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ATTRACTORS IN HYPERSPACE

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Abstract

Given a map Φ defined on bounded subsets of the (base) metric space X and with bounded sets as its values, one can follow the orbits $A, \Phi(A), \Phi^2(A), \ldots,$ of nonempty, closed, and bounded sets A in X. This is the system (Φ, X). On the other hand, the same orbits can be viewed as trajectories of points in the hyperspace $X^\sharp$ of nonempty, closed, and bounded subsets of X. This is the system $(\Phi, X^\sharp)$. We study the existence and properties of global attractors for both (Φ, X) and $(\Phi, X^\sharp)$. We give very basic conditions on Φ, stated at the level of the base space X, that are necessary and sufficient for the existence of a global attractor for (Φ, X). Continuity is not among those conditions, but if Φ is continuous in a certain sense then the attractor and the ω-limit sets are Φ-invariant. If (Φ, X) has a global attractor, then $(\Phi, X^\sharp)$ has a global attractor as well. Every point of the global attractor of $(\Phi, X^\sharp)$ is a compact set in X, and the union of all the points of that attractor is the global attractor of (Φ, X).

Keywords
hyperspace, global attractors, dynamical systems, iterated functions systems.

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1 Introduction

A global compact attractor represents the aggregate of long-term regimes in a dissipative system. This important notion was introduced to describe regimes more general than a stable equilibrium or a periodic orbit, and to capture the inherent stability of the aggregate. The very definition of global attractor involves the orbits of bounded sets, not just of individual points. To be more specific, let X be a metric space representing the states of the system, and let $S : X \to X$ be a map representing the one time-step evolution. We denote by (S, X) the discrete time semi-dynamical system generated by iterations of S on X. The global compact attractor of (S, X) is the minimal compact set $K \subset X$ that attracts all bounded sets $B \subset X$ in the sense that for every bounded B and every open neighborhood U of K, $S^n(B) \subset U$ for all sufficiently large n. Of a large literature on the theory of global attractors we refer to [26, 35] as closest to our approach (for the historic roots in ordinary and retarded differential equations see [16], for the first global attractor in partial differential equations see [25], see also [6, 36] for further developments and applications). In the case X is a complete metric space and the map S is continuous and bounded, the necessary and sufficient conditions for the existence of a global compact attractor for (S, X) are known and can be stated as follows: one, there exists a bounded global absorbing set (i.e., a bounded set $\mathbf{B}$ such that for any bounded set A, $S^n(A) \subset \mathbf{B}$ for all sufficiently large n), and two, for every bounded sequence x_k in X and every increasing sequence of integers $n_k \nearrow +\infty$, the sequence $S^{n_k}(x_k)$ is relatively compact. The first property we associate with dissipativity, and the second – with compactness. The global attractor[1], K, is invariant under S, i.e., $S(K) = K$, and it is the maximal compact set with this property. This invariance is important, especially in the case of irreversible processes, because on the attractor the trajectories can be extended back in time, sometimes even extended uniquely as is the case with the Navier-Stokes equations, see [25]. We should mention that although the discussion so far dealt with discrete time systems, historically the theory was first developed for continuous time systems. In fact, time can be any nontrivial additive subgroup of real numbers ($S^{t_2} \circ S^{t_1} = S^{t_1+t_2}$ is needed), and it is easy to see how to move from discrete to this more general case.

A large source of examples of dissipative systems with global attractors is the ordinary and partial differential equations describing real life dissipative processes. For an evolution equation, the maps $S^t : x \mapsto S^t(x)$ give the solution at time t for every initial condition x. The theory of global attractors has been extended to multi-valued S^t starting from the paper [5] and advancing to more general and more recent work [33, 12], see also references therein. The maps S^t act from X to one or the other space of subsets of X and define a generalized semi-flow (in particular, $S^{t_1+t_2}(x) \subset S^{t_2} \circ S^{t_1}(x)$). Such generalized semi-flows arise, e.g., as solution maps for differential inclusions. The theory of global compact attractors for multi-valued semi-flows is developed within the conceptual framework that works in the single-valued case. The global compact attractor is a compact set in X that attracts all bounded sets; its existence is proved under some "dissipativity" and "compactness" assumptions as in the single-valued case.

[1] From now on, we will incorporate compactness into this notion.

The iterated function systems (IFSs) is another fertile source of multi-valued maps and attractors, see e.g. [17, 7, 8]. The setting is as follows. Consider a finite number of maps $f_j : X \to X$, $j = 1, \ldots, N$, and define the Hutchinson (multi-valued) map $F : x \mapsto f_1(x) \cup \cdots \cup f_N(x)$. Extend F to a map on the sets as the closure of the union of the images of the set under all of f_j,

$$F(A) = \overline{\bigcup_j f_j(A)}\,.$$

Originally, [17], each map f_j was assumed to be a strict contraction, and the focus was on the unique invariant compact set, i.e., K such that $F(K) = K$. This invariant compact set is called the fractal (in the setting of IFSs), or, in [8], the attractor, and the focus of the theory is on the structural properties of the fractal (self-similarity, dimension, etc.). The contraction requirement can be weakened to some extent, but not too much if the uniqueness of the fractal viewed as an invariant compact is of importance. In [32], the fractal was identified with the set that is the ω-limit set of its neighborhood; such sets were called attractors in [32]. It was noticed later (see, e.g., [28, 3]) that the fractal, K, is the global attractor for the iterations of the map F (i.e., K attracts every bounded set B; in fact, for contractions, $F^n(B)$ converges to K in the Hausdorff metric). However, the simple theory of global attractors has not been applied to the IFSs until recently. In [19], as part of a larger program, we prove the following result. Assume that, one, there is a bounded global absorbing set $\mathbf{B} \subset X$, i.e., for any bounded set A, $F^n(A) \subset \mathbf{B}$ for all sufficiently large n, and two, each map f_j is continuous, bounded, and ψ-condensing with respect to some measure of noncompactness on X (again, these are the dissipativity and compactness assumptions). Then the IFS has a global compact attractor. This attractor, K, is the minimal compact set that attracts all bounded sets, and it is the maximal compact set such that $F(K) = K$. A measure of noncompactness (mnc) is a function on the sets that "measures" how far those sets are from being compact. For example, the Hausdorff mnc of a set A is the infimum of those $\epsilon > 0$ for which there is a finite ϵ-net in X for A. Given an mnc ψ, a map is ψ-condensing if it reduces the ψ-value of the sets except for the compact sets on which ψ is 0. See Section 2 for details on mnc's.

We should mention relations as a separate instance of multi-valued maps. A relation on X is a subset of $X \times X$. Every relation F defines a multi-valued function $x \mapsto F(x) = \{y \in X : (x, y) \in F\}$. For a set $A \subset X$ its image under F is $F(A) = \{y \in X : \exists x \in A \text{ such that } (x, y) \in F\}$. Again, one can study the dynamics of iterations of relations and its global attractors. For compact X this has been done in [32].

All examples so far were of one-to-many maps. In this paper we deal with the maps defined from the start on the sets, allowing, in principle, the set $\Phi(A)$ to be larger than the union of $\Phi(\{x\})$ when x runs through A. The maps are defined on all nonempty, bounded subsets of the base space X with values in the same class of subsets. Each map Φ generates orbits of bounded sets: A, $\Phi(A)$, $\Phi^2(A)$, ... When viewed at the level of the space X, this is not a (semi-) dynamical system. However, this is a valid and useful dynamical structure. We denote it (Φ, X). Another possibility is to view the sets as points of a hyperspace. The collection of all nonempty, bounded subsets of X is not a convenient choice of a hyperspace because of its topological and metric properties. For us the hyperspace will be the collection of all nonempty, closed, bounded subsets of X. This hyperspace will be denoted $X^\sharp$. The Hausdorff distance,

$$d^\sharp(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\},$$

makes $(X^\sharp, d^\sharp)$ a complete metric space (because (X,d) is assumed to be complete). The hyperspace carries the inclusion partial order. Consider a map $\Phi : X^\sharp \to X^\sharp$ that respects the partial order. We are interested in the dynamics generated by the iterations of Φ on $X^\sharp$. Denote this semi-dynamical system by $(\Phi, X^\sharp)$. When does it possess a global attractor? Of course, one could re-state the known results for the single-valued map Φ on $X^\sharp$. This would require checking, for example, the "compactness" conditions on Φ in the hyperspace, which may be difficult and impractical, since in practice Φ may be defined at the level of the base space X, as in the case of IFSs. Keeping this in mind, we are looking for reasonable conditions on Φ at the base space level and draw conclusions about the dynamics in the hyperspace.

The existence of a bounded global absorbing set in $X^\sharp$ is a necessary condition for the existence of a global compact attractor. If $\mathscr{B}$ is a global absorbing set in $X^\sharp$, the merger of the sets comprising $\mathscr{B}$, $\mathscr{B}^\flat := \bigcup_{C\in\mathscr{B}} C$, is a bounded subset of X, and it is absorbing in the sense that for every $A \in X^\sharp$ the inclusion $\Phi^n(A) \subset \mathscr{B}^\flat$ is true for all sufficiently large n. So, we make the assumption about Φ that there is a closed, bounded set $\mathbf{B} \subset X$ that absorbs all closed, bounded subsets of X. Now, to state the "compactness" property of Φ, we assume there is a measure of noncompactness ψ on X, with just a few very basic properties, so that $\psi(\Phi^n(A)) \to 0$ for any closed and bounded $A \subset X$. We say then that Φ is asymptotically ψ-condensing. This property is not unreasonable. We prove that if $S : X \to X$ is a continuous single-valued map such that the system (S, X) possesses a global attractor, then the multi-valued map $S^\sharp$ defined as $S^\sharp(A) = \overline{\bigcup_{x\in A} S(A)}$ is asymptotically ψ-condensing for any mnc ψ.

Suppose Φ maps nonempty, bounded sets into nonempty, bounded sets and preserves the partial order (inclusions). We prove that the existence of a bounded global absorbing set together with the property of being asymptotically ψ-condensing for some mnc ψ, is necessary and sufficient for the system (Φ, X) to possess a global attractor. In the context of global attractors, if Φ is asymptotically ψ-condensing for one mnc, it will automatically be asymptotically condensing with respect to any mnc, as our Lemma 25 shows. Also, we prove that the existence of a global attractor for (Φ, X) implies the existence of a global attractor for the discrete semi-dynamical system $(\Phi, X^\sharp)$. Let $K \subset X$ be the global attractor of (Φ, X) and let $\mathscr{K} \subset X^\sharp$ be the global attractor of $(\Phi, X^\sharp)$. Every point of $\mathscr{K}$ is a compact subset of K and the merger $\mathscr{K}^\flat$ is equal to K. In general, $\mathscr{K}$ is a proper subset of $K^\sharp$, the space of all nonempty, closed subsets of K. Notice that no continuity of Φ is mentioned, because we do not need continuity for the existence of the global attractor. (In fact, continuity is not needed in the single-valued case either. It seems, this has been overlooked. We analyze this situation in Section 4.) However, without additional assumptions, the attractor we get is not invariant under Φ. We cannot even claim that $\Phi(\mathscr{K}) \subset \mathscr{K}$. A reasonable choice of a sufficient condition for invariance, $\Phi(\mathscr{K}) = \mathscr{K}$, is this (Lemma 15): if $A_n \in X^\sharp$ and $A_n \to A$ in $X^\sharp$, and if A is *compact*, then $\Phi(A_n)$ converges to $\Phi(A)$ in $X^\sharp$. As a corollary, we prove that if a single-valued continuous and bounded map $S : X \to X$ gives rise to the system (S, X) which has the global attractor $K \subset X$, then the system $(S^\sharp, X^\sharp)$ also has a global attractor. In addition, if $\mathscr{K}$ is the attractor of $(S^\sharp, X^\sharp)$, then $\mathscr{K}^\flat = K$ and $K^\sharp = \mathscr{K}$ (i.e., in this case, $\mathscr{K}$ is made of all nonempty closed subsets of K).

One should always think of how abstract assumptions could be verified in applications. Our requirement that Φ be asymptotically ψ-condensing, one would think should hold true if Φ is ψ-condensing, i.e., $\psi(\Phi(A)) < \psi(A)$ for any non-compact A, and $\psi(\Phi(A)) = \psi(A) = 0$, if A is compact. This is what works in the single-valued case. However, in the multi-valued case, we do not know whether this is true without an additional assumption on Φ. One possible requirement is this: for every $A \in X^\sharp$, and for every compact subset $L \subset \Phi(A)$, there is a compact subset $L' \subset A$ such that $L \subset \Phi(L')$. A slightly different assumption was used for similar purposes in [31]. Thus, if Φ is ψ-condensing and has the above property, it is asymptotically ψ-condensing, and we can use our results about global attractors. We discuss two weaker assumptions in Section 6. One of the assumptions applies directly to the IFSs with condensing S_j.

We started this paper as an attachment to our paper [21] on variable time-step dynamics with choice. One of the issues we discuss there is global attractors for certain versions of IFSs. We wanted to include convergence in the hyperspace and quickly realized that we needed to explain more and more results on the convergence of sets, on measures of noncompactness, and on attractors for single-valued maps, that we could not find at all or could find only parts of in dispersed publications. We decided to write about a more general than IFSs subject – the attractors of set-valued maps, and include all the auxiliary facts. We try to make presentation self-contained and give the proofs especially when some pieces are needed in the main part of the paper.

In Section 2 we set some notation for the hyperspace, discuss measures of noncompactness, and refresh some facts on the convergence of sets. Section 3 deals with the main subject of the paper – attractors of multi-valued maps. The consequences of the results obtained in Section 3 for the attractor of the map $S^\sharp$ in the hyperspace are explained in Section 4. The results on attractors in the hyperspace for IFSs are presented in Section 5. The multi-valued condensing maps are compared with asymptotically condensing maps in Section 6.

2 Limits of sets. General notation and facts

- Throughout this paper, X is a complete metric space with metric d. We use lower case letters to denote points in X and upper case letters to denote subsets in X.
- For $A \subset X$, and $x \in X$, $d(x, A) = \inf_{y \in A} d(x, y)$.

- $B_r(x_0)$ is the open ball of radius r centered at x_0, $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$.
- $\mathcal{O}_r(A)$ is the open r-neighborhood of A, $\mathcal{O}_r(A) = \{x \in X : d(x, A) < r\} = \bigcup_{y \in A} B_r(y)$.
- The closure of a set A in X is denoted $\overline{A}$.
- $e(A, B)$ denotes the excess of A over B, i.e., $e(A, B) = \sup_{x \in A} d(x, B)$.
- We say that the *set B attracts the sets A_n* iff $e(A_n, B) \to 0$.
- The Hausdorff distance between the sets A and B is $d^\sharp(A, B) = \max\{e(A, B), e(B, A)\}$.
- $X^\sharp$ is the space of nonempty, closed, bounded subsets of X. We endow $X^\sharp$ with with the Hausdorff metric $d^\sharp$. The symbols for set-theoretic operations in $X^\sharp$ will have $^\sharp$ attached to them.
- The points of $X^\sharp$ will be denoted A, B, etc. The sets in $X^\sharp$ will be denoted $\mathscr{A}$, $\mathscr{B}$, etc. The symbol for the union of sets in $X^\sharp$ is $\bigcup^\sharp$, the symbol for the intersection is $\bigcap^\sharp$, and the closure of a subset in $X^\sharp$ uses over-line with the $^\sharp$. Thus, we write $\mathscr{A} \bigcup^\sharp \mathscr{B}$, $\mathscr{A} \bigcap^\sharp \mathscr{B}$, and $\overline{\mathscr{A}}^\sharp$. For the Hausdorff distance between the sets in $X^\sharp$ we use the symbol $d^{\sharp\sharp}$, thus
$$d^{\sharp\sharp}(\mathscr{A}_1, \mathscr{A}_2) = \max\{\sup_{A \in \mathscr{A}_1} d^\sharp(A, \mathscr{A}_2), \sup_{A \in \mathscr{A}_2} d^\sharp(A, \mathscr{A}_1)\}$$
- Sometimes we identify the points in $X^\sharp$ with one-point sets as in the following example: $\mathscr{A} = \bigcup^\sharp_{C \in \mathscr{A}} C$.
- The operation $\flat$ is used to make a set in X from a set in $X^\sharp$ as follows: $\mathscr{A}^\flat = \displaystyle\bigcup_{C \in \mathscr{A}} C$.
- The operation $\sharp$ is used to make a set in $X^\sharp$ out of a closed set in X as follows:
$$A^\sharp = \bigcup^\sharp_{C \in X^\sharp,\, C \subset A} C\,.$$

The following facts are an easy exercise in the Hausdorff distance.

Lemma 1

(1) $d^\sharp(A, B) = d^\sharp(A, \overline{B})$*; if* $d^\sharp(A, B) = 0$*, then* $\overline{A} = \overline{B}$*.*

(2) *If B is a nonempty, closed, bounded set in X, then $B^\sharp$ is a bounded set in $X^\sharp$ (though it need not be closed).*

(3) *If $\mathscr{A}$ is a nonempty, bounded subset of $X^\sharp$, then $\mathscr{A}^\flat$ is a bounded subset of X.*

Given a sequence of sets A_n, one can define the so-called *limit superior* of this sequence:

$$\mathrm{Ls}A_n = \bigcap_n \overline{\bigcup_{m\geq n} A_m}\,.$$

This notion goes back to Painlevé and Hausdorff, as explained in [24, §29]. Even if $\mathrm{Ls}A_n$ is not empty, it is not true, in general, that $\mathrm{Ls}A_n$ attracts A_n. However, the construct itself is very useful.

Lemma 2

(1) *If* $A_n \in X^\sharp$ *and* $A \subset X$ *are such that* $\lim_n d^\sharp(A_n, A) = 0$*, then* $\overline{A} \in X^\sharp$ *and*

$$\overline{A} = \bigcap_n \overline{\bigcup_{m\geq n} A_m} = \{x \in X : \exists n_k \nearrow \infty\, \exists x_k \in A_{n_k} \text{ such that } x = \lim x_k\}\,. \tag{1}$$

(2) *The space* $(X^\sharp, d^\sharp)$ *is complete.*

(3) *If* $A \subset X$ *is compact, then* $A^\sharp$ *is compact in* $X^\sharp$*.*

Proof. Denote $B = \bigcap_n \overline{\bigcup_{m\geq n} A_m}$. Clearly, B is closed. Assume $A_n \in X^\sharp$ and $A_n \to A$ in the Hausdorff metric. If $x \in A$, then for every $\epsilon > 0$ and for all sufficiently large n, $d(x, A_n) < \epsilon$, hence there exist $x_n \in A_n$ with $d(x, x_n) < \epsilon$. This shows that $A \subset B$. In the opposite direction, if $x \in B$, then there is $m_k \nearrow +\infty$ and there exist $x_{m_k} \in A_{m_k}$ such that $x_{m_k} \to x$. Hence, $d(x, A_{m_k}) \to 0$ and we have $d(x, A) \leq d(x, A_{m_k}) + d^\sharp(A_{m_k}, A) \to 0$. Thus, $x \in \overline{A}$. Since $d^\sharp(A, A_N) < 1$ for some N, and A_N is bounded, $\overline{A}$ is bounded.

The proof of completeness of $X^\sharp$ goes as follows (see [24, §33, IV]). Suppose A_n is Cauchy in $X^\sharp$ and form the set $A = \bigcap_n \overline{\bigcup_{m\geq n} A_m}$. Let $N(k)$ be a strictly increasing integer sequence such that $d^\sharp(A_n, A_m) < 2^{-k}$ when $n, m \geq N(k)$. Clearly, $A \subset \overline{\bigcup_{m\geq N(k)} A_m} \subset \mathcal{O}_{2^{-k+2}} A_{N(k)}$.

In the opposite direction, fix an integer $k > 0$, pick any $x_0 \in A_{N(k)}$, and then pick successively $x_m \in A_{N(k+m)}$, $m = 1, 2, \ldots$, such that $d(x_m, x_{m+1}) < 2^{-k-m+1}$. Because x_m is Cauchy, there exists $x = \lim x_m$, which must belong to A, hence A is not empty. At the same time, $d(x_0, x) < 2^{-k+1}$. Thus, $\sup_{x_0 \in A_{N(k)}} d(x_0, A) < 2^{-k+2}$.

The proof of Lemma 2(3) relies on the fact that if $V = \{x_1, \ldots, x_N\} \subset A$ is an ϵ-net for A, then $V^\sharp$ is an ϵ-net for $A^\sharp$. That $A^\sharp$ is closed when A is, follows from the previous step, Lemma 2(2). $\square$

Next, we introduce measures of noncompactness (mnc for short). For us, an mnc ψ is a real-valued function defined on bounded subsets of X that measures how far the sets are from being (relatively) compact. There are mnc's associated with the names of Hausdorff, Kuratowski, Istrăţescu, Gol'denštein-Markus, Sadovskii, and others, and there are methods of constructing new mnc's. A very good discussion of the properties of the common mnc's and the methods to build new mnc's is in [30]. For our purposes any of the standard mnc's will do. For example, the Hausdorff mnc defined as

$$\chi(A) = \inf\{r > 0 \,:\, A \subset \mathcal{O}_r(Y), \text{ for some finite set } Y \subset X\}$$

would work. The Kuratowski mnc defined as

$$\alpha(A) = \inf\{r > 0 \,:\, A = \bigcup_{i=1}^{N_r} A_i, \text{ where } \operatorname{diam}(A_i) \le r,\ 1 \le i \le N_r < \infty\}$$

would work as well. However, all we need in this paper is an mnc with just a few properties. Some results require even less, but we make a definition that serves the whole paper.

Definition 1 *A function ψ defined on nonempty bounded subsets of X with values in $[0, +\infty)$ is a measure of noncompactness on X iff*

mnc(i) $\psi(A) = 0$ *iff A is relatively compact;*

mnc(ii) *If $A_1 \subset A_2$, then $\psi(A_1) \le \psi(A_2)$;*

mnc(iii) $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$ *;*

mnc(iv) *ψ is Lipschitz continuous, i.e., there exists a constant $c_\psi > 0$ such that*

$$|\psi(A_1) - \psi(A_2)| \le c_\psi\, d^\sharp(A_1, A_2)\,.$$

As a corollary of mnc(iv), $\psi(\bar{A}) = \psi(A)$ for any A. Also, the fact that $\psi(B)$ is finite for every bounded B agrees with mnc(iv) because, for any $x \in B$,

$$\psi(B) = \psi(B) - \psi(\{x\}) \le c_\psi \operatorname{diam}(B)\,.$$

In what follows a weaker property than mnc(iv) will do just as fine:

mnc(iv*) ψ is continuous in the Hausdorff metric and has a modulus of continuity, m_ψ, defined on some small interval $[0, \epsilon_*]$ as follows:

$$m_\psi(\delta) = \sup\{|\psi(A) - \psi(B)| \,:\, \text{bounded } A, B \text{ such that } d^\sharp(A, B) \le \delta\}\,.$$

Definition 2 *A multi-valued map Φ from the collection of nonempty bounded subsets of X into itself is ψ-condensing (condensing with respect to the mnc ψ) iff $\psi(\Phi(A)) \le \psi(A)$ for any bounded A, and $\psi(\Phi(A)) < \psi(A)$ if $\psi(A) > 0$, i.e., if $\overline{A}$ is not compact. A single-valued map $S : X \to X$ is ψ-condensing iff the multi-valued map $S : A \mapsto S(A)$ is ψ-condensing.*

For applications it is good to know that examples of ψ-condensing single-valued maps include all strict contractions (i.e., $d(S(x), S(y)) \le \gamma d(x, y)$ for some $\gamma \in [0, 1)$ and all x, y), and all compact maps $X \to X$. Also, in a Banach space, any map of the form *strict contraction + compact* is ψ-condensing for any ψ.

The following theorem is a version of the Kuratowski generalization of Cantor's result, [24, §34]. The result is known, but we could not find one reference with both assertions. The first part is in [31, Theorem 3.1] and the second follows from [27, Prop. 3(viii)].

Theorem 3 *Assume $A_1 \supset A_2 \supset \ldots$ is a nested sequence of nonempty, closed, bounded subsets in X. Assume that $\psi(A_n) \to 0$ for some mnc ψ. Then,*

(1) *the set $A = \bigcap_n A_n$ is nonempty and compact;*

(2) *$A_n \to A$ in $X^\sharp$.*

Proof. Any sequence $\{x_k\}$ such that $x_k \in A_k$ has the property $\psi(\{x_k\}) = 0$ because, for any n, $\{x_k\} \subset A_n$ except for a finite number of points, and hence $\psi(\{x_k\}) \leq \psi(A_n) \to 0$. Thus, any such sequence has a convergent subsequence, which implies A is a nonempty compact.

Suppose A_n is not a Cauchy sequence in $X^\sharp$. Then there exist an $\epsilon_0 > 0$ and a monotone sequence $k_n \nearrow 0$, $k_n > n$, such that $d^\sharp(A_n, A_{k_n}) > \epsilon_0$. Because $A_n \supset A_{k_n}$, for every n there exists $x_n \in A_n$ such that $d(x_n, A_{k_n}) > \epsilon_0$. The sequence x_n cannot have a convergent subsequence, because if it had and $x_{n_m} \to x_\infty$, we would be able to find M such that $d(x_{n_\ell}, x_{n_m}) < \epsilon_0/2$ for all $\ell, m \geq M$, and this would imply $d(x_{n_\ell}, A_{k_{n_\ell}}) < \epsilon_0/2$, a contradiction. But we have just shown that $\{x_n\}$ is relatively compact. Therefore, A_n is Cauchy, and $A_n \to A$ by Lemma 2. □

It is convenient to have a different description of the condition $\lim_n \psi(A_n) = 0$.

Lemma 4 *Let $A_1 \supset A_2 \supset \ldots$ be a nested sequence of nonempty, closed, and bounded sets. The following conditions are equivalent:*

1) $\lim_n \psi(A_n) = 0$.

2) *Every sequence x_k, where $x_k \in A_{n_k}$, $n_k \nearrow +\infty$, has a convergent subsequence.*

Proof. Let the first condition be satisfied. For any n, $\{x_k\} \subset A_n$ except for at most a finite number of x_k's. Hence, $\psi(\{x_k\}) \leq \psi(A_n)$, and therefore $\psi(\{x_k\}) = 0$, which implies the second condition.

Now suppose the second condition is satisfied. Note that if a set C has a finite ϵ-net $Q \subset C$, then $d^\sharp(C, Q) \leq \epsilon$, and therefore, by the property mnc(iv),

$$\psi(C) \leq c_\psi \, \epsilon \, .$$

If $\psi(A_n)$ does not converge to 0, then there is subsequence n_k and an $\epsilon > 0$ such that

$$\psi(A_{n_k}) > c_\psi \, \epsilon \, .$$

Hence, the sets A_{n_k} do not have finite ϵ-nets. Take a point $x_1 \in A_{n_1}$. There exists $x_2 \in A_{n_2}$ such that $d(x_1, x_2) > \epsilon/2$. There exists $x_3 \in A_{n_3}$ such that $d(x_3, \{x_1, x_2\}) > \epsilon/2$, and so on. The sequence $\{x_n\}$ is not totally bounded. However, by construction, $x_k \in A_{n_k}$, and by assumption, the sequence $\{x_k\}$ must have a convergent subsequence. A contradiction. □

Lemma 5 *Let $A \subset X$ be compact and let B_n be a sequence of sets attracted by A, i.e., $\lim_n e(B_n, A) = 0$. Then*

(1) *The sequence* $\overline{B_n}$ *has a subsequence converging in the Hausdorff metric. Its limit is a compact subset of* A.

(2) $\psi(B_n) \to 0$;

(3) $\psi(\overline{\bigcup_{m\geq n} B_m}) \to 0$.

(4) $B = \bigcap_n \overline{\bigcup_{m\geq n} B_m}$ *is a nonempty compact.*

(5) $\overline{\bigcup_{m\geq n} B_m} \to B$ *in* $X^\sharp$.

Proof. To prove the first statement, choose a sequence $n_k \nearrow +\infty$ such that $\overline{B_{n_k}} \subset \mathcal{O}_{2^{-k}}(A)$. Because A is compact, we can define the sets

$$A_k = \{y \in A \,:\, \exists x \in B_{n_k} \text{ such that } d(x,y) = d(x,A)\}\,.$$

Obviously, $d^\sharp(B_{n_k}, A_k) \leq 2^{-k}$. Since $A^\sharp$ is compact by Lemma 2(3), the sequence $\overline{A_k}$ has a convergent subsequence with a limit in $A^\sharp$. The corresponding subsequence of B_{n_k} will have the same limit.

The second statement follows from the fact that every subsequence of $\{B_n\}$ has a subsequence that converges to a compact set. If B_{n_k} is that subsequence and C is its compact limit, then $\psi(B_{n_k}) = \psi(B_{n_k}) - \psi(C) \leq c_\psi d^\sharp(B_{n_k}, C) \to 0$.

The third statement follows from the second because A attracts the sets $\overline{\bigcup_{m\geq n} B_m}$. The fourth and the fifth follow from Theorem 3. $\square$

Next, we discuss the limits of sets under single-valued continuous maps. Let $S : X \to X$ be a continuous single-valued map. Assume that S is a bounded map, i.e., S maps bounded sets to bounded sets. Suppose A_n is a convergent sequence in $X^\sharp$, and $A = \lim A_n$. The question is: Does the sequence of bounded sets $S(A_n)$ have a limit in some meaningful sense? and how is the limit related to $S(A)$?

Lemma 6 *Let* $A_n, A \in X^\sharp$ *and* $A_n \to A$ *in* $X^\sharp$. *Let* $S : X \to X$ *be a continuous, bounded map. Then*

(1) $\overline{S(A)} = \bigcap_n \overline{\bigcup_{m\geq n} S(A_m)}$

(2) $e(S(A_n), \overline{S(A)}) = \sup_{y_n \in S(A_n)} \inf_{y \in \overline{S(A)}} d(y_n, y) \to 0$.

If, in addition, A *is compact, then*

(3) $S(A) = \bigcap_n \overline{\bigcup_{m\geq n} S(A_m)}$ *and* $\overline{S(A_n)} \to S(A)$ *in* $X^\sharp$.

Proof. Since $A = \lim_n A_n$, we have $A = \bigcap_n \overline{\bigcup_{m\geq n} A_m}$. Consider the set $\Omega = \bigcap_n \overline{\bigcup_{m\geq n} S(A_m)}$ and compare it with $S(A)$. If $z \in S(A)$, then $z = S(x)$ where $x \in \bigcap_n \overline{\bigcup_{m\geq n} A_m}$, i.e., $x = \lim_k x_{n_k}$ with $x_{n_k} \in A_{n_k}$. But then $S(x) = \lim_k S(x_{n_k})$ by continuity. This means that, first, the set Ω is not empty, and second, $S(A) \subset \Omega$, and hence, $\overline{S(A)} \subset \Omega$. On the other hand, if $y \in \Omega$, then $y = \lim_k S(x_{n_k})$ for some sequence $x_{n_k} \in A_{n_k}$. Since $A_{n_k} \to A$, we have $d(x_{n_k}, A) \to 0$. Since A is closed, there are $y_k \in A$ such that $d(x_{n_k}, A) = d(x_{n_k}, y_k)$, and hence, $d(x_{n_k}, y_k) \to 0$. Then $d(S(x_{n_k}), S(y_k)) \to 0$ and as a result $d(S(x_{n_k}), S(A)) \to 0$. Hence, $y \in \overline{S(A)}$. This implies $\Omega \subset \overline{S(A)}$, and hence, $\overline{S(A)} = \Omega$.

The second statement follows from the fact that Ω attracts the sets $S(A_n)$. If this were not true, there would be an $\epsilon_0 > 0$ and a sequence $x_{n_k} \in A_{n_k}$ such that $d(S(x_{n_k}), \Omega) \geq \epsilon_0$. But then, choosing the sequence $y_k \in A$ as above, we would have $d(x_{n_k}, y_k) \to 0$ and consequently $d(S(x_{n_k}), S(A)) \to 0$, a contradiction.

Now assume that A is compact. It remains to show that $e(S(A), S(A_n)) \to 0$. If this is not the case, there is an $\epsilon_0 > 0$ and a sequence $x_k \in A$ such that

$$d(S(x_k), S(A_{n_k})) > \epsilon_0$$

for some $n_k \nearrow \infty$. We may assume that $x_k \to x$ in A. However, since $A_{n_k} \to A$, there is a sequence $y_k \in A_{n_k}$ such that $d(x_k, y_k) \to 0$, and therefore $d(S(x_k), S(A_{n_k})) \leq d(S(x_k), S(y_k)) \to 0$, a contradiction. $\square$

3 Attractors for multivalued maps

In this section the main actor is a map Φ defined on nonempty bounded subsets of X with values in subsets of X. The first two assumptions are these:

Φ1: Φ maps bounded sets into bounded sets.
Φ2: If $A \subset B$, then $\Phi(A) \subset \Phi(B)$.

When working within $X^\sharp$, we assume in addition that
$\Phi 1^\sharp$: Φ maps $X^\sharp$ into itself.

We consider two dynamics associated with the iterations of Φ. The first dynamics is the dynamics of bounded subsets of X. We denote it by (Φ, X). (Here we do not need the map Φ to be closed.)

Definition 3 *K is the global attractor of the system (Φ, X) iff 1) K attracts every bounded set, i.e., $e(\Phi^n(B), K) \to 0$ for every bounded $B \subset X$; 2) K is compact; 3) K is minimal with the properties 1) and 2).*

Definition 4 **B** *is a global absorbing set for (Φ, X) iff* **B** *is bounded and, for every bounded A there exists $N(A) \geq 0$ such that $\Phi^n(A) \subset \mathbf{B}$ for all $n \geq N(A)$.*

For the system (Φ, X), the ω-limit set of A is defined as

$$\omega^\flat(A) = \bigcap_n \overline{\bigcup_{m\geq n} \Phi^m(A)}\,. \tag{2}$$

The second dynamics generated by the iterations of Φ is a discrete time dynamics on $X^\sharp$. This semi-dynamical system will be denoted $(\Phi, X^\sharp)$ (and here we assume $\mathbf{\Phi 1}^\sharp$). For a set $\mathscr{L} \subset X^\sharp$, $\Phi(\mathscr{L})$ is the set $\biguplus^\sharp_{A\in\mathscr{L}} \Phi(A)$. When viewed on $X^\sharp$, Φ is a single-valued map; we can extend it to sets in $X^\sharp$ in a natural fashion:

$$\Phi(\mathscr{A}) = \biguplus^\sharp_{C\in\mathscr{A}} \Phi(C)\,.$$

Then all the notions just discussed apply. In order to emphasize that the ω-limit sets of $(\Phi, X^\sharp)$ lie in the hyperspace $X^\sharp$, we use the notation $\omega^\sharp$. Thus, the ω-limit set of a bounded set $\mathscr{L} \subset X^\sharp$ is the following set:

$$\omega^\sharp(\mathscr{L}) = \{B \in X^\sharp : \exists\, n_k \nearrow \infty,\ \exists\, A_k \in \mathscr{L} \text{ such that } B = \lim_k \Phi^{n_k}(A_k)\}\,.$$

An equivalent description is this:

$$\omega^\sharp(\mathscr{L}) = \bigcap^\sharp_n \overline{\biguplus^\sharp_{m\geq n} \Phi^m(\mathscr{L})}^\sharp\,. \tag{3}$$

We are interested in the ω-limit sets and attractors of these two dynamics.

Theorem 7 *The system (Φ, X) has a global attractor iff the following two conditions are satisfied.*

A1: *There exists a global absorbing set.*

A2: *For some mnc ψ, Φ is asymptotically condensing in the sense that $\psi(\Phi^n(A)) \to 0$ for all $A \in X^\sharp$.*

Suppose K is the global attractor of (Φ, X) and let $\mathbf{B}$ be any global absorbing set. Then $K = \omega^\flat(\mathbf{B})$.

Proof. Suppose K is the global attractor of (Φ, X). Clearly, the 1-neighborhood of K is a global absorbing set. Let A be a bounded set. Then $\lim_n e(\Phi^n(A), K) = 0$. That Φ is asymptotically ψ-condensing (for any mnc ψ) follows from Lemma 5(2).

In the opposite direction, let $\mathbf{B}$ be a global absorbing set, and assume that Φ is asymptotically ψ-condensing. Under these assumptions we prove the following result.

Lemma 8 *For any bounded $A \subset X$,*

(1) $\omega^\flat(A)$ *is a nonempty, compact set in* X*;*

(2) $\omega^\flat(A)$ *attracts* A*;*

(3) $\omega^\flat(A) = \omega^\flat(\Phi(A))$.

For any $A \in X^\sharp$,

(4) $\omega^\sharp(A)$ *is a nonempty, compact set in* $X^\sharp$*;*

(5) $\omega^\sharp(A) \subset \left(\omega^\flat(A)\right)^\sharp$.

(6) *If* $A \subset B$*, then* $\left(\omega^\sharp(A)\right)^\flat \subset \left(\omega^\sharp(B)\right)^\flat$ *and* $\omega^\flat(A) \subset \omega^\flat(B)$.

Proof. Notice that, for any integer $N \geq 0$,

$$\omega^\flat(A) = \bigcap_{n>N} \overline{\bigcup_{m\geq n} \Phi^m(A)} = \bigcap_{k\geq 1} \overline{\bigcup_{m\geq N+k} \Phi^m(A)} = \bigcap_{k\geq 1} \overline{\bigcup_{\ell\geq k} \Phi^\ell\left(\Phi^N(A)\right)}\,.$$

We see immediately that $\omega^\flat(A) = \omega^\flat\left(\Phi^N(A)\right)$. Also, by changing indices in (2), we obtain

$$\omega^\flat(A) = \bigcap_n \overline{\bigcup_{k=0}^{\infty} \Phi^n\left(\Phi^k(A)\right)}\,.$$

Since $\bigcup_k \Phi(A_k) \subset \Phi(\bigcup_k A_k)$ due to monotonicity, we have $\psi\left(\overline{\bigcup_{k=0}^{\infty} \Phi^n\left(\Phi^k(A)\right)}\right) =$

$$\psi\left(\bigcup_{k=0}^{\infty} \Phi^n\left(\Phi^k(A)\right)\right) \leq \psi\left(\Phi^n\left(\bigcup_{k=0}^{\infty} \Phi^k(A)\right)\right) = \psi\left(\Phi^n(\Gamma)\right) \to 0\,, \tag{4}$$

because $\Gamma = \bigcup_{k=0}^{\infty} \Phi^k(A)$, the full trajectory of A, is bounded due to our assumption **A1** and the boundedness of the map Φ. By Theorem 3, $\omega^\flat(A)$ is a nonempty compact, and $\overline{\bigcup_{k=0}^{\infty} \Phi^n\left(\Phi^k(A)\right)} \to \omega^\flat(A)$ in $X^\sharp$. A quick consequence of this is that $\omega^\flat(A)$ attracts A. By Lemma 5(1), there is a sequence $n_k \nearrow \infty$ such that $\Phi^{n_k}(A)$ converges to a compact subset of $\omega^\flat(A)$. This implies nonemptiness of $\omega^\sharp(A)$. On the other hand, any convergent subsequence of $\{\Phi^n(A)\}$ will have its limit in $\omega^\flat(A)$. This means that

$$\omega^\sharp(A) \subset \left(\omega^\flat(A)\right)^\sharp\,.$$

Being a closed subset of a compact set in $X^\sharp$ (see Lemma 1) makes $\omega^\sharp(A)$ compact (in $X^\sharp$).

That $\omega^\flat(A) \subset \omega^\flat(B)$ is obvious. To show that $(\omega^\sharp(A))^\flat \subset (\omega^\sharp(B))^\flat$, let $D \in \omega^\sharp(B)$. Then $\exists\, n_k \nearrow \infty$ such that $D = \lim_k \Phi^{n_k}(B)$. Consider the sequence $\{\Phi^{n_k}(A)\}$. This sequence has a convergent subsequence $\{\Phi^{n_{k_l}}(A)\}$ whose limit $C \in \omega(A)$. But, since $\Phi^{n_{k_l}}(A) \subset \Phi^{n_{k_l}}(B)$, we have $C \subset D$. Hence, $(\omega^\sharp(A))^\flat = \bigcup_{C\in\omega^\sharp(A)} C \subset (\omega^\sharp(B))^\flat = \bigcup_{D\in\omega^\sharp(B)} D$. $\square$

Proof of Theorem 7, continued. Consider the ω-limit set of the global absorbing set, $K = \omega^\flat(\mathbf{B})$. The ω-limit set of any bounded A is a subset of K as the following computation shows:

$$\omega^\flat(A) = \bigcap_k \overline{\bigcup_{\ell \geq k} \Phi^{\ell + N(A)}(A)} \subset \bigcap_k \overline{\bigcup_{\ell \geq k} \Phi^{\ell}(\mathbf{B})}\,.$$

Hence, K attracts every bounded set. K is compact by Lemma 8(1). If M is any compact set attracting every bounded set, then $\omega^\flat(A) \subset M$ for any bounded A because for every $\epsilon > 0$

$$\overline{\bigcup_{m \geq n} \Phi^m(A)} \subset \mathcal{O}_\epsilon(M)$$

for all large enough n thanks to assumption **A1**. Thus, $K = \omega^\flat(\mathbf{B}) \subset M$, which proves minimality of K and completes the proof of Theorem 7. □

We next turn to the dynamics in the hyperspace.

Theorem 9 *Suppose system (Φ, X) has a global attractor and Φ satisfies $\mathbf{\Phi 1}^\sharp$. Then system $(\Phi, X^\sharp)$ has a global attractor as well. For any closed global absorbing set $\mathbf{B}$, $\mathscr{K} = \omega^\sharp(\mathbf{B}^\sharp)$ is the global attractor of system $(\Phi, X^\sharp)$. Every point of $\mathscr{K}$ is a compact subset of X. If K is the global attractor of (Φ, X), then $\mathscr{K}^\flat = K$ and $\mathscr{K} \subset^\sharp K^\sharp$.*

Remark 10 *The inclusion $\mathscr{K} \subset^\sharp K^\sharp$ may indeed be strict. [Here is a trivial example. Let X be the finite set $\{1, 2\}$ with $d(i, j) = 1$ only if $i \neq j$. Define Φ by the rule $\Phi(A) = X$ for any nonempty $A \subset X$. Then $\mathscr{K}$ is the point $\{1, 2\}$ of the three-point set $X^\sharp$ while $K = X$ and $K^\sharp = X^\sharp$.] However, there are situations where $\mathscr{K} = K^\sharp$, see Theorem 20 below.*

The proof of Theorem 9 will rely on a few lemmas. We assume the conditions **A1** and **A2** to be satisfied.

Lemma 11 *Let $\mathscr{L}$ be a bounded subset of $X^\sharp$. Then $\omega^\sharp(\mathscr{L})$ is a nonempty, compact subset of $X^\sharp$.*

Proof. Recall that

$$\omega^\sharp(\mathscr{L}) = \bigcap_n{}^\sharp \; \overline{\bigcup_{m \geq n}{}^\sharp \bigcup_{A \in \mathscr{L}}{}^\sharp \; \Phi^m(A)}^\sharp\,. \tag{5}$$

This set is not empty because $\omega^\sharp(A)$ is not empty for any $A \in \mathscr{L}$ by Lemma 8. Next, if $C \in \omega^\sharp(\mathscr{L})$, $C = \lim_k \Phi^{n_k}(A_k)$ for some sequence of points $A_k \in \mathscr{L}$. By Lemma 8, for some subsequence $m_\ell = n_{k_\ell}$ of n_k, the limit $\lim_\ell \Phi^{m_\ell}(\mathscr{L}^\flat)$ exists in $X^\sharp$, and C is a compact subset (in X) of that limit. Hence, $\omega^\sharp(\mathscr{L})$ is compact as a closed subset of the compact set $\big(\omega^\flat(\mathscr{L}^\flat)\big)^\sharp$. □

Lemma 12 *Let $\mathscr{L}$ be a bounded subset of $X^\sharp$. For any sequence $A_n \in \mathscr{L}$ and any sequence $m_n \nearrow +\infty$, the sequence $\Phi^{m_n}(A_n)$ has a convergent in $X^\sharp$ subsequence. Its limit is a compact set in X.*

Proof. Denote

$$\mathscr{M} = \bigcup_n^{\sharp} A_n\,,$$

and consider the set (nonempty and compact)

$$\omega^\flat(\mathscr{M}^\flat) = \bigcap_m \overline{\bigcup_{k\geq m} \Phi^k(\mathscr{M}^\flat)}\,.$$

The closed sets $\Phi^{m_n}(A_n)$ are subsets of the sets $\overline{\bigcup_{k\geq m} \Phi^k(\mathscr{M}^\flat)}$, and the latter sets converge to $\omega^\flat(\mathscr{M}^\flat)$ (see (4) and Lemma 3). That the sequence $\Phi^{m_n}(A_n)$ has a convergent subsequence in $X^\sharp$ then follows from Lemma 5(1). $\square$

A few facts about ω-limit sets for $(\Phi, X^\sharp)$ can be established without additional assumptions.

Lemma 13 *Let $\mathscr{L}$ and $\mathscr{N}$ be bounded subsets of $X^\sharp$. Then*

(1) *If $\mathscr{L} \subset \mathscr{N}$ then $\omega^\sharp(\mathscr{L}) \subset \omega^\sharp(\mathscr{N})$;*

(2) *$\omega^\sharp(\mathscr{L} \bigcup^\sharp \mathscr{N}) = \omega^\sharp(\mathscr{L}) \bigcup^\sharp \omega^\sharp(\mathscr{N})$;*

(3) *$\omega^\sharp(\mathscr{L})$ attracts $\mathscr{L}$, i.e., for any $\epsilon > 0$, $\Phi^n(\mathscr{L}) \subset \mathcal{O}^\sharp_\epsilon\left(\omega^\sharp(\mathscr{L})\right)$ when n is large enough;*

(4) *$\omega^\sharp(\mathscr{L}) \subset \omega^\sharp(\mathbf{B}^\sharp)$;*

(5) *Every point of $\omega^\sharp(\mathscr{L})$ is a compact subset of X.*

Proof. The first three properties are valid for any semi-dynamical system with non-empty ω-limit sets. The last assertion of the lemma can be proved as follows. Every point $C \in \omega^\sharp(\mathscr{L})$ is a limit of a sequence of the form $\Phi^{n_k}(A_k)$, where $A_k \in \mathscr{L}$. Since the compact set $\omega^\flat(\mathscr{L}^\flat)$ attracts $\mathscr{L}$, C is compact by Lemma 5. We can write $C = \lim_k \Phi^{m_k}\left(\Phi^N(A_k)\right))$ where N is so large, that $\Phi^N(A_k) \in \mathbf{B}^\sharp$ for all k. Hence $C \in \omega^\sharp(\mathbf{B}^\sharp)$. $\square$

Proof of Theorem 9. It is clear that $\mathbf{B}^\sharp$ is an absorbing set for $(\Phi, X^\sharp)$. Lemma 12 is the second ingredient needed to apply Theorem 18 and obtain the existence of the global attractor for $(\Phi, X^\sharp)$. By Lemma 13, the ω-limit sets of all bounded subsets of $X^\sharp$ lie in $\mathscr{K} = \omega^\sharp(\mathbf{B}^\sharp)$. Hence, $\mathscr{K}$ is the global attractor.

Every point $C \in \mathscr{K}$ is a limit of the form $C = \lim_k \Phi^{n_k}(A_k)$ and therefore $C \subset \omega^\flat(A)$ where $A = \bigcup_m A_m$. Since $\omega^\flat(A) \subset K$, we have $C \subset K$. Hence, $\mathscr{K}^\flat \subset K$. To show that $\mathscr{K}^\flat = K$, we proceed as follows. There is a sequence $n_k \nearrow \infty$ such that the sets $\Phi^{n_k}\left(\bigcup_\ell \Phi^\ell(\mathbf{B})\right)$ converge in the Hausdorff metric to some compact set $C \in \mathscr{K}$, and $C \subset$

K, of course. On the other hand, $\overline{\bigcup_{m\geq n}\Phi^m(\mathbf{B})} \to K$ as $n \to \infty$. At the same time, $\bigcup_\ell \Phi^{n_k+\ell}(\mathbf{B}) \subset \Phi^{n_k}\left(\bigcup_\ell \Phi^\ell(\mathbf{B})\right)$ implies $K \subset C$. $\square$

In general, it is possible that $(\Phi, X^\sharp)$ has a global attractor and (Φ, X) has not: take X to be an infinite dimensional Banach space and define Φ that maps every bounded set to the fixed closed ball. However, the situation is different if X has the property that every closed, bounded set is compact (e.g., if X is a Montel space, see [15]).

Theorem 14 *Suppose X has the property that every closed, bounded set in X is compact. If Φ satisfies $\mathbf{\Phi 1}^\sharp$ and if system $(\Phi, X^\sharp)$ has the global attractor $\mathscr{K}$, then $K = \mathscr{K}^\flat$ is the global attractor for (Φ, X).*

Proof. By Lemma 1(3), $\mathscr{K}^\flat$ is bounded. To see that it is closed, let $(x_k) \subset \mathscr{K}^\flat$ be a convergent sequence with a limit $x \in X$. There exists a sequence (A_k) such that $x_k \in A_k$ and $A_k \in \mathscr{K}$. As $\mathscr{K}$ is compact, there exists a subsequence A_{k_l} which converges to some $A \in \mathscr{K}$. Since $x_{k_l} \to x$ and $e(x_{k_l}, A) \leq d^\sharp(A_{k_l}, A) \to 0$, it follows that $x \in A \subset \mathscr{K}^\flat$. Hence, $\mathscr{K}^\flat$ is compact. To see that it attracts all bounded sets is easy. In fact, every sequence $\Phi^{n_k}(A_k)$ has a convergent subsequence with limit $C \in \mathscr{K}$, i.e., $C \subset \mathscr{K}^\flat$. But then $e(\Phi^{n_k}(A_k), \mathscr{K}^\flat) \to 0$ as $k \to \infty$. Thus, (Φ, X) must have a global attractor. That this attractor is $\mathscr{K}^\flat$ follows from Theorem 9. $\square$

In our setting, some familiar properties of ω-limit sets may require additional assumptions on Φ. One such property concerns the ω-limit set of the closure (in the Hausdorff metric) of a bounded set in $X^\sharp$. In order to have $\omega(\mathscr{L}) = \omega(\overline{\mathscr{L}}^\sharp)$ we need some form of continuity of Φ in $X^\sharp$. Another important property that requires continuity of Φ is the invariance of ω-limit sets, $\Phi(\omega(\mathscr{L})) = \omega(\mathscr{L})$. It turns out that the continuity requirement needed for the latter property is somewhat weaker than for the former. We state the requirements separately.

$\mathbf{\Phi 3a}$: If $A_k, A \in X^\sharp$ and A is compact, then

$$A_k \to A \quad \Longrightarrow \quad \Phi(A_k) \to \Phi(A)\,.$$

$\mathbf{\Phi 3b}$: If $A_k, A \in X^\sharp$, then

$$A_k \to A \quad \Longrightarrow \quad \Phi(A_k) \to \Phi(A)\,.$$

Lemma 15 *Suppose Φ has properties $\mathbf{\Phi 1}$, $\mathbf{\Phi 2}$, and $\mathbf{\Phi 3a}$, and assume that the hypotheses $\mathbf{A1}$ and $\mathbf{A2}$ are valid. Then, for any bounded set $\mathscr{L} \subset X^\sharp$,*

$$\Phi(\omega^\sharp(\mathscr{L})) = \omega^\sharp(\Phi(\mathscr{L})) = \omega^\sharp(\mathscr{L})\,.$$

Proof. Any point B of $\omega^\sharp(\mathscr{L})$ is a limit of the form $B = \lim \Phi^{m_k}(A_k)$, where $m_k \nearrow +\infty$ and A_k is a sequence in $\mathscr{L}$. In addition, B is compact as a subset of X. Using property **Φ3a** we obtain $\Phi(B) = \lim \Phi^{m_k+1}(A_k)$. Note that $\lim \Phi^{m_k+1}(A_k)$ is simultaneously a point of $\omega^\sharp(\mathscr{L})$ and a point of $\omega^\sharp(\Phi(\mathscr{L}))$. Thus,

$$\Phi(\omega^\sharp(\mathscr{L})) \subset \omega^\sharp(\mathscr{L}), \quad \Phi(\omega^\sharp(\mathscr{L})) \subset \omega^\sharp(\Phi(\mathscr{L})).$$

On the other hand, the sequence $\Phi^{m_k-1}(A_k)$ has a convergent subsequence by Lemma 12. Working with this subsequence we obtain $B = \lim \Phi^{m_k-1}(\Phi(A_k)) = \Phi(\lim \Phi^{m_k-1}(A_k))$, which proves the inclusions

$$\omega^\sharp(\mathscr{L}) \subset \omega^\sharp(\Phi(\mathscr{L})), \quad \omega^\sharp(\mathscr{L}) \subset \Phi(\omega^\sharp(\mathscr{L})).$$

If $C \in \omega^\sharp(\Phi(\mathscr{L}))$, then $C = \lim \Phi^{m_k}(\Phi(A_k))$, and it is clear that $C \in \omega^\sharp(\mathscr{L})$, and we are done. □

Lemma 16 *Suppose Φ has properties* **Φ1**, **Φ2**, *and* **Φ3b**, *and assume that the hypotheses* **A1** *and* **A2** *are valid. Then, for any bounded set $\mathscr{L} \subset X^\sharp$,*

$$\omega^\sharp(\mathscr{L}) = \omega^\sharp(\overline{\mathscr{L}}^\sharp).$$

Proof. By monotonicity, $\omega^\sharp(\mathscr{L}) \subset \omega^\sharp(\overline{\mathscr{L}}^\sharp)$. On the other hand, if $B \in \omega^\sharp(\overline{\mathscr{L}}^\sharp)$, then $B = \lim_k \Phi^{n_k}(B_k)$, where $n_k \nearrow +\infty$ and $B_k \in \overline{\mathscr{L}}^\sharp$. For each B_k we can write $B_k = \lim_\ell A_\ell^k$, where $A_\ell^k \in \mathscr{L}$. Choose ℓ_k so that $d^\sharp(\Phi^{n_k}(B_k), \Phi^{n_k}(A_{\ell_k}^k)) < 2^{-k}$. The sequence $\Phi^{n_k}(A_{\ell_k}^k)$ has a convergent subsequence, call its limit C. For this subsequence,

$$d^\sharp(B, C) \leq d^\sharp(B, \Phi^{n_k}(B_k)) + d^\sharp(\Phi^{n_k}(B_k), \Phi^{n_k}(A_{\ell_k}^k)) + d^\sharp(\Phi^{n_k}(A_{\ell_k}^k), C) \to 0.$$

By construction, $C \in \omega^\sharp(\mathscr{L})$. □

Theorem 17 *If Φ has properties* **Φ1**, **Φ2**, *and* **Φ3a** *and the assumptions* **A1** *and* **A2** *are satisfied, then the global attractor, $\mathscr{K} = \omega^\sharp(\mathbf{B}^\sharp)$, of the system $(\Phi, X^\sharp)$ is invariant under Φ, i.e., $\Phi(\mathscr{K}) = \mathscr{K}$. This, in turn, implies that $\mathscr{K}$ is the union (in $X^\sharp$) of bounded two-sided trajectories of Φ (a two-sided trajectory is a sequence A_n, $n \in \mathbb{Z}$, such that $\Phi(A_n) = A_{n+1}$). Also, $\mathscr{K}$ is the maximal Φ-invariant compact subset of $X^\sharp$.*

4 Attractors and hyperattractors for single-valued maps

In this section we derive a few corollaries for attractors of single-valued maps. Let $S : X \to X$ be a single-valued map. We assume that S is bounded. We do not assume it to be continuous for now. We will write (S, X) for both the semi-dynamical system on X generated by the iterations of S in the traditional sense and the semi-dynamical system on the bounded subsets of X as discussed in the previous section. The hypotheses **Φ1** and **Φ2** are obviously satisfied for the map $\Phi(A) = S(A)$. As an immediate corollary of Theorem 7 and Lemma 4, we obtain this result.

Theorem 18 *Suppose S is a bounded map. For the system (S, X) to possess a global attractor it is necessary and sufficient that the following two conditions be satisfied:*

1) *there exists a global absorbing set;*

2) *for every bounded sequence $\{x_k\}$ and every strictly increasing integer sequence n_k, the sequence $S^{n_k}(x_k)$ has a convergent subsequence.*

The invariance properties of the attractor and the ω-limit sets require continuity of the map S.

Proposition 19 *Let S be a bounded map such that (S, X) possesses a global attractor. If, in addition, S is continuous, then $S(\omega^\flat(A)) = \omega^\flat(A)$ for any bounded A. Also, $S(K) = K$, and the global attractor K is the maximal S-invariant compact.*

Proof. If $x \in \omega^\flat(A)$, then $x = \lim_k S^{n_k+1}(x_k)$ for some $x_k \in A$ and some $n_k \nearrow \infty$. There is a convergent subsequence of $S^{n_k}(x_k)$, and its limit lies necessarily in $\omega^\flat(A)$. The equation $\lim_k S^{n_k+1}(x_k) = S(\lim_k S^{n_k}(x_k))$ proves $\omega^\flat(A) = S(\omega^\flat(A))$. There cannot be a compact S-invariant set larger than K because K wouldn't be able to attract it, which it must do. $\square$

We now move to the hyperspace.

Theorem 20 *Suppose $S : X \to X$ is a bounded map such that the discrete semi-dynamical system (S, X) possesses a global attractor.*

(1) *If the map S is closed, then the system $(S, X^\sharp)$ possesses a global attractor.*

(2) *If the map S is continuous, then the semi-dynamical system generated by the map*

$$S^\sharp : A \mapsto \overline{S(A)} \tag{6}$$

on the space $X^\sharp$ possesses a global attractor.

(3) *If the map S is continuous, if K is the global attractor of (S, X), and if $\mathscr{K}$ is the global attractor of $(S^\sharp, X^\sharp)$, then*

$$S(K) = K, \quad S^\sharp(\mathscr{K}) = \mathscr{K}, \quad K = \mathscr{K}^\flat, \quad \mathscr{K} = K^\sharp.$$

Proof. The first statement follows from Theorem 9. To prove the second statement, we first notice that for a continuous S the following equality holds for all bounded A and all $n \geq 1$:

$$S^{\sharp^n}(A) = \overline{S^n(A)}. \tag{7}$$

Now consider the system $(S^\sharp, X^\sharp)$ with a single-valued map $S^\sharp : X^\sharp \to X^\sharp$. This system has a global absorbing set: take $\mathbf{B}^\sharp$, where $\mathbf{B} = \overline{\mathcal{O}_1(K)}$ and K is the global attractor of (S, X). In addition, if A_k is a bounded sequence in $X^\sharp$ and $n_k \nearrow \infty$, the sequence

$S^{\sharp n_k}(A_k)$ has a convergent subsequence. Indeed, consider the bounded set $A = \bigcup_k A_k$ and the corresponding compact set $\omega^\flat(A)$. Since $\omega^\flat(A)$ attracts A, i.e., $e(S^n(A), \omega^\flat(A)) \to 0$, we have $e(S^{n_k}(A_k), \omega^\flat(A)) \to 0$ as well. By Lemma 5(1), the sequence $\overline{S^{n_k}(A_k)} = S^{\sharp n_k}(A_k)$ (we have used (7)) has a convergent subsequence in $X^\sharp$. The existence of a global attractor for $(S^\sharp, X^\sharp)$ now follows from Theorem 18.

Let S be continuous. Then $S(K) = K$ by Proposition 19. Lemma 6 shows that $S^\sharp$ has the property **Φ3a**, and Theorem 17 implies the invariance of the attractor $\mathscr{K}$ under $S^\sharp$. The fact that $S^\sharp(\mathscr{K}) = \mathscr{K}$ means that 1) for every $A \in \mathscr{K}$, $S(A) \in \mathscr{K}$; and 2) for every $A \in \mathscr{K}$ there exists a $B \in \mathscr{K}$ such that $A = S(B)$. This implies that the set $K_0 = \mathscr{K}^\flat$ is invariant under S. Indeed,

$$K_0 = \bigcup_{A \in \mathcal{K}} A = \bigcup_{B \in \mathcal{K}} S(B) = S\left(\bigcup_{B \in \mathcal{K}} B\right) = S(K_0)\,.$$

K_0 is compact by Lemma 2(3). Since every invariant compact is an element of $\mathscr{K}$, K_0 is maximal, and hence $K_0 = K$ by Proposition 19. Now, consider the set $K^\sharp$. It is clear that $S(K^\sharp) \subset K^\sharp$. In the opposite direction, if $A \in K^\sharp$, then $B = \{y \in K \,:\, S(y) \in A\}$ is compact and $S(B) = A$. Thus, $K^\sharp$ is $S^\sharp$-invariant. Every $A \in \mathscr{K}$ is a compact subset of $\mathscr{K}^\flat = K$, and hence $A \in K^\sharp$. Thus $K^\sharp = \mathscr{K}$. The theorem is proved. □

5 Example: IFS

Let (X, d) be a complete metric space and let S_j be a family of maps $X \to X$ indexed by a compact metric space $\mathscr{J}$ with a metric $d_{\mathscr{J}}$. The hypotheses on the maps S_j are these.

H1 Each map S_j is bounded and continuous.

H2 There is a mnc ψ on X such that each map S_j is ψ-condensing.

H3 For every closed, bounded set $A \subset X$, the maps S_j, when restricted to A, depend uniformly continuously on j. More precisely, given a closed, bounded A, for every $\epsilon > 0$ there exists $\delta(A; \epsilon) > 0$ such that

$$\sup_{x \in A} d(S_{j_1}(x), S_{j_2}(x)) \leq \epsilon$$

provided $d_{\mathscr{J}}(j_1, j_2) \leq \delta(A; \epsilon)$.

NB: If $\mathscr{J}$ is a finite set, **H3** is satisfied automatically.

H4 There exists a closed, bounded absorbing set $\mathbf{B} \subset X$ such that for every closed, bounded $A \subset X$ there exists $N(A) > 0$ such that $S_{j_n} \circ S_{j_{n-1}} \circ \cdots \circ S_{j_1}(A) \subset \mathbf{B}$, for any $n \geq N(A)$ and any choice of $j_k \in \mathscr{J}$.

Define the map $\Phi : X^\sharp \to X^\sharp$ as follows:

$$\Phi(A) = \overline{\bigcup_{j \in \mathscr{J}} S_j(A)}\,. \tag{8}$$

Lemma 21 *The map Φ is ψ-condensing, i.e., $\psi(\Phi(B)) \le \psi(B)$ with strict inequality when B is not compact.*

Proof. Let $B \in X^\sharp$. Set $\epsilon = 1/2$ and let $\delta^0 = \delta(B; \epsilon^0)$ be as in **H3**. Denote by $\mathscr{J}(\delta^0) = \{j_1^0, \dots, j_R^0\}$ a finite δ^0-net in $\mathscr{J}$. Using the property mnc(iv) of ψ, we estimate

$$\begin{aligned} &\left|\psi\left(\bigcup_{j\in\mathscr{J}} S_j(B)\right) - \psi\left(\bigcup_{j'\in\mathscr{J}(\delta^0)} S_{j'}\right)\right| \\ &\le c_\psi\, d^\sharp\left(\bigcup_{j\in\mathscr{J}} S_j(B), \bigcup_{j'\in\mathscr{J}(\delta^0)} S_{j'}(B)\right) \le c_\psi\, \epsilon^0. \end{aligned} \tag{9}$$

Using the properties mnc(ii) and mnc(iii) of ψ, we obtain

$$\begin{aligned} &\psi\left(\Phi(B)\right) = \psi\left(\bigcup_{j\in\mathscr{J}} S_j(B)\right) \le \psi\left(\bigcup_{j'\in\mathscr{J}(\delta^0)} S_{j'}(B)\right) + c_\psi\, \epsilon^0 \\ &= \psi(S_{j^0}(B)) + c_\psi\, \epsilon^0\,, \end{aligned} \tag{10}$$

for some $j^0 \in \mathscr{J}(\delta^0)$ such that

$$\psi(S_{j^0}(B)) = \max_{j'\in\mathscr{J}(\delta^0)} \psi(S_{j'}(B))\,.$$

Similarly, we can find $j^1 \in \mathscr{J}(\delta^1) \subset \mathscr{J}$ such that

$$\psi\left(\Phi(B)\right) \le \psi(S_{j^1}(B)) + c_\psi\, \epsilon^1\,.$$

Proceeding in this fashion, let $j^k \in \mathscr{J}$ be the sequence of indices such that for every $k \ge 0$, we have

$$\psi\left(\Phi(B)\right) \le \psi(S_{j^k}(B)) + c_\psi\, \epsilon^k\,.$$

Since $\mathscr{J}$ is a compact set, we can find a convergent subsequence $j^{k_l} \to j^* \in \mathscr{J}$. Then we have

$$\psi(S_{j^k}(B)) + c_\psi\, \epsilon^k \to \psi(S_{j^*}(B))$$

as $k \to \infty$. Therefore,

$$\psi\left(\Phi(B)\right) \le \psi(S_{j^*}(B)) \le \psi(B).$$

The second inequality follows from the assumption **H2**, and this inequality is strict if B is not compact. Lemma is proved. $\square$

Lemma 22 *The map Φ has the property* **Φ3a**.

Proof. The following inequality (its proof is straightforward) will be useful. Let $\mathscr{N}_1$ and $\mathscr{N}_2$ be two subsets of $X^\sharp$. Then

$$d^\sharp\left(\mathscr{N}_1^\flat, \mathscr{N}_2^\flat\right) \;\leq\; d^{\sharp\sharp}(\mathscr{N}_1, \mathscr{N}_2)\,. \tag{11}$$

Let A_n be a sequence in $X^\sharp$ converging to A. We have

$$d^\sharp(\Phi(A_n), \Phi(A)) = d^\sharp(\overline{\bigcup_{j\in\mathscr{J}} S_j(A_n)}, \overline{\bigcup_{k\in\mathscr{J}} S_k(A)}) = d^\sharp(\bigcup_{j\in\mathscr{J}} S_j(A_n), \bigcup_{k\in\mathscr{J}} S_k(A))\,. \tag{12}$$

Pick an $\epsilon > 0$. For all sufficiently large n, A_n will lie in the closed 1-neighborhood of A; call this neighborhood V. Thanks to **H3**, there is a finite $\delta(V;\epsilon)$-net in $\mathscr{J}$, call it $\mathscr{J}_\epsilon$, such that

$$\inf_{j'\in\mathscr{J}_\epsilon} \sup_{x\in V} d_X(S_j(x), S_{j'}(x)) \leq \epsilon$$

for any $j \in \mathscr{J}$. This implies, in particular, that for any closed set $C \subset V$ and for any j there is a $j' \in \mathscr{J}_\epsilon$ such that

$$d^\sharp(S_j(C), S_{j'}(C)) \leq \epsilon\,.$$

Indeed, for j' closest to j, we have

$$\sup_{x\in C} \inf_{y\in C} d(S_j(x), S_{j'}(y)) \leq \sup_{x\in C} d(S_j(x), S_{j'}(x)) \leq \epsilon\,,$$

and the same is true with the roles of x and y interchanged.

We use this information to obtain

$$d^\sharp\left(\bigcup_{k\in\mathscr{J}} S_k(C), \bigcup_{k'\in\mathscr{J}_\epsilon} S_{k'}(C)\right) \leq \epsilon\,,$$

for any closed set $C \subset V$. Thus, we have

$$d^\sharp(\bigcup_{j\in\mathscr{J}} S_j(A_n), \bigcup_{k\in\mathscr{J}} S_k(A)) \leq d^\sharp(\bigcup_{k\in\mathscr{J}_\epsilon} S_k(A_n), \bigcup_{k'\in\mathscr{J}_\epsilon} S_{k'}(A)) + 2\epsilon\,. \tag{13}$$

In view of (11), we have

$$d^\sharp(\bigcup_{k\in\mathscr{J}_\varkappa} S_k(A_n), \bigcup_{k'\in\mathscr{J}_\varkappa} S_{k'}(A)) \leq d^{\sharp\sharp}(\mathscr{M}_n, \mathscr{M})\,,$$

where

$$\mathscr{M}_n = \biguplus_{k\in\mathscr{J}_\varkappa}^{\sharp} S_k(A_n)\,, \quad \mathscr{M} = \biguplus_{k\in\mathscr{J}_\varkappa}^{\sharp} S_k(A)\,.$$

Now,

$$d^{\sharp\sharp}(\mathscr{M}_n, \mathscr{M}) \leq \sup_{k\in\mathscr{J}_\varkappa} d^\sharp(S_k(A_n), S_k(A))\,. \tag{14}$$

Since the set $\mathscr{J}_\epsilon$ is finite, there is a $k_* \in \mathscr{J}_\epsilon$ such that

$$\sup_{k\in\mathscr{J}_\epsilon} d^\sharp(S_k(A), S_k(A_n)) = d^\sharp(S_{k_*}(A), S_{k_*}(A_n)) \tag{15}$$

for infinitely many n. If A is compact, by Lemma 6(3), for any k,

$$d^\sharp(S_k(A), S_k(A_n)) \to 0\,.$$

Combining this with (12),(13), (14), and (15), we conclude that $d^\sharp(\Phi(A_n), \Phi(A)) \to 0$, proving the property **Φ3a**. □

Theorem 23 *Assume that the maps S_j satisfy the hypotheses* **H1** *through* **H4**. *Let Φ be defined by formula* (8). *Then the semi-dynamical system* $(\Phi, X^\sharp)$ *possesses a global attractor. This attractor, $\mathscr{K}$, is the minimal compact set in $X^\sharp$ that attracts all bounded subsets of $X^\sharp$. In addition, $\Phi(\mathscr{K}) = \mathscr{K}$, and $\mathscr{K}$ is the maximal compact subset of $X^\sharp$ with this property.*

Proof. From its definition (8), the map Φ has the properties **Φ1** and **Φ2**. The hypothesis **H4** guarantees the property **A1** for Φ. The property **Φ3a** is proved in Lemma 22. Assumption **A2**, i.e., the fact that Φ is asymptotically ψ-condensing, follows from Lemma 21 and from Lemma 27 that we prove in the next section. Now, all the claims follow from Theorem 17. □

The map Φ will have the property **Φ3b** if we impose in addition the following assumption on all S_j:

H5: For every closed, bounded set $V \subset X$ and for every $j \in \mathscr{J}$, the map S_j is uniformly continuous on V, i.e., for every $\epsilon > 0$, there exists a $\delta = \delta(j, V, \epsilon) > 0$ such that $d(S_j(x), S_j(y)) < \epsilon$ provided $x, y \in V$ and $d_X(x, y) < \delta$.

Lemma 24 *Assume that* **H5** *is satisfied. If A_n is a sequence in $X^\sharp$ converging to $A \in X^\sharp$, then $S_j(A_n) \to S_j(A)$ for any $j \in \mathscr{J}$. As a corollary, the map Φ has the property* **Φ3b**.

The proof is straightforward and we omit it.

6 On asymptotically condensing maps

We start by showing that the hypothesis **A2** does not depend on the particular choice of the mnc.

Lemma 25 *Suppose Φ has the properties* **Φ1**, **Φ2**, *and the assumption* **A1** *is satisfied. If Φ is asymptotically condensing with respect to some mnc ψ_0, then Φ is asymptotically condensing with respect to any mnc ψ.*

Proof. The assumptions guarantee that for every $A \in X^\sharp$, the set $\omega^\flat(A) = \bigcap_n \overline{\bigcup_{m\geq n} \Phi^m(A)}$ is a nonempty compact, see Lemma 8. The proof of that Lemma relies on the fact that $\psi_0\left(\bigcup_{m\geq n} \Phi^m(A)\right) \to 0$. By Lemma 4, this is equivalent to the fact that any sequence of points of the form $x_k \in \bigcup_{m\geq n_k} \Phi^m(A)$ has a convergent subsequence. This statement is not attached to any mnc. We use Lemma 4 to conclude that $\psi\left(\bigcup_{m\geq n} \Phi^m(A)\right) \to 0$ for any given ψ with the properties mnc(i)-(iv). Thanks to mnc(ii) it follows that $\psi(\Phi^n(A)) \to 0$, and we are done. □

For applications, it is important to have a collection of sufficient conditions for a map to be asymptotically condensing. It is clear, that if Φ maps bounded sets into compact sets, it is asymptotically condensing. Another example is a single-valued contraction.

Lemma 26 *Any (single-valued) strict contraction $S : X \to X$ is asymptotically ψ-condensing.*

Proof. Let $q < 1$ be the contraction constant of S. By the Banach fixed point theorem, there exists a unique $z \in X$ such that $S(z) = z$, and moreover, $z = \lim_n S^n(x)$ for any $x \in X$. In fact, $\{z\}$ is the global attractor of (S, X) because, given a bounded set A,

$$e(S^n(A), z) = \sup_{x\in A} d(S^n(x), z) = \sup_{x\in A} d(S^n(x), S^n(z)) \leq q^n\, e(A, z) \to 0,$$

and similarly, $e(z, S^n(A)) = \inf_{x\in A} d(z, S^n(x)) \leq q^n d(z, A) \to 0$. Thus, $S^n(a) \to \{z\}$, and by Lemma 5, $\psi(S^n(A)) \to 0$. □

In the single-valued setting, compact maps and strict contractions are examples of a more general class of condensing maps. Recall that a map, $S : X \to X$, is ψ-condensing if $\psi(S(A)) \leq \psi(A)$ for any bounded A, with strict inequality if $\psi(A) > 0$. The proof of the fact that ψ-condensing implies asymptotically ψ-condensing for single-valued maps may be found in [1, Lemmas 1.6.10, 1.6.11] for mnc's having properties mnc(i)-(iv), see [1, Remark 1.6.13]. In his PhD thesis, Massatt considered multi-valued maps and more general mnc's. The result below is a variation on his result, cf. [31, Theorems 4.1 and 4.2].

• We consider a multi-valued map $\Phi : X^\sharp \to X^\sharp$. We assume that Φ is ψ-condensing with respect to some mnc ψ with the properties mnc(i)-(iv). Thus, $\psi(\Phi(B)) \leq \psi(B)$ for any bounded B, and the inequality is strict if $\psi(B) > 0$. In addition, we require one of the following two properties ("compact sets" in **Φ4a** are replaced by "finite sets" in **Φ4b**).

Φ4a(**b**): For every $D \in X^\sharp$, there is a sequence of sets $B_n \subset \Phi^n(D)$, $n = 1, 2, \ldots$, such that

1) $d^\sharp(B_n, \Phi^n(D)) \to 0$;
2) for $n = 1, 2, \ldots$, for every compact set (**b** : finite set) $L \in B_n$ there is a compact set (**b** : finite set) $L' \subset B_{n-1}$ such that

$$L \subset \Phi(L')\ . \tag{16}$$

A ready example of a map that satisfies **Φ4b** is the Hutchinson-Barnsley map.

Lemma 27 *Under the hypotheses* **H1** *through* **H4** *of Section 5, the Hutchinson-Barnsley map* $\Phi(D) = \overline{\bigcup_{\mathscr{J}} S_j(D)}$ *has the property* **Φ4b**.

Indeed, define

$$\hat{\Phi}(A) = \bigcup_{\mathscr{J}} S_j(A)$$

and notice that $\Phi^n(D) = \overline{\hat{\Phi}^n(D)}$ due to the continuity of each map S_j. Clearly, Φ has the property **Φ4b** with the choice $B_n = \hat{\Phi}^n(D)$. □

Proposition 28 *Let* Φ *satisfy the assumptions* **Φ1**, **Φ2**, **Φ4a** *or* **Φ4b**, *and* **A1**. *Assume that* Φ *is* ψ*-condensing. Then, for every* $D \in X^\sharp$, $\psi(\Phi^n(D)) \to 0$.

Proof. We work with assumption **Φ4a**. The same proof works with **Φ4b**, just change the "compact sets" to "finite sets".

Our goal is to show that any sequence of the form $x_k \in \Phi^{n_k}(D)$ has a convergent subsequence, i.e., $\psi(\{x_k\}) = 0$. Once this is done, we see that any sequence of the form $y_k \in \bigcup_{\ell \geq m_k} \Phi^\ell(D)$ has a convergent subsequence. By Lemma 4, this implies $\psi\left(\bigcup_{\ell \geq m} \Phi^\ell(D)\right) \to 0$, and hence, $\psi(\Phi^m(D)) \to 0$, as stated.

We can replace the sequence $x_k \in \Phi^{n_k}(D)$ by a sequence $x'_k \in B_{n_k} \subset \Phi^{n_k}(D)$ such that $d(x'_k, x_k) \to 0$.

Now, instead of sequences $x'_k \in B_{n_k}$ we consider sequences in the hyperspace $X^\sharp$, which have the following form:

$$\mathscr{Y} = \biguplus_{n=0}^{\infty}{}^{\sharp}\, Y(n)\,,$$

where each $Y(n)$ is either an empty set or a compact subset of B_n, with the assumption that $Y(n)$ is nonempty for infinitely many n. We will prove that for each such $\mathscr{Y}$ the set $\mathscr{Y}^\flat \subset X$ is relatively compact.

To begin with, due to assumption **A1**, each set $\mathscr{Y}^\flat$ lies in the absorbing set **B** except, maybe, for a finite number of compact subsets. Hence, thanks to the properties mnc(i) and mnc(iii), $\psi(\mathscr{Y}^\flat) \leq \psi(\mathbf{B}) < \infty$, and there is a finite number

$$a = \sup_{\mathscr{Y}} \ \psi\left(\mathscr{Y}^\flat\right)\,.$$

If $\mathscr{Y}_m$, $m = 0, 1, \ldots$, is a sequence for which the corresponding values $\psi(\mathscr{Y}_m^\flat)$ approach a, the sequence

$$\mathscr{Z} = \biguplus_m{}^{\sharp}\, Z(m) := \biguplus_m{}^{\sharp} \bigcup_{0 \leq k \leq m} Y_k(m)$$

delivers the maximum value a,

$$\psi\left(\mathscr{Z}^\flat\right) = a\,,$$

because, except for a finite number of compact sets, each set $(\mathscr{Y}_m)^\flat$ is a subset of $\mathscr{Z}^\flat$.

We now prove that $\psi\left(\mathscr{Z}^\flat\right) = 0$. To this end, consider each nonempty $Z(n)$, where n is large enough so that $Z(n) \subset \mathbf{B}$. By $\mathbf{\Phi 4}$ there is a compact set $L(n-1) \subset B_{n-1}$ such that $\Phi(L(n-1)) \supset Z(n)$. This defines a new sequence $\mathscr{L}$. Except for a finite number of compacts,

$$\mathscr{Z}^\flat \subset \Phi\left(\mathscr{L}^\flat\right) .$$

As a result, since Φ is ψ-condensing,

$$a = \psi\left(\mathscr{Z}^\flat\right) \leq \psi\left(\Phi\left(\mathscr{L}^\flat\right)\right) \leq \psi\left(\mathscr{L}^\flat\right) \leq a\,,$$

and hence, $\psi\left(\Phi\left(\mathscr{L}^\flat\right)\right) = \psi\left(\mathscr{L}^\flat\right)$, which implies $\psi\left(\mathscr{L}^\flat\right) = 0$, and then $a = 0$. Our claim is proved. □

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