{Z}$-rank of Q. Let $Q_v = \{q \in Q : v(q) \geq 0\}$. This is a monoid, so we can form the monoid ring $\mathbb{Z}Q_v$. We then associate to every finitely generated $\mathbb{Z}Q$-module A the set

$$\Sigma_A = \{[v] \in S(Q) : A \text{ is finitely generated as a } \mathbb{Z}Q_v\text{-module } \}.$$

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The $\mathbb{Z}Q$-module A is then said to be *m-tame* if every m-point subset of $\Sigma_A^c = S(Q) \setminus \Sigma_A$ lies in an open hemisphere of $S(Q)$. Equivalently, $v_1 + \ldots + v_m \neq 0$ for any $[v_1], \ldots, [v_m] \in \Sigma_A^c$.

A group G is said to be of type FP_m if there is a $\mathbb{Z}G$-projective resolution of the trivial module $\mathbb{Z}$ where the modules are finitely generated in dimensions $\leq m$. In particular a group is of type FP_1 if and only if it is itself finitely generated. Bieri and Strebel [13] proved that the properties of being of type FP_2 and being finitely presented are equivalent for metabelian groups. It is not known whether this is true for all soluble groups, but it is certainly not true for groups in general: for example, the right-angled Artin groups in [6] are of type FP_2 but not finitely presented. In addition to this, Bieri and Strebel showed that G is finitely presented if and only if the corresponding $\mathbb{Z}Q$-module A is 2-tame. These results led to the FP_m-conjecture of Bieri and Groves.

The FP_m-Conjecture. [9] Suppose we have the short exact sequence (1) and let m be a positive integer. Then G is of type FP_m if and only if A is m-tame as a $\mathbb{Z}Q$-module.

Both directions of the FP_m conjecture remain open for $m > 2$ but a number of specific cases have been proved. In [9] it was shown that if G is of type FP_m then $A \otimes_{\mathbb{Z}} K$ is m-tame as a KQ-module for every field K. In [1] Hans Åberg established the full FP_m-conjecture for the case where G has finite Prüfer rank. Kochloukova [18] extended Åberg's methods to show that the 'only-if' part of the conjecture holds true if either the additive group of A is torsion or if G is the split extension of A by Q. In [18] it is also shown that the full conjecture is true for the case where A is torsion and of Krull dimension 1 as a $\mathbb{Z}Q$-module. More recently Bieri and Harlander [12] proved the FP_3-conjecture for the case where G is the split extension of A by Q.

Baumslag's example [2] of a finitely presented metabelian group with a free abelian normal subgroup of infinite rank was the group G_1 generated by the 2×2-matrices

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, s = \begin{pmatrix} 1+x & 0 \\ 0 & 1 \end{pmatrix}, t = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

over $\mathbb{Q}(x)$. This is a 3-generator 3-relator group with presentation

$$G_1 = \langle a, s, t : [s,t] = [a^t, a] = 1, a^s = aa^t \rangle.$$

Fix a positive integer $n > 1$. To this group we add the generators

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i+x & 0 \\ 0 & 1 \end{pmatrix}$$

for all $2 \leq i \leq n$. Using methods similar to Baumslag's it can be shown that the resulting metabelian group G_n has a finite presentation and that its derived subgroup is free abelian of infinite rank. We know that G_n is isomorphic to the split extension $A_n \rtimes Q_n$, where

$$A_n = \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, \frac{1}{n!}\right]$$

and Q_n is the free abelian group with basis $\{q_{-1}, q_0, q_1, \ldots, q_n\}$ and an action $\circ$ on A_n given by

$$\begin{aligned} a \circ q_{-1} &= (n!)a \\ a \circ q_0 &= xa \\ a \circ q_1 &= (1+x)a \\ &\vdots \\ a \circ q_n &= (n+x)a \end{aligned}$$

for all $a \in A$. This action turns A_n into a cyclic $\mathbb{Z}Q_n$-module. Now A_n and Q_n embed as subgroups of G_n and so we can think of this action as conjugation in G_n by the fact that

$$(1, a \circ q) = (q, 0)^{-1}(1, a)(q, 0).$$

The $\mathbb{Z}Q_n$-module A_n was first studied by Baumslag and Stammbach [5], who showed that the exterior power $\bigwedge_{\mathbb{Z}}^i A_n \cong H_i(A_n, \mathbb{Z})$ is a non-free finitely generated $\mathbb{Z}Q_n$-module for $1 \leq i \leq n$, is a free $\mathbb{Z}Q_n$-module of finite rank for $i = n+1$ and is free of infinite rank for $i \geq n+2$. By studying the action of G_n on a subcomplex of a product of trees indexed by a finite set of discrete (i.e. image $\mathbb{Z}$) characters of G_n we can show that G_n is a group of type FP_{n+1}. In fact we prove a more general result from which this can be deduced as a corollary.

Theorem A. *Let $Q = Q_0 \times Q_1 \times \ldots \times Q_l$ be a finitely generated free abelian group, where $Q_0 = \langle q_{-1} \rangle$ and Q_i is a free abelian group with basis $\{q_{i,j}\}_{0 \leq j \leq z_i}$ for $1 \leq i \leq l$. Let A be a finitely generated (right) $\mathbb{Z}Q$-module and assume that the action of $\mathbb{Z}Q$ on A factors through an action of a quotient $M = M_0 \otimes M_1 \otimes \ldots \otimes M_l$, where $M_i = \mathbb{Z}Q_i/I_i$, $I_0 = \langle q_{-1} - k \rangle$ where k is some positive integer and, for $1 \leq i \leq l$, I_i is generated as an ideal by $\{q_{i,j} - f_{i,j}\}_{0 \leq j \leq z_i}$, where for fixed i the $f_{i,j}$ are irreducible non-constant monic polynomials in $\mathbb{Z}[q_{i,0}]$ that are pairwise coprime in $\mathbb{Z}[q_{i,0}, 1/k]$*

and $f_{i,0} = q_{i,0}$*. Assume further that* A *is free as an* M*-module. Then the split extension* G *of* A *by* Q *is of type* FP_m*, where* $m = \min\{rk(Q_i) : 1 \le i \le l\}$*.*

Corollary. *The group* G_n *is of type* FP_{n+1}*.*

Proof. Note that no two of the polynomials $x, 1+x, \ldots, n+x$ generate a proper ideal in $\mathbb{Z}[x, 1/n!]$. Take $l = 1$, $z_1 = n$, $q_{1,j} = q_j$, $f_{1,j} = x + j$ and $k = n!$. Then $M = M_0 \otimes M_1 \cong A_n$. □

The method of our proof originated with Åberg [1] and closely follows that of a similar result by Groves and Kochloukova ([17], Theorem 5), of which ours is a generalization. In essence we have taken their result and showed it to be true when we invert an appropriate integer k as well as inverting a set of polynomials that are pairwise coprime (in the sense that no two of them lie in a proper ideal of $\mathbb{Z}[q_{i,0}]$.) The set of characters that we choose in our proof has the property that the set of equivalence classes of the restrictions of each character to Q is contained in Σ^c_A. After constructing a G-tree for each character, and taking the product X of these, we prove three properties of an appropriate subspace Y of X:

(i) G acts co-compactly on Y;
(ii) Y is $(m-1)$-connected;
(iii) the stabilizers in G of cells in Y are of type FP_m.

That G is of type FP_m is then implied by a criterion of Brown [15]. Hence the group G_n is of type FP_{n+1}. It follows from Kochloukova's 'only if' result [18] that A_n is $(n+1)$-tame as a $\mathbb{Z}Q_n$-module, but by our choice of characters we see that it is not $(n+2)$-tame and so G_n is not of type FP_{n+2}. In this way we have further evidence that the FP_m-conjecture may be true. Theorem B below aims to provide a short direct way of establishing that A_n is $(n+1)$-tame and thus giving self-contained evidence which supports the conjecture.

Consider again $A_n = \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, 1/n!\right]$. Since A_n is $(n+1)$-tame it follows from ([9], p. 377) that $\bigotimes^i_{\mathbb{Z}} A_n$, as well as $\bigwedge^i_{\mathbb{Z}} A_n$, is a finitely generated $\mathbb{Z}Q_n$-module for $1 \le i \le n+1$. In [9] it was shown that finite generation of $\bigotimes^i_K M$, finite generation of $\bigwedge^i_K M$ and m-tameness are all equivalent whenever $1 \le i \le m$, K is a field, Q is a finitely generated abelian group and M is a finitely generated KQ-module. This is not true in general when $K = \mathbb{Z}$. However since we know that A_n possesses all three properties for $m = n+1$ we would like to be able to prove directly (without appealing to the Bieri-Groves conjecture) that A_n is $(n+1)$-tame. For A_1 we can use the method of [14], §5, since A_1 has principal annihilator ideal, to show that $\Sigma^c_{A_1}$ is a 3-point subset of the circle and that none of the points are antipodal. For $n = 2$ we use the characterisation ([13], Proposition 2.1) of a finitely generated Q_v-module in terms of its centralizer and also the Bieri-Groves formula ([10], Theorem 8.1) to calculate $\Sigma^c_{A_2}$ explicitly as a 1-dimensional subset of the 3-sphere. We have not been able to give a complete description of $\Sigma^c_{A_n}$ for $n > 2$ but the following result shows that it is indeed always $(n+1)$-tame.

Theorem B. *Let v_i be the character of Q_n defined by $v_i(q_i) = 1, v_i(q_j) = 0$ for $i \neq j$, $-1 \leq i \leq n$. These form a basis for $\mathrm{Hom}(Q, \mathbb{R})$ and so an arbitrary character is of the form $v = k_1 v_{-1} + k_0 v_0 + \ldots + k_n v_n$ for some $k_i \in \mathbb{R}$. Then $[v] \in \Sigma^c_{A_n}$ implies that either (i) $k_{-1} > 0$ or (ii) $k_{-1} = 0$ and $(k_0, k_1, \ldots, k_n)$ is a row of the matrix*

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & & \\ 0 & 0 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}.$$

Moreover all $[v]$ with $(k_0, \ldots, k_n)$ as in (ii) are elements of $\Sigma^c_{A_n}$. In particular A_n is $(n+1)$-tame.

One consequence of the Bieri-Strebel theorems is that whether or not a metabelian group G is finitely presented depends only upon the Q-module A, and not on the extension class in $H^2(Q, A)$. In particular, G is finitely presented if and only if the split extension $A \rtimes Q$ is finitely presented. Using the Lyndon-Hochschild-Serre spectral sequence we have been able to show that $H^2(Q_n, A_n) = 0$ and so *every* extension of A_n by Q_n is split. In general we have the following.

Theorem C. *Let A, Q and the positive integer k be as in Theorem A with $l = 1$, and let $z_1 = n$, $q_{1,j} = q_j$ and $f_{1,j} = f_j$. Then $H^2(Q, A) = \mathbb{Z}/(k-1)\mathbb{Z}$ whenever $f_j(1) \equiv 1 \mod (k-1)$ for all $1 \leq j \leq n$, and is trivial otherwise.*

1. Character trees

Suppose that a group G is the split extension of a finitely generated free abelian group Q by a finitely generated Q-module A. For each non-zero character $v : G \to \mathbb{R}$ we can construct a tree associated to v. In this paper we shall make some assumptions concerning the character v. First, that it is discrete i.e. its image is $\mathbb{Z}$. Furthermore, we assume that $v(A) = 0$ and that Q has a basis consisting of a basis of $\mathrm{Ker}\ (v|_Q)$ together with an element q_v such that $v(q_v) = 1$.

Choose a set of elements $a_1, \dots, a_d$ which generates A as a right $\mathbb{Z}Q$-module. Define Q_v to be the submonoid $\{q \in Q : v(q) \geq 0\}$ of Q and A_v to be the $\mathbb{Z}Q_v$-submodule of A generated by $a_1, \dots, a_d$. Fix some negative integer β and let $G(v)$ be the subgroup of G generated by $A_v \circ {q_v}^{\beta}$ and Ker $(v|_Q)$. Note that $G(v) \cap A = A_v \circ {q_v}^{\beta}$ and that since $q_v \in Q_v$ we have ${q_v}^{-1} G(v) q_v \subseteq G(v)$. Hence $G(v) q_v \subseteq q_v G(v)$. Define $G(v)^+$ to be the submonoid of $G(v)$ generated by q_v and the elements of $G(v)$. Then we have

$$G(v)^+ = \bigcup_{z \geq 0} {q_v}^z G(v).$$

The $\supseteq$ direction is easy; that $G(v)^+ \subseteq \bigcup_{z\geq 0} {q_v}^z G(v)$ relies on the observation that if $y = {q_v}^{k_1} g_1 {q_v}^{k_2} g_2 \dots {q_v}^{k_m} g_m$ is an arbitrary element of $G(v)^+$, with the k_i nonnegative integers and the g_i elements of $G(v)$, then

$$y \in {q_v}^{k_1+k_2+\dots k_m} G(v).$$

Let T_0 be the right G-set $G/G(v) = \{G(v)g : g \in G\}$ with action given by right multiplication. We define an ordering $\leq$ on T_0 by $G(v)g \leq G(v)h$ if and only if $h \in G(v)^+ g$.

Lemma 1.1. *$\leq$ is a partial order on T_0.*

Proof. It is straightforward to check that $\leq$ is well-defined and transitive so we can move straight on to proving that, given $G(v)g \leq G(v)h$ and $G(v)h \leq G(v)g$, we have $G(v)g = G(v)h$ i.e. that $gh^{-1} \in G(v)$. Write $(G(v)^+)^{-1}$ for the set of inverses in G of the elements of $G(v)^+$. Since $gh^{-1} = (hg^{-1})^{-1} \in (G(v)^+)^{-1}$ we have $gh^{-1} \in G(v)^+ \cap (G(v)^+)^{-1}$. But $G(v)^+ \cap (G(v)^+)^{-1} = G(v)$ because each element of $G(v)$ is a generator for $G(v)^+$ and $q_v \notin (G(v)^+)^{-1}$. □

As $G(v) \subseteq$ Ker v, the character $v : G \to \mathbb{Z}$ factors through a map $h_v : T_0 \to \mathbb{Z}$ i.e. $v(g) = h_v(G(v)g)$ for all $g \in G$. We call two elements $\alpha \leq \gamma$ of T_0 *neighbours* if $h_v(\gamma) - h_v(\alpha) = 1$ and link every pair $\alpha \leq \gamma$ of neighbours with a closed unit interval. We obtain a graph Γ_v with vertex set T_0 and in which two vertices are joined by an edge if and only if the corresponding elements of T_0 are neighbours. We think of the edges as being the constructed unit intervals and extend $h_v : T_0 \to \mathbb{Z}$ to an $\mathbb{R}$-linear map $h_v : \Gamma_v \to \mathbb{R}$ by setting $h_v((1-t)\alpha + t\gamma) = (1-t)h_v(\alpha) + t h_v(\gamma) = h_v(\alpha) + t$ whenever $\alpha \leq \gamma$ are neighbours.

If $\alpha \leq \gamma$ are neighbours then αg and γg are also neighbours. We make G act on Γ_v by setting $((1-t)\alpha + t\gamma) * g = (1-t)(\alpha g) + t(\gamma g)$. For any $g \in G$ we define $L_{g,v}$ to be the subgraph of Γ_v spanned by the set of vertices $\{G(v){q_v}^z g\}_{z\in\mathbb{Z}}$. Any two of these vertices are distinct, since assuming otherwise forces us to conclude that $q_v \in G(v)$, a contradiction. Note that $G(v){q_v}^z g$ and $G(v){q_v}^{z+1}g$ are neighbours and so $L_{g,v}$ is a line. Indeed two vertices $G(v){q_v}^m g$ and $G(v){q_v}^n g$ in $L_{g,v}$ are neighbours if and only if $|m-n| = 1$. The image under h_v of a point $(1-t)G(v){q_v}^z g + tG(v){q_v}^{z+1}g$ on the edge joining $G(v){q_v}^z g$ and $G(v){q_v}^{z+1}g$ in $L_{g,v}$ is $n+z+t$, where $n = v(g)$. Hence the restriction $h_v : L_{g,v} \to \mathbb{R}$ is bijective: for a fixed integer n, every non-integral real number r can be written in precisely one way as

$$r = n + z + t$$

where z is some integer and $t \in (0,1)$. If $m \in \mathbb{Z}$ then $m = n + (m-n-1) + 1 = n + (m-n) + 0$ but the pairs $(m-n-1, 1)$ and $(m-n, 0)$ for (z,t) give the same point in $L_{g,v}$.

Let $K = \text{Ker}\ (v\ |_Q)$. Then since ${q_v}^z K = \{q \in Q : v(q) = z\}$ for any integer z we have

$$Q = \bigcup_{z \in \mathbb{Z}} {q_v}^z K.$$

Hence every element $g \in G = QA$ can be expressed in the form $g = q{q_v}^z a$ where $q \in K$, $a \in A$. But $K \subseteq G(v)$ and so we conclude that

$$G = \bigcup_{z \in \mathbb{Z}} G(v){q_v}^z A. \tag{2}$$

It follows immediately that

$$\Gamma_v = \bigcup_{a \in A} L_{a,v}.$$

We write $[(a,r)]$ where $a \in A$ and $r \in \mathbb{R}$ for the unique element of $L_{a,v}$ whose image under h_v is r. By the definition of the G-action on Γ_v we have, for any two elements $a, b \in A$,

$$[(a,r)] * b = [(a+b, r)].$$

Lemma 1.2 ([18], Lemma 2.1)**.** *For every $a, b \in A$ we have $L_{a,v} \cap L_{b,v} = \{[(a,r)] = [(b,r)] : r \leq z_0\}$, where $z_0 = \sup\{z : a - b \in A_v \circ {q_v}^{\beta+z}\}$. In other words, the intersection of any two lines labelled by different elements of A is either a line or a ray i.e. it may have a defined starting point.*

Proof. Let r be a non-integral real number. Then r can be written uniquely as $z+t$ for some integer z and $t \in (0,1)$. If a, b are elements of A then $[(a,r)] = [(b,r)]$ if and only if $G(v){q_v}^z a = G(v){q_v}^z b$; i.e., if and only if $(a-b) \circ {q_v}^{-z} \in G(v) \cap A = A_v \circ {q_v}^{\beta}$. The result follows from the fact that if $z \leq z_0$ then $a - b \in A_v \circ {q_v}^{z_0+\beta} \subseteq A_v \circ {q_v}^{z+\beta}$, since $A_v \circ q_v \subseteq A_v$. □

From this we conclude that Γ_v is a tree.

2. Åberg's complex

We now choose V to be a set of $n+2$ discrete characters of G, where n is some natural number. Suppose that the characters in V satisfy the assumptions of the previous section. Define $X = \prod_{v\in V} \Gamma_v$ and a map $h : X \to \mathbb{R}^{n+2}$ given by

$$h\left(\prod_{v\in V}[(a_v, r_v)]\right) = \prod_{v\in V} h_v([(a_v, r_v)]) = \prod_{v\in V} r_v.$$

Let $f : Q \otimes_{\mathbb{R}} \mathbb{R} \to \mathbb{R}^{n+2}$ be the $\mathbb{R}$-linear map extending the map

$$q \mapsto \prod_{v\in V} v(q)$$

and let W be the image of f. We then define

$$Y = \{x \in X : h(x) \in W\}.$$

We can put CW-complex structures on X and its subspace Y (see [18], page 1181), and define the diagonal action of G on X by

$$\left(\prod_{v\in V}[(a_v, r_v)]\right) * g = \prod_{v\in V}[(a_v, r_v)] * g.$$

Lemma 2.1 ([18], Lemma 2.2)**.** *The subspace Y of X is invariant under this G-action.*

Proof. Let $\prod_{v\in V}[(a_v, r_v)]$ be a point of Y and let g be an element of G. Then $\left(\prod_{v\in V}[a_v, r_v]\right) * g = \prod_{v\in V}[(a'_v, r_v + v(g))]$ where $G(v){q_v}^{\lceil r_v\rceil} A_v g = G(v){q_v}^{v(g)+\lceil r_v\rceil} a'_v$ for some element a'_v of A and where $\lceil r_v\rceil$ is the least integer greater than or equal to r_v. If $q = \pi(g)$ then $v(g) = v(q)$ and so $\prod_{v\in V} v(g) \in W$. Hence $\prod_{v\in V}(r_v + v(g)) \in W$ and so

$$\prod_{v\in V}[(a'_v, r_v + v(g))] \in Y.$$

□

3. The construction of the set of characters V

From now on we are in the situation described in Theorem A. We shall write q_j for $q_{1,j}$, f_j for $f_{1,j}$ and $z_1 = n$ so that Q_1 has rank $n+1$. Hence $n+1 \geq m$. Let $\widetilde{M} = M_0 \otimes M_1$; then

$$\widetilde{M} \cong \mathbb{Z}\left[q_0, {q_0}^{-1}, f_1^{-1}, \ldots, f_n^{-1}, k^{-1}\right]$$

where q_j acts as multiplication by f_j for $j > 1$ and q_{-1} acts as multiplication by the positive integer k. Let $\widetilde{Q} = Q_0 \times Q_1$ and let $\phi : \mathbb{Z}\widetilde{Q} \to \widetilde{M}$ be the ring homomorphism sending q_{-1} to k and q_i to f_i for $0 \leq i \leq n$. The restriction of ϕ to $\widetilde{Q}$ is a group homomorphism τ from the free abelian group $\widetilde{Q}$ to the group of units $(\widetilde{M})^\times$ of $\widetilde{M}$.

Lemma 3.1. *τ is injective.*

Proof. Suppose $q = {q_{-1}}^{l_{-1}} {q_0}^{l_0} \ldots {q_n}^{l_n} \in \operatorname{Ker} \tau$. We then have

$$k^{l_{-1}} q_0^{l_0} f_1^{l_1} \ldots f_z^{l_n} = 1.$$

Suppose that $l_{i_1}, \ldots, l_{i_k}$ are precisely the positive powers occurring on the LHS and $l_{j_1}, \ldots, l_{j_t}$ are the negative powers. Then, assuming without loss that l_{-1} is positive, we get

$$k^{l_{-1}} f_{i_1}^{l_{i_1}} \ldots f_{i_k}^{l_{i_k}} = f_{j_1}^{l_{j_1}} \ldots f_{j_t}^{l_{j_t}}.$$

Since $\mathbb{Z}[q_0]$ is a unique factorization domain, and $q_0, f_1, \ldots, f_n$ are all irreducible, then by writing k as a product of its prime factors we see that the powers on both sides must all be zero. Hence $q = 1$.

□

So we may identify $\widetilde{Q}$ with its isomorphic copy in $(\widetilde{M})^\times$. We shall think of $\widetilde{M}$ as a subring of the field of rational functions $\mathbb{Q}(q_0)$.

Definition 3.2. Let R be a non-trivial commutative ring and let $\mathbb{R}_\infty$ denote the set of real numbers together with an additional point ∞. Then a *valuation* on R is a map $v : R \to \mathbb{R}_\infty$ satisfying $v(0) = \infty$, $v(1) = 0$, $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in R$.

We then define v_i, for each $0 \leq i \leq n$, to be the f_i-adic valuation on $\mathbb{Q}(q_0)$; that is, the unique valuation on $\mathbb{Q}(q_0)$ that is zero on $\mathbb{Q} \setminus \{0\}$ and satisfies $v_i(f_i) = 1$. In addition we define, for polynomials $g, h \in \mathbb{Z}[q_0]$,

$$w(g/h) = \deg(h) - \deg(g)$$

and it is easily seen that this too is a valuation. So w and the v_i are group homomorphisms into the group of rational integers $\mathbb{Z}$ and their restrictions to $\widetilde{Q}$ provide us with a set V of $n+2$ discrete characters of $\widetilde{Q}$. If in addition we define $v(Q_i) = 0$ when $i > 1$ for each $v \in V$ then we get a set of discrete characters of the group Q. Note that $v(Q_0) = 0$. Each $v \in V$ can be extended to a character of G via composition with the natural projection $\pi : G \twoheadrightarrow Q$.

Using the method described in Section 2 we can now construct a tree Γ_v corresponding to each $v \in V$. This construction depends on the choice of an element $q_v \in Q$ satisfying $v(q_v) = 1$. Here we fix $q_{v_i} = q_i$ for $0 \leq i \leq n$ and $q_w = {q_0}^{-1}$. In addition we set

$$\beta = -\left(2 + \sum_{i=0}^{n} d_i\right)$$

where $d_i = \deg(f_i)$. We must also choose a set of generators $a_1, \ldots, a_d$ of A as a Q-module that is a basis of A as a free M-module. With this we have built $n+2$ trees $\Gamma_w, \Gamma_{v_0}, \ldots, \Gamma_{v_n}$ and can then get the Åberg complex Y described in Section 2.

4. G acts cocompactly on Y

Our aim is to prove three important properties of the CW-complex Y:

(i) G acts co-compactly on Y;
(ii) Y is $(m-1)$-connected, that is Y is path connected and the homotopy groups

$$\pi_1(Y), \ldots, \pi_{m-1}(Y)$$

are trivial; and
(iii) the stabilizers in G of cells in Y are of type FP_m.

We can then apply Brown's criterion below to see that G is of type FP_m.

Brown's criterion. ([15], Proposition 1.1) *Let G be a group acting on a CW-complex Y via permutation of the cells and such that for every cell the stabilizer of the cell fixes the vertices pointwise. Assume further that Y is $(m-1)$-acyclic, the stabilizers in G of cells of dimension $i \leq m$ are always of type FP_{m-i} and G acts co-compactly on Y. Then G is of homological type FP_m.*

By the Hurewicz theorem the fact that Y is $(m-1)$-connected implies that it is $(m-1)$-acyclic.

First we prove (i). Observe that, as a real vector space, W is spanned by elements in $\mathbb{Z}^{n+2}$ of the form

$$(-(d_0m_0 + d_1m_1 + \ldots + d_nm_n), d_0m_0, d_1m_1, \ldots, d_nm_n),$$

since the elements of $\widetilde{Q}$ are of the form ${q_{-1}}^{m_{-1}}{q_0}^{m_0} \ldots {q_n}^{m_n}$, and so

$$W = \{(y_{-1}, y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+2} : y_{-1} + \sum_{i=0}^{n} d_i y_i = 0\}.$$

Definition 4.1. For a subset B of $\prod_{v\in V} \mathbb{R}$ we define

$$[[B]] = \left\{ \prod_{v\in V} \lceil b_v \rceil : \prod_{v\in V} b_v \in B \right\}$$

where $\lceil b_v \rceil$ denotes the least integer greater than or equal to b_v.

Lemma 4.2 ([17], Lemma 7)**.** *If $\prod_{v\in V} s_v \in [[W]]$, then*

$$0 \le s_w + \sum_{i=0}^{n} d_i s_{v_i} < 1 + \sum_{i=0}^{n} d_i.$$

Hence Q acts co-finitely on $[[W]]$.

Proof. Let $(s_w, s_{v_0}, \ldots, s_{v_n}) \in [[W]]$. Then there exists an element $(r_w, r_{v_0}, \ldots, r_{v_n}) \in W$ such that $r_v \le s_v < r_v + 1$ for all $v \in V$. Hence

$$0 = \sum_{v\in V} r_v \le \sum_{v\in V} s_v < (r_w + 1) + \sum_{i=0}^{n} d_i(r_{v_i} + 1) = 1 + \sum_{i=0}^{n} d_i.$$

Now observe that $q \in Q$ acts on $\mathbb{R}^{n+2}$ via translation with $\prod_{v\in V} v(q)$. Since $v(Q_i) = 0$ for $i > 1$ we can suppose $q = q_{-1}^{k_{-1}}{q_0}^{k_0} \ldots {q_n}^{k_n}$, then

$$\begin{aligned}(s_w, s_{v_0}, \ldots, s_{v_z}) \circ q &= (s_w + w(q), s_{v_0} + v_0(q), \ldots, s_{v_z} + v_z(q)) \\ &= (s_w - (k_0 + \ldots + k_n), s_{v_0} + k_0, \ldots, s_{v_n} + k_n).\end{aligned}$$

We see from this that Q acts transitively on the last $n+1$ coordinates of points in $[[W]]$: we can get any integers in these coordinates by appropriate choice of $k_0, k_1, \ldots, k_n$. Hence each Q-orbit in $[[W]]$ contains some point with the last $n+1$ coordinates equal to zero, so we can take $s_{v_0} = s_{v_1} = \ldots = s_{v_n} = 0$. But then the above inequalities bound s_w as an integer in the interval $[0, 1 + \sum_{i=0}^{n} d_i)$ and so there can be only finitely many Q-orbits. □

For $v \in V$ let $(Q_1)_v$ be the submonoid of Q_1 of all elements q with $v(q) \ge 0$ and let $(M_1)_v$ be the $\mathbb{Z}(Q_1)_v$-submodule of M_1 generated by the image of 1_{Q_1} in M_1.

Lemma 4.3 ([17], Lemma 6). *For $v \in V$ we have $(M_1)_v = \{m \in M_1 : v(m) \geq 0\}$.*

Proof. Observe that, since $M_1 = \mathbb{Z}[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}]$, any element $m \in M_1$ can be written as

$$m = h(q_0, q_0^{-1}) f_1^{\alpha_1} \ldots f_n^{\alpha_n} = 1.h(q_0, q_0^{-1}) q_1^{\alpha_1} \ldots q_n^{\alpha_n} = 1.h(q_0, q_0^{-1}) q,$$

say, where the α_i are some integers, $h \in \mathbb{Z}[q_0, q_0^{-1}]$ and no f_i is a factor of h for $i \geq 1$. Then, since v is a valuation on $\mathbb{Q}(q_0)$, we have

$$v(m) = v(h(q_0, q_0^{-1})q) = \min\{v(q_0^i q)\}$$

where i runs through the powers of q_0 appearing in the monomials of h. But each $q_0^i q \in (Q_1)_v$ if and only if $m \in (M_1)_v$ and so $v(m) \geq 0$ if and only if $m \in (M_1)_v$. □

Theorem 4.4 (c.f. [17], Theorem 6.). *For every $\prod_{v \in V} s_v$ in $[[W]]$ and every set $\{a_v \in A : v \in V\}$, there exists an element $a \in A$ such that $\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a, s_v)]$. Consequently for every $\prod_{v \in V} r_v \in W$ such that $[[\prod_{v \in V} r_v]] = \prod_{v \in V} s_v$ one has $\prod_{v \in V}[(a_v, r_v)] = \prod_{v \in V}[(a, r_v)]$.*

Proof. Since A is free as an M-module, it is sufficient to prove that the result holds when A is cyclic as an M-module. So suppose $A = M$. Then A is a cyclic $\mathbb{Z}Q$-module (since $M = \mathbb{Z}Q/I$ for some ideal I) and so is generated by the image of 1_Q in $\mathbb{Z}Q$. Hence by definition A_v is the $\mathbb{Z}Q_v$-submodule of A generated by the image of 1_Q in $M = A$, that is

$$A_v = M_0 \otimes (M_1)_v \otimes M_2 \otimes \ldots \otimes M_l.$$

We now observe that if the theorem is true for any G-translate of $\prod_{v \in V}[(a_v, s_v)]$, then it is true for $\prod_{v \in V}[(a_v, s_v)]$ itself. To see this, note that $(\prod_{v \in V}[(a_v, s_v)]) * g = (\prod_{v \in V} G(v) q_v^{s_v} a_v) * g = \prod_{v \in V}[(a'_v, s_v + v(g))]$, where $G(v) q_v^{s_v} a_v g = G(v) q_v^{s_v + v(g)} a'_v$ for some $a'_v \in A$. Suppose that there exists $a' \in A$ with

$$\prod_{v \in V}[(a', s_v + v(g))] = \prod_{v \in V}[(a'_v, s_v + v(g))].$$

Then $(\prod_{v \in V}[(a'_v, s_v + v(g))]) * g^{-1} = (\prod_{v \in V}[(a', s_v + v(g))]) * g^{-1}$, i.e.

$$\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a'', s_v)]$$

where $G(v) {q_v}^{s_v} a'' = G(v) {q_v}^{s_v + v(g)} a' g^{-1}$ for some $a'' \in A$. Now, since Q acts transitively on the last $z+1$ coordinates of $[[W]]$, we can find a Q-translate of $\prod_{v \in V} s_v$ in which each of these last $z+1$ coordinates is zero. By Lemma 4.2 we have

$$0 \leq s_w < -\beta.$$

Without affecting this latter property we can also find an A-translate of $\prod_{v\in V}[(a_v, s_v)]$ in which $a_w = 0$, namely $(\prod_{v\in V}[(a_v, s_v)]) * (-a_w)$. Thus we may assume that $s_v + \beta < 0$ for each $v \in V$ and that $a_w = 0$.

Let $a \in A$. Then

$$\prod_{v\in V}[(a_v, s_v)] = \prod_{v\in V}[(a, s_v)]$$

if and only if

$$a - a_v \in A_v \circ {q_v}^{s_v+\beta} \text{ for each } v \in V.$$

As the q_v act trivially on M_0 and on each M_i with $i > 1$ it will suffice to consider the case in which each $a_v \in M_1$ and we seek $a \in M_1$. Here we are identifying M_1 with the subring $\mathbb{Z} \otimes M_1 \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \ldots \otimes \mathbb{Z}$ of $M = M_0 \otimes M_1 \otimes \ldots \otimes M_l$. Now $s_v + \beta < 0$ for each v and so, since $A_v \circ q_v \subseteq A_v$, we have $A_v \subseteq A_v \circ {q_v}^{s_v+\beta}$. Therefore by Lemma 4.3 it will suffice to find an $a \in A$ so that $v(a - a_v) \geq 0$ for each v.

Let $F(q_0) = \prod_{i=0}^{n} f_i(q_0)$. Then, for some t, we have $F^t a_v \in \mathbb{Z}[q_0]$ for every $v \in \{v_0, v_1, \ldots, v_n\}$. Since the f_i are all pairwise co-prime as elements of $\mathbb{Z}[q_0, 1/k]$ we can apply the Chinese Remainder Theorem for commutative rings to get a unique solution mod $F^t\mathbb{Z}[q_0, 1/k]$ to the congruences

$$a' \equiv a_{v_i} F^t \mod f_i^t \mathbb{Z}[q_0, 1/k]$$

for $0 \leq i \leq n$. This $a' \in A$ must have degree less than that of F^t. Set $a = a'F^{-t}$. Then

$$v_i(a - a_{v_i}) = v_i(a'F^{-t} - a_{v_i}) = v_i(F^{-t}(a' - a_{v_i}F^t) = -t + v_i(a' - a_{v_i}F^t)$$

and since ${f_i}^t$ divides $a' - a_{v_i}F^t$ in $\mathbb{Z}[q_0, 1/k]$ we have $v_i(a - a_{v_i}) \geq 0$. Recall that $a_w = 0$. Thus

$$w(a - a_w) = w(a) = w(a'F^{-t}) = \deg(F^t) - \deg(a') > 0.$$

This completes the proof.

□

Theorem 4.5. ([17], Theorem 7.) *The group G acts co-compactly on Y.*

Proof. Let $\prod_{v\in V}[(a_v, r_v)]$ be a point of Y. Then by Theorem 4.4 there exists $a \in A$ such that $\prod_{v\in V}[(a_v, s_v)] = \prod_{v\in V}[(a, s_v)]$, where $\lceil r_v \rceil = s_v$. Then

$$\begin{aligned}\prod_{v\in V}[(a_v, s_v)] &= \prod_{v\in V}[(a, s_v)] \\ &= \prod_{v\in V} G(v){q_v}^{s_v} a \\ &= \left(\prod_{v\in V} G(v){q_v}^{s_v}\right) * a \\ &= \prod_{v\in V}[(0_A, s_v)] * a\end{aligned}$$

and so

$$\prod_{v\in V}[(a_v, r_v)] = \prod_{v\in V}[(0_A, r_v)] * a.$$

Hence $Y/A \cong \{\prod_{v\in V}[(0_A, r_v)] : \prod_{v\in V} r_v \in W\}$, where Y/A is the set of equivalence classes of the points of Y under the action of A. But $\{\prod_{v\in V}[(0_A, r_v)] : \prod_{v\in V} r_v \in W\} \cong W$ via the map h, and $q \in Q$ acts on W via translation with $\prod_{v\in V} v(q)$. The quotient space W/Q is compact, since it is the continuous image of $Q \otimes \mathbb{R}/Q \cong \mathbb{R}^{n+2}/\mathbb{Z}^{n+2}$ under the map induced by $f : Q \otimes \mathbb{R} \to \mathbb{R}^{n+2}$. Hence $Y/G \cong (Y/A)/Q \cong W/Q$ is compact. □

Lemma 4.6. *Y is $(m-1)$-connected.*

Proof. By ([1], Proposition III.3.3) it is enough to show that every set of m elements of V lies in an open halfspace of $\mathrm{Hom}(Q, \mathbb{R}) \cong \mathbb{R}^z$, where z is the rank of Q. This is proved in [17], Lemma 8. □

5. Stabilizers in G of cells in Y

It remains to show that if P is the stabilizer in G of a cell in Y then P is of type FP_m. The following lemma implies that it is enough to prove that the stabilizer of a vertex $\prod_{v\in V} G(v)g_v$ in X lying in $h^{-1}[[W]]$ is of type FP_m.

Lemma 5.1. ([18], Lemma 2.9.) *If Γ is a cell of Y then the stabilizer of Γ in G coincides with the stabilizer in G of a vertex of X lying in $h^{-1}([[W]])$.*

Proof. Let Γ be the intersection of an i-subcell of the $(n+2)$-cell $J = \{\prod_{v\in V}[(a_v, r_v)] : s_v \le r_v \le s_v + 1\}$ in X with Y, let $\gamma_1, \ldots, \gamma_t$ be the vertices (0-subcells) of Γ and let $g \in G$ stabilize Γ. Then g permutes the vertices, say $\gamma_i * g = \gamma_{\rho(i)}$ for some permutation ρ. If $\gamma_i = \prod_{v\in V}[(a_v, s_{v,i})]$ we have $h_v(\gamma_i * g) = s_{v,i} + v(g)$ and so $s_{v,\rho(i)} = s_{v,i} + v(g)$. Summing over all i gives

$$\sum_{i=1}^{t}(s_{v,i} + v(g)) = \sum_{i=1}^{t} s_{v,\rho(i)} = \sum_{i=1}^{t} s_{v,i}.$$

Hence $v(g) = 0$ for all $v \in V$ and so $s_{v,i} = s_{v,\rho(i)}$ i.e. $\gamma_i = \gamma_{\rho(i)}$. Thus g stabilizes the vertices $\gamma_1, \ldots, \gamma_t$ and hence stabilizes Γ pointwise.

Note that $g \in G$ stabilizes a point $\prod_{v\in V}[(a_v, r_v)]$ of X if and only if g stabilizes the vertex $\prod_{v\in V}[(a_v, s_v)]$, where $\prod_v s_v = [[\prod_v r_v]]$, i.e. $s_v \in \mathbb{Z}$ and $s_v - r_v \in [0, 1)$ for all v. Then $\{g \in G : g \text{ stabilizes } \Gamma \text{ pointwise}\} = \{g \in G : g \text{ stabilizes all points from the set } \Lambda\}$, where

$$\Lambda = \left\{ \prod_{v\in V}[(a_v, s_v)] : \text{ there exists a point } \prod_{v\in V}[(a_v, r_v)] \text{ in } \Gamma \text{ such that } \prod_v s_v = \left[\left[\prod_v r_v\right]\right] \right\}.$$

So each element of Λ is a vertex of the $(n + 2)$-cell J. List the elements of Λ as $T_1, \ldots, T_r$, where $T_j = \prod_{v\in V}[(a_v, s_{v,j})]$, and suppose that $h(T_j) = [[h(S_j)]]$ for some points $S_1, \ldots, S_r$ of Γ. We claim that the point $T = \prod_{v\in V}[(a_v, s_v + 1)]$ belongs to Λ. Indeed $(1/r)(S_1 + \ldots + S_r)$ is a point of Γ and $[[h((1/r)(S_1 + \ldots + S_r))]] = h(T)$. Thus $g \in G$ stabilizes Γ pointwise if and only if it stabilizes the vertex T, and T lies in $h^{-1}[[W]]$. □

Observe that the stabilizer of a vertex $\prod_{v\in V} G(v)g_v$ of X is

$$\bigcap_{v\in V} {g_v}^{-1}G(v)g_v,$$

and that the stabilizer of a G-translate $\prod_{v\in V} G(v)g_v h$ of $\prod_{v\in V} G(v)g_v$ is

$$\bigcap_{v\in v}(g_v h)^{-1}G(v)g_v h = h^{-1}\left(\bigcap_{v\in V} {g_v}^{-1}G(v)g_v\right)h.$$

Since the two stabilizers are conjugate, they are isomorphic, and so it suffices to work with any vertex within each G-orbit. By Theorem 4.4 and equation (2) (on page 6) we get

$$\prod_{v\in V} G(v)g_v = \prod_{v\in V} G(v){q_v}^{s_v}a = \left(\prod_{v\in V} G(v){q_v}^{s_v}\right) * a$$

and so $\prod_{v\in V} G(v){q_v}^{s_v}$ and $\prod_{v\in V} G(v)g_v$ lie in the same G-orbit. Thus we can assume that $a_v = 0$ for all $v \in V$ and, given that Q acts transitively on the last $n + 2$

coordinates of $[[W]]$, we can assume that $s_v = 0$ for $v \neq w$. That is, we can assume that $g_v = 1$ if $v \neq w$ and that $g_w = {q_w}^{s_w}$, with $0 \leq s_w < -\beta$ by Lemma 4.2.

Then an element aq of G with $a \in A$ and $q \in Q$ lies in the stabilizer P of a vertex $(G(w){q_w}^{s_w}, G(v_0), G(v_1), \dots, G(v_n))$ if and only if

$$a \in A \cap ({q_w}^{-s_w} G(w) q_w^{s_w}), a \in G(v_i) \cap A = A_{v_i} \circ {q_{v_i}}^{\beta} \text{ and } q \in G(v) \text{ for all } v \in V.$$

Now $A \cap ({q_w}^{-s_w} G(w) q_w^{s_w}) = (A \cap G(w)) \circ {q_w}^{s_w}$. To see this, let $a = {q_w}^{-s_w} g {q_w}^{s_w}$ for some $g \in G(w)$. Since ${q_w}^{s_w} G(w) {q_w}^{s_w} \subseteq G(w)$ we have $a \in G(w) \cap A$ and so $a = {q_w}^{-\beta} a_w {q_w}^{\beta}$ for some $a_w \in A_w$. Hence

$${q_w}^{-s_w} g {q_w}^{s_w} = {q_w}^{-\beta} a_w {q_w}^{\beta}$$

and so

$$g = {q_w}^{s_w - \beta} a_w {q_w}^{s_w + \beta}.$$

Thus $g \in A$. For the reverse inclusion note that if $a \in A \cap G(w)$ then $a \circ {q_w}^{s_w} = {q_w}^{-s_w} a {q_w}^{s_w} \in A \cap ({q_w}^{-s_w} G(w) q_w^{s_w})$. Since $(A \cap G(w)) \circ {q_w}^{s_w} = A_w \circ {q_w}^{s_w + \beta}$ we can restate the above conditions for an element of G to lie in P: $aq \in P$ if and only if

$$a \in A_w \circ {q_w}^{s_w + \beta} \cap A_{v_0} \circ {q_{v_0}}^{\beta} \cap \ldots \cap A_{v_n} \circ {q_{v_n}}^{\beta} \text{ and } q \in G(v) \text{ for all } v \in V.$$

The condition on $q \in Q$ is satisfied precisely when $q \in \operatorname{Ker} v$ for all v, i.e. whenever $q \in Q_0 \times Q_2 \times \ldots \times Q_l$. Since A is free as an M-module it suffices to consider the case when A is cyclic as an M-module, and so we can take $A = M$. Then $b \in A_v$ for all $v \neq w$ exactly if $b \in \mathbb{Z}[q_0, 1/k] \otimes M_2 \ldots \otimes M_l$. Let $\tilde{q} = q_0 q_1 \ldots q_n \in Q_1$. Then since $a \in \bigcap_{0 \leq i \leq n} A_{v_i} \circ {q_{v_i}}^{\beta}$ we have $a \circ (\tilde{q})^{-\beta} \in A_{v_i}$ for $0 \leq i \leq n$ and so $a \circ (\tilde{q})^{-\beta} \in \mathbb{Z}[q_0, 1/k] \otimes M_2 \ldots \otimes M_l$. Write the component of $a \circ (\tilde{q})^{-\beta}$ in $\mathbb{Z}[q_0, 1/k]$ as $g(q_0)$. Now

$$a \in A_w \circ {q_0}^{-(s_w + \beta)} \subseteq \{b \in A : w(b) \geq s_w + \beta\}.$$

It follows that $s_w + \beta \sum_{i=0}^{n} d_i \leq w(a \circ (\tilde{q})^{-\beta}) < 0$, and so the degree of $g(q_0)$ is bounded above by $-(s_w + \beta \sum_{i=0}^{n} d_i) = d$. Hence the component $g(q_0) \circ (\tilde{q})^{\beta}$ of a that is a polynomial in $\mathbb{Z}[q_0, 1/k]$ is a $\mathbb{Z}$-linear combination of the elements

$$(k)^{l_j} {q_0}^{j + \beta} {f_1}^{\beta} \ldots {f_n}^{\beta}$$

where $0 \leq j \leq d$ and the l_j are the powers of k appearing in the monomials of g. Thus $P \cap \widetilde{M}$ is a free $\mathbb{Z}[1/k]$-module of finite rank $d+1$ and so $P \cap A = P \cap (\mathbb{Z}[q_0, 1/k] \otimes M_2 \otimes \ldots \otimes M_l)$ is a finitely generated free $M' = \mathbb{Z}[1/k] \otimes M_2 \otimes \ldots \otimes M_l$-module. Hence, P is the split extension of a free $\mathbb{Z}[1/k] \otimes M_2 \otimes \ldots \otimes M_l$-module by $Q_0 \times Q_2 \times \ldots \times Q_l$.

We now perform induction on l to show that P is always of type FP_m and so G is of type FP_m, where $m = \min\{rk(Q_i) : 1 \leq i \leq l\}$.

Theorem 5.2. *Every stabilizer P in G of a cell in the Åberg complex Y is of type FP_m, and so G is of type FP_m.*

Proof. First suppose $l = 1$. Then P is the split extension of a free $\mathbb{Z}[1/k]$-module of finite rank $d+1$ by the infinite cyclic group Q_0. Since the group $\mathbb{Z}[1/k] \rtimes Q_0$ has the presentation $\langle x, t : t^{-1}xt = x^k \rangle$ we deduce that $P = (P \cap A) \rtimes Q_0$ has the presentation

$$\langle H, t : t^{-1}x_i t = {x_i}^k \text{ for } 0 \leq i \leq d \rangle$$

where $H = \langle x_0, \ldots, x_d \rangle$ is free abelian group of rank $d+1$. Hence P is an HNN-extension of H with stable letter t, and the base group H and associated subgroups are free abelian of finite rank. It follows from ([7], Proposition 2.13(b)) that P is of type FP_∞ and so is certainly of type FP_m. Hence by Brown's criterion G is of type FP_m.

We now assume that Theorem A holds when M' is a tensor product of $\mathbb{Z}[1/k]$ and $l-1$ other components. In particular then P is of type $FP_{m'}$, where $m' = \min\{rk(Q_i) : 2 \leq i \leq l\}$, and so P is of type FP_m, since $m \leq m'$. This completes the proof of Theorem A. □

Hence we have proved that G_n is of type FP_{n+1}. To see that this statement is sharp, we appeal to the fact ([18], Theorem B) that the 'only if' direction of the Bieri-Groves conjecture holds in the split extension case. This implies that A is $(n+1)$-tame. However it is not $(n+2)$-tame, since

$$w + v_0 + v_1 + \ldots + v_n = 0$$

and $[w], [v_0], \ldots, [v_n] \in \Sigma_A^c$ by ([13], Proposition 2.1), since, by Lemma 4.3, $A \neq A_v$ for all $v \in V$.

6. Calculating the sets Σ_A^c

The following definition is taken from [10], Chapter 2. Let R be a non-trivial commutative ring and $v : R \to \mathbb{R}_\infty$ a valuation: that is, a map satisfying $v(0) = \infty$, $v(1) = 0$ and $v(ab) = v(a) + v(b)$, $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in R$. Let Q be a finitely generated abelian group of $\mathbb{Z}$-rank n and A an algebra over the group ring RQ via a ring homomorphism $\phi : RQ \to A$. Then $\Delta_A^v(Q) \subseteq \mathrm{Hom}(Q, \mathbb{R})$ is defined to be the set of all real characters χ of Q with the property that there is a valuation $w : A \to \mathbb{R}_\infty$ such that $w\phi|_R = v$ and $w\phi|_Q = \chi$. Theorem 8.1 in [10] then in theory allows us to determine $\Sigma_{A_n}^c$ for the Baumslag groups $A_n \rtimes Q_n$ by first calculating

$\Delta^v_{A_n}(Q_n)$ in the cases where v is either the trivial valuation on $\mathbb{Z}$ or any of the p-adic valuations on $\mathbb{Z}$, where $p \leq n$ is any prime dividing $n!$. We have been able to give a complete description of $\Sigma^c_{A_n}$ only when $n \leq 2$. However Theorem B provides a direct proof, without appealing to the cohomology of G_n, that A_n is $(n+1)$-tame.

First note that the characters $v_{-1}, v_0, v_1, \ldots, v_n$ form a basis for $\mathrm{Hom}(Q_n, \mathbb{R})$ as a real vector space and so an arbitrary character of Q_n can be written in the form

$$v = k_{-1}v_{-1} + k_0v_0 + k_1v_1 + \ldots + k_nv_n,$$

$k_{-1}, \ldots, k_n \in \mathbb{R}$. We can extend v to a valuation of the group ring $\mathbb{Z}Q_n$ by defining $v(0) = \infty$ and $v(\lambda) = \min\{v(q) : q \in \ \mathrm{supp}(\lambda)\}$, where $0 \neq \lambda \in \mathbb{Z}Q_n$.

Define the centralizer $C(A)$ of a Q-module A to be the set

$$C(A) = \{\lambda \in \mathbb{Z}Q : \lambda a = a \text{ for all } a \in A\}.$$

We then use the following criterion of Bieri and Strebel [13] to determine whether or not A_n is finitely generated as a module over the valuation monoid Q_v for a given v.

Proposition 6.1. ([13], Proposition 2.1) *Let A be a finitely generated Q-module and $v : Q \to \mathbb{R}$ a non-trivial character. Then A is finitely generated over Q_v if and only if there is $\lambda \in C(A)$ with $v(\lambda) > 0$. Moreover if A is finitely generated over Q_v then any set generating A as a Q-module generates A as a Q_v-module.*

If $k_{-1} < 0$ then $v(\lambda) > 0$ if $\lambda = (n!)q_{-1}^{-1}$. If $k_i, k_j > 0$ for some $i \neq j$ we can take $\lambda = \frac{n!}{i-j}(q_jq_{-1}^{-1} - q_iq_{-1}^{-1})$. If at most one of the $k_j > 0$ then choose some $k_i < 0$ and take $\lambda = q_j{q_i}^{-1} + (i-j)q_i^{-1}$.

So we have established that the set of equivalence classes of characters induced by the trivial valuation on $\mathbb{Z}$ is $\Delta^0_{A_n}(Q_n) = \{[v_0], [v_1], \ldots, [v_n], [w]\}$. This completes the proof of Theorem B.

When $n = 2$ we have been able to establish precisely when A_n is finitely generated as a Q_v-module for v with $k_{-1} > 0$. We do this by checking which characters v are induced by valuations on A. It turns out that $[v] \in \Sigma^c_{A_n}$ whenever (k_{-1}, k_0, k_1, k_2) is of the form $(a, 0, b, 0)$, $(a, a, 0, b)$ $(b \geq a > 0)$, $(a, b, 0, a)$ $(b \geq a > 0)$, $(a, b, 0, b)$ $(a \geq b > 0)$ or $(a, -b, -b, -b)$ $(a, b > 0)$, for $a, b \in \mathbb{R}$. A general principle is that from the 0-th coordinate on no n-tuple will contain adjacent positive coordinates.

7. Group extensions

Suppose we are in the $l = 1$ case of Theorem A, and write $q_j = q_{1,j}$, $f_j = f_{1,j}$ and $z_1 = n$. Suppose also that A is a cyclic M-module; then $A \cong \mathbb{Z}[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}, k^{-1}]$. We shall use the Lyndon-Hochschild-Serre spectral sequence to calculate the second cohomology group $H^2(Q, A)$. First of all note that we have a short exact sequence

$$\langle q_{-1} \rangle = B \longrightarrow Q \longrightarrow C = \langle q_0, q_1, \ldots, q_n \rangle.$$

Since B is infinite cyclic $H^0(B, A) = 0 = H^n(B, A)$ for $n \geq 2$, and $H^1(B, A) = H^0(B, A) = (\mathbb{Z}/(k-1)\mathbb{Z})[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}]$. So in order to calculate $H^2(Q, A)$ we must first calculate $H^1(C, H^1(B, A))$. We get another LHS-spectral sequence via the short exact sequence

$$\langle q_0 \rangle = X \longrightarrow C \longrightarrow C' = \langle q_1, \ldots, q_n \rangle$$

Now $H^0(X, H^1(B, A)) = 0$ since only the identity element in $H^1(B, A)$ is fixed under the action of X. However $H^1(X, H^1(B, A)) = (\mathbb{Z}/(k-1)\mathbb{Z})[f_1^{-1}(1), \ldots, f_n^{-1}(1)]$. The group $H^1(C, H^1(B, A))$, and thus $H^2(Q, A)$, is then given by the fixed points in $(\mathbb{Z}/(k-1)\mathbb{Z})[f_1^{-1}(1), \ldots, f_n^{-1}(1)]$ under the action of C'. If

$$f_1(1) \equiv f_2(1) \equiv \ldots \equiv f_n(1) \equiv 1 \mod (k-1)$$

then this is $\mathbb{Z}/(k-1)\mathbb{Z}$ since each q_j then acts as multiplication by 1 ($1 \leq j \leq n$). Otherwise, it is trivial. Thus we have proved Theorem C.

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