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REPORT No. 35, 2011/2012, spring
ISSN 1103-467X
ISRN IML-R- -35-11/12- -SE+spring

THE SIMPLICIAL VOLUME OF 3-MANIFOLDS WITH BOUNDARY

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Abstract. We provide sharp lower bounds for the simplicial volume of compact 3-manifolds in terms of the simplicial volume of their boundaries. As an application, we compute the simplicial volume of several classes of 3-manifolds, including handlebodies and products of surfaces with the interval. Our results provide the first exact computation of the simplicial volume of a compact manifold whose boundary has positive simplicial volume. For the proofs, we use pseudomanifolds to represent integral cycles that approximate the simplicial volume, introduce a topological straightening for aspherical, boundary irreducible manifolds and compute the exact value of the Δ-complexity of products of surfaces with the interval. Finally, we also prove a partial converse of a result by the last two authors regarding the simplicial volume of hyperbolic manifolds with geodesic boundary.

Introduction

The simplicial volume is an invariant of manifolds introduced by Gromov in his seminal paper [Gro82]. If M is a connected, compact, oriented manifold with (possibly empty) boundary, then the simplicial volume of M is the infimum of the sum of the absolute values of the coefficients over all singular chains representing the real fundamental cycle of M (see Section 1). It is usually denoted by $\|M\|$ if M is closed, and by $\|M, \partial M\|$ if $\partial M \neq \emptyset$. If M is open, the fundamental class and the simplicial volume of M admit analogous definitions in the context of homology of *locally finite* chains, but in this paper we will restrict our attention to compact manifolds: unless otherwise stated, henceforth every manifold is assumed to be compact. Observe that the simplicial volume of an oriented manifold does not depend on its orientation and that it is straightforward to extend the definition also to nonorientable or disconnected manifolds: if M is connected and nonorientable, then its simplicial volume is equal to one half of the simplicial volume of its orientable double covering, and the simplicial volume of any manifold is the sum of the simplicial volumes of its components.

Even if it depends only on the homotopy type of a manifold, the simplicial volume is deeply related to the geometric structures that a manifold

Michelle Bucher was supported by Swiss National Science Foundation project PP00P2-128309/1. The authors thank the Institute Mittag-Leffler in Djursholm, Sweden, for their warm hospitality during the preparation of this paper.

can carry. For example, closed manifolds which support negatively curved Riemannian metrics have nonvanishing simplicial volume, while the simplicial volume of flat or spherical manifolds is null (see *e.g.* [Gro82]). Several vanishing and nonvanishing results for the simplicial volume are available by now, but the exact value of nonvanishing simplicial volumes is known only in a very few cases. If M is (the natural compactification of) a complete finite-volume hyperbolic n-manifold without boundary, then a celebrated result by Gromov and Thurston implies that the simplicial volume of M is equal to the Riemannian volume of M divided by the volume v_n of the regular ideal geodesic n-simplex in hyperbolic space (see [Gro82, Thu79] for the compact case and *e.g.* [Fra04, FP10, FM11, BBI] for the cusped case). The only other exact computation of nonvanishing simplicial volume is for the product of two closed hyperbolic surfaces or more generally manifolds locally isometric to the product of two hyperbolic planes [BK08]. Building on these examples, more values for the simplicial volume can be obtained by surgery or by taking connected sums or amalgamated sums over submanifolds with amenable fundamental group, however not by taking products.

For hyperbolic manifolds with geodesic boundary, it is proved by Jungreis [Jun97] that if M is such a manifold and $\partial M \neq \emptyset$, then $\|M, \partial M\|$ strictly exceeds $\mathrm{Vol}(M)/v_n$, and the last two authors showed that there exist, in any dimension, examples for which $\mathrm{Vol}(M)/\|M, \partial M\|$ is arbitrarily close to v_n [FP10]. These results were the sharpest estimates so far for the simplicial volume of manifolds whose boundary has positive simplicial volume. We provide here the first exact computations of $\|M, \partial M\|$ for classes of 3-manifolds for which $\|\partial M\| > 0$.

The simplicial volume of 3-manifolds with boundary. If M is a connected oriented n-manifold with boundary, then the usual boundary map takes any relative fundamental cycle of M to the sum of fundamental cycles of the components of ∂M. As a consequence, for any n-manifold M we have

$$\|M, \partial M\| \geq \frac{\|\partial M\|}{n+1} \ . \tag{1}$$

In particular, if $\|\partial M\| > 0$, then $\|M, \partial M\| > 0$. We improve this bound in Proposition 2.7 by replacing the factor $n+1$ by $n-1$ when $n \geq 2$, after observing that good cycles for the simplicial volume do not have more than $n-1$ faces in the boundary.

The main result of this paper concerns 3-dimensional manifolds, and provides a sharp lower bound for $\|M, \partial M\|$ in terms of $\|\partial M\|$:

Theorem 1. *Let M be a 3-manifold. Then there is a sharp inequality*

$$\|M, \partial M\| \geq \frac{3}{4}\|\partial M\| \ .$$

The fact that the bound of Theorem 1 is sharp is an immediate consequence of Theorem 2. Theorems 1 and 2 are proved in Section 3. We will see in Theorem 4 that in the case of a boundary irreducible aspherical 3-manifold, the constant 3/4 can be improved to 5/4.

Stable Δ-complexity and simplicial volume. If M is an n-manifold, we denote by $\sigma(M)$ the Δ-*complexity* of M, *i.e.* the minimal number of tetrahedra in a triangulation of M. We employ here the word "triangulation" in a loose sense, as is customary in geometric topology: a triangulation is the realization of M as the gluing of finitely many n-simplices via some simplicial pairing of their codimension-1 faces. It is easy to see that the inequality $\|M, \partial M\| \leq \sigma(M)$ holds (see *e.g.* [FFM, Proposition 0.1] or the discussion in the proof of Theorem 5 in Section 5). The simplicial volume is multiplicative with respect to finite coverings, while for every degree d covering $\widehat{M} \stackrel{d}{\to} M$ we have

$$\sigma(\widehat{M}) \leqslant d \cdot \sigma(M),$$

which is very often a strict inequality. The *stable* Δ-*complexity* $\sigma_\infty(M)$ of M is defined by setting

$$\sigma_\infty(M) = \inf_{\widehat{M} \stackrel{d}{\to} M} \left\{ \frac{\sigma(\widehat{M})}{d} \right\}$$

where the infimum is taken over all finite coverings $\widehat{M} \stackrel{d}{\to} M$ of any finite degree d. The definition of the stable Δ-complexity, which was introduced by Milnor and Thurston in [MT77], is made to be multiplicative with respect to finite coverings. The inequality $\|M, \partial M\| \leq \sigma(M)$ and the multiplicativity of the simplicial volume with respect to finite coverings imply that

$$\|M, \partial M\| \leq \sigma_\infty(M) \tag{2}$$

for every n-manifold M. It has recently been established [FFM] that this inequality is strict for closed hyperbolic manifolds of dimension ≥ 4.

The simplicial volume of handlebodies. Every Seifert manifold with nonempty boundary has a finite covering which is the product of a surface with a circle. Such a covering admits in turn nontrivial self-coverings, and has therefore null stable Δ-complexity. As a consequence, for every Seifert manifold with nonempty boundary both the stable Δ-complexity and the simplicial volume vanish. In particular, the inequality (2) is an equality. (The same is true for closed Seifert manifolds with infinite fundamental group.) Non zero examples where the simplicial volume equals the stable Δ-complexity are provided by the following result.

Theorem 2. *Let M be a Seifert manifold with nonempty boundary, and let N be obtained by performing a finite number of* 1*-handle additions on M. Then*

$$\|N, \partial N\| = \sigma_\infty(N) = \frac{3}{4}\|\partial N\| \ .$$

For every $g \in \mathbb{N}$ let us denote by H_g the orientable handlebody of genus g. We easily have $\|H_0, \partial H_0\| = \|H_1, \partial H_1\| = 0$. Since H_1 is a Seifert manifold and H_g can be obtained by performing $g-1$ handle additions on H_1, Theorem 2 implies the following:

Corollary 3. *For every $g \geq 2$, the equalities*

$$\|H_g, \partial H_g\| = \sigma_\infty(H_g) = \frac{3}{4}\|\partial H_g\| = 3(g-1)$$

hold.

This improves the bounds

$$\frac{4}{3}(g-1) \leq \|H_g, \partial H_g\| \leq \sigma_\infty(H_g) \leq 3(g-1)$$

exhibited by Kuessner [Kue03]. Note that the upper bound also follows from the computation of the Δ-complexity of the handlebody $\sigma(H_g) = 3g-2$ established by Jaco and Rubinstein [JR].

Aspherical manifolds with aspherical π_1-injective boundary. Recall that a connected manifold M is *aspherical* if $\pi_i(M) = 0$ for every $i \geq 2$, or, equivalently, if the universal covering of M is contractible. If M is disconnected, we say that M is aspherical if every connected component of M is. Moreover, we say that M is *boundary irreducible* if for every connected component B of ∂M the inclusion $B \hookrightarrow M$ induces an injective map on fundamental groups (we borrow this terminology from the context of 3-manifold topology, and use it also in the higher dimensional case).

The estimate provided by Theorem 1 may be improved in the case of boundary irreducible aspherical manifolds. More precisely, we prove the following:

Theorem 4. *Let M be a boundary irreducible aspherical 3-manifold. Then there is a sharp inequality*

$$\|M, \partial M\| \geq \frac{5}{4}\|\partial M\| \ .$$

The equality is realized by products of surfaces with intervals (Corollary 6) for which we first compute the Δ-complexity. Both theorems and their corollary will be proven in Section 5.

Theorem 5. *Let S_g be a closed orientable surface of genus $g \geq 1$ and let $M_g = S_g \times [0,1]$. Then*

$$\sigma(M_g) = 10(g-1) + 6 \ .$$

There are remarkably few examples of exact computations of Δ-complexity of manifolds. The first family of examples is given by surfaces, Jaco, Rubinstein and Tillmann computed the Δ-complexity of an infinite family of lens spaces [JRT09], and the Δ-complexity of handlebodies is computed by Jaco and Rubinstein [JR]. Moreover, a census of closed 3-manifolds up to

Δ-complexity 9 and 10 may be deduced from the results in [MP01] and [Mar06]. Our Theorem 5 provides the exact computation of Δ-complexity for a new infinite family of examples. It might be worth mentioning that, in the case of manifolds with boundary, the minimal number of simplices in *ideal* triangulations of manifolds, rather than in (loose) triangulations, has been computed for several families of 3-manifolds.

Corollary 6. *Let S_g be a closed orientable surface of genus $g \geq 2$ and let $M_g = S_g \times [0,1]$. Then*

$$\|M_g, \partial M_g\| = \sigma_\infty(M_g) = \frac{5}{4}\|\partial M_g\| \ .$$

Hyperbolic manifolds with geodesic boundary. We have already mentioned that, if M is a closed hyperbolic n-manifold, then

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n} \ ,$$

where v_n denotes the volume of the regular ideal simplex in hyperbolic n-space, while in the case of hyperbolic manifolds with nonempty geodesic boundary, $n \geq 3$, Jungreis proved [Jun97] that this equality does not hold by showing that

$$\|M, \partial M\| > \frac{\mathrm{Vol}(M)}{v_n} \ . \tag{3}$$

In Section 6 we provide a quantitative version of Jungreis' result in the case when $n \geq 4$. More precisely, we prove the following:

Theorem 7. *Let $n \geq 4$. Then there exists a constant $\eta_n > 0$ depending only on n such that*

$$\frac{\|M, \partial M\|}{\mathrm{Vol}(M)} \geq \frac{1}{v_n} + \eta_n \cdot \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)} \ .$$

It is well-known that $\|M, \partial M\| = \mathrm{Vol}(M)/v_2 = \mathrm{Vol}(M)/\pi$ for every hyperbolic surface with geodesic boundary M, so Theorem 7 cannot be true in dimension 2. The 3-dimensional case is still open.

Theorem 7 states that, if $n \geq 4$, then $\mathrm{Vol}(M)/\|M, \partial M\|$ cannot approach v_n unless the $(n-1)$-dimensional volume of ∂M is small with respect to the volume of M. On the other hand, it is known that $\mathrm{Vol}(M)/\|M, \partial M\|$ indeed approaches v_n if $\mathrm{Vol}(\partial M)/\mathrm{Vol}(M)$ is small. In fact, the following result is proved in [FP10] for $n \geq 3$: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\mathrm{Vol}(M)}{\|M, \partial M\|} \geq v_n - \varepsilon$$

for every hyperbolic n-manifold M with nonempty geodesic boundary such that

$$\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)} < \delta \ .$$

Note that in particular, the ratio between $\|M, \partial M\|$ and $\mathrm{Vol}(M)$ does not depend only on the dimension of M. Putting together this result with Theorem 7, we obtain, for $n \geq 4$, a complete characterization of hyperbolic n-manifolds with geodesic boundary whose simplicial volume is close to the bound given by Inequality (3):

Corollary 8. *Let $n \geq 4$, and let M_i be a sequence of hyperbolic n-manifolds with nonempty geodesic boundary. Then*

$$\lim_{i\to\infty} \frac{\mathrm{Vol}(M_i)}{\|M_i, \partial M_i\|} = v_n \quad \Longleftrightarrow \quad \lim_{i\to\infty} \frac{\mathrm{Vol}(\partial M_i)}{\mathrm{Vol}(M_i)} = 0 \ .$$

Hyperbolic 3-manifolds with geodesic boundary. Every hyperbolic manifold with geodesic boundary is aspherical and boundary irreducible. Therefore, even if Theorem 7 is still open in dimension 3, Theorem 4 may be exploited to show that, if ∂M is big with respect to $\mathrm{Vol}(M)$, then indeed the simplicial volume of M is bounded away from $\mathrm{Vol}(M)/v_3$. Let us briefly introduce some families of examples for which the bound provided by Theorem 4 is sharper than Jungreis' bound (3).

For every $g \geq 2$ let $\overline{\mathcal{M}}_g$ be the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$ (so ∂M, if orientable, is the closed orientable surface of genus g). Recall that for every 3-manifold with boundary M the equality $\chi(\partial M) = 2\chi(M)$ holds, and in particular $\chi(\partial M)$ is even. Therefore, the union $\bigcup_{g\geq 2} \overline{\mathcal{M}}_g$ coincides with the set of hyperbolic 3-manifolds with connected geodesic boundary.

For every $g \geq 2$ we denote by $\mathcal{M}_g$ the set of 3-manifolds with boundary M that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$ (see Section 6 for the definition of ideal triangulation). Every element of $\mathcal{M}_g$ has connected boundary and supports a hyperbolic structure with geodesic boundary (which is unique by Mostow rigidity), hence $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ (see Proposition 6.11, which lists some facts proved in [Miy94, FMP03]). Furthermore, Miyamoto proved in [Miy94] that elements of $\mathcal{M}_g$ are exactly the ones having the smallest volume among the elements of $\overline{\mathcal{M}}_g$. In particular, $\mathcal{M}_g$ is nonempty for every $g \geq 2$. The eight elements of $\mathcal{M}_2$ are exactly the smallest hyperbolic manifolds with nonempty geodesic boundary [KM91, Miy94].

Recall that the simplicial volume and the Riemannian volume of hyperbolic 3-manifolds with nonempty geodesic boundary are not related by a universal proportionality constant. Nevertheless, it is reasonable to expect that these invariants are closely related to each other. Therefore, we make here the following:

Conjecture 9. For $g \geq 2$, the elements of $\mathcal{M}_g$ are exactly the ones having the smallest simplicial volume among the elements of $\overline{\mathcal{M}}_g$. Moreover, the eight elements of $\mathcal{M}_2$ are the hyperbolic manifolds with nonempty geodesic boundary having the smallest simplicial volume.

At the moment, no precise computation of the simplicial volume of hyperbolic 3-manifold with nonempty geodesic boundary is known. The following result is proved in Section 6, and may provide an approach to the above conjecture.

Theorem 10. *Let M be a hyperbolic 3-manifold with nonempty geodesic boundary. Then*

$$\|M, \partial M\| \geq \frac{\mathrm{Vol}(M)}{v_3} + \frac{v_3 - G}{2(3v_3 - 2G)} \left(7\|\partial M\| - 4\frac{\mathrm{Vol}(M)}{v_3} \right)$$

where $G \approx 0.916$ is Catalan's constant (see Section 6).

Together with Miyamoto's results about volumes of hyperbolic manifolds with geodesic boundary [Miy94], Theorem 10 implies the following (see Section 6):

Corollary 11. *If $M \in \overline{\mathcal{M}}_2$, then $\|M, \partial M\| \geq 6.461 \approx 1.615 \cdot \|\partial M\|$. If $M \in \overline{\mathcal{M}}_3$, then $\|M, \partial M\| \geq 10.882 \approx 1.360 \cdot \|\partial M\|$. If $M \in \overline{\mathcal{M}}_4$, then $\|M, \partial M\| \geq 15.165 \approx 1.264 \cdot \|\partial M\|$.*

As we will see in Section 6, the corollary shows that Theorem 10 indeed improves Jungreis' Inequality (3) and Theorem 4 in some cases. More precisely we will show that if $M \in \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$, the bounds provided by Theorem 10 and Corollary 11 coincide, and are sharper than the bounds provided by Inequality (3) and Theorem 4, while if $M \in \mathcal{M}_g$, $g \geq 5$, then the bound for $\|M, \partial M\|$ provided by Theorem 4 is sharper than the ones given by Inequality (3) and Theorem 10.

Structure of the paper. We recall the first properties of simplicial volume in Section 1 and in particular give a proof of the folklore fact that the rational simplicial volume is equal to the (real) simplicial volume. This in turn allows us to restrict to integral cycles and we will describe a geometric realization of integral cycles by pseudomanifolds in Section 2. Theorems 1 and 2 are then proved in Section 3. Theorems 4 and 5 and Corollary 6 are proved in Section 5 after a treatment of straightening procedures for boundary irreducible aspherical manifolds is carried out in Section 4. Finally, Section 6 is devoted to hyperbolic manifolds with geodesic boundary and in particular to the proofs of Theorems 7 and 10 and Corollary 11.

1. Simplicial volume

Let us first fix some notations. Let X be a topological space and $Y \subseteq X$ a (possibly empty) subspace of X. Let R be a normed ring. In this paper only the cases $R = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$ are considered, where each of these rings is endowed with the norm given by the absolute value. For $i \in \mathbb{N}$ we denote by $S_i(X)$ the set of singular i-simplices in X, by $C_i(X; R)$ the module of singular i-chains over R, and we set as usual $C_i(X, Y; R) = C_i(X; R)/C_i(Y; R)$. We observe that the R-module $C_i(X, Y; R)$ is free and admits the preferred basis given by the classes of the singular simplices whose image is not contained in Y. Therefore, we will often identify $C_i(X,Y;R)$ with the free R-module generated by $S_i(X) \setminus S_i(Y)$. In particular, for $z \in C_i(X,Y;R)$, it will be understood from the equality $z = \sum_{k=1}^n a_k\sigma_k$ that $\sigma_k \neq \sigma_h$ for $k \neq h$, and $\sigma_k \notin S_i(Y)$ for every k, while we could also write $z = \sum_{k=1}^n a_k[\sigma_k]$ in $C_i(X;R)/C_i(Y;R)$. We denote by $H_*(X,Y;R)$ the singular homology of the pair (X,Y) with coefficients in R, *i.e.* the homology of the complex $(C_*(X,Y;R), d_*)$, where d_* is the usual differential.

We endow the R-module $C_i(X,Y;R)$ with the L^1-norm defined by

$$\left\| \sum_\sigma a_\sigma \sigma \right\|_R = \sum_\sigma |a_\sigma| \ ,$$

where σ ranges over the simplices in $S_i(X)\setminus S_i(Y)$. We denote simply by $\|\cdot\|$ the norm $\|\cdot\|_{\mathbb{R}}$. The norm $\|\cdot\|_R$ descends to a seminorm on $H_*(X,Y;R)$, which is still denoted by $\|\cdot\|_R$ and is defined as follows: if $[\alpha] \in H_i(X,Y;R)$, then

$$\|[\alpha]\|_R = \inf\{\|\beta\|_R,\ \beta \in C_i(X,Y;R),\ d\beta = 0,\ [\beta] = [\alpha]\} \ .$$

Note that although $\|\cdot\|_{\mathbb{Z}}$ is often called a seminorm in the literature, it is technically not so as it is not multiplicative in general (see below). The *real* singular homology module $H_*(X,Y;\mathbb{R})$ and the seminorm on $H_*(X,Y;\mathbb{R})$ will be simply denoted by $H_*(X,Y)$ and $\|\cdot\|$ respectively.

Simplicial volume. If M is a connected oriented n-manifold with (possibly empty) boundary ∂M, then we denote by $[M,\partial M]_R$ the fundamental class of the pair $(M,\partial M)$ with coefficients in R. The following definition is due to Gromov [Gro82, Thu79]:

Definition 1.1. The *simplicial volume* of M is

$$\|M,\partial M\| = \|[M,\partial M]_{\mathbb{R}}\| = \|[M,\partial M]_{\mathbb{R}}\|_{\mathbb{R}} \ .$$

The *rational*, respectively *integral*, simplicial volume of M is defined as $\|M,\partial M\|_{\mathbb{Q}} = \|[M,\partial M]_{\mathbb{Q}}\|_{\mathbb{Q}}$, resp. $\|M,\partial M\|_{\mathbb{Z}} = \|[M,\partial M]_{\mathbb{Z}}\|_{\mathbb{Z}}$.

Just as in the real case, the rational and the integral simplicial volume may be defined also when M is disconnected or nonorientable. Of course we have the inequalities $\|M,\partial M\| \leq \|M,\partial M\|_{\mathbb{Q}} \leq \|M,\partial M\|_{\mathbb{Z}}$. It is folklore that $\|M,\partial M\| = \|M,\partial M\|_{\mathbb{Q}}$ and we provide here a complete proof of this fact.

Proposition 1.2. *For every n-manifold M, the real and rational simplicial volumes are equal,*

$$\|M,\partial M\| = \|M,\partial M\|_{\mathbb{Q}} \ .$$

Proof. We have to show that $\|M,\partial M\|_{\mathbb{Q}} \leq \|M,\partial M\|$. Let $\varepsilon > 0$ be fixed, and let $z = \sum_{i=1}^k a_i\sigma_i$ be a real fundamental cycle for M such that $\|z\| =$

$\sum_{i=1}^k |a_i| \leq \|M, \partial M\| + \varepsilon$. We set

$$H_{\mathbb{R}} = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{i=1}^k x_i \sigma_i \text{ is a relative cycle} \right\} \subseteq \mathbb{R}^k \ .$$

Of course, $H_{\mathbb{R}}$ is a linear subspace of $\mathbb{R}^k$. Since $H_{\mathbb{R}}$ is defined by a system of equations with integral coefficients, if $H_{\mathbb{Q}} = H_{\mathbb{R}} \cap \mathbb{Q}^k$, then $H_{\mathbb{Q}}$ is dense in $H_{\mathbb{R}}$. As a consequence, we may find sequences of rational coefficients $\{\alpha_i^j\}_{j\in\mathbb{N}} \subseteq \mathbb{Q}$, $i = 1, \ldots, k$ such that $z^j = \sum_{i=1}^k \alpha_i^j \sigma_i$ is a rational cycle for every $j \in \mathbb{N}$, and $\lim_j \alpha_i^j = a_i$ for every $i = 1, \ldots, k$. This implies in particular that $\lim_j \|z^j\|_{\mathbb{Q}} = \|z\|$, so we are left to show that the z^j's may be chosen among the representatives of the rational fundamental class of M.

Let $\lambda_j \in \mathbb{Q}$ be defined by $[z^j] = \lambda_j \cdot [M, \partial M]$ (such a λ_j exists because $[M, \partial M]$ lies in the image of $H_n(M, \partial M; \mathbb{Q})$ in $H_n(M, \partial M; \mathbb{R})$ under the change of coefficients homomorphism). The Universal Coefficient Theorem provides a real cocycle $\varphi\colon C_n(M, \partial M; \mathbb{R}) \to \mathbb{R}$ such that $\varphi(z) = 1$. Observe that $\varphi(z^j) = \lambda_j$, so from $\lim_j \alpha_i^j = a_i$ we deduce that $\lim_j \lambda_j = \lim_j \varphi(z^j) = \varphi(z) = 1$. For large j we may thus define $w^j = \lambda_j^{-1} \cdot z^j \in C_n(M, \partial M; \mathbb{Q})$, and by construction w^j represents the rational fundamental class of M. Finally, we have

$$\lim_j \|w^j\|_{\mathbb{Q}} = \lim_j \frac{\|z^j\|_{\mathbb{Q}}}{\lambda_j} = \|z\| \leq \|M, \partial M\| + \varepsilon \ ,$$

which finishes the proof of the proposition. □

The integral simplicial volume does not behave as nicely as the rational or real simplicial volume. For example, it follows from the definition that $\|M\|_{\mathbb{Z}} \geq 1$ for every manifold M. Therefore, the integral simplicial volume cannot be multiplicative with respect to finite coverings (otherwise it should vanish on manifolds that admit finite nontrivial self-coverings, as S^1). Another defect is that the L^1-seminorm on integral homology is not really a seminorm, since the equality $\|n \cdot \alpha\|_{\mathbb{Z}} = |n| \cdot \|\alpha\|_{\mathbb{Z}}$, for $n \in \mathbb{Z}$, $\alpha \in H_*(X, Y; \mathbb{Z})$, may not hold. Indeed, it is easy to see that $\|n \cdot [S^1]\|_{\mathbb{Z}} = 1$ for every $n \in \mathbb{Z} \setminus \{0\}$.

Despite these facts, we will use integral cycles extensively, as they admit a clear geometric interpretation in terms of pseudomanifolds (see Section 2). In order to follow this strategy, we need the following easy results.

Lemma 1.3. *Let M be connected and oriented, and let $\varepsilon > 0$ be given. Then, there exists an integral cycle $z \in C_n(M, \partial M; \mathbb{Z})$ such that*

$$\frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M, \partial M\| + \varepsilon \ ,$$

where $[z] = d \cdot [M, \partial M]_{\mathbb{Z}}$ and $d > 0$ is an integer.

Proof. Since $\|M, \partial M\| = \|M, \partial M\|_{\mathbb{Q}}$, a rational cycle $z' \in C_n(M, \partial M; \mathbb{Q})$ exists such that $[z']_{\mathbb{Q}} = [M, \partial M]_{\mathbb{Q}}$ and $\|z'\|_{\mathbb{Q}} \leq \|M, \partial M\| + \varepsilon$. Of course there exists $d \in \mathbb{N} \setminus \{0\}$ such that $z = d \cdot z'$ lies in $C_n(M, \partial M; \mathbb{Z})$. The integral cycle z satisfies the desired properties. $\square$

Remark 1.4. The statements and the proofs of Proposition 1.2 and Lemma 1.3 hold more generally after replacing the fundamental class $[M, \partial M]_{\mathbb{Q}}$ by any rational homology class. In other words, for every $i \in \mathbb{N}$ the change of coefficients map $H_i(M, \partial M; \mathbb{Q}) \to H_i(M, \partial M; \mathbb{R})$ is norm-preserving.

Lemma 1.5. *Let M be connected and oriented, and let z be an integral n-dimensional cycle such that $[z] = d \cdot [M, \partial M]_{\mathbb{Z}}$, where $d > 0$ is an integer. Then*

$$\|\partial M\| \leq \frac{\|\partial z\|_{\mathbb{Z}}}{d} \ .$$

Proof. The chain z/d is a real (in fact, rational) fundamental cycle for M, so the class $[(\partial z)/d] \in H_{n-1}(\partial M; \mathbb{R})$ is the sum of the real fundamental classes of the components of ∂M, and $\|\partial M\| \leq \|(\partial z)/d\| = \|\partial z\|_{\mathbb{Z}}/d$. $\square$

Finally, let us list some elementary properties of the simplicial volume which will be needed later.

Proposition 1.6 ([Gro82])**.** *Let M, N be connected oriented manifolds of the same dimension, and suppose that either M, N are both closed, or they both have nonempty boundary. Let $f\colon N \to M$ be a map of degree d. Then*

$$\|N, \partial N\| \geq |d| \cdot \|M, \partial M\| \ .$$

The following well-known result describes the simplicial volume of closed surfaces. In fact, the same statement also holds for connected surfaces with boundary.

Proposition 1.7 ([Gro82])**.** *Let S be a closed surface. Then*

$$\|S\| = \max\{0, -2\chi(S)\} \ .$$

Let S, S' be (possibly disconnected) orientable surfaces without boundary. We say that S' is obtained from S by an *elementary tubing* if S' is obtained from S by removing two disjoint embedded disks and gluing an annulus to the resulting boundary components in such a way that the resulting surface is orientable. We say that S' is obtained by tubing from S if it is obtained from S via a finite sequence of elementary tubings. An immediate application of Proposition 1.7 yields the following:

Corollary 1.8. *Let S, S' be (possibly disconnected) orientable surfaces without boundary, and suppose that S' is obtained from S by tubing. Then*

$$\|S'\| \geq \|S\| \ .$$

2. Representing integral cycles

We now recall the well-known notion of n-pseudomanifold.

Definition 2.1. Let $n \in \mathbb{N}$. An *n-dimensional pseudomanifold* P consists of a finite number of copies of the standard n-simplex, a choice of pairs of simplex $(n-1)$-dimensional faces such that each face appears in at most one of these pairs, and an affine identification between the faces of each pair. It is *orientable* if orientations on the simplices of P may be chosen in such a way that the affine identifications between the paired faces (endowed with the induced orientations) are all orientation-reversing. A face which does not belong to any pair of identified faces is a *boundary* face.

We denote by $|P|$ the *topological realization* of P, *i.e.* the quotient space of the union of the simplices by the equivalence relation generated by the identification maps. We say that P is connected if $|P|$ is. We denote by $\partial|P|$ the image in $|P|$ of the boundary faces of P, and we say that P is *without boundary* if $\partial|P| = \emptyset$.

A *k-dimensional face* of $|P|$ is the image in $|P|$ of a k-dimensional face of a simplex of P. Usually, we refer to 1-dimensional, resp. 0-dimensional faces of P and $|P|$ as to *edges*, resp. *vertices* of P and $|P|$.

Observe that we do not require the topological realization of a pseudomanifold to be connected. In this way, the boundary of a pseudomanifold is itself a pseudomanifold (see below).

It is well-known that, if P is an n-dimensional pseudomanifold, $n \geq 3$, then $|P|$ does not need to be a manifold. However, in the 3-dimensional orientable case, singularities may occur only at vertices (and it is not difficult to construct examples where they indeed occur). Let us be more precise, and state the following well-known result (see *e.g.* [Hat02, pages 108-109]):

Lemma 2.2. *Let P be an orientable n-dimensional pseudomanifold, and let $V_k \subseteq |P|$ be the union of the k-dimensional faces of $|P|$. Then $|P| \setminus V_{n-3}$ is an orientable manifold. In particular, if P is an orientable 2-dimensional pseudomanifold without boundary, then $|P|$ is an orientable surface without boundary.*

If P is nonorientable, then $|P|$ is a manifold outside its faces of codimension two (see again [Hat02, pages 108-109]), and in the 2-dimensional case $|P|$ is still a surface (see *e.g.* [Thu97, Exercise 1.3.2(b)] for the case without boundary).

The boundary of a pseudomanifold. Let us prove that the boundary of the topological realization of an (orientable) n-dimensional pseudomanifold P is naturally the topological realization of an (orientable) $(n-1)$-dimensional pseudomanifold without boundary, that will be denoted by ∂P.

Let Ω be the set of the boundary $(n-1)$-dimensional faces of the simplices of P. If e, e' are distinct codimension-1 faces of elements of Ω (so e, e' are $(n-2)$-dimensional simplices), then we pair e and e' if and only if they have the same image in $|P|$. Observe that each $(n-2)$-dimensional face of each simplex in Ω is paired to exactly one other $(n-2)$-dimensional face. If e is paired to e', then there exists a unique affine diffeomorphism between e

and e' which identifies exactly those pairs of points which have the same image in $\partial|P| \subseteq |P|$. We have thus defined an $(n-1)$-dimensional pseudomanifold without boundary ∂P whose simplices are exactly the boundary $(n-1)$-dimensional faces of P. It follows from the construction that $|\partial P|$ is canonically homeomorphic to $\partial|P|$. If P is oriented, then we may define an orientation on ∂P simply by putting on any simplex of Ω the orientation induced by the corresponding n-simplex of P.

Lemma 2.2 now implies the following:

Proposition 2.3. *Let P be an orientable 3-dimensional pseudomanifold. Then every connected component of $\partial|P|$ is an orientable closed surface.*

The pseudomanifold associated to an integral cycle. Let M be an oriented connected n-dimensional manifold with (possibly empty) boundary ∂M. It is well-known that every integral relative cycle on $(M, \partial M)$ can be represented by a map from a suitable pseudomanifold to M. Let us describe this procedure in detail in the case we are interested in, *i.e.* in the case of n-dimensional integral cycles (see also [Hat02, pages 108-109]).

Let $z = \sum_{i=1}^{k} \varepsilon_i \sigma_i$ be an n-dimensional relative cycle in $C_n(M, \partial M; \mathbb{Z})$, where σ_i is a singular n-simplex on M, and $\varepsilon_i = \pm 1$ for every i (note that here we do not assume that $\sigma_i \neq \sigma_j$ for $i \neq j$). We construct an n-pseudomanifold associated to z as follows. Let us consider k distinct copies $\Delta^n_1, \ldots, \Delta^n_k$ of the standard n-simplex Δ^n. For every i we fix an identification between Δ^n_i and Δ^n, so that we may consider σ_i as defined on Δ^n_i. For every $i = 1, \ldots, k$, $j = 0, \ldots, n$, we denote by F^i_j the j-th face of Δ^n_i, and by $\partial^i_j \colon \Delta^{n-1} \to F^i_j \subseteq \Delta^n_i$ the usual face inclusion. We say that the distinct faces F^i_j and $F^{i'}_{j'}$ form a *canceling pair* if $\sigma_i|_{F^i_j} = \sigma_{i'}|_{F^{i'}_{j'}}$ and $(-1)^j \varepsilon_i + (-1)^{j'} \varepsilon_{i'} = 0$. This is equivalent to say that, when computing the boundary ∂z of z, the pair of $(n-1)$-simplices arising from the restrictions of σ_i and $\sigma_{i'}$ to F^i_j and $F^{i'}_{j'}$ cancel each other.

Let us define a pseudomanifold P as follows. The simplices of P are $\Delta^n_1, \ldots, \Delta^n_k$, and we identify the faces belonging to a maximal collection of canceling pairs. Note that such a family is not uniquely determined, see Example 2.5. If F^j_i, $F^{j'}_{i'}$ are paired faces, we identify them via the affine diffeomorphism $\partial^{j'}_{i'} \circ (\partial^j_i)^{-1} \colon F^j_i \to F^{j'}_{i'}$. We observe that P is orientable: in fact, we can define an orientation on P by endowing Δ^n_i with the standard orientation of Δ^n if $\varepsilon_i = 1$, and with the reverse orientation if $\varepsilon_i = -1$.

By construction, the maps $\sigma_1, \ldots, \sigma_k$ glue up to a well-defined continuous map $f \colon |P| \to M$. For every $i = 1, \ldots, k$, let $\hat{\sigma}_i \colon \Delta^n \to |P|$ be the singular simplex obtained by composing the identification $\Delta^n \cong \Delta^n_i$ with the quotient map with values in $|P|$, and let us set $z_P = \sum_{i=1}^{k} \varepsilon_i \hat{\sigma}_i$. The following result immediately follows from the definitions:

Lemma 2.4. *The chain z_P is a relative cycle in $C_n(|P|, \partial|P|; \mathbb{Z})$ and the map f_* induced by $f\colon (|P|, \partial|P|) \to (M, \partial M)$ on integral singular chains sends z_P to $f_*(z_P) = z$.*

Example 2.5. Let $M = S^3$ and $q \in S^3$ be any fixed point. Consider the integral cycle given by $z = \sigma$, where $\sigma\colon \Delta^3 \to S^3$ is any map which restricts to a homeomorphism $\Delta^3 \setminus \partial\Delta^3 \simeq S^3 \setminus \{q\}$ and maps the whole boundary $\partial\Delta^3$ onto q. Then, there are two possible pseudomanifolds associated to z, corresponding to the pairings (F_0, F_1), (F_2, F_3) and (F_0, F_3), (F_1, F_2), where $F_0, \ldots, F_3$ are the 2-dimensional faces of Δ^3. Also observe that *any* choice for the affine diffeomorphisms identifying the faces of Δ^3 would allow us to define a continuous map from the resulting pseudomanifold $|P|$ to S^3. However, not every choice for these identifications would ensure that the corresponding chain $z_P \in C_3(|P|; \mathbb{Z})$ is a cycle.

Approximating real cycles via pseudomanifolds. As before, let M be a connected oriented n-manifold. By Lemma 1.3, the simplicial volume of M can be computed from *integral* cycles. The following proposition shows that such cycles may be represented by n-pseudomanifolds with additional properties.

If P is an n-dimensional pseudomanifold, we denote by $c(P)$ the number of n-simplices of P. Of course, if P is associated to the integral cycle z, then $c(P) = \|z\|_\mathbb{Z}$ and $c(\partial P) = \|\partial z\|_\mathbb{Z}$ (the obvious inequality $\|\partial z\|_\mathbb{Z} \leq c(\partial P)$ is an equality since by definition the pairings defining P correspond to a *maximal* set of canceling pairs of faces).

Proposition 2.6. *Let $n \geq 2$ and $\varepsilon > 0$ be fixed. Then, there exists a relative integral n-cycle $z \in C_n(M, \partial M; \mathbb{Z})$ with associated pseudomanifold P such that the following conditions hold:*

(1) $[z] = d \cdot [M, \partial M]_\mathbb{Z}$ in $H_n(M, \partial M; \mathbb{Z})$ for some integer $d > 0$, and

$$\frac{\|z\|_\mathbb{Z}}{d} \leq \|M, \partial M\| + \varepsilon \ ;$$

(2) P is connected;

(3) every simplex of P has at most $n-1$ boundary faces, and in particular

$$\|\partial z\|_\mathbb{Z} \leq (n-1)\|z\|_\mathbb{Z} \ .$$

Proof. By Lemma 1.3, we may choose an integral cycle satisfying condition (1).

Let us suppose that the pseudomanifold P associated to z is disconnected. Then P decomposes into a finite collection of connected pseudomanifolds $P_1, \ldots, P_k$ such that $c(P_1) + \ldots + c(P_k) = c(P)$. Each P_i represents an integral cycle z_i, and if $d_i \in \mathbb{Z}$ is defined by the equation $[z_i] = d_i[M, \partial M]_\mathbb{Z}$, then $d_1 + \ldots + d_k = d$. Therefore, there exists $i_0 \in \{1, \ldots, k\}$ such that $d_{i_0} \neq 0$ and $c(P_{i_0})/|d_{i_0}| \leq c(P)/d$. After replacing z with $\pm z_{i_0}$, we may suppose that our cycle z satisfies conditions (1) and (2).

As for condition (3), first note that since P is now connected, if it has a simplex with $n+1$ boundary faces, then P consists of a single simplex Δ^n with no identifications between its faces. In particular, z consists of a single singular simplex $\sigma : \Delta^n \to M$. Let z_0 be the chain on $\Delta^n = |P|$ obtained by coning $\partial\Delta^n$ to the barycenter $b = (1, \ldots, 1) \in \Delta^n$ of Δ^n. More precisely, $z_0 = \sum_{i=0}^{n} (-1)^i [b, e_0, \ldots, \widehat{e_i}, \ldots, e_n]$, where $[b, e_0, \ldots, \widehat{e_i}, \ldots, e_n] : \Delta^n \to \Delta^n$ denotes the affine simplex with vertices $b, e_0, \ldots, \widehat{e_i}, \ldots, e_n$. Clearly, its integral norm is equal to $\|z_0\|_{\mathbb{Z}} = \|\partial z_0\|_{\mathbb{Z}} = n+1$. Let $f_0 : (\Delta^n, \partial\Delta^n) \to (\Delta^n, \partial\Delta^n)$ be a map of degree $\geq n+1$ and consider the integral cycle $z' = (\sigma \circ f_0)_*(z_0)$. Note that the corresponding pseudomanifold P' is the coning of $\partial\Delta^n$. In particular, each of its $n+1$ simplices has exactly one boundary face and $\|\partial z'\|_{\mathbb{Z}} = n+1 = \|z'\|_{\mathbb{Z}}$ which validates condition (3). Observe also that P' is clearly connected. Furthermore, as z represented $d \cdot [M, \partial M]_{\mathbb{Z}}$, the relative cycle z' represents $d \cdot \deg(f_0) \cdot [M, \partial M]_{\mathbb{Z}}$ and

$$\frac{\|z'\|_{\mathbb{Z}}}{d \cdot \deg(f_0)} = \frac{n+1}{d \cdot \deg(f_0)} \leq \frac{1}{d} = \frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M, \partial M\| + \varepsilon \ .$$

It remains to see that P can be chosen not to have any simplex with n boundary faces. Suppose that P contains a simplex Δ_1^n having exactly one *non*-boundary face. Then Δ_1^n is adjacent to another simplex Δ_2^n of P. Roughly speaking, we remove from z the singular simplex σ_1 corresponding to Δ_1^n, and modify the singular simplex corresponding to Δ_2^n by suitably "expanding" it to compensate the removal of σ_1. More precisely, if $z = \sum_{i=1}^{k} \varepsilon_i \sigma_i$, then there exist a map $f\colon (|P|, \partial|P|) \to (M, \partial M)$ and singular simplices $\hat{\sigma_i}\colon \Delta^n \to |P|$, $i = 1, \ldots, k$ such that $\sigma_i = f \circ \hat{\sigma}_i$ for every $i = 1, \ldots, k$. Let us denote by $F^1 \subseteq \Delta_1^n$, $F^2 \subseteq \Delta_2^n$ the pair of faces glued in $|P|$, and by $\varphi\colon F^1 \to F^2$ their affine identification. Let Q be the space obtained by gluing Δ_1^n and Δ_2^n along φ. We observe that the inclusion $\Delta_1^n \sqcup \Delta_2^n \hookrightarrow \Delta_1^n \sqcup \ldots \sqcup \Delta_k^n$ induces a well-defined map $\theta\colon Q \to |P|$, and we fix a homeomorphism $\psi\colon \Delta^n \to Q$ that restricts to the identity on $\partial\Delta_2^n \setminus F^2$ (recall that an identification $\Delta^n \cong \Delta_2^n$ is fixed from the very beginning). Finally, we define the singular simplex $\sigma_2'\colon \Delta^n \to M$ by setting $\sigma_2' = f \circ \theta \circ \psi$.

Let us set $z' = \varepsilon_2 \sigma_2' + \sum_{i=3}^{k} \varepsilon_i \sigma_i$, and let P' be the pseudomanifold obtained by removing from P the simplex Δ_1^n (and ignoring the only pairing involving a face of Δ_1^n). It is readily seen that z' is still a relative cycle in $C_n(M, \partial M; \mathbb{Z})$. What is more, we have $[z'] = [z] = d \cdot [M, \partial M]_{\mathbb{Z}}$ in $H_n(M, \partial M; \mathbb{Z})$. By construction, z' admits P' as associated pseudomanifold, $|P'|$ is connected and $c(P') = c(P) - 1$. As a consequence, the cycle z' satisfies (1) and (2), and $c(P') < c(P)$. If z' still has some simplex with exactly n boundary faces of codimension one, then we may iterate our procedure until we get a cycle satisfying (1) and (2), and having no simplices with n boundary faces. Since at every step the number of simplices of the associated pseudomanifold decreases, this iteration must come to an end. □

The following result is an easy consequence of Proposition 2.6:

Proposition 2.7. *If M is an n-manifold, where $n \geq 2$, then*

$$\|M, \partial M\| \geq \frac{\|\partial M\|}{n-1} \ .$$

Proof. We may assume that M is connected and oriented. Let $\varepsilon > 0$, and choose an integral cycle z as described in Proposition 2.6. Then by Lemma 1.5 we get

$$\|\partial M\| \leq \frac{\|\partial z\|_{\mathbb{Z}}}{d} \leq \frac{(n-1)\|z\|_{\mathbb{Z}}}{d} \leq (n-1)(\|M, \partial M\| + \varepsilon) \ ,$$

which proves the proposition since ε is arbitrary. □

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Let M be a 3-manifold. We want to prove that

$$\|M, \partial M\| \geq \frac{3}{4}\|\partial M\| \ .$$

We may assume that M is connected and oriented. If $\|\partial M\| = 0$, there is nothing to prove, so we may assume that $\|\partial M\| > 0$.

Let $\varepsilon > 0$ be given. We choose an integral cycle $z \in C_3(M, \partial M; \mathbb{Z})$ with associated pseudomanifold P that satisfies all the properties described in Proposition 2.6. Recall that $z_P \in C_3(|P|, \partial|P|; \mathbb{Z})$ is the relative cycle represented by the (signed) sum of the simplices of P, and that P comes with a map $f\colon (|P|, \partial|P|) \to (M, \partial M)$ such that $f_*(z_P) = z$.

For $i = 0, \ldots, 4$, let us denote by t_i the number of 3-simplices of P having exactly i boundary 2-faces. Our choice for z implies that $t_3 = t_4 = 0$, so Lemma 1.5 implies that

$$d \cdot \|\partial M\| \leq \|\partial z\|_{\mathbb{Z}} = c(\partial P) = t_1 + 2t_2 \ . \tag{4}$$

Let $S_1, \ldots, S_h$ be the boundary components of M, and for every $i = 1, \ldots, h$, let $Y_i^1, \ldots, Y_i^{k_i}$ be the connected components of $\partial|P|$ that are taken into S_i by f. Since P is orientable, each Y_i^j is by Proposition 2.3 a closed orientable surface. Let d_i^j be the degree of the map $f|_{Y_i^j}\colon Y_i^j \to S_i$. Since $[f_*(\partial z_P)] = [\partial f_*(z_P)] = [\partial z]$ we have $\sum_{j=1}^{k_i} d_i^j = d$ for every $i = 1, \ldots, h$, whence

$$\sum_{j=1}^{k_i} \|Y_i^j\| \geq d \cdot \|S_i\| \tag{5}$$

(see Proposition 1.6).

Let us consider the space H obtained by removing from $|P|$ a closed tubular neighbourhood of edges and vertices. Of course, H is an orientable handlebody, so in particular ∂H is an orientable surface. If we denote by $Z_1, \ldots, Z_r$ the boundaries of regular neighbourhoods of the vertices of $|P| \setminus \partial|P|$, then each Z_l is an orientable surface, and ∂H is obtained from the union of the Y_i^j's and the Z_l's by tubing. Putting together Corollary 1.8 and inequality (5), this implies that

$$\|\partial H\| \geq \sum_{i=1}^{h}\sum_{j=1}^{k_i}\|Y_i^j\| + \sum_{l=1}^{r}\|Z_l\| \geq \sum_{i=1}^{h} d\cdot\|S_i\| = d\cdot\|\partial M\| \ . \tag{6}$$

Let us denote by Γ the graph dual to P. Since vertices and edges of Γ correspond respectively to 3-simplices and pairs of identified faces of P, we compute

$$\chi(\Gamma) = (t_0+t_1+t_2) - \frac{4t_0+3t_1+2t_2}{2} = \frac{-2t_0-t_1}{2} \ .$$

The handlebody H retracts onto Γ. As a consequence, if g is the genus of H we obtain

$$1-g = \chi(H) = \chi(\Gamma) = \frac{-2t_0-t_1}{2} \ . \tag{7}$$

As we are assuming that $\|\partial M\| > 0$, Equation (6) implies that $g \geq 2$, and can be rewritten as

$$4g-4 \geq d\cdot\|\partial M\| \ . \tag{8}$$

Putting together Equations (7) and (8) we obtain

$$4t_0+2t_1 \geq d\cdot\|\partial M\| \ . \tag{9}$$

Using inequalities (4) and (9) we get

$$4(t_0+t_1+t_2) = 2(t_1+2t_2)+(4t_0+2t_1) \geq 3d\cdot\|\partial M\| \ ,$$

whence

$$\frac{3}{4}\|\partial M\| \leq \frac{t_0+t_1+t_2}{d} = \frac{c(P)}{d} = \frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M,\partial M\| + \varepsilon \ . \tag{10}$$

Since ε is arbitrary, this concludes the proof of Theorem 1. □

Proof of Theorem 2. If M is a 3-manifold with nonempty boundary, we say that N is obtained from M by adding a 1-handle if $N = M\cup_f H$, where $H = D^2\times[0,1]$ is a solid cylinder and $f\colon D^2\times\{0,1\}\to\partial M$ is a homeomorphism onto the image.

Proposition 3.1. *Let M be any 3-manifold with nonempty boundary, and let N be obtained from M by adding a 1-handle. Then*

$$\sigma(N) \leq \sigma(M)+3, \qquad \sigma_\infty(N) \leq \sigma_\infty(M)+3 \ .$$

Proof. Let T be a copy of the standard 2-simplex, and fix an identification of the added 1-handle with the prism $T\times[0,1]$. For $i=0,1$, we denote by $\partial_i M$ the boundary component of ∂M glued to $T\times\{i\}$. Observe that we may have $\partial_0 M = \partial_1 M$. The prism $T\times[0,1]$ can be triangulated by 3 simplices, in such a way that $T\times\{0\}$ and $T\times\{1\}$ appear as boundary faces of the triangulation. Let $\mathcal{T}$ be a minimal triangulation of M. Every component of ∂M inherits from $\mathcal{T}$ a triangulation with at least two triangles, so we may choose distinct boundary faces T_0, T_1 of $\mathcal{T}$ and affine identifications

$\varphi_i\colon T\times\{i\}\to T_i$ in such a way that N is homeomorphic to the space obtained by gluing M and $T\times[0,1]$ along the φ_i's. This space is endowed with a triangulation with $\sigma(M)+3$ tetrahedra, and this proves that

$$\sigma(N)\leq\sigma(M)+3\ .$$

As for the stable Δ-complexity, if $f\colon\widehat{M}\to M$ is any covering of degree d, then it is immediate that f extends to a covering $\overline{f}\colon\widehat{N}\to N$ of degree d, where $\widehat{N}$ is obtained from $\widehat{M}$ by adding d 1-handles. We thus get

$$\frac{\sigma(\widehat{N})}{d}\leq\frac{\sigma(\widehat{M})+3d}{d}=\frac{\sigma(\widehat{M})}{d}+3\ .$$

Since the covering $f\colon\widehat{M}\to M$ was arbitrary, this implies that

$$\sigma_\infty(N)\leq\sigma_\infty(M)+3\ ,$$

and concludes the proof of Proposition 3.1. □

We can now easily conclude the proof of Theorem 2. In fact, if M is a Seifert manifold with nonempty boundary then we know from the introduction that $\sigma_\infty(M)=0$. Therefore, if N is obtained by consecutively adding h handles on M, Proposition 3.1 implies that

$$\sigma_\infty(N)\leq\sigma_\infty(M)+3h=3h\ . \tag{11}$$

On the other hand, every boundary component of M has null Euler characteristic, and no boundary component of any manifold obtained by adding 1-handles to M has positive Euler characteristic, so Proposition 1.7 implies that $\|\partial N\|=4h$. Putting together this inequality with Inequality (11), and recalling that stable Δ-complexity always bounds the simplicial volume from above, we get

$$\|N,\partial N\|\leq\sigma_\infty(N)\leq\frac{3}{4}\|\partial N\|\ . \tag{12}$$

Finally, Theorem 1 implies that all the inequalities in (12) are in fact equalities, which finishes the proof of Theorem 2. □

4. Boundary irreducible aspherical manifolds

This section sets the foundations for the proof of Theorem 4 presented in the next section, that implies that the estimate provided by Theorem 1 may be improved in the case of boundary irreducible aspherical 3-manifolds. We begin by showing how this additional hypothesis can be exploited to construct a topological straightening for simplices. Even though we are mostly interested in the 3-dimensional case, we describe the straightening procedure in the general n-dimensional case.

A topological straightening. The *straightening procedure* for simplices was introduced by Thurston in [Thu79], in order to bound from below the simplicial volume of hyperbolic manifolds. If M admits a nonpositively curved Riemannian metric, then one may associate to every singular simplex σ in M a *straight* simplex $\mathrm{str}(\sigma)$, which is uniquely determined by the vertices of σ and the homotopy classes (relative to the endpoints) of the edges of σ. The straightening map, which is defined by linearly extending str to singular chains, establishes an isometric isomorphism between the usual singular homology of M and the homology of the complex of *straight* chains. In practice, this implies that the simplicial volume may be computed just by looking at straight chains, that verify interesting additional geometric properties.

There is no hope for extending the straightening procedure to generic manifolds: for example, if M is simply connected, then straight simplices in M should depend only on their vertices, and this implies that the homology of straight chains of M should vanish in positive degree. However, in this subsection we show that, if M is aspherical, boundary irreducible, and has aspherical boundary, then the pair $(M, \partial M)$ admits a topological relative straightening.

Before going into the details of our definition of relative straightening, let us briefly recall some useful tools from homological algebra. If G is a group and R is a commutative ring (as usual, we confine ourselves to the cases $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$), we denote by RG the group ring freely generated by G over R. If (C_*, ∂_*) is an RG-complex (*i.e.* a complex of RG-modules such that $\partial_n \colon C_n \to C_{n-1}$ is a map of RG-modules for every $n \in \mathbb{N}$), then we denote by C_n^G the quotient of C_n by the RG-submodule generated by the elements of the form $v - g \cdot v$, for $v \in C_n$, $g \in G$. The differential ∂_n induces a differential $\overline{\partial}_n \colon C_n^G \to C_{n-1}^G$, and we denote by $H_*(C_*^G)$ the homology of the complex $(C_*^G, \overline{\partial}_*)$.

Let us now come back to our specific context. Until the end of the section, we denote by M a boundary irreducible aspherical manifold such that ∂M is also aspherical. Let $p \colon \widetilde{M} \to M$ be the universal covering of M, and observe that $\widetilde{M}$ is contractible. Since M is boundary irreducible, the restriction of p to any connected component of $\partial \widetilde{M}$ is a universal covering of a component of ∂M. Moreover, since ∂M is aspherical, every component of $\partial \widetilde{M}$ is contractible.

Let us fix an identification of $\pi_1(M)$ with the group Γ of the covering automorphisms of $p \colon \widetilde{M} \to M$. Of course, every element of Γ acts on the pair $(\widetilde{M}, \partial \widetilde{M})$, so each of the complexes

$$C_*(\widetilde{M}; R), \quad C_*(\partial \widetilde{M}; R), \quad C_*(\widetilde{M}, \partial \widetilde{M}; R)$$

is endowed with a structure of $R\Gamma$-complex. By the very definitions we also have isometric isomorphisms

$$\begin{array}{rcl} H_*(C_*^\Gamma(\widetilde{M};R)) & \cong & H_*(M;R), \\ H_*(C_*^\Gamma(\partial\widetilde{M};R)) & \cong & H_*(\partial M;R), \\ H_*(C_*^\Gamma(\widetilde{M},\partial\widetilde{M};R)) & \cong & H_*(M,\partial M;R)\ . \end{array}$$

Let us introduce the complexes of straight chains. For every $n \in \mathbb{N}$ we define $SC_n(\widetilde{M};R)$ as the free R-module generated by the $(n+1)$-tuples in $\widetilde{M}^{n+1}$. We say that an $(n+1)$-tuple in $\widetilde{M}^{n+1}$ is a *boundary* $(n+1)$-tuple if there exists a connected component of $\partial\widetilde{M}$ that contains all its elements, and we define $SC_n(\partial\widetilde{M};R) \subseteq SC_n(\widetilde{M};R)$ as the free R-submodule generated by the boundary $(n+1)$-tuples. We also set $SC_n(\widetilde{M},\partial\widetilde{M};R) = SC_n(\widetilde{M};R)/SC_n(\partial\widetilde{M};R)$. The diagonal action of Γ on $\widetilde{M}^{n+1}$ endows each of $SC_n(\widetilde{M};R)$, $SC_n(\partial\widetilde{M};R)$ and $SC_n(\widetilde{M},\partial\widetilde{M};R)$ with a natural structure of $R\Gamma$-module. The map

$$\widetilde{M}^{n+1} \ \ni\ (x_0,\ldots,x_n) \mapsto \sum_{i=0}^n (-1)^i (x_0,\ldots,\widehat{x}_i,\ldots,x_n)\ \in SC_{n-1}(\widetilde{M};R)$$

extends to an $R\Gamma$-map $SC_n(\widetilde{M};R) \to SC_{n-1}(\widetilde{M};R)$. This endows $SC_*(\widetilde{M};R)$ with a structure of $R\Gamma$-complex. It is clear that $SC_*(\partial\widetilde{M};R)$ is an $R\Gamma$-subcomplex of $SC_*(\widetilde{M};R)$, so that $SC_*(\partial\widetilde{M};R)$ and $SC_*(\widetilde{M},\partial\widetilde{M};R)$ are also endowed with structures of $R\Gamma$-complexes.

The map which takes any singular k-simplex into the $(k+1)$-tuple of its vertices extends to an $R\Gamma$-map

$$r_*\colon C_*(\widetilde{M};R) \to SC_*(\widetilde{M};R)\ ,$$

which induces in turn $R\Gamma$-maps (still denoted by r_*)

$$r_*\colon C_*(\partial\widetilde{M};R) \to SC_*(\partial\widetilde{M};R), \quad r_*\colon C_*(\widetilde{M},\partial\widetilde{M};R) \to SC_*(\widetilde{M},\partial\widetilde{M};R)\ .$$

The following lemma constructs a right inverse for r_*.

Lemma 4.1. *There exists a $R\Gamma$-chain map of complexes*

$$i_*\colon SC_*(\widetilde{M};R) \to C_*(\widetilde{M};R)$$

such that the following conditions hold:

(1) for every $(x_0,\ldots,x_k) \in \widetilde{M}^{k+1}$ the chain $i_*(x_0,\ldots,x_k)$ consists of a single simplex with vertices $x_0,\ldots,x_k$; in particular, the composition

$$r_* \circ i_*\colon SC_*(\widetilde{M};R) \to C_*(\widetilde{M};R)$$

is equal to the identity;

(2) *i_* takes any boundary $(k+1)$-tuple into a simplex whose image is completely contained in $\partial\widetilde{M}$ (whence in a connected component of $\partial\widetilde{M}$); in particular, i_* induces $R\Gamma$-chain maps (still denoted by i_*)*

$$\begin{array}{rcl} i_* & : & SC_*(\partial\widetilde{M};R)\to C_*(\partial\widetilde{M};R),\\ i_* & : & SC_*(\widetilde{M},\partial\widetilde{M};R)\to C_*(\widetilde{M},\partial\widetilde{M};R)\ . \end{array}$$

Proof. Let us choose a set of representatives $V_k(\partial\widetilde{M})$ for the action of Γ on $\partial\widetilde{M}^{k+1}$, and extend it to a set of representatives $V_k(\widetilde{M})$ for the action of Γ on $\widetilde{M}^{k+1}$. Then $SC_k(\widetilde{M})$ (resp. $SC_k(\partial\widetilde{M})$) is canonically identified with the free $R\Gamma$-module with basis $V_k(\widetilde{M})$ (resp. $V_k(\partial\widetilde{M})$).

Let us define i_k by induction on k as follows. In order to satisfy condition (1), we define i_0 by setting $i_0(x)=\sigma_x$ for every $x\in\widetilde{M}$, where σ_x is the 0-simplex whose image is equal to x.

If $k\geq 1$, we first define i_k on $V_k(\widetilde{M})$ as follows. Since $\widetilde{M}$ is contractible, for every $(x_0,\ldots,x_k)\in V_k(\widetilde{M})$ there exists a singular simplex $\sigma_{(x_0,\ldots,x_k)}$ such that $\partial\sigma_{(x_0,\ldots,x_k)}=i_{k-1}(\partial(x_0,\ldots,x_k))$. If $(x_0,\ldots,x_k)$ is a boundary $(k+1)$-tuple, then there exists a connected component $\widetilde{B}$ of $\partial\widetilde{M}$ such that $x_i\in\widetilde{B}$ for every $i=0,\ldots,k$. Our inductive hypothesis ensures that the image of each $(k-1)$-dimensional singular simplex appearing in $i_{k-1}(\partial(x_0,\ldots,x_k))$ is contained in $\widetilde{B}$. By asphericity of $\widetilde{B}$, we may then choose $\sigma_{(x_0,\ldots,x_k)}$ in such a way that its image is contained in $\widetilde{B}$. We may finally define i_k on the whole $SC_k(\widetilde{M};R)$ by linearly extending (over $R\Gamma$) the map

$$(x_0,\ldots,x_k)\mapsto\sigma_{(x_0,\ldots,x_k)}\ ,$$

which finishes the proof of the lemma. □

The composition $i_*\circ r_*$ defines maps

$$C_*(\widetilde{M};R)\to C_*(\widetilde{M};R)\,,\qquad C_*(\widetilde{M},\partial\widetilde{M};R)\to C_*(\widetilde{M},\partial\widetilde{M};R)\ .$$

With an abuse, we denote both such maps by $\widetilde{\mathrm{str}}_*$. Being Γ-equivariant, the maps $\widetilde{\mathrm{str}}_*$ induce maps

$$C_*(M;R)\to C_*(M;R)\,,\qquad C_*(M,\partial M;R)\to C_*(M,\partial M;R)\ ,$$

that will be both denoted by str_*.

Lemma 4.2. *The map*

$$\mathrm{str}_*\colon C_*(M,\partial M;R)\to C_*(M,\partial M;R)$$

is homotopic to the identity.

Proof. It is sufficient to show that the $R\Gamma$-chain map

$$\widetilde{\mathrm{str}}_*\colon C_*(\widetilde{M},\partial\widetilde{M};R)\to C_*(\widetilde{M},\partial\widetilde{M};R)$$

is homotopic to the identity via an $R\Gamma$-homotopy. We construct an $R\Gamma$-map $T_*\colon C_*(\widetilde{M};R)\to C_{*+1}(\widetilde{M};R)$ such that $T_{k-1}\partial_k+\partial_{k+1}T_k=\mathrm{Id}_k-i_k\circ r_k$ and

$T_k(C_k(\partial\widetilde{M};R)) \subseteq C_{k+1}(\partial\widetilde{M};R)$ for every $k\in\mathbb{N}$ (where we understand that $T_{-1}\partial_0=0$ since $\partial_0=0$).

Since $r_0 i_0=\mathrm{Id}_0$ we may set $T_0=0$. Then we define T_k inductively on k as follows. We choose a set of representatives $W_k(\partial\widetilde{M})$ for the action of Γ on the set of singular k-simplices with values in $\partial\widetilde{M}$, and extend it to a set of representatives $W_k(\widetilde{M})$ for the action of Γ on the set of singular k-simplices with values in $\widetilde{M}$.

Let us take $\sigma\in W_k(\widetilde{M})$ and consider the cycle

$$z(\sigma)=T_{k-1}(\partial_k\sigma)+\sigma-i_k(r_k(\sigma)) .$$

Using the inductive hypothesis $T_{k-2}\partial_{k-1}+\partial_k T_{k-1}=\mathrm{Id}_{k-1}-i_{k-1}\circ r_{k-1}$ one checks that $\partial_k z(\sigma)=0$. Recall that $k\geq 1$ and that $\widetilde{M}$ is contractible, whence acyclic. Therefore, the cycle $z(\sigma)$ is a boundary, and there exists $z'(\sigma)\in C_k(\widetilde{M};R)$ such that $\partial_{k+1}z'(\sigma)=z(\sigma)$. Furthermore, if $\sigma\in W_k(\partial\widetilde{M})$, then the fact that every component of $\partial\widetilde{M}$ is contractible implies that $z'(\sigma)$ may be chosen to belong to $C_{k+1}(\partial\widetilde{M};R)$. We may now define T_k by linearly extending (over $R\Gamma$) the map

$$\sigma\mapsto z'(\sigma) ,$$

and this concludes the proof. □

The map str_* may be regarded as a topological (relative) straightening: just as the straightening defined by Thurston in the context of hyperbolic manifolds, the map str_* is homotopic to the identity, and its image contains a representative for each class of singular simplices of M, where two simplices are considered equivalent if they share the same vertices and the homotopy classes (relative to the endpoints) of the edges. Accordingly, simplices that lie in the image of str_* are called *straight*.

Lemma 4.3. *Let σ be a straight k-simplex, $k\geq 2$ with image in M, and suppose that σ is not supported on ∂M. Then:*

(1) at most one $(k-1)$-face of σ lies on ∂M;
(2) if $k=3$ and no 2-face of σ lies on ∂M, then at most two edges of σ lie on ∂M;
(3) if $k=3$ then there exist at most three edges of σ in ∂M.

Proof. (1) Let $\widetilde{\sigma}$ be a fixed lift of σ to $\widetilde{M}$. If σ has two faces on ∂M, then the vertices of $\widetilde{\sigma}$ are contained in the same connected component of $\partial\widetilde{M}$. Since σ is straight, this implies that $\widetilde{\sigma}$ is supported on $\partial\widetilde{M}$, so σ is supported on ∂M.

(2) Suppose that σ has at least three edges on ∂M. Since $k=3$, the union of the corresponding edges of $\widetilde{\sigma}$ is connected, so at least three vertices of $\widetilde{\sigma}$ lie on the same connected component of $\partial\widetilde{M}$. Since σ is straight, this implies that at least one face of σ lies on ∂M.

(3) If four edges of σ lie on ∂M, then as in (2), the union of the corresponding edges of $\widetilde{\sigma}$ is connected. But the vertices of these four edges of the 3-simplex are all the vertices of the 3-simplex, which all lie on the same connected component of $\partial\widetilde{M}$, implying that σ is supported on ∂M. □

The fact that M supports a relative straightening can be used to improve Proposition 2.6 as follows:

Proposition 4.4. *Suppose that M is an aspherical and boundary irreducible n-manifold, $n \geq 2$, and that ∂M is also aspherical. Let $\varepsilon > 0$ be fixed. Then, there exists a relative integral cycle $z \in C_n(M, \partial M; \mathbb{Z})$ with associated pseudomanifold P such that the following conditions hold:*

(1) $[z] = d \cdot [M, \partial M]_{\mathbb{Z}}$ in $H_n(M, \partial M; \mathbb{Z})$ for some integer $d > 0$, and

$$\frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M, \partial M\| + \varepsilon \ ;$$

(2) every singular simplex appearing in z is straight;

(3) every simplex of P has at most one $(n-1)$-dimensional boundary face;

(4) if $n = 3$, then every simplex of P without 2-dimensional boundary faces has at most two edges contained in $\partial|P|$ and every simplex has at most three edges in $\partial|P|$.

Proof. Let z' be an integral cycle satisfying conditions (1) and (2) of Proposition 2.6, and set $z = \mathrm{str}_n(z') \in C_n(M, \partial M; \mathbb{Z})$. Point (1) descends from the fact that the straightening operator is norm nonincreasing and homotopic to the identity, while point (2) is obvious. Points (3) and (4) follow from point (2) and Lemma 4.3. □

Proposition 4.4 implies the following:

Proposition 4.5. *Let M be a boundary irreducible aspherical n-manifold, $n \geq 2$, and assume that ∂M is also aspherical. Then*

$$\|M, \partial M\| \geq \|\partial M\| \ .$$

Proof. Take $\varepsilon > 0$. If z is chosen as in Proposition 4.4, then $\|\partial z\|_{\mathbb{Z}} \leq \|z\|_{\mathbb{Z}}$, so

$$\|\partial M\| \leq \frac{\|\partial z\|_{\mathbb{Z}}}{d} \leq \frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M, \partial M\| + \varepsilon \, ,$$

which proves the proposition since ε is arbitrary. □

5. Proofs of Theorems 4 and 5 and Corollary 6

Let us first concentrate our attention on the proof of Theorem 4. So, we suppose that M is an aspherical boundary irreducible 3-manifold. As usual, we may also suppose that M is oriented. In order to exploit the machinery introduced in the preceding section, we first reduce to the case when ∂M is also aspherical. So, let us suppose that a component S of ∂M is a sphere. Since M is aspherical, S is homotopically trivial, whence homologically trivial, in M. This implies that $\partial M = S$, so $\|\partial M\| = 0$ and the conclusion of Theorem 4 is trivially satisfied. (In fact, using the Poincaré conjecture one can prove that the only aspherical 3-manifold with at least one spherical boundary component is the ball.) Therefore, henceforth we suppose that ∂M is also aspherical.

We denote by z the cycle provided by Proposition 4.4, and by P the associated pseudomanifold. As usual, let $z_P \in C_3(|P|, \partial|P|; \mathbb{Z})$ be the relative cycle represented by the (signed) sum of the simplices of P, and let $f\colon (|P|, \partial|P|) \to (M, \partial M)$ be such that $f_*(z_P) = z$. Also recall from Lemma 2.2 that the space $\partial|P|$ is an orientable surface. If $[\partial P]_{\mathbb{Z}}$ is the sum of the integral fundamental classes of the components of ∂P, then $[\partial P]_{\mathbb{Z}} = [\partial z_P]$ (we prefer here the notation $[\partial P]_{\mathbb{Z}}$ rather than the heavier $[\partial |P|]_{\mathbb{Z}}$). Therefore, the equality $f_*(z_P) = d \cdot [M, \partial M]_{\mathbb{Z}}$ implies that

$$(f|_{\partial|P|})_*([\partial P]_{\mathbb{Z}}) = d \cdot [\partial M]_{\mathbb{Z}} \ ,$$

where $[\partial M]_{\mathbb{Z}}$ is the sum of the integral fundamental classes of the components of ∂M. This equality implies that

$$(f|_{\partial|P|})_*([\partial P]) = d \cdot [\partial M] \ , \tag{13}$$

where $[\partial P]$ (resp. $[\partial M]$) is the sum of the *real* fundamental classes of the components of $\partial|P|$ (resp. of ∂M).

Let Ω_i, for $i = 0, \ldots, 4$, be the set of simplices of P having exactly i boundary 2-faces. As usual, we denote by t_i the number of elements of Ω_i. By Proposition 4.4 we have $\Omega_2 = \Omega_3 = \Omega_4 = \emptyset$, so

$$t_2 = t_3 = t_4 = 0, \qquad \|z\|_{\mathbb{Z}} = t_0 + t_1 \ .$$

Since $\partial|P|$ admits a triangulation with t_1 triangles we have $t_1 \geq \|\partial|P|\|$. Also, Equation (13) implies that $\|\partial|P|\| \geq d\|\partial M\|$, so

$$t_1 \geq d \cdot \|\partial M\| \ . \tag{14}$$

In order to prove Theorem 4 we now need to bound t_0 from below. Let us first introduce some definitions. We say that an edge e of the 2-dimensional pseudomanifold ∂P is *nice* if e is the edge of at least one simplex in Ω_0. We also say that an edge of ∂P is *bad* if it is not nice.

Lemma 5.1. *Let e be a bad edge of ∂P, let $T_j, T_{j'}$ be the triangles of ∂P adjacent to e, and let Δ^3_j (resp. $\Delta^3_{j'}$) be the simplex of P containing T_j (resp. $T_{j'}$). If F_j (resp. $F_{j'}$) is the 2–face of Δ^3_j (resp. $\Delta^3_{j'}$) such that $e = F_j \cap T_j = F_{j'} \cap T_{j'}$, then $F_j, F_{j'}$ are glued to each other in P.*

Proof. Let $\overline{\Delta^3}$ be the simplex of P glued to Δ^3_j along F_j. Denote by $\overline{F}$ the face of $\overline{\Delta^3}$ paired to F_j. We consider separately the following cases.

(1) Suppose that $\overline{\Delta^3} \neq \Delta^3_j$ and $\overline{\Delta^3} \neq \Delta^3_{j'}$. Since e is bad, at least one face $\overline{F}$ of $\overline{\Delta^3}$ has to lie on $\partial|P|$. Moreover, the conditions $\overline{\Delta^3} \neq \Delta^3_j$, $\overline{\Delta^3} \neq \Delta^3_{j'}$ imply that $\overline{F}$ cannot contain e, because otherwise e would be adjacent to 3

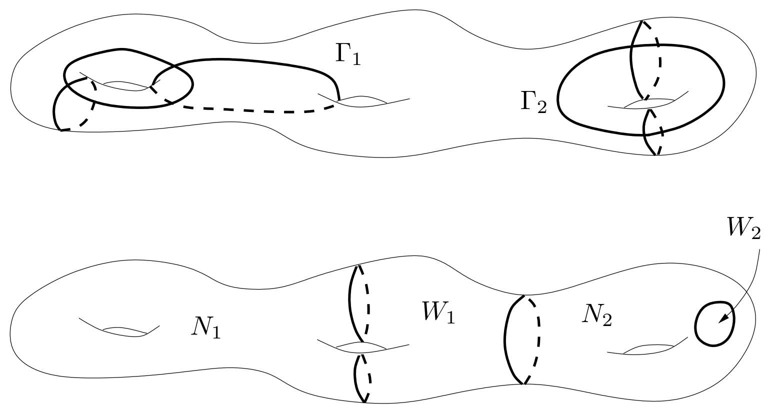

Figure 1: N_i is a regular neighbourhood of Γ_i on $\partial|P|$ for $i = 1, 2$.

boundary faces of P (counted with multiplicities). Therefore, $\overline{\Delta^3}$ contains four edges which lie on $\partial|P|$, and this contradicts point (3) of Lemma 4.3.

(2) Suppose that $\overline{\Delta^3} = \Delta^3_j$. Since Δ^3_j has at most three edges on $\partial|P|$, the unique boundary edge of F_j is paired to the unique boundary edge of $\overline{F}$. As a consequence we have $\overline{F} = F_{j'}$, as desired (note that in this case we have $T_j = T_{j'}$ and $\Delta^3_j = \Delta^3_{j'}$).

(3) Suppose that $\overline{\Delta^3} = \Delta^3_{j'}$. Using that $\Delta^3_{j'}$ has at most three edges on $\partial|P|$, one may easily show that $\overline{F} = F_{j'}$, whence the conclusion. □

We denote by $\Gamma \subseteq \partial|P|$ the union of all the nice edges of ∂P. Then, Γ is a (possibly disconnected) graph in $\partial|P|$.

Let Γ_i, for $i = 1, \ldots, s$ be the connected components of Γ. For each i we denote by $N_i = N(\Gamma_i)$ a closed regular neighbourhood of Γ_i in $\partial|P|$, chosen in such a way that $N_i \cap T$ is a regular neighbourhood of $\Gamma_i \cap T$ for every triangle T of ∂P. We also set $N = \cup_{i=1}^s N_i$ and $W = \overline{\partial|P| \setminus N}$. Finally, we denote by $W_1, \ldots, W_r$ the components of W (see Figure 1).

We will prove that $f|_{\partial|P|}$ is homotopic to a map which is constant on each W_i. Since $f|_{\partial|P|}$ has degree d, this will imply that N has to be sufficiently complicated, and this will prove in turn that the number of nice edges of ∂P cannot be too small.

Lemma 5.2. *Let $\gamma \subseteq W$ be a loop. Then $f(\gamma)$ is null-homotopic in ∂M.*

Proof. Let W_i be the component of W containing γ. Every edge that intersects W_i is bad, so Lemma 5.1 implies that the inclusion $W_i \hookrightarrow \partial|P|$ extends to an inclusion $C(W_i) \hookrightarrow |P|$, where $C(W_i)$ is the topological cone over W_i. Since $C(W_i)$ is contractible, this implies in turn that the loop γ is null-homotopic in P.

Let $i\colon \partial|P| \to |P|$ and $j\colon \partial M \to M$ be the inclusions. Since $i(\gamma)$ is null-homotopic in $|P|$, the loop $j(f(\gamma)) = f(i(\gamma))$ is null-homotopic in M.

Due to the boundary irreducibility of M, this implies in turn that $f(\gamma)$ is null-homotopic in ∂M. $\square$

Corollary 5.3. *There exists a map $g\colon \partial|P| \to \partial M$ homotopic to $f|_{\partial|P|}$ and such that $g|_{W_i}$ is constant for every $i = 1, \ldots, r$.*

Proof. Each component W_i of W is a compact orientable surface. Since ∂M is aspherical, by Lemma 5.2 the map $f|_{W_i}$ can be homotoped to a constant map $g_i\colon W_i \to \partial M$ via a homotopy $H_i\colon W_i \times [0,1] \to \partial M$ such that $H_i(x,0) = g_i(x)$ and $H_i(x,1) = f(x)$ for every $x \in W_i$. We now need to define a global map $g\colon \partial|P| \to \partial M$ such that $g|_{W_i} = g_i$ for every i.

We piecewise define g as follows. For every component γ of ∂W we denote by N_γ the component of N containing γ. We also fix a collar $C(\gamma) \cong \gamma \times [0,1]$ of γ in N_γ, in such a way that γ is identified with $\gamma \times \{0\} \subseteq C(\gamma)$ and all the chosen collars are disjoint, and we set

$$N' = \overline{N \setminus \bigcup_{\gamma \subseteq \partial W} C(\gamma)} \ .$$

Of course, N' is homeomorphic to N. More precisely, there exists a homeomorphism $t\colon N \to N'$ such that the composition $N \to N' \hookrightarrow N$ is homotopic to the identity of N, and $t(x,0) = (x,1)$ for every $x \in \gamma \subseteq C(\gamma) \cong \gamma \times [0,1]$, where γ is any component of $\partial N = \partial W$. We set $g|_{N'} = f|_N \circ t^{-1}$. It remains to properly define g on the annuli $C(\gamma)$. To this aim, if γ is any component of ∂W_i and $(x,s) \in \gamma \times [0,1] \cong C(\gamma)$, then we set $g(x,s) = H_i(x,s)$. It is easy to check that the resulting map g is well-defined, continuous and homotopic to f. $\square$

The following proposition provides the key step in the proof of Theorem 4. We denote by E_{nice} the number of nice edges of ∂P.

Proposition 5.4. *We have*

$$2 \cdot E_{\text{nice}} \geq 4 \cdot \sharp\{\text{connected components of } \partial M \text{ not } \cong S^2\} + d \cdot \|\partial M\| \ .$$

Proof. For every $i = 1, \ldots, s$, let S_i be the closed orientable surface obtained from N_i by collapsing to a point each connected component of ∂N_i (we understand that distinct components of ∂N_i give rise to distinct points). If e_i is the number of nice edges of Γ_i, then

$$\chi(S_i) \geq 1 + \chi(N_i) = 1 + \chi(\Gamma_i) \geq 2 - e_i \ .$$

Summing over i, we obtain

$$E_{\text{nice}} \geq 2 \cdot s - \sum_{i=1}^{s} \chi(S_i) \ .$$

Since for closed oriented surfaces S the simplicial volume is equal to $\|S\| = -2\chi(S)$ unless S is homeomorphic to the 2-sphere in which case $\|S^2\| =$

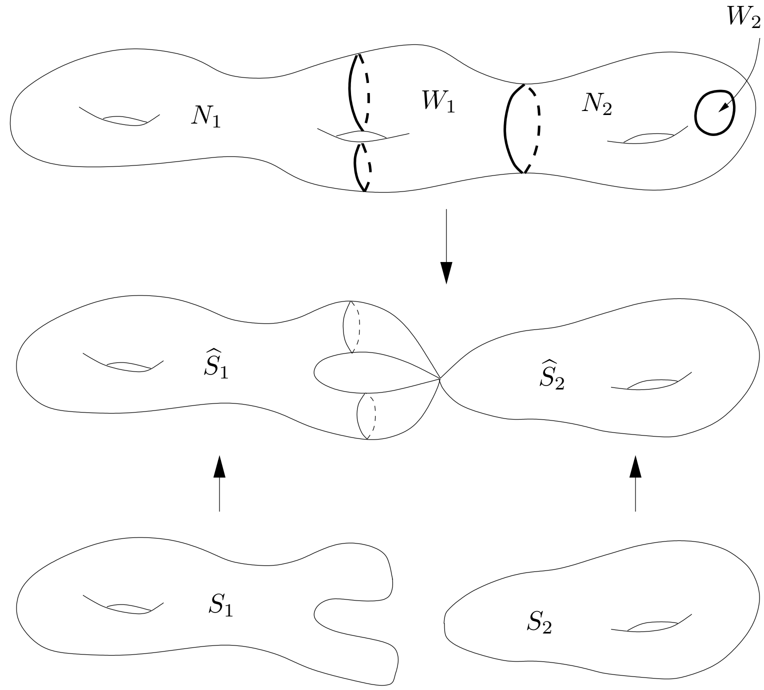

Figure 2: The construction described in Proposition 5.4.

$2-\chi(S^2)$, the latter inequality can be rewritten as

$$2 \cdot E_{\text{nice}} \geq 4 \cdot \sharp\{i \mid S_i \ncong S^2\} + \sum_{i=1}^{s} \|S_i\| \ .$$

It remains to show that

$$\sharp\{i \mid S_i \ncong S^2\} \geq \sharp\{\text{connected components of } \partial M \text{ not } \cong S^2\}$$

and

$$\sum_{i=1}^{s} \|S_i\| \geq d \cdot \|\partial M\| \ .$$

Let $\widehat{S}$ be the space obtained from $\partial|P|$ by collapsing to a point each connected component of W (again, we understand that distinct components of W give rise to distinct points), and let us denote by $\pi\colon \partial|P| \to \widehat{S}$ the quotient map (see Figure 2). We observe that S_i canonically projects onto $\widehat{S}$ and we denote by $p : \cup_{i=1}^{s} S_i \to \widehat{S}$ the resulting map.

Let us consider the map $g\colon \partial|P| \to \partial M$ provided by Corollary 5.3. Being constant on the components of W, the map g induces a map $\widehat{g}$ on $\widehat{S}$ and by precomposition by p a map $\widehat{g}_S$ on $\cup_{i=1}^{s} S_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\partial|P| \arrow[d, "\pi"] \arrow[dr, "g"] & \\
\widehat{S} \arrow[r, "\widehat{g}"'] & \partial M \\
\cup_{i=1}^s S_i \arrow[u, "p"] \arrow[ur, "\widehat{g}_S"'] &
\end{tikzcd}$$

The orientation on $\partial|P|$ induces an orientation on each N_i, whence on each S_i, endowing $\cup_{i=1}^s S_i$ and also $\widehat{S} = p(\cup_{i=1}^s S_i)$ with a fundamental class. Since g is homotopic to $f|_{\partial|P|}$, it follows from the commutativity of the previous diagram and the fact that $\pi_*([\partial|P|]) = [\widehat{S}] = p_*([\cup_{i=1}^s S_i])$ that

$$(\widehat{g}_S)_*([\cup_{i=1}^s S_i]) = \widehat{g}_*([\widehat{S}]) = g_*([\partial|P|]) = d \cdot [\partial M] \ , \tag{15}$$

where the last equality is a consequence of Equation (13). Thus, for every connected component M_0 of ∂M not homeomorphic to S^2 there exists at least one S_i not homeomorphic to S^2 and mapped to M_0 by $\widehat{g}_S$, proving the first desired inequality. Finally, since $(\widehat{g}_S)_*$ is norm nonincreasing, we obtain from (15) the second desired inequality

$$d \cdot \|\partial M\| = \|(\widehat{g}_S)_*([\cup_{i=1}^s S_i])\| \leq \sum_{i=1}^s \|S_i\| \ ,$$

which finishes the proof of the proposition. □

To conclude the proof of Theorem 4 note that by definition every nice edge of $\partial|P|$ is contained in at least one simplex in Ω_0. Moreover, by point (4) of Proposition 4.4 every simplex in Ω_0 has at most two edges on $\partial|P|$, so the inequality $t_0 \geq E_{\text{nice}}/2$ holds. Putting this inequality together with Proposition 5.4 and Inequality (14) we get

$$d(\|M, \partial M\| + \varepsilon) \geq \|z\| = t_0 + t_1 \geq d \cdot \frac{\|\partial M\|}{4} + d \cdot \|\partial M\| = \frac{5d}{4}\|\partial M\| \ ,$$

which proves the theorem since ε is arbitrary. □

Proof of Theorem 5. We show that the Δ-complexity of the product $M_g = S_g \times [0,1]$ of a surface of genus $g \geq 1$ with an interval is equal to

$$\sigma(M_g) = 10 \cdot (g-1) + 6 \ .$$

For the inequality $\sigma(M_g) \leq 10 \cdot (g-1) + 6$ we exhibit a topological triangulation of M_g with the prescribed amount of top dimensional simplices. Let us realize S_g as the space obtained by gluing the sides of a $4g$-gon, as described in Figure 3. We denote by p_i and e_i, $i = 0, \dots, 4g-1$, respectively the vertices and the sides of the $4g$-gon, in such a way that p_i and p_{i+1} are the vertices of e_i. We subdivide the $4g$-gon into $4g-2$ triangles $T_1, \dots, T_{4g-2}$ with p_0 as a common vertex, so that e_i is the side of T_i opposite to p_0 for $i = 1, \dots, 4g-2$.

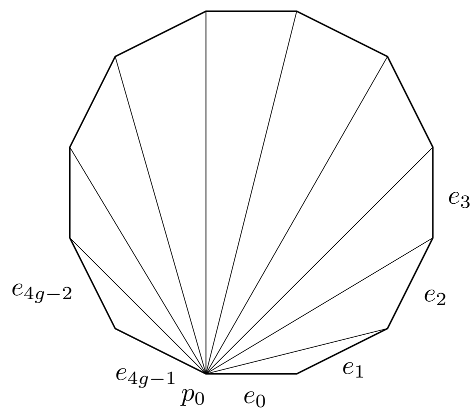

Figure 3: The triangulation of S_g in the case $g = 3$.

By taking the product with $[0,1]$, the triangulation of S_g just described defines a decomposition of M_g into $4g-2$ triangular prysms $Q_1, \ldots, Q_{4g-2}$ such that $Q_i = T_i \times [0,1]$ for every $i = 1, \ldots, 4g-2$.

For $i = 1, \ldots, 4g-2$, let $V_i \subseteq Q_i$ be the tetrahedron with vertices $(p_0, 0)$, $(p_0, 1)$, $(p_i, 1)$, $(p_{i+1}, 1)$ and let $W_i = \overline{Q_i \setminus V_i}$ (see Figure 4). Then, each W_i is a pyramid with rectangular basis $F_i = e_i \times [0,1]$ and vertex $(p_0, 0)$. We are going to suitably decompose the union of the W_i's in order to get a triangulation of M_g with $10g-4$ tetrahedra.

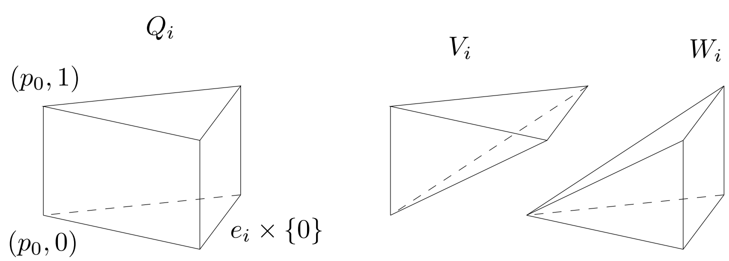

Figure 4: The prysm Q_i decomposes into the union of a tetrahedron V_i and a pyramid W_i.

Let us first observe that the rectangles F_0 and F_2 are identified in M_g. We are going to keep the V_i's in our final triangulation of M_g, and this choice already defines a subdivision of F_0 into two triangles. This forces us to subdivide W_2 into two tetrahedra W_2', W_2'' in such a way that the induced triangulation of F_2 matches the triangulation of F_0. In the same way we subdivide W_{4g-3} into two tetrahedra W_{4g-3}', W_{4g-3}'' so to match the corresponding subdivision of F_{4g-1}.

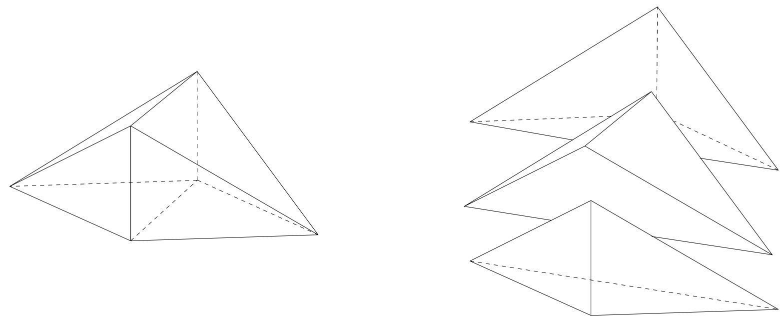

Figure 5: A decomposition of an octahedron into three tetrahedra.

Let us consider the remaining pyramids W_i, $i \notin \{0, 2, 4g-3, 4g-1\}$. These pyramids are glued in pairs along their rectangular faces. More precisely, if e_i is paired to e_j in the triangulation of S_g we started with, where $i, j \notin \{0, 2, 4g-3, 4g-1\}$, then W_i is glued to W_j along the identified faces $F_i = F_j$. We denote by Z_{ij} the octahedron that results from such a gluing. We have thus decomposed M_g into the union of $4g+2$ tetrahedra $V_1, \ldots, V_{4g-2}$, W_2, W_2', W_{4g-3}', W_{4g-3}'' and $2g-2$ octahedra. Each of these octahedra may be triangulated by 3 simplices as described in Figure 5, and this shows that M_g may indeed be triangulated with $10g-4$ tetrahedra.

It remains to prove the other inequality $\sigma(M_g) \geq 10 \cdot (g-1) + 6$. Let T be a triangulation of M_g with simplices $\Delta^3_1, \ldots, \Delta^3_N$. We need to show that $N \geq 10(g-1) + 6$.

We start by choosing a straightening operator on $(M_g, \partial M_g)$ with additional symmetry. To do so, we endow Σ_g with a hyperbolic structure and consider on Σ_g the barycentric straightening, associating to any singular simplex $\sigma\colon \Delta^q \to \Sigma_g$, for $q \geq 0$, a straightened simplex $\mathrm{str}_{\mathrm{bar}}(\sigma)$ (see [Rat94, Chapter 11] for details). Note that the barycentric straightening has the property that it does not depend, in constant curvature, on the order of the vertices of σ (in contrast to the straightening obtained by geodesic coning). Consider the resulting product straightening on $\Sigma_g \times [0,1]$, denoted by str, where on the interval $[0,1]$ we consider the affine straightening. Note that str still has the property that it does not depend on the order of the vertices.

For every $i = 1, \ldots, N$, we fix an orientation-preserving parameterization $\sigma_i\colon \Delta^3 \to \Delta^3_i$, and we say that a simplex of T is *inessential* if the image of $\mathrm{str}(\sigma_i)$ lies on ∂M_g, and *essential* otherwise. We order the simplices of T so that $\Delta^3_1, \ldots, \Delta^3_{N_0}$ are essential, and $\Delta^3_{N_0+1}, \ldots, \Delta^3_N$ are not. We now define an orientable pseudomanifold P as follows. The simplices of P bijectively correspond to the essential simplices of T, and gluings in P correspond to gluings between essential simplices of T. (This does not mean that P is identified with a subset of M, since, for example, two essential simplices of

T may share an edge because they are glued to the same inessential simplex, so they may intersect in M, while being disjoint in P.)

We define a map $\mathrm{str}_T\colon |P| \to M_g$ which corresponds to the simultaneous straightening of all the essential simplices of T. To define the map str_T on $|P|$, we choose, for every $p \in |P|$, an $i \in \{1, \ldots, N_0\}$ such that $p \in |\Delta_i^3| \subseteq |P|$, and we set $\mathrm{str}_T(p) = \mathrm{str}(\sigma_i)(q)$, where $q \in \Delta_i^3 \cong \Delta^3$ is any point which gets identified with p under the natural map $\Delta_i^3 \to |\Delta_i^3| \subseteq |P|$. Of course, if the point p belongs to the 2-skeleton of $|P|$, then there may be several choices of i and possibly also for $q \in \Delta^3$ (recall that distinct faces of Δ_i^3 may be identified in $|P|$). However, since our straightening does not depend on the order of the vertices, one may easily check that str_T is indeed well-defined and continuous. Let us further see that str_T is a map of pairs

$$\mathrm{str}_T\colon (|P|, \partial|P|) \to (M_g, \partial M_g)\ .$$

If F is any boundary face of $|P|$, then either F corresponds to a boundary face of T, or it corresponds to a face of T which is glued to an inessential simplex of T. In the first case we deduce that $\mathrm{str}_T(F) \subseteq \partial M_g$ from the fact that straightening preserves the space of singular simplices supported on ∂M_g. In the second case it is sufficient to observe that faces of inessential simplices are supported on ∂M_g.

Let us analyze the action of str_T on fundamental cycles. We first point out that, in general, we cannot assume that the sum $\sum_{i=1}^N \sigma_i$ is a relative cycle in $C_n(M_g, \partial M_g; \mathbb{Z})$. In fact, if two 2-faces of T are identified in M_g, then it may happen that the corresponding faces of the σ_i's, when considered as singular 2-simplices, differ by the precomposition with a nontrivial affine automorphism of the standard 2-simplex, and do not cancel each other in the algebraic boundary of $\sum_{i=1}^N \sigma_i$. This problem can be fixed by alternating each σ_i as follows. For any singular 3-simplex σ we define the chain

$$\mathrm{alt}(\sigma) = \frac{1}{(4)!} \sum_{\tau \in \mathfrak{S}_4} (-1)^{\mathrm{sgn}(\tau)} \sigma \circ \overline{\tau},$$

where $\overline{\tau}$ is the unique affine diffeomorphism of the standard 3-simplex Δ^3 corresponding to the permutation τ of the vertices of Δ^3. Now it is immediate that the real chain $z_M^{\mathbb{R}} = \mathrm{alt}(\sigma_1) + \ldots + \mathrm{alt}(\sigma_N)$ is a cycle which represents the relative fundamental class of M_g. (Note in passing that this construction may be exploited to prove the inequality $\|M, \partial M\| \leqslant \sigma(M)$ stated in the introduction, which holds for every 3-manifold M). Since we know that the straightening operator induces the identity on homology, the cycle $\mathrm{str}(z_M^{\mathbb{R}})$ also represents the relative fundamental class of M.

The cycle $\mathrm{str}(z_M^{\mathbb{R}})$ can be realized as the push-forward of a relative cycle in $C_n(P, \partial P; \mathbb{R})$ via str_T. To see this, we denote by $\hat{\sigma}_i\colon \Delta^3 \to |P|$ the singular simplex corresponding to $\sigma_i\colon \Delta^3 \to M_g$, $i = 1, \ldots, N_0$. Here also, the sum of the $\hat{\sigma}_i$'s does not provide in general an integral relative cycle for $(P, \partial P)$, so we have to recur to the real relative cycle $z_P^{\mathbb{R}} = \mathrm{alt}(\hat{\sigma}_1) + \ldots + \mathrm{alt}(\hat{\sigma}_{N_0})$. By contruction, the chains $(\mathrm{str}_T)_*(z_P^{\mathbb{R}})$ and $\mathrm{str}(z_M^{\mathbb{R}})$ differ just by a linear combination of simplices supported on ∂M_g. As a consequence, they define the same element of $C_n(M_g, \partial M_g; \mathbb{R})$, so $(\mathrm{str}_T)_*(z_P^{\mathbb{R}})$ is a relative fundamental cycle of M_g.

Since P is an orientable 3-dimensional pseudomanifold, every connected component of $\partial|P|$ is a closed orientable surface. If we denote by $[\partial P]$ the sum of the real fundamental classes of the components of $\partial|P|$, then our previous considerations imply that $(\mathrm{str}_T)_*([\partial P])$ is equal to the sum of the fundamental classes of the boundary components $\partial_0 M_g$, $\partial_1 M_g$ of M_g. In particular, for $i = 0, 1$, there exist components $\partial_i|P|$ of $\partial|P|$ such that the restriction $\mathrm{str}_T|_{\partial_i|P|} \colon \partial_i|P| \to \partial_i M_g$ has positive degree. This implies that $g_i \geq g$, where g_i is the genus of $\partial_i|P|$. As usual, we denote by t_j the number of simplices of P with exactly j boundary faces, and we recall that $t_2 = t_3 = t_4 = 0$, so that $N_0 = t_0 + t_1$. Since the Δ-complexity of a closed surface of genus $\overline{g}$ is $4\overline{g} - 2$ we get

$$t_1 \geq (4g_0 - 2) + (4g_1 - 2) \geq 8g - 4 \ . \tag{16}$$

Just as in the computations leading to Theorem 4, we now need to bound t_0 from below. To this aim we exploit Proposition 5.4 with $d = 1$. Actually, the proposition is stated for pseudomanifolds associated to integral cycles, but the proof carries through without changes in our present setting of a pseudomanifold P with a map of pairs

$$\mathrm{str}_T \colon (|P|, \partial|P|) \to (M_g, \partial M_g)$$

sending the real fundamental class of $(|P|, \partial|P|)$ to the real fundamental class of $(M_g, \partial M_g)$. Keeping notations and terminology from above, we denote by E_{nice} the number of nice edges of P, and we recall that $t_0 \geq E_{\mathrm{nice}}/2$. Since M_g has two boundary components each of which is not homeomorphic to S^2, Proposition 5.4 implies that

$$t_0 \geq \frac{E_{\mathrm{nice}}}{2} \geq \frac{8 + \|\partial M_g\|}{4} = 2g \ .$$

Putting together this inequality with Inequality (16) we get

$$N \geq N_0 = t_0 + t_1 \geq 2g + 8g - 4 = 10g - 4 \ ,$$

which concludes the proof. □

Proof of Corollary 6. Using that the stable Δ-complexity bounds the simplicial volume from above and applying Theorem 4, we have the inequalities

$$\sigma_\infty(M_g) \geq \|M_g, \partial M_g\| \geq \frac{5}{4}\|\partial M_g\| = 10(g-1) \ .$$

It remains to see that the stable Δ-complexity of M_g is smaller than or equal to $10(g-1)$. For every $d \geq 2$ the manifold M_g admits a covering of degree d whose total space is homeomorphic to $M_{g'}$, where $g' = d(g-1)+1$. By Theorem 5, this implies that

$$\sigma_\infty(M_g) \leq \frac{\sigma(M_{d(g-1)+1})}{d} \leq \frac{10d(g-1)+6}{d} = 10(g-1) + \frac{6}{d} \ .$$

Since d is arbitrary, the corollary is proved. □

6. Hyperbolic manifolds with geodesic boundary

In the context of hyperbolic manifolds, the straightening procedure introduced in Section 4 admits a useful geometric description, which dates back to Thurston [Thu79]: the universal covering of a hyperbolic n-manifold with geodesic boundary is a convex subset of the hyperbolic space $\mathbb{H}^n$, and the support of any straight simplex is just the image of a geodesic simplex of $\mathbb{H}^n$ via the universal covering projection. As a consequence, to compute the simplicial volume of a hyperbolic manifold with geodesic boundary we may restrict to considering only cycles supported by (projections of) geodesic simplices.

Geodesic simplices. Let $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$ be the usual compactification of the hyperbolic space $\mathbb{H}^n$. We recall that every pair of points of $\overline{\mathbb{H}^n}$ is connected by a unique geodesic segment (which has infinite length if any of its endpoints lies in $\partial\mathbb{H}^n$). A subset in $\overline{\mathbb{H}^n}$ is *convex* if whenever it contains a pair of points it also contains the geodesic segment connecting them. The *convex hull* of a set A is defined as usual as the intersection of all convex sets containing A. A *(geodesic) k-simplex* Δ in $\overline{\mathbb{H}^n}$ is the convex hull of $k+1$ points in $\overline{\mathbb{H}^n}$, called *vertices*. We say that a k-simplex is:

- *ideal* if all its vertices lie in $\partial\mathbb{H}^n$,
- *regular* if every permutation of its vertices is induced by an isometry of $\mathbb{H}^n$,
- *degenerate* if it is contained in a $(k-1)$-dimensional subspace of $\overline{\mathbb{H}^n}$.

As above, we denote by v_n the volume of the regular ideal simplex in $\overline{\mathbb{H}^n}$. The following result characterizes hyperbolic geodesic simplices of maximal volume, and plays a fundamental role in the study of the simplicial volume of hyperbolic manifolds:

Theorem 6.1 ([HM81, Pey02])**.** *Let Δ be an n-simplex in $\overline{\mathbb{H}^n}$. Then $\mathrm{Vol}(\Delta) \leqslant v_n$, with equality if and only if Δ is ideal and regular.*

Let Δ be a nondegenerate geodesic n-simplex, and let E be an $(n-2)$-dimensional face of Δ. The *dihedral angle* $\alpha(\Delta, E)$ of Δ at E is defined as follows: let p be a point in $E \cap \mathbb{H}^n$, and let $H \subseteq \mathbb{H}^n$ be the unique 2-dimensional geodesic plane which intersects E orthogonally in p. We set $\alpha(\Delta, E)$ to be equal to the angle in p of the polygon $\Delta \cap H$ of $H \cong \mathbb{H}^2$. Observe that this definition is independent of p.

From the computation of the dihedral angle of the regular ideal geodesic n-simplex, together with the fact that geodesic simplices of almost maximal volume are close in shape to regular ideal simplices, one deduces:

Lemma 6.2. *Let $n \geqslant 4$. Then, there exists $\varepsilon_n > 0$, depending only on n, such that the following condition holds: if $\Delta \subseteq \overline{\mathbb{H}^n}$ is a geodesic n-simplex* *such that* $\mathrm{Vol}(\Delta) \geqslant (1-\varepsilon_n)v_n$ *and* α *is the dihedral angle of* Δ *at any of its* $(n-2)$*-faces, then*

$$2 < \frac{\pi}{\alpha} < 3 \ .$$

We refer the reader to [FFM, Lemma 2.16] for a proof.

Geometric straightening and volume form. Let us come back to the definition of straightening for simplices in hyperbolic manifolds. Henceforth we denote by M a hyperbolic manifold with geodesic boundary. As usual, we also assume that M is oriented.

The universal covering $\widetilde{M}$ of M is a convex subset of $\mathbb{H}^n$ bounded by a countable family of disjoint geodesic hyperplanes (see *e.g.* [Koj90]). If $\sigma\colon \Delta^k \to \widetilde{M}$ is a singular k-simplex, then we may define the simplex $\widetilde{\mathrm{str}}_k(\sigma)$ as follows: set $\widetilde{\mathrm{str}}_k(\sigma)(v) = \sigma(v)$ on every vertex v of Δ^k, and extend using barycentric coordinates (see [Rat94, Chapter 11]) or by an inductive cone construction (which exploits the fact that any pair of points in $\widetilde{M}$ is joined by a unique geodesic, that continuously depends on its endpoints – see *e.g.* [FP10, Section 3.1] for full details). The image of $\widetilde{\mathrm{str}}_k(\sigma)$ is the geodesic simplex spanned by the vertices of σ. This map is indeed a straightening in the sense of Section 4, and defines therefore a map

$$\mathrm{str}_*\colon C_*(M,\partial M;R) \to C_*(M,\partial M;R)$$

which is homotopic to the identity.

Let $\sigma\colon \Delta^n \to M$ be a smooth n-simplex, and let ω be the volume form of M. We set

$$\mathrm{Vol}_{\mathrm{alg}}(\sigma) = \int_{\Delta^n} \sigma^*(\omega) \ .$$

Since straight simplices are smooth, the map

$$C_n(M,\partial M;\mathbb{R}) \to \mathbb{R}, \qquad \sum_{i=1}^{n} a_i\sigma_i \mapsto \sum_{i=1}^{n} a_i \mathrm{Vol}_{\mathrm{alg}}(\mathrm{str}_n(\sigma_i))$$

is well-defined. This map is a relative cocycle that represents the volume coclass on M (see *e.g.* [FP10, Section 4] for the details). Therefore, if $z = \sum_{i=1}^{h} a_i\sigma_i \in C_n(M,\partial M;\mathbb{Z})$ is an integral cycle supported by straight simplices such that $[z] = d\cdot[M,\partial M]_{\mathbb{Z}}$, then

$$\sum_{i=1}^{h} a_i \mathrm{Vol}_{\mathrm{alg}}(\sigma_i) = d \cdot \mathrm{Vol}(M) \ . \tag{17}$$

Let us rewrite z as follows:

$$z = \sum_{i=1}^{N} \varepsilon_i\sigma_i \ ,$$

where $\varepsilon_i = \pm 1$ for every $i = 1,\dots,N$. Note that we do not assume that $\sigma_i \neq \sigma_j$ for $i \neq j$. Let P be the pseudomanifold associated to z, and recall that the simplices $\Delta^n_1,\dots,\Delta^n_N$ of P are in bijection with the σ_i's. An identification of Δ_i^n with the standard n-simplex is fixed for every $i = 1, \ldots, N$, so that we may consider σ_i as a map defined on Δ_i^n. We set

$$\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = \varepsilon_i \mathrm{Vol}_{\mathrm{alg}}(\sigma_i) \ , \tag{18}$$

and we say that Δ_i^n is *positive* (resp. *degenerate*, *negative*) if $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) > 0$ (resp. $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = 0$, $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) < 0$). Equation (17) may now be rewritten as follows:

$$\sum_{i=1}^{N} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = d \cdot \mathrm{Vol}(M) \ . \tag{19}$$

If $\widetilde{\sigma}_i$ is any lift of σ_i to $\widetilde{M} \subseteq \mathbb{H}^n$, then Δ_i^n is degenerate if and only if the image of $\widetilde{\sigma}_i$ is. Since $|\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)|$ is just the volume of the image of $\widetilde{\sigma}_i$, by Theorem 6.1 we have

$$|\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)| \leq v_n \ .$$

If Δ_i^n is nondegenerate and F is an $(n-2)$-face of Δ_i^n, then we define the angle of Δ_i^n at F as the angle of the image of $\widetilde{\sigma}_i$ at $\widetilde{\sigma}_i(F)$.

Lemma 6.3. *Let F be an $(n-2)$-face of ∂P, and let $\Delta_{i_1}^n, \ldots, \Delta_{i_k}^n$ be the simplices of P that contain F (taken with multiplicities). For every $j = 1, \ldots, k$ we also suppose that $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) > 0$, so in particular $\Delta_{i_j}^n$ is nondegenerate, and has a well-defined angle α_{i_j} at F. Then*

$$\sum_{j=1}^{k} \alpha_{i_j} = \pi \ .$$

Proof. Up to choosing suitable lifts $\widetilde{\sigma}_{i_j}$ of the σ_{i_j}'s, we may glue the $\widetilde{\sigma}_{i_j}$'s in order to develop the union of the $\Delta_{i_j}^n$'s into $\widetilde{M} \subseteq \mathbb{H}^n$. Since the $(n-1)$-faces of ∂P sharing F are developed into two adjacent $(n-1)$-geodesic simplices in $\partial \widetilde{M}$, this implies at once that a suitable algebraic sum of the α_{i_j}'s is equal either to 0 or to π. In order to conclude it is sufficient to show that the condition $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) > 0$ implies that all the signs in this algebraic sum are positive (this implies in particular that the sum is itself positive, whence equal to π).

To prove the last statement, it is sufficient to check that, if $\Delta_{i_{j_1}}^n$, $\Delta_{i_{j_2}}^n$ are adjacent in P along their common $(n-1)$-face V and the lifts $\widetilde{\sigma}_{i_{j_1}}$, $\widetilde{\sigma}_{i_{j_2}}$ coincide on V, then the images of $\widetilde{\sigma}_{i_{j_1}}$ and $\widetilde{\sigma}_{i_{j_2}}$ lie on different sides of $\widetilde{\sigma}_{i_{j_1}}(V) = \widetilde{\sigma}_{i_{j_2}}(V)$. Let us set for simplicity $j_1 = 1$ and $j_2 = 2$, and for $j = 1, 2$ let $\varepsilon'_{i_j} = 1$ if V is the k-th face of $\Delta_{i_j}^n$ and k is even, and $\varepsilon'_{i_j} = -1$ otherwise. It is easily checked that the images of $\widetilde{\sigma}_{i_1}$ and $\widetilde{\sigma}_{i_2}$ lie on different sides of $\widetilde{\sigma}_{i_1}(V) = \widetilde{\sigma}_{i_1}(V)$ if and only if the quantities

$$\varepsilon'_{i_1} \mathrm{Vol}_{\mathrm{alg}}(\sigma_{i_1}) \, , \qquad \varepsilon'_{i_2} \mathrm{Vol}_{\mathrm{alg}}(\sigma_{i_2})$$

have opposite sign. However, since V corresponds to a canceling pair, we have $\varepsilon_{i_1}\varepsilon'_{i_1} + \varepsilon_{i_2}\varepsilon'_{i_2} = 0$, so the conclusion follows from the positivity of $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_1}^n)$ and $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_2}^n)$. ∎

Proof of Theorem 7. Throughout this subsection we suppose that $\dim M = n \geq 4$. The idea of the proof is as follows: Lemma 6.2 implies that no dihedral angle of a geodesic n-simplex of almost maximal volume can be a submultiple of π. Together with Lemma 6.3, this implies that any fundamental cycle M must contain simplices whose support has small volume (that is, smaller than $(1-\varepsilon_n)v_n$). In fact, the weights of these simplices in any fundamental cycle may be bounded from below by the simplicial volume of the boundary of M, and this will finally yield the estimate needed in Theorem 7. Let us now provide the detailed computations.

Let $I = \{1, \ldots, N\}$ and let

$$z = \sum_{i \in I} \varepsilon_i \sigma_i$$

be an integral n-cycle satisfying the conditions of Proposition 4.4, where $\varepsilon_i = \pm 1$ for every $i \in I$. Let P be a pseudomanifold associated to z, and let Δ_i^n, $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)$ be defined as in the previous subsection.

We choose ε_n as in Proposition 6.2 and set

$$I_{\mathrm{small}} = \{i \in I \mid \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) \leq (1-\varepsilon_n)v_n\}, \qquad N_{\mathrm{small}} = \# I_{\mathrm{small}} \ .$$

Lemma 6.4. *We have*

$$N_{\mathrm{small}} \geq \frac{d}{n+1} \|\partial M\| \ .$$

Proof. We start by showing that every $(n-2)$-face of ∂P is contained in at least one small n-simplex Δ_i^n of P, with $i \in I_{\mathrm{small}}$, corresponding to some σ_i. Indeed, let F be an $(n-2)$-face of ∂P and let $\Delta_{i_1}^n, \ldots, \Delta_{i_k}^n$ be the n-simplices of P containing F. Suppose by contradiction that $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) \geq (1-\varepsilon_n)v_n$ for every $j = 1, \ldots, k$. Let σ_{i_j} be the straight simplex corresponding to $\Delta_{i_j}^n$. Our assumptions imply that the dihedral angle α_{i_j} of $\sigma_{i_j}(\Delta_{i_j}^n)$ at F is well-defined. Moreover, Lemma 6.3 gives

$$\sum_{j=1}^{k} \alpha_{i_j} = \pi \ ,$$

which contradicts Lemma 6.2.

Of course, a small simplex could have several $(n-2)$-faces in the boundary, but since an n-simplex has exactly $(n+1)n/2$ faces of codimension two, we can bound the number of small simplices by the number of $(n-2)$-dimensional faces in ∂P,

$$N_{\mathrm{small}} \geq \frac{2}{(n+1)n} \sharp\{(n-2)\text{−faces in } \partial P\}.$$

An $(n-1)$-simplex has exactly n faces of codimension one. Moreover, since ∂P is an $(n-1)$-dimensional pseudomanifold without boundary, every $(n-2)$-face of ∂P is shared by exactly two $(n-1)$-simplices, so the number of

$(n-2)$-faces of ∂P is equal to $(n/2)c(\partial P)$, where $c(\partial P)$ is the number of $(n-1)$-simplices of ∂P. The inequality

$$c(\partial P) = \|\partial z\|_{\mathbb{Z}} \geq d \cdot \|\partial M\|$$

concludes the proof of the lemma. □

To conclude the proof of Theorem 7, note that by Equation (19) we have

$$\begin{aligned} d \cdot \mathrm{Vol}(M) &= \sum_{i \in I} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = \sum_{i \in I_{\mathrm{small}}} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) + \sum_{i \in I \setminus I_{\mathrm{small}}} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) \\ &\leq N_{\mathrm{small}}(1-\varepsilon_n)v_n + (N - N_{\mathrm{small}})v_n = (N - N_{\mathrm{small}} \cdot \varepsilon_n)v_n \ . \end{aligned}$$

Putting together this inequality with Lemma 6.4 and the inequality $N = \|z\|_{\mathbb{Z}} \leq d(\|M, \partial M\| + \varepsilon)$ we get

$$d \cdot \mathrm{Vol}(M) \leq \left(d(\|M, \partial M\| + \varepsilon) - \frac{d \cdot \varepsilon_n}{n+1} \|\partial M\| \right) v_n \ .$$

As ε is arbitrary, after dividing each side of this inequality by $d \cdot \mathrm{Vol}(M)$ and reordering, we get

$$\frac{\|M, \partial M\|}{\mathrm{Vol}(M)} \geq \frac{1}{v_n} + \frac{\varepsilon_n \cdot \|\partial M\|}{(n+1)\mathrm{Vol}(M)} = \frac{1}{v_n} + \frac{\varepsilon_n \mathrm{Vol}(\partial M)}{(n+1)v_{n-1}\mathrm{Vol}(M)} \ ,$$

which finishes the proof of Theorem 7. □

Some estimates on volumes of hyperbolic 3-simplices. In order to provide lower bounds on the simplicial volume of hyperbolic 3-manifolds with geodesic boundary we first need to analyze some properties of volumes of hyperbolic 3-simplices. An essential tool for computing such volumes is the *Lobachevsky function* $L \colon \mathbb{R} \to \mathbb{R}$ defined by the formula

$$L(\theta) = -\int_0^\theta \log|2 \sin u| \, du \ .$$

In a nondegenerate ideal 3-simplex, opposite sides subtend isometric angles, the sum of the angles of any triple of edges sharing a vertex is equal to π and the simplex is determined up to isometry by its dihedral angles. The following result is proved by Milnor in [Thu79, Chapter 7], and plays a fundamental role in the computation of volumes of hyperbolic 3-simplices.

Proposition 6.5. *Let Δ be a nondegenerate ideal simplex with angles α, β, γ. Then*

$$\mathrm{Vol}(\Delta) = L(\alpha) + L(\beta) + L(\gamma) \ .$$

Moreover,

$$\mathrm{Vol}(\Delta) \leq 3L(\pi/3) = v_3 \approx 1.014942 \ ,$$

*where the equality holds if and only if $\alpha = \beta = \gamma = \pi/3$ (*i.e.* Δ is regular).*

We say that a nondegenerate geodesic simplex with nonideal vertices Δ is:

- *1-obtuse* if it has at least one nonacute dihedral angle,

- *2-obtuse* if there exist two edges of Δ which share a vertex and subtend nonacute dihedral angles,
- *3-obtuse* if there exists a face F of Δ such that each edge of F subtends a nonacute dihedral angle.

Lemma 6.6. *There do not exist* 3*-obtuse geodesic simplices. Moreover, if* Δ *is a* 2*-obtuse geodesic simplex, then*

$$\mathrm{Vol}(\Delta) \leq \frac{v_3}{2} \ .$$

Proof. Let F be a face of a non degenerate geodesic simplex Δ. Let H be the geodesic plane containing F and let $\pi\colon \mathbb{H}^3 \to H$ denote the nearest point projection. Let $v \in \overline{\mathbb{H}^n}$ be the vertex of Δ not contained in H.

For every edge e of F, the geodesic line containing e divides H into two regions. Note that the angle at e is acute if and only if the projection $\pi(v)$ of the last vertex point belongs to the region containing F. Consider the three geodesic lines containing the three edges of F. Since no point in H can simultaneously be contained in the region of H bounded by each of these geodesics and not containing F, it follows that Δ cannot be 3-obtuse.

Suppose now that two of the edges of F subtend nonacute dihedral angles and consider the four regions of H delimited by the two corresponding geodesics. Denote by v_0 the vertex of F given as the intersection of these two geodesics. Note that $\pi(v)$ belongs to the region opposite to the region containing F. Denote by r the reflection along H. Set $v' = r(v)$ and $\Delta' = r(\Delta)$. The convex hull of Δ and Δ' is equal to the convex hull of F,v and v'. Let $\widehat{\Delta}$ be the geodesic simplex with vertices v, v' and the two vertices of F opposite to v_0. Since v_0 belongs to $\widehat{\Delta}$ (see Figure 6) it follows that $\Delta \cup \Delta' \subset \widehat{\Delta}$ and hence

$$v_3 \geq \mathrm{Vol}(\widehat{\Delta}) \geq \mathrm{Vol}(\Delta \cup \Delta') = 2\mathrm{Vol}(\Delta) \ ,$$

which finishes the proof of the lemma. $\square$

Recall that

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \approx 0.915965$$

is Catalan's constant.

Lemma 6.7. *If* Δ *is a* 1*-obtuse geodesic simplex, then*

$$\mathrm{Vol}(\Delta) \leq G \ .$$

Proof. Suppose first that Δ is a 1-obtuse ideal geodesic simplex. Let α, β, γ be its three dihedral angles with $\alpha \geq \pi/2$ and $\beta + \gamma = \pi - \alpha$. Using Proposition 6.5, we conclude that when $\alpha \geq \pi/2$ is fixed, the maximum volume $\mathrm{Vol}(\Delta) = L(\alpha) + L(\beta) + L(\gamma)$ is attained at $\beta = \gamma = (\pi - \alpha)/2$. Another easy computation based on Proposition 6.5 implies that, under the assumption that $\alpha \geq \pi/2$, the quantity $L(\alpha) + 2L((\pi - \alpha)/2)$ attains its maximum at $\alpha = \pi/2$. Therefore, we may conclude that

$$\mathrm{Vol}(\Delta) \leq L(\pi/2) + 2L(\pi/4) = G \ ,$$

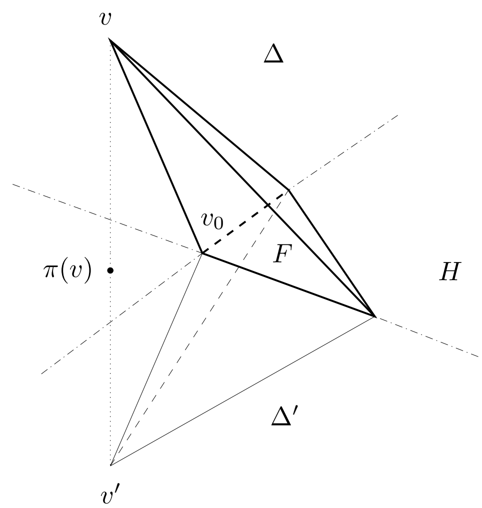

Figure 6: A 2-obtuse simplex Δ and its mirror copy $\overline{\Delta}$.

where the last equality is proved in [Thu79, Chapter 7].

Let now Δ be a 1-obtuse non ideal geodesic simplex. The lemma will follow once we exhibit a 1-obtuse ideal geodesic simplex $\overline{\Delta}$ with $\Delta \subset \overline{\Delta}$. Let v_1, v_2 be the vertices on the edge e subtending the nonacute angle. Two of the vertices of $\overline{\Delta}$ will be the two endpoints w_1, w_2 of the geodesic through v_1, v_2. Let v, v' be the two remaining vertices of Δ and denote by F, F' the two faces of Δ opposite to v' and v respectively. Let w, respectively w', be vertices on the boundary of the hyperplane containing F, resp. F', and such that the convex hull of v_1, v_2, w, resp. v_2, v_2, w', contains v, resp. v'. (For example, pick w, resp. w', as the endpoint of the geodesic through v_1 and v, resp. v'.) Let $\overline{\Delta}$ be the ideal geodesic simplex with vertices w_1, w_2, w, w'. As it contains all the vertices of Δ, the simplex Δ is indeed contained in $\overline{\Delta}$. Furthermore, it is still 1-obtuse as its dihedral angle on the edge with endpoints w_1, w_2 is equal to the dihedral angle of Δ at the edge with endpoints v_1, v_2. $\square$

Proof of Theorem 10. Let z be the integral cycle provided by Proposition 4.4, let P be the associated pseudomanifold, and let $\Delta_1^3, \ldots, \Delta_N^3$ be the simplices of P. In Equation (18) a well-defined algebraic volume $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^3)$ is associated to every Δ_i^3, in such a way that the equality

$$d \cdot \mathrm{Vol}(M) = \sum_{i=1}^{N} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^3) \tag{20}$$

holds. We also say that Δ_i^n is 1-, 2- or 3-obtuse if the corresponding geodesic simplex in $\mathbb{H}^n$ is (by Lemma 6.6, the last possibility cannot hold in fact).

Let Ω_i, $i = 0, \ldots, 4$, be the set of simplices of P having exactly i boundary 2-faces. As usual, we denote by t_i the number of elements of Ω_i. By Proposition 4.4 we have $\Omega_2 = \Omega_3 = \Omega_4 = \emptyset$, so that

$$t_2 = t_3 = t_4 = 0, \qquad \|z\|_{\mathbb{Z}} = t_0 + t_1 = N \ .$$

We denote by $t_{1,n}$ the number of nonpositive simplices in Ω_1 (*i.e.* simplices with nonpositive volume), and by $t_{1,1}$ (resp. $t_{1,2}$) the number of 1-obtuse (resp. 2-obtuse) positive simplices in Ω_1. Recall from Section 5 that an edge e of the 2-dimensional pseudomanifold ∂P is *nice* if it is contained in at least one element of Ω_0, and *bad* otherwise. We denote by E_{bad} (resp. E_{nice}) the number of bad (resp. nice) edges of ∂P.

Lemma 6.8. *We have*

$$3t_{1,n} + 2t_{1,2} + t_{1,1} \geq E_{\text{bad}} \ .$$

Proof. Let e be a bad edge and suppose that $\Delta^3_{i_1}, \Delta^3_{i_2}$ are the simplices of P incident to e (see Lemma 5.1). Also suppose that $\Delta^3_{i_1}$ and $\Delta^3_{i_2}$ are both positive and let α_{i_j} be the dihedral angle of $\Delta^3_{i_j}$ at e. By Lemma 6.3 we have $\alpha_{i_1} + \alpha_{i_2} = \pi$. As a consequence, either $\Delta^3_{i_1}$ or $\Delta^3_{i_2}$ (or both) has a nonacute angle along e. We have thus shown that at every bad edge of P there is (at least) one incident simplex that either is nonpositive or has a nonacute angle at an edge of its boundary face. Since we know that no simplex of P can be 3-obtuse, the conclusion follows from an obvious double counting argument. $\square$

Proposition 6.9. *We have*

$$\|M, \partial M\| + \varepsilon \geq \frac{\mathrm{Vol}(M)}{v_3} + \left(1 - \frac{G}{v_3}\right) \frac{E_{\text{bad}}}{d} \ .$$

Proof. Since $v_3 \geq 3(v_3 - G)$ and $v_3/2 \geq 2(v_3 - G)$, by Equation (20) and Lemmas 6.6, 6.7 and 6.8 we have

$$\begin{aligned} d \cdot \mathrm{Vol}(M) &\leq (t_0 + t_1 - t_{1,1} - t_{1,2} - t_{1,n})v_3 + G t_{1,1} + t_{1,2}\frac{v_3}{2} \\ &= (t_0 + t_1)v_3 - (v_3 - G)t_{1,1} - t_{1,2}\frac{v_3}{2} - t_{1,n}v_3 \\ &\leq (t_0 + t_1)v_3 - (v_3 - G)(t_{1,1} + 2t_{1,2} + 3t_{1,n}) \\ &\leq (t_0 + t_1)v_3 - (v_3 - G)E_{\text{bad}} \ . \end{aligned}$$

Now the conclusion follows from the inequality $t_0 + t_1 = \|z\|_{\mathbb{Z}} \leq d(\|M, \partial M\| + \varepsilon)$ (see Proposition 4.4). $\square$

Proposition 6.10. *We have*

$$\|M, \partial M\| + \varepsilon \geq \frac{7}{4}\|\partial M\| - \frac{E_{\text{bad}}}{2d} \ .$$

Proof. Just as in the proof of Theorem 1 and using the fact that $\partial|P|$ is the union of a finite number of closed orientable surfaces, an easy application of Proposition 1.6 shows that $\|\partial|P|\| \geq d \cdot \|\partial M\|$. Since ∂M is decomposed into t_1 triangles, this implies that

$$t_1 \geq d \cdot \|\partial M\| \ . \tag{21}$$

For the number E_{nice} of nice edges of the triangulation ∂P of $|\partial P|$ we have the obvious equality $E_{\text{nice}} = (3/2)t_1 - E_{\text{bad}}$. By definition, every nice edge is contained in a simplex in Ω_0, and by point (4) of Proposition 4.4 any such simplex has at most 2 edges on ∂P, so

$$t_0 \geq \frac{E_n}{2} = \frac{3}{4}t_1 - \frac{E_{\text{bad}}}{2} \ . \tag{22}$$

Together with Proposition 4.4, Inequalities (21) and (22) imply that

$$d(\|M, \partial M\| + \varepsilon) \geq t_0 + t_1 \geq \frac{7}{4}t_1 - \frac{E_{\text{bad}}}{2} \geq \frac{7d}{4} \cdot \|\partial M\| - \frac{E_{\text{bad}}}{2} \ ,$$

which finishes the proof of the proposition. □

We are now ready to prove Theorem 10. In fact, if we set $k_0 = E_{\text{bad}}/(2d)$, then putting together Propositions 6.9 and 6.10 we get

$$\|M, \partial M\| + \varepsilon \geq \max \left\{ \frac{\operatorname{Vol}(M)}{v_3} + 2\left(1 - \frac{G}{v_3}\right) k_0, \ \frac{7}{4} \cdot \|\partial M\| - k_0 \right\} \ ,$$

whence

$$\|M, \partial M\| + \varepsilon \geq \min_{k \geq 0} \max \left\{ \frac{\operatorname{Vol}(M)}{v_3} + 2\left(1 - \frac{G}{v_3}\right) k, \ \frac{7}{4} \cdot \|\partial M\| - k \right\} \ .$$

If $(7/4)\|\partial M\| \leq \operatorname{Vol}(M)/v_3$, then the statement of Theorem 10 is an obvious consequence of Jungreis' inequality (3). Otherwise, the right-hand side of the inequality above is equal to

$$\frac{\operatorname{Vol}(M)}{v_3} + \frac{v_3 - G}{2(3v_3 - 2G)} \left(7\|\partial M\| - 4\frac{\operatorname{Vol}(M)}{v_3} \right)$$

which finishes the proof of Theorem 10 since ε is arbitrary. □

Small hyperbolic manifolds with geodesic boundary. We start by recalling some results from [FMP03] and [Miy94]. An ideal triangulation of a 3-manifold M is a homeomorphism between M and $|P| \setminus V(|P|)$, where P is a 3-pseudomanifold and $V(|P|)$ is a regular open neighbourhood of the vertices of $|P|$. In other words, it is a realization of M as the space obtained by gluing some topological *truncated tetrahedra*, *i.e.* tetrahedra with neighbourhoods of the vertices removed (see Figure 7).

As in the introduction, let $\mathcal{M}_g$, $g \geq 2$, be the class of 3-manifolds with boundary M that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$ (so $\chi(\partial M) = 2 - 2g$). We also denote by $\overline{\mathcal{M}}_g$ the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$. For $g \geq 2$, let $\Delta_g \subseteq \mathbb{H}^3$ be the regular truncated

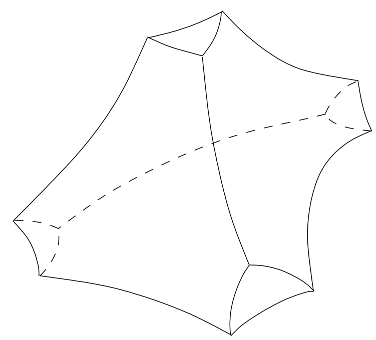

Figure 7: A truncated tetrahedron.

tetrahedron of dihedral angle $\pi/(3g)$ (see *e.g.* [Koj90, FP04] for the definition of hyperbolic truncated tetrahedron). It is proved in [KM91] that

$$\begin{aligned} \mathrm{Vol}(\Delta_g) &= 8L\left(\frac{\pi}{4}\right) - 3\int_0^{\frac{\pi}{3g}} \operatorname{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right) \mathrm{d}t \\ &= 4G - 3\int_0^{\frac{\pi}{3g}} \operatorname{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right) \mathrm{d}t \ . \end{aligned} \tag{23}$$

The following result lists some known properties of manifolds belonging to $\mathcal{M}_g$. The last point implies that $\mathcal{M}_g$ coincides with the set of the elements of $\overline{\mathcal{M}}_g$ of smallest volume.

Proposition 6.11 ([FMP03, Miy94])**.** *Let $g \geq 2$. Then:*

(1) the set $\mathcal{M}_g$ is nonempty;
(2) the boundary of every element of $\mathcal{M}_g$ is connected, so $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$;
(3) every element of $\mathcal{M}_g$ admits a hyperbolic structure with geodesic boundary (which is unique up to isometry by Mostow Rigidity Theorem);
(4) if $M \in \mathcal{M}_g$, then M decomposes into the union of g copies of Δ_g, so in particular $\mathrm{Vol}(M) = g\mathrm{Vol}(\Delta_g)$;
(5) if $M \in \overline{\mathcal{M}}_g$, then $\mathrm{Vol}(M) \geq g\mathrm{Vol}(\Delta_g)$.

Items (2) and (3) and (4) are proved in [FMP03], items (1) and (5) in [Miy94].

Proposition 6.12. *Fix $g \geq 2$. Then all the elements of $\mathcal{M}_g$ share the same simplicial volume.*

Proof. Take $M_1, M_2 \in \mathcal{M}_g$, and let us consider the universal coverings $\widetilde{M}_1$ and $\widetilde{M}_2$. Both $\widetilde{M}_1$ and $\widetilde{M}_2$ are obtained as the union in $\mathbb{H}^3$ of a countable family of adjacent copies of Δ_g, and this easily implies that $\widetilde{M}_1$ and

$\widetilde{M}_2$ are isometric to each other. Since the isometry group of $\widetilde{M}_1$ is discrete, this fact can be used to show that M_1 and M_2 are commensurable, *i.e.* there exists a hyperbolic 3-manifold with geodesic boundary M' that is the total space of finite coverings $p_1 \colon M' \to M_1$ and $p_2 \colon M' \to M_2$ (see [Fri06, Lemma 2.4]). Since the Riemannian volume and the simplicial volume are multiplicative with respect to finite coverings, this implies in turn that $\|M_1, \partial M_1\|/\mathrm{Vol}(M_1) = \|M_2, \partial M_2\|/\mathrm{Vol}(M_2)$, which finishes the proof since $\mathrm{Vol}(M_1) = \mathrm{Vol}(M_2)$. $\square$

Let us prove Corollary 11 and see that for $M \in \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$, the bounds provided by the corollary are indeed sharper than Jungreis' Inequality (3) and Theorem 4.

- If $M \in \overline{\mathcal{M}}_2$, then $\|\partial M\| = 4$ and $\mathrm{Vol}(M) \geq 2\mathrm{Vol}(\Delta_2) \approx 6.452$. Applying Theorem 10 we get
$$\|M, \partial M\| \geq 6.461 \approx 1.615 \cdot \|\partial M\|.$$
Also observe that, if $M \in \mathcal{M}_2$, then $\mathrm{Vol}(M)/v_3 \approx 6.357$.
- If $M \in \overline{\mathcal{M}}_3$, then $\|\partial M\| = 8$ and $\mathrm{Vol}(M) \geq 3\mathrm{Vol}(\Delta_3) \approx 10.429$. Applying Theorem 10 we get
$$\|M, \partial M\| \geq 10.882 \approx 1.360 \cdot \|\partial M\|.$$
Also observe that, if $M \in \mathcal{M}_3$, then $\mathrm{Vol}(M)/v_3 \approx 10.274$.
- If $M \in \overline{\mathcal{M}}_4$, then $\|\partial M\| = 12$ and $\mathrm{Vol}(M) \geq 4\mathrm{Vol}(\Delta_4) \approx 14.238$. Applying Theorem 10 we get
$$\|M, \partial M\| \geq 15.165 \approx 1.264 \cdot \|\partial M\|.$$
If $M \in \mathcal{M}_4$, then $\mathrm{Vol}(M)/v_3 \approx 14.097$.

Finally, let us take $M \in \mathcal{M}_g$, $g \geq 5$ and show that the bound for $\|M, \partial M\|$ provided by Theorem 4 is sharper than the ones given by Inequality (3) and Theorem 10. Note that it is sufficient to show that
$$\frac{5}{4}\|\partial M\| > \frac{7\|\partial M\|(v_3 - G) + 2\mathrm{Vol}(M)}{2(3v_3 - 2G)} > \frac{\mathrm{Vol}(M)}{v_3} \ .$$
Using that $\|\partial M\| = 4(g-1)$ and $\mathrm{Vol}(M) = g\mathrm{Vol}(\Delta_g)$, after some straightforward algebraic manipulations the first inequality and the second inequality may be rewritten respectively as follows:
$$\left(1 - \frac{1}{g}\right)(v_3 + 4G) > \mathrm{Vol}(\Delta_g) \ , \qquad 7\left(1 - \frac{1}{g}\right)v_3 > \mathrm{Vol}(\Delta_g) \ .$$
We know from Equation (23) that $\mathrm{Vol}(\Delta_g) < 4G$. Therefore, for every $g \geq 5$ we have
$$\left(1 - \frac{1}{g}\right)(v_3 + 4G) \geq \frac{4}{5}(v_3 + 4G) > 4G > \mathrm{Vol}(\Delta_g)$$
and
$$7\left(1 - \frac{1}{g}\right)v_3 \geq \frac{28}{5}v_3 > 4G > \mathrm{Vol}(\Delta_g) \ ,$$

whence the conclusion.

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