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SMALL SPECTRAL RADIUS AND PERCOLATION CONSTANTS ON NON-AMENABLE CAYLEY GRAPHS

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Abstract. Motivated by the Benjamini-Schramm nonunicity of percolation conjecture we study the following question. For a given finitely generated non-amenable group Γ, does there exist a generating set S such that the Cayley graph (Γ, S), without loops and multiple edges, has nonunique percolation, i.e., $p_c(\Gamma, S) < p_u(\Gamma, S)$? We explore nonamenability numerics, such as the isoperimetric constant and the spectral radius, on various growing generating sets in the group.

1. Introduction

Let Γ be a finitely generated group and let S be a finite symmetric generating set in Γ.

The *isoperimetric constant* (or *the Cheeger constant*) of Γ with respect to S is

$$\phi(\Gamma, S) = \inf_{F \subset \Gamma} \frac{\Sigma_{s \in S} |sF \backslash F|}{|F|}.$$

where the infimum is taken over all finite subsets F of Γ.

Equivalently we can write

$$\phi(\Gamma, S) = \inf_{F \subset \Gamma} \frac{|\partial_E F|}{|F|},$$

where the infimum is taken over all finite subsets F of Γ and where $\partial_E F$ denotes the boundary of F, i.e., the set of all edges connecting F to its complement in the Cayley graph of G with respect to S.

The isoperimetric constant $\phi(\Gamma, S)$ normalized by the size of the generating set is often called the *conductance* constant of Γ with respect to S:

$$h(\Gamma, S) = \frac{1}{|S|} \phi(\Gamma, S).$$

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Note that we can also define a different isoperimetric constant $\phi_V(\Gamma, S)$ by considering the boundary ∂_V understood as the set of vertices at distance 1 from F. We have

$$\phi_V(\Gamma, S) \leq \phi(\Gamma, S) \leq |S|\phi_V(\Gamma, S) \ .$$

Let $\lambda : \Gamma \to B(l_2\Gamma)$ be the left-regular representation of Γ. The spectral radius of Γ with respect to S is

$$\rho(S, \Gamma) = \frac{1}{|S|}||\Sigma_{g\in S}\lambda(g)||.$$

Then from [8] we have the following characterization of amenability:

(1) Γ is amenable,
(2) $\rho(\Gamma, S) = 1$, for some (iff for every) finite generating set $S \subseteq \Gamma$,
(3) $\phi(\Gamma, S) = \phi_V(\Gamma, S) = h(\Gamma, S) = 0$, for some (iff for every) finite generating set $S \subseteq \Gamma$.

Note that the conductance constant and the spectral radius are connected by the following inequalities (due to Mohar [11]):

$$\frac{|S|(1-\rho(\Gamma, S))}{|S|-1} \leq h(\Gamma, S) \leq \sqrt{1-\rho(\Gamma, S)^2}. \tag{1}$$

Equivalently

$$1 - \frac{h(\Gamma, S)(|S|-1)}{|S|} \leq \rho(\Gamma, S) \leq \sqrt{1-h(\Gamma, S)^2}. \tag{2}$$

For $0 \leq p \leq 1$ consider the Bernoulli bond percolation on the Cayley graph (Γ, S). Namely, an edge (g, gs) is open with probability p and closed with probability $1-p$; connected components of the subgraph spanned by the open edges are called open clusters. Let $\theta(p)$ be the probability that the origin belongs to an infinite open cluster and $\xi(p)$ be the probability that there exists exactly one infinite open cluster. The critical percolation constants of (Γ, S) are defined as follows:

$$p_c(\Gamma, S) = \sup\{p : \theta(p) = 0\}, p_u(\Gamma, S) = \inf\{p : \xi(p) = 1\}.$$

The critical constant p_c satisfies the following inequality [2]:

$$p_c(\Gamma, S) \leq \frac{1}{\phi(\Gamma, S) + 1}.$$

If G is amenable, then $p_c = p_u$ ([4]), and Benjamini and Schramm conjectured that this property characterizes amenability (of quasi-transitive locally finite graphs). For Cayley graphs the conjecture says:

Conjecture 1 ([2])**.** *If Γ is a non-amenable group generated by a finite set S then*

$$p_c(\Gamma, S) < p_u(\Gamma, S).$$

In [3], the conjecture was proved for planar quasi-transitive graphs. It was also shown that if G is an almost transitive graph, then there exists a constant $k = k(G)$ such that for the product of G with a regular tree of degree higher than k, $p_c < p_u$.

Benjamini and Schramm also proved the following sufficient condition that serves as a basis for several proofs of special cases of the conjecture.

Theorem 2 ([2]). *If*

$$\rho(\Gamma, S) p_c(\Gamma, S)|S| \; < \; 1, \tag{3}$$

then $p_c(\Gamma, S) < p_u(\Gamma, S)$.

In particular, Pak and the second author proved a weak version of Benjamini-Schramm conjecture for non-amenable groups by showing

Theorem 3 ([12]). *For every non-amenable group* Γ *and any symmetric finite generating set* S *there exists a positive integer* k *such that* $p_c(\Gamma, S^{(k)}) < p_u(\Gamma, S^{(k)})$.

Let us also mention the following sufficient condition

Proposition 4 ([12]). *If* $\rho(\Gamma, S) < \frac{1}{2}$, *then* $p_c(\Gamma, S) < p_u(\Gamma, S)$.

In [1], Benjamini, Nachmias and Peres gave an estimate on $p_c(G, S)$ in terms of $\rho(G, S)$, $|S|$ and the girth of the graph, that, when combined with (3), gives a sufficient condition for $p_c(G, S) < p_u(G, S)$ in terms of the girth of the graph.

There have also been advances towards Benjamini-Schramm conjecture through the theory of measurable equivalence relations, notably the following sufficient condition proven by Gaboriau

Theorem 5 ([6]). *For any locally finite transitive unimodular graph* G *the first* l_2*-Betti number,* $\beta_1^{(2)}(G)$, *of* G *(i.e. of any closed transitive group of automorphisms of* G*) is strictly greater then* 0, *then* $p_c(G) < p_u(G)$. *More precisely we have the following estimate:*

$$\beta_1^{(2)}(G) \leq \frac{1}{2}(p_u - p_c).$$

It has also been shown that if Γ has $cost(\Gamma) > 1$ then for every generating set S of Γ we have $p_c(\Gamma, S) < p_u(\Gamma, S)$, ([7]). Moreover, a group Γ has fixed price if and only if the Benjamini-Schramm conjecture is true for all generating sets S, ([10]).

It is important to note that the generating set $S^{(k)}$ in Theorem 3 is the k-th power of the set S understood as a multiset, so that the corresponding Cayley graph has lots of multiple edges. It is thus natural to look for a proof that every non-amenable group has a generating set for which $p_c < p_u$, where only "simple" generating sets are allowed. Here *simple* means that the corresponding Cayley graph is a graph without loop or multiple edge. In particular one can try to prove Theorem 3 with $S^{(k)}$ replaced by S^k, the k-th power of the set S with elements taken without repetitions. These are the questions we are going to discuss in this paper.

It was observed already in [12] that if Γ contains a free group on two generators then there exist generating sets $(A_l)_l$ such that $\rho(\Gamma, A_l) \to 0$ and thus for l big enough we have $p_c(\Gamma, A_l) < p_u(\Gamma, A_l)$ by Proposition 4.

In view of this application to the nonunicity of percolation problem, it would be interesting to know whether the spectral radius can always be made arbitrarily small in a finitely generated non-amenable group, in other words, whether nonamenability is equivalent to the following stronger property:

Definition 6. *Let Γ be a non-amenable group. Suppose that there exists a sequence of generating sets $\{A_l\}_l \in \mathbb{N} \subset \Gamma$ such that*

$$\rho(\Gamma, A_l) \to 0, \text{ as } l \to \infty.$$

We then say that Γ has infinitesimally small spectral radius (with respect to the family of generating sets $(A_l)_l$.)

By Proposition 4, having infinitesimally small spectral radius on generators $(A_l)_l$ implies that for big l, there is nonunicity of percolation on Cayley graphs (Γ, A_l).

Question 7. *(1) Does every non-amenable group have infinitesimally small spectral radius with respect to some family of generating sets?*
(2) With respect to some family of the form $\{S^k\}_k$?
(3) With respect to $\{S^k\}_k$ for any generating set S?

Note that if there is an element g of infinite order in Γ, then the spectral radius can be made arbitrarily close to 1 by adding to any given generating set bigger and bigger powers of g. In Section 3 we show that the spectral radius is bounded away from 1 uniformly on S^k, as $k \to \infty$, for arbitrary S. It follows from (2) that infinitesimally small spectral radius can be equivalently reformulated as conductance constant being arbitrarily close to 1, over some family of generating sets in a group. Our **Theorem 18** then says that the conductance constant is bounded away from 0 uniformly on $(S^k)_k$. In Section 4 we discuss a new condition equivalent to nonamenability, which implies in particular that in a non-amenable group the isoperimetric constant can be made arbitrarily close to one.

It is easy to check that if a subgroup of a group has infinitesimally small spectral radius then the group has. As noted above, groups with free subgroups have infinitesimally small spectral radius. This result is extended to other classes of non-amenable groups in Section 2 below, where we show that the spectral radius goes to 0 on certain sequences of generating sets in the group, and therefore the nonuniqueness of percolation conjecture holds at least on some generating sets, for groups with nontrivial non-amenable quotients, and for direct products $\Gamma \times \mathbb{Z}_d$ with non-amenable Γ and d big enough (see Corollary 14 and Corollary 15 in Section 2). We also show the following theorem (Theorem 13 in Section 2).

Theorem 8. *If Γ contains an infinite normal subgroup N such that Γ/N is non-amenable, then Γ has infinitesimally small spectral radius on some family of generating sets. Thus the nonunicity of percolation holds for these generating sets.*

For a finitely generated non-amenable group, the question about the behaviour of the spectral radius on Cayley graphs of Γ with respect to S^k remains very much open. One sufficient condition for the spectral radius to be infinitesimally small on generators S^k is Property (RD).

Proposition 9 ([12])**.** *Suppose Γ is a finitely generated non-amenable group with Property (RD). Then Γ has infinitesimally small spectral radius with respect to the sequence $(S^k)_k$ for any finite symmetric generating set S.*

In Section 2 below we also show the following (see Corollary 17).

Theorem 10. *Let Γ be a nonameanble group and S a finite generating set in Γ. Assume that*

$$\frac{\rho(\Gamma, S) \cdot |S|}{gr(\Gamma, S)} < 1, \tag{4}$$

where $gr(\Gamma, S)$ denotes the rate of exponential growth of the Cayley graph $\Gamma, S)$. Then $\rho(\Gamma, S^k) \to 0$ when $k \to \infty$.

Observe that the estimate $p_c(\Gamma, S) \leq gr(\Gamma, S)^{-1}$ combined with the condition (3) implies that (4) is enough to guarantee $p_c(\Gamma, S^k) < p_u(\Gamma, S^k)$ for big enough k. Theorem 10 shows that (4) implies a stronger property.

More generally, for any non-amenable quasi-transitive graph, it would be interesting to know how the spectral radius changes when edges are added to the graph so as to connect all vertices inside bigger and bigger balls.

Question 11. *Let G be a quasi-transitive locally finite non-amenable graph. For every $k \geq 1$, define G_k by adding to G edges connecting any two vertices at distance $\leq k$ in G. What is the asymptotics of $\rho(G_k)$ as $k \to \infty$? Is it true that $\rho(G_k) \to 0$?*

Finally, in view of Proposition 4 and for the sake of completeness, let us also mention the following related open question:

Question 12. *Let Γ be a non-amenable group. Does there exist a generating set S in Γ with $\rho(\Gamma, S) < 1/2$?*

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2. Spectral radius and nonunicity of percolation

In this Section we investigate the property of infinitesimally small spectral radius and draw conclusions about nonunicity of percolation.

Theorem 13. *If Γ contains an infinite normal subgroup N such that Γ/N is non-amenable, then Γ has infinitesimally small spectral radius.*

Proof. Let $q : \Gamma \to \Gamma/N$ be the canonical quotient map. It was proved by Kesten that

$$\rho(S, \Gamma) \leq \rho(q(S), q(\Gamma)).$$

Since Γ/N is non-amenable group, there exists a finite set $S \subseteq \Gamma/N$ such that $\rho(S, \Gamma/N) \leq C < 1$. Thus $\frac{1}{|S|^n}||\Sigma_{g\in S_n}\lambda(g)|| \leq C^n$, where the set $S_n = \{\alpha_\omega \omega : \omega \in S^n\}$, here α_ω is the number of elements in the free group $\mathbb{F}_{|S|}$ which are equal to ω under the canonical quotient map $\mathbb{F}_{|S|} \to \langle S \rangle$. Denote by $\hat{S}^n$ a lifting of the set S^n to Γ such that $|S^n| = |\hat{S}^n|$. For each $\omega \in S^n$ choose α_ω different elements in N: $\{g_{\omega,1}, \dots, g_{\omega,\alpha_\omega}\}$. Therefore we can lift S_n to Γ preserving cardinality and the spectral radius:

$$\hat{S}_n = \bigcup_{\omega \in \hat{S}^n} \{g_{\omega,1}\omega, g_{\omega,2}\omega, \dots, g_{\omega,\alpha_\omega}\omega\}$$

Thus we have $\rho(\hat{S}_n, \Gamma) \leq C^n$ which implies the statement. □

Corollary 14. *Let Γ be a discrete group that contains an infinite normal subgroup N such that Γ/N is non-amenable. Then there exists a finite set $S \subset \Gamma$ such that*

$$p_c(\Gamma, S) < p_u(\Gamma, S).$$

As a direct modification of the proof of the Theorem 13 and the Proposition 4 we have the following.

Corollary 15. *Let* Γ *be a non-amenable finitely generated group. Then there exists* $d = d(\Gamma, S)$ *such that* $p_c(\Gamma', S') < p_u(\Gamma', S')$ *for* $\Gamma' = \Gamma \times \mathbb{Z}_d$ *with a generating set* S'.

In a Cayley graph of a finitely generated group Γ with respect to a generating set S, denote by $B_k(\Gamma, S)$ the ball of radius k, $k \in \mathbb{N}$. Denote by $gr(\Gamma, S) = \lim_{n\to\infty} |B_k(\Gamma, S)|^{\frac{1}{k}}$ the rate of exponential growth of Γ with respect to the generating set S, strictly bigger than 1 for any S in a non-amenable Γ.

Lemma 16. *Let* Γ *be a group generated by a finite set* S. *Then*

$$\rho(\Gamma, S^k) \leq \frac{|S|^k}{|S^k|}(\rho(\Gamma, S))^k,$$

Proof. Let X be a finite subset of Γ and $A = \sum_{g\in X} \alpha_g \lambda(g) \in \mathbb{C}[\Gamma]$, then we have

$$\|A\| = \lim_{p\to\infty} \tau((A^*A)^p)^{\frac{1}{2p}},$$

where τ is the standard trace on $\mathbb{C}[\Gamma]$, i.e. τ is linear functional such that $\tau(g) = 0$ if $g \neq 1$ and $\tau(1) = 1$. Therefore for every $\beta_g \in \mathbb{N}$, in particular if β_g is the multiplicity of the element g in the multiset $S^{(k)}$, we have

$$\begin{aligned} || \sum_{g\in S^k} \lambda(g)|| \leq & || \sum_{g\in S^k} \beta_g \lambda(g)|| \\ = & || \sum_{g\in S^k} \lambda(g)||. \end{aligned}$$

Therefore we have

$$\begin{aligned} \rho(\Gamma, S^k) \leq & \frac{\rho(\Gamma, S^{(k)}) \cdot |S|^k}{|S^k|} \\ \leq & \frac{(\rho(\Gamma, S) \cdot |S|)^k}{|S^k|}. \end{aligned}$$

□

Corollary 17. *If there exists a generating set* S *such that*

$$\frac{\rho(\Gamma, S) \cdot |S|}{gr(\Gamma, S)} < 1,$$

then Γ has an infinitesimally small spectral radius.

Proof. Assume now that there exists a generating set S such that

$$\frac{\rho(\Gamma, S) \cdot |S|}{gr(\Gamma, S)} < 1.$$

The cardinality $|S^k|$ is equal to the cardinality of the k-th ball in the Cayley graph of Γ with respect to S, therefore the assumption implies that there exists $k_0 \in \mathbb{N}$ and $C < 1$ such that for every $k \geq k_0$ we have

$$\frac{\rho(\Gamma, S) \cdot |S|}{\sqrt[k]{|S^k|}} \leq C < 1.$$

Therefore, $\rho(\Gamma, S^k) \to 0$, when $k \to \infty$. □

3. Asymptotics of spectral radius and conductance constant along balls

In this section we study the asymptotic behavior of the spectral radius and of the conductance constant on Cayley graphs with respect to the generating sets S^k, $k \geq 1$, for any S. Since both constants are related by inequalities (1) and (2), in our case it is sufficient to study only one of them.

Theorem 18. *Let Γ be non-amenable group generated by a finite set S. Then there is a constant $0 < C < 1$ such that for every $k \in \mathbb{N}$ we have*

$$\rho(\Gamma, S^k) \leq C.$$

In terms of conductance constant we have that there exists some $0 < C' < 1$ such that

$$h(\Gamma, S^k) \geq C'$$

for every $k \in \mathbb{N}$.

Proof. To reach a contradiction, assume that $\rho(\Gamma, S^k) \to 1$ on some subsequence, then $h(\Gamma, S^k) \to 0$. In terms of conductance constant this means that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and a finite set $F \subseteq \Gamma$ such that

$$\sum_{s \in B_k} |sF \backslash F| \leq \varepsilon |B_k| \cdot |F|,$$

where, as above, B_k denotes the ball of radius k in the Cayley graph of (Γ, S), i.e., $B_k = S^k$. Fix ε and let F and k are given by inequality above. Consider the function $f = \sum_{g \in B_k} \chi_{gF}$ and let Λ_k be the sphere of radius k. Then $\|f\|_{l_1(\Gamma)} = |F| \cdot |B_k|$. Note that for every $h \in S$ we have

$|hB_k \backslash B_k| = |B_k \backslash hB_k|$ and $hB_k \backslash B_k \subseteq \Lambda_{k+1}$ and $B_k \backslash hB_k \subseteq \Lambda_k$. Thus for $h \in S$ we have

$$\begin{aligned}
\| \sum_{g \in B_k} \chi_{hgF} - \sum_{g \in B_k} \chi_{gF} \|_1 &= \| \sum_{g \in hB_k} \chi_{gF} - \sum_{g \in B_k} \chi_{gF} \|_1 \\
&= \| \sum_{g \in hB_k \backslash B_k} \chi_{gF} - \sum_{g \in B_k \backslash hB_k} \chi_{gF} \|_1 \\
&= \| \sum_{g \in hB_k \backslash B_k} (\chi_{gF} - \chi_F) - \sum_{g \in B_k \backslash hB_k} (\chi_{gF} - \chi_F) \|_1 \\
&\leq \sum_{g \in \Lambda_{k+1}} \|\chi_{gF} - \chi_F\|_1 + \sum_{g \in \Lambda_k} \|\chi_{gF} - \chi_F\|_1 \\
&= \sum_{g \in \Lambda_{k+1}} |gF \Delta F| + \sum_{g \in \Lambda_k} |gF \Delta F| \\
&\leq \sum_{g \in B_{k+1}} |gF \Delta F| \\
&\leq 2\varepsilon |B_{k+1}| \cdot |F| \\
&\leq 2\varepsilon \frac{|B_{k+1}|}{|B_k|} \cdot \|f\|_1 \\
&\leq 2\varepsilon |S| \cdot \|f\|_1.
\end{aligned}$$

Normalizing f we obtain a positive function $f \in l_1(\Gamma)$ with $\|f\|_1 = 1$ and $\|hf - f\| \leq \varepsilon'$ for every $h \in S$. Thus Γ is amenable. □

4. Isoperimetric constants of non-amenable groups

In this section we show a new characterization of amenability that can be viewed as positive evidence towards the conjecture that every non-amenable group has infinitesimally small spectral radius. As an application we also have a new estimate on the isoperimetric constants of the group.

We will need the following lemma which is well known, but we include it for completeness.

Lemma 19. *Let $\pi : \Gamma \to B(H)$ be a unitary representation of a discrete group Γ and there exists unit vector $\xi \in H$ such that $\|\pi(g)\xi - \xi\| \leq C < \sqrt{2}$ for every $g \in \Gamma$. Then π has an invariant vector.*

Proof. Note that

$$Re(<\pi(g)\xi,\xi>) = 1 - \frac{1}{2}||\pi(g)\xi - \xi||^2$$
$$\geq 1 - \frac{C^2}{2} = C' > 0.$$

Let $V = \overline{conv\{\pi(g)\xi : g \in \Gamma\}}$ then V is $\pi(\Gamma)$-invariant and

$$Re(<\theta,\xi>) \geq C' \text{ for every } \theta \in V.$$

Let $\nu \in V$ be the unique element of V that has minimal norm, then $Re(<\nu,\xi>) \geq C'$ and $\nu \neq 0$. Since π is a unitary representation, we have that ν is invariant under the action of $\pi(\Gamma)$. □

Theorem 20. *A finitely generated group Γ is amenable if and only if there exists a constant $C < 2$ such that for every finite set $S \subset \Gamma$ there exists a finite set $F \subset \Gamma$ such that*

$$|sF\Delta F| \leq C|F|, \text{ for every } s \in S.$$

Proof. The existence of $C \leq 2$ that satisfy the condition of the theorem for amenable group Γ follows from Følner's criteria.

To prove the converse fix a finite set S and let F be a finite set of Γ such that

$$|sF\Delta F| \leq C|F|, \text{ for every } s \in S.$$

Consider $\xi_F = \frac{1}{\sqrt{|F|}}\chi_F$, we have $||\lambda(s)\xi_F - \xi_F|| \leq \sqrt{C}$ for every $s \in S$. Let S_i be an increasing sequence of sets in Γ with $\Gamma = \cup S_i$ and let $\lambda_\omega : G \to B(l_2(\Gamma)^\omega)$ be an ultra-limit of the left-regular representation acting on an ultra power of the Hilbert space $l_2(\Gamma)$. Then for the vector $\xi = (\xi_{F_i})_{i\in\mathbb{N}}$ we have that $||\lambda_\omega(g)\xi - \xi|| \leq \sqrt{C}$ for every $g \in G$. By Lemma 19 we have that λ_ω has an invariant vector. Thus λ has a sequence of almost invariant vectors, therefore Γ is amenable. □

As a direct application of the Theorem 22 we have the following corollary.

Corollary 21. *Let Γ be a non-amenable group then for every $\varepsilon > 0$ there exists a finite set $S \subset \Gamma$ such that*

$$\phi(\Gamma, S) \geq \phi(\Gamma, S) \geq 1 - \varepsilon.$$

Remark 22. *If the condition equivalent to amenability in Theorem 22 could be strengthen to say that there exists a constant $C < 2$ such that for every finite set $S \subset \Gamma$ there exists a finite set $F \subset \Gamma$ such that*

$$\Sigma_{s\in S}|sF\Delta F| \leq C|F| \cdot |S|,$$

then it would imply that for every non-amenable group the conductance constant is arbitrary close to 1 *on some finite sets and thus the group has infinitesimally small spectral radius, by Mohar's inequalities.*

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