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Variable Time Step Dynamics With Choice

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Abstract

We develop a simple and general approach to study long term behavior of deterministic systems that switch regimes and have dwell times of variable length. We investigate the results of all possible as well as restricted switchings. To analyze all these situations, we introduce the notions of variable time step dynamics with choice and variable time step iterated function systems. We establish general sufficient conditions for the existence of global compact attractors of such systems and describe how they are related. To study the variable time step iterated function systems with restricted choice, we introduce the set-trajectories and point-trajectories generating systems and study their global attractors. A few examples illustrate the abstract theory.

Keywords
Global attractors, dynamical systems, switched systems, iterated functions systems, dwell time.

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1 Introduction

We are interested in time evolution of systems that can and do switch their modes (regimes) of operation at discrete moments of time. The intervals between switching may, in general, vary. The number of modes (regimes) may be finite or infinite. Such systems are very common in life. In control theory the systems we talk about are often called hybrid, see [10, 11]. The switching times and the switching of the regimes may be deterministic or random. Here we will discuss the deterministic case only. A few examples may be helpful.

The first example is a switched system, i.e., a special case of a hybrid system governed by a finite set of systems of ordinary differential equations,

$$\dot{x} = f_{w(n)}(x) \quad \text{on the interval } [t_n, t_{n+1}), \tag{1}$$

where $t_0 < t_1 < \ldots$ are the switching times, and each $f_{w(n)}$ is taken from a finite set of functions $\{f_0, \ldots, f_{N-1}\}$. Here $w(\cdot)$ is a regime switching function, it maps the non-negative integers into the label set of the available regimes, $\mathcal{J} = \{0, 1, \ldots, N-1\}$. Denote the states at the regime switching times by $x_n = x(t_n)$ and the time intervals between the switching (dwell time) by $h(n) = t_{n+1} - t_n$. We can write the transition from x_n to x_{n+1} symbolically as $x_{n+1} = S_{w(n)}^{h(n)}(x_n)$. Here $S_{w(n)}^{h(n)}$ is a transformation that solves system (1), i.e., for every y, $S_{w(n)}^{h(n)}(y)$ is the solution $x(t)$ of the system $\dot{x} = f_{w(n)}(x)$ with the initial condition $x(0) = y$, evaluated at time $t = h(n)$.

The second example is the Navier-Stokes equations in a bounded domain Ω in $\mathbb{R}^2$ with the Dirichlet boundary condition,

$$\begin{aligned} &\mathbf{v}_t - \nu_{w(n)}\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \mathbf{f}_{w(n)}(x), \quad \operatorname{div}\mathbf{v} = 0, \quad \text{in } [t_n, t_{n+1}) \times \Omega\,, \\ &\mathbf{v}(t,x) = 0 \quad \text{for } x \in \partial\Omega\,. \end{aligned} \tag{2}$$

Here we allow both the viscosity coefficient, ν, and the external forces, $\mathbf{f}$, instantaneously switch their values.

The third example is a "discrete" version of (1),

$$x_{n+1} = x_n + h(n)\, f_{w(n)}(x_n)\,, \tag{3}$$

where $h(n)$ is a variable, in general, time step. (In fact, (3) does not have to be related to (1) and could have a totally independent origin.) Again, we can write the transformation from the state x_n to the next state, x_{n+1}, as $x_{n+1} = S_{w(n)}^{h(n)}(x_n)$, but now $S_{w(n)}^{h(n)}$ is given explicitly by the right hand side of (3).

The presence of the time step, $h(n)$, in the equation may be more intricate as our fourth example shows:

$$x_{n+1} = x_n + r_{w(n)}\sqrt{h(n)} \mod 1\,. \tag{4}$$

Here $r_{w(n)}$ takes on values $+1$ or -1.

Note the following difference between the first two and the last two examples. In the first and second examples, the solution maps with fixed parameters do satisfy the semi-group property, $S_a^{\tau_1} \circ S_a^{\tau_2} = S_a^{\tau_1+\tau_2}$, whereas in the third and fourth examples we have $S_a^{\tau_1} \circ S_a^{\tau_2} \neq S_a^{\tau_1+\tau_2}$.

The switching times and the regime switching function can be given in advance, or can be generated step-by-step depending on external or internal information. In the latter case, $h(n+1)$ and $w(n+1)$ may depend on the state x_n, on the regime $w(n)$, or even on x_{n-1}, $w(n-1)$, etc. The rule generating $h(n+1)$ and $w(n+1)$ can be viewed as a control law.

In the present report we address the long term behavior of systems of the type just described. In particular, we are interested whether or not a system possesses a global attractor. A global attractor is a minimal compact subset of the state space that attracts all bounded sets (and not just individual points). This is an important notion because in practice the states of the system are known only approximately. If the global attractor consists of one point, this point (state) is automatically asymptotically stable. The global attractors of nonlinear dissipative systems may have very complicated geometry (as, e.g., the Lorenz attractor).

There is a large literature on asymptotic stability (of one state, the origin, $x = 0$) of both linear and nonlinear switched systems, see e.g. [29, 34] and references there. An unorthodox application of switched systems to space travel is described in [8]. Outside of engineering applications, we should mention an interesting paper [32] on using switching controls in epidemic models. A number of papers are devoted to attractors of switched systems, mostly of the form (1) in the finite-dimensional setting, see e.g. [13, 14, 22].

In [17, 18, 41] we developed a simple and general approach to dynamics of fixed time step switching systems. We use the term *dynamics with choice* because we handle systems more general than (1) and (3). In this paper we extend our approach to deal with variable time step. Also, in [17, 18, 41] we analyzed the connection between the dynamics with choice $x_{n+1} = S_{j_n}(x_n)$ and the corresponding iterated function system (IFS, for short) defined by the maps S_j via the Hutchinson-Barnsley map $F(x) = \bigcup_{j\in\mathcal{J}} S_j(x)$ and its extension to sets. We showed how fractals of IFSs appear in the attractors in dynamics with choice. Our treatment does not require the maps S_j be contractive.

It will help to be more specific in describing what the present paper is about. We do this in the next section.

2 Dynamics with Choice

Consider a set X (the state space) and a collection of maps $S_j^\tau : X \to X$. The lower index, j, indicates the chosen regime. All available regimes are labeled by the points of some set $\mathcal{J}$. The upper index, τ, is interpreted as the dwell time. The values of τ are taken from some set $\mathcal{I} \subset (0, +\infty)$. Thus, $S_j^\tau(x)$ is interpreted as the result of the regime j acting on the state x over the time period τ. Switching regimes and dwell times leads to the dynamics

$$x_{n+1} = S_{j_n}^{\tau_n}(x_n)\,. \tag{5}$$

We would like to be able to work with all the trajectories $x_0, x_1, x_2, \dots$ generated this way with arbitrary $j_n \in \mathcal{J}$ and $\tau_n \in \mathcal{I}$. One can think of several different mathematical formalizations of this dynamics. We will use two. Denote by $\Sigma_{\mathcal{J}}$ (respectively, $\Sigma_{\mathcal{I}}$) the space of maps from the non-negative integers $\mathbb{Z}_{\geq 0}$ into $\mathcal{J}$ (respectively, $\mathcal{I}$). Sometimes it is convenient to view a map, $w \in \Sigma_{\mathcal{J}}$, as a one-sided infinite string (word) of symbols $w(0)$, $w(1)$, $w(2)$, ... from $\mathcal{J}$; we write $w = w(0)w(1)w(2)\dots$. Sometimes we refer to w as a regime switching function. In a similar fashion, $h \in \Sigma_{\mathcal{I}}$ can be viewed as $h(0)h(1)h(2)\dots$, and we call h a dwell time function, or a time step function. The sets $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{I}}$ are equipped with a shift operator, σ, that acts as follows: $\sigma(w)(n) = w(n+1)$. [In the language of symbolic dynamics, $\Sigma_{\mathcal{J}}$ is a full one-sided shift over the alphabet $\mathcal{J}$, see [21, 30].] Now, we view equation (5) as a "non-autonomous system" and apply the well-known skew-product construction, [39], to obtain an "autonomous system."

Definition 1. *The variable time step dynamics with choice on X associated with the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, is the discrete time dynamics generated on the product space $\mathfrak{X} = X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$ by the iterations of the map*

$$\mathfrak{S} : (x, w, h) \mapsto (S_{w(0)}^{h(0)}(x), \sigma(w), \sigma(h))\,. \tag{6}$$

If the time step is fixed, i.e., the interval $\mathcal{I}$ is a single number, τ, then there is no need in showing it in the notation S_j^τ and we write simply S_j. Also, there is no need in using $\Sigma_{\mathcal{I}}$. The above definition then reduces to the dynamics generated on the space $X \times \Sigma_{\mathcal{J}}$ by the iterations of the map

$$\mathfrak{S} : (x, w) \mapsto (S_{w(0)}(x), \sigma(w))\,. \tag{7}$$

This is the *dynamics with choice* we introduced and studied in [17, 18, 41].

The second formalization of the deterministic dynamics (5) is motivated by iterated function systems (abbreviation: IFS – singular, IFSs – plural), [15, 6, 7]. The traditional IFS generated by the maps S_j (there is no variable time step there) on X can be defined by a single multi-valued Hutchinson map

$$\mathcal{F} : x \mapsto \bigcup_{j \in \mathcal{J}} S_j(x)\,.$$

This map is extended to one or the other family of subsets of X, e.g., the space of all subsets of X, or the space of all nonempty subsets of X, or the space of all nonempty and bounded subsets, or the space of all nonempty compact subsets, etc. A space of subsets is called a **hyperspace**. The iterations of $\mathcal{F}$ define a discrete dynamics on the chosen hyperspace. However, in the traditional approach to IFSs, the dynamics *per se* seems to be of little interest compared to the special set called attractor or fractal. Traditionally, $X = \mathbb{R}^d$, the set $\mathcal{J}$ is finite, and the maps S_j are contractions satisfying the Lipschitz condition with a common constant < 1. The IFS fractal, K, is the unique compact subset invariant under $\mathcal{F}$. In other words, K is the unique fixed point of $\mathcal{F}$ on the hyperspace. The fractal structure of K manifests itself in the equations

$$K = \bigcup_{j=1}^{N} K^j\,, \quad K^i(= S_i(K)) = S_i(K^1) \cup \cdots \cup S_i(K^N)\,, \quad i = 1, \ldots, N\,. \tag{8}$$

This setting has been generalized in several directions but so that one or the other fixed point theorem could be applied, see [2, p. 666] for references. In our papers [17, 18] we took a different approach. We identified K with the X component of the global attractor of the dynamics with choice generated by the maps S_j on X. The move from the fixed point theorems to the realm of global attractors has allowed us to considerably relax the assumptions on X, $\mathcal{J}$, and the maps S_j. In particular, the maps S_j are no longer required to be contractive. We prove that the attractor of the dynamics with choice (when it exists) has the form $K \times \Sigma_{\mathcal{J}}$, where K is the attractor of the IFS, i.e., it is the minimal compact subset of X that attracts all bounded sets in the sense that for every bounded $B \subset X$ and every open neighborhood U of K, $\mathcal{F}^n(B) \subset U$ for all sufficiently large n. In [19] we go further. We consider $\mathcal{F}$ as a map on the appropriate hyperspace $X^\sharp$, consider the dynamics $\mathcal{F}$ generates on $X^\sharp$, and prove that the semi-dynamical system $(\mathcal{F}, X^\sharp)$ possesses a global compact attractor (in $X^\sharp$). It turns out that the union of the sets comprising this attractor results in K, the IFS fractal. When the maps S_j are contractions, the global attractor of $(\mathcal{F}, X^\sharp)$ is $\{K\}$, a singleton. For more general maps, the global attractor may have more structure, and this offers new possibilities.

In the variable time step case, we propose three different versions of IFSs related to (5). We choose the hyperspace $X^\sharp$ (of all nonempty, bounded, and closed subsets of the complete metric space X). For the first version, define the maps F^τ, $\tau \in \mathcal{I}$, acting on $X^\sharp$ according to the rule

$$F^\tau(A) = \overline{\bigcup_{j \in \mathcal{J}} S_j^\tau(A)}\,. \tag{9}$$

Definition 2 (VTSIFS-1)**.** *The variable time step iterated function system on X associated with the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, is the discrete time dynamics generated on the product space $X^\sharp \times \Sigma_{\mathcal{I}}$ by the iterations of the map*

$$\mathfrak{F}^{\mathcal{I}} : (A, h) \mapsto (F^{h(0)}(A), \sigma(h))\,. \tag{10}$$

Another possibility is to define the maps

$$F_j(A) = \overline{\bigcup_{\tau \in \mathcal{I}} S_j^\tau(A)} \tag{11}$$

and define the corresponding IFS as the discrete time dynamics generated on the product space $X^\sharp \times \Sigma_{\mathcal{J}}$ by the iterations of the map

$$\mathfrak{F}_{\mathcal{J}} : (A, u) \mapsto (F_{u(0)}(A), \sigma(u))\,. \tag{12}$$

Definition 3 (VTSIFS-2)**.** *The variable time step iterated function system on X associated with the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, is the discrete time dynamics generated on the product space $X^\sharp \times \Sigma_{\mathcal{J}}$ by the iterations of the map $\mathfrak{F}_{\mathcal{J}}$.*

In addition, we propose another useful variant of a variable time step IFS.

Definition 4 (VTSIFS-3)**.** *The variable time step iterated function system associated with the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, is the discrete time dynamics generated on the hyperspace $X^\sharp$ by the iterations of the map*

$$\mathcal{F} : A \mapsto \overline{\bigcup_{\tau \in \mathcal{I}} F^\tau(A)}\,. \tag{13}$$

In this paper we show that the proposed notions of the variable time step dynamics with choice and of the associated IFSs are viable. We use them to study in a unified way the existence of global compact attractors in each of the systems given by four definitions and to establish the relationships between the corresponding attractors. In the forthcoming paper [20] we give another application of the formalism: we study what kind of dynamics results when the dwell times shrink to zero.

Once again, the assumptions on the maps S_j^τ are at the level of the base space X. The conclusions we draw for the dynamics of IFSs are at the level of the hyperspace $X^\sharp$. This is one of the main difficulties we overcome.

Our setting for the variable time step dynamics with choice can be easily adjusted to describe the situations where there are restrictions on the allowed sequences of switching regimes and/or dwell times. Mathematically this means that the full shifts $\Sigma_{\mathcal{J}}$ and/or $\Sigma_{\mathcal{I}}$ are replaced by some subshifts, say, $\Lambda_{\mathcal{J}}$ and $\Lambda_{\mathcal{I}}$. Recall that a subshift is a closed and shift-invariant subset of the shift. To give a few examples, take $\mathcal{J} = \{0, 1\}$, a two-symbol set. Then $\Sigma_{\mathcal{J}}$ is the set of all binary sequences. An example of a subshift is the set of all binary sequences with no two consecutive 1's and no two consecutive 0's, i.e., $\Lambda_{\mathcal{J}}$ consists of two sequences, $10101\ldots$ and $01010\ldots$. The restriction from $\Sigma_{\mathcal{J}}$ to $\Lambda_{\mathcal{J}}$ then simply means that the two regimes must alternate. Another example of a subshift of the same shift is the set of all binary sequences with no two consecutive 1's (i.e., the word 11 is forbidden anywhere in the sequence) – this is the so-called golden mean shift. The subshifts defined by (a finite number of) forbidden words form the class of subshifts of finite type. The subshift of all binary sequences containing at most one 1 is not of finite type. It belongs to a larger class of sofic subshifts. Consider a directed finite graph such that through each vertex passes at least one infinite path, and label each edge by symbols 0 or 1. Then to an infinite path starting from any vertex corresponds an infinite sequence of 0's and 1's. The set of all sequences generated this way (while the graph and its labeling are fixed) is a sofic subshift of $\Sigma_{\{0,1\}}$, and all sofic subshifts are obtained by choosing a graph and its labeling. There are subshifts that are not sofic. For an in-depth discussion of shifts see [30].

The definition of a variable time step dynamics with restricted choice is this.

Definition 5. *The variable time step dynamics with choice on X associated with the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, and guided by the subshifts $\Lambda_{\mathcal{J}}$ and $\Lambda_{\mathcal{I}}$, is the discrete time dynamics generated on the product space $X \times \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}}$ by the iterations of the map* (6).

Defining restricted IFSs is a little trickier. There are several possibilities. One possibility is to consider the systems $(\mathfrak{F}, X^\sharp \times \Lambda_{\mathcal{I}})$ and $(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Lambda_{\mathcal{J}})$ leaving the same definitions (9) and (11) and forgetting the corresponding subshifts $\Lambda_{\mathcal{J}}$ or $\Lambda_{\mathcal{I}}$ respectively. The VTSIFS-3 system then will have no natural analogue.

We should mention that in the traditional setting there is a meaningful definition of the IFS fractal associated with a subshift $\Lambda \subset \Sigma_{\mathcal{J}}$, see [5]. Consider a finite number of contractions S_j on X. As shown in [15], there is a well-defined map $p : \Sigma_{\mathcal{J}} \to X$, where

$$p(w) = \lim_{n\to\infty} S_{w(0)} \circ S_{w(1)} \circ \cdots \circ S_{w(n-1)}(x)\,, \tag{14}$$

and x is any point of X. This map is continuous. It is easy to see that $S_i\,(p(w)) = p(i.w)$ and conclude that $K = p(\Sigma_{\mathcal{J}})$ is the IFS fractal. Given a subshift Λ, one can *define* the corresponding fractal K_Λ to be $p(\Lambda)$, see [7]. This time the IFS fractal K_Λ satisfies the equations

$$K_\Lambda = \bigcup_{j=1}^{N} K_\Lambda^j\,, \quad K_\Lambda^j = \{x \in K_\Lambda \,:\, S_j(x) \in K_\Lambda\}\,. \tag{15}$$

In the more general setting of dynamics with choice, when the maps S_j are not contractions, the construction with the map p does not work. Also, the global attractor, $\mathfrak{M}_\Lambda$, of the system $(\mathfrak{S}, X \times \Lambda)$ is not in general a product of a set in X and Λ (as it was in the case of $\Sigma_{\mathcal{J}}$). Nevertheless, in [17], we defined K_Λ as the projection of the global attractor on X, $K_\Lambda = \{x \in X \,:\, (x,w) \in \mathfrak{M}_\Lambda \text{ for some } w \in \Lambda\}$, and showed that it admits a decomposition of the form (15). (However, this K_Λ is not, in general, equal to $p(\Lambda)$: it is equal to $p(\Lambda')$ where Λ' is the subshift dual to Λ. We explain this in detail in Section 5.3.)

In this paper we propose a different approach to restricted IFSs. Consider the maps $\Theta_\Lambda^{[n]}$, $n = 1, 2, \ldots$, acting on the subsets of X as follows:

$$\Theta_\Lambda^{[n]}(A) = \overline{\bigcup_{w\in\Lambda_{\mathcal{J}}} \bigcup_{h\in\Lambda_{\mathcal{I}}} S_{w[n]}^{h[n]}(A)}\,. \tag{16}$$

These maps do not satisfy the strict semi-group property. Instead, they have the property that

$$\Theta_\Lambda^{[m+n]}(A) \subset \Theta_\Lambda^{[m]}\left(\Theta_\Lambda^{[n]}(A)\right) \tag{17}$$

for any bounded set $A \subset X$ and all $m, n \geq 1$. Every time there is a collection of maps like that, we can study the dynamics of sets in the base space X, or we can study the dynamics of points in the hyperspace $X^\sharp$. The former dynamics we refer to as a **set-trajectories generating system** and the latter we call a **point-trajectories generating system**. [The notation in the case of the maps $\Theta_\Lambda^{[n]}$ will be $\left(\{\Theta_\Lambda^{[n]}\}, X\right)$ and $\left(\{\Theta_\Lambda^{[n]}\}, X^\sharp\right)$, respectively.]

We study briefly such systems associated with a sequence of maps $\Theta^{[n]}$ defined on the subsets of X. The notion of a set-trajectories generating system generalizes the notion of a multi-valued flow (or m-flow) considered by Melnik and Valero in [37]. We do not require that the maps on the sets originate from the set-valued maps on the points of the sets: we do not have the equality $\Theta^{[n]}(A) = \bigcup_{x \in A} \Theta^{[n]}(x)$ in the abstract setting, and it does not hold for our maps (16). Lifting the dynamics to the hyperspace level is new both conceptually and technically. Even though the space X or the space $X^\sharp$ with the maps $\Theta_\Lambda^{[1]}$, $\Theta_\Lambda^{[2]}$, ... are not semi-dynamical systems, the notions of a trajectory, of stability and attractivity, all make sense. Under a few natural assumptions we are able to develop a theory of global attractors for such systems. We apply this theory to the restricted variable time step IFSs and show that under the same general assumptions on X, $\mathcal{J}$, $\mathcal{I}$, and the maps S_j^τ, the systems corresponding to any choice of the subshifts $\Lambda_\mathcal{J}$ and $\Lambda_\mathcal{I}$ have global attractors. In the case when the maps S_j^τ are contractions, it turns out that the attractor of the set-trajectories generating system $\big(\{\Theta^{[n]}, X\}\big)$ is not the image of the subshift $\Lambda = \Lambda_\mathcal{J} \times \Lambda_\mathcal{I}$ under the mapping p, but rather the image of the **dual to Λ subshift** Λ'. We could not find in the literature the notion relating the two subshifts, so we had to invent it and we called it duality, see Section 5.4.

Our assumptions on X, $\mathcal{J}$, $\mathcal{I}$, and the maps S_j^τ stay the same through the theoretical sections 4 and 5. We set the notation and specify the assumptions in Section 3. We should mention that our assumptions are very mild. We require of S_j^τ what one would normally check for a single map to make sure that the corresponding discrete semi-dynamical system has a global attractor. We prove the existence of global attractors for the variable time step dynamics with choice and for different versions of IFSs in Section 4. There, we also show how different attractors are related to each other. In Section 5 we establish similar results for dynamics with restricted choice. In Section 6 we consider an example of the Navier-Stokes system in a two-dimensional bounded domain. The kinematic viscosity and the external forces are allowed to switch, and the dwell times are bounded from above and the lower limit on the inter-switching times is positive. We show how the standard *a priori* estimates imply our theoretical assumptions. The results on global attractors then follow from the theorems of Section 4.

In applications, some of our assumptions are easier to check than the others. In Section 7 we show that the assumption we associate with dissipativity may be very sensitive to the choice of the interval $\mathcal{I}$. We give an example of a switched system on the two-dimensional cylinder such that for one choice of the interval of allowed dwell times, the variable time step dynamics with choice has a global attractor, while for a different $\mathcal{I}$, there is no global attractor.

The abstract theory we advance in Sections 4 and 5 uses certain basic results on convergence of sets, measures of noncompactness, and attractors for multi-valued maps. These results are collected in the appendix, Section 8.

3 Basic notation, definitions, and assumptions

Consider a discrete time dynamics generated on a metric space Y by iterations of a map $G : Y \to Y$.

Definition 6. *The global attractor of the semi-dynamical system (G, Y) is the minimal compact set $\mathfrak{A} \subset Y$ that attracts all bounded sets. The latter means that for every $A \subset Y$ of finite diameter, and for any open neighborhood U of $\mathfrak{A}$, there exists N such that $G^n(A) \subset U$ for all $n \geq N$.*

The definitions in the previous section present the variable time step dynamics (5) in the form of a discrete semi-dynamical system. The theory of attractors for continuous time as well as for discrete semi-dynamical systems is well developed, see e.g. [26, 40] and a brief account in [16]. We will apply this theory in our situation. To this end we first need to specify the basic assumptions on the state space X and the maps S_j^τ.

• X is a complete metric space with metric d_X. We use letters $A, B, \ldots$ to denote subsets of X, and $\overline{A}$ denotes the closure of A in X. For $A \subset X$, and $x \in X$, $d(x, A) = \inf_{y \in A} d(x, y)$. $B_r(x_0)$ is the open ball of radius r centered at x_0, $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$. $\mathcal{O}_r(A)$ is the open r-neighborhood of A, $\mathcal{O}_r(A) = \{x \in X : d(x, A) < r\} = \bigcup_{y \in A} B_r(y)$. $e(A, B)$ denotes the excess of A over B, i.e., $e(A, B) = \sup_{x \in A} d(x, B)$. We say that the *set B attracts the sets A_n* iff $e(A_n, B) \to 0$. The Hausdorff distance between the sets A and B is $d^\sharp(A, B) = \max\{e(A, B), e(B, A)\}$.

• $\mathcal{J}$ is a metric compact with metric $d_{\mathcal{J}}$.

• $\mathcal{I} = [a, b]$, where $0 < a \leq b$.

• The spaces $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{I}}$ consist of infinite one-sided strings of symbols from $\mathcal{J}$ and $\mathcal{I}$ respectively. The spaces $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{I}}$ will be equipped with the metrics $d_{\Sigma_{\mathcal{J}}}$ and $d_{\Sigma_{\mathcal{I}}}$, where

$$d_{\Sigma_{\mathcal{J}}}(u, v) = \sum_{n=0}^{\infty} 2^{-n} \, d_{\mathcal{J}}(u(n), v(n)) \,,$$

and a similar formula for $d_{\Sigma_{\mathcal{I}}}$. If the alphabet $\mathcal{J}$, or $\mathcal{I}$, or both are finite, the corresponding metrics are discrete. The metric spaces $\Sigma_{\mathcal{I}}$ and $\Sigma_{\mathcal{J}}$ are compact. The shift, σ, is continuous in each space.

(A1) For every $j \in \mathcal{J}$ and every $\tau \in \mathcal{I}$, the map $S_j^\tau : X \to X$ is continuous and bounded.

(A2) On X, there is a measure of noncompactness, ψ_X, such that each map S_j^τ is ψ_X-condensing.

Definition 7. *We call a function $\psi : 2^X \to [0, +\infty]$ a measure of noncompactness (mnc, for short) on X iff*

mnc(i) $\psi(A) = 0$ *iff A is relatively compact;*

mnc(ii) *If* $A_1 \subset A_2$*, then* $\psi(A_1) \leq \psi(A_2)$ *;*

mnc(iii) $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$ *;*

mnc(iv) *There exists a constant* $c(\psi) > 0$ *such that*

$$|\psi(A_1) - \psi(A_2)| \leq c(\psi)\, d^\sharp(A_1, A_2)\,.$$

In fact, a weaker than mnc(iv) condition would be sufficient for everything we do in this paper:

mnc(iv*) ψ is continuous in the Hausdorff metric and has a modulus of continuity, m_ψ, defined on some small interval $[0, \epsilon_*]$ as follows:

$$m_\psi(\delta) = \sup\{|\psi(A) - \psi(B)| \,:\, \text{bounded } A, B \text{ such that } d^\sharp(A, B) \leq \delta\}\,.$$

• Note that $\psi(\overline{A}) = \psi(A)$. Also, if C is compact, then $\psi(A \cup C) = \psi(A)$.

Our definition of mnc (cf. [35]) contains the properties each of which will be used at some point in the paper. The Kuratowski mnc (the Kuratowski measure of noncompactness of a set A is the infimum of the numbers $\epsilon > 0$ such that A admits a finite cover by sets of diameter less than ϵ) and other familiar mnc's share the properties mnc(i)-(iv). For an interesting discussion of mnc's and their properties see [33].

Definition 8. *A map* $S : X \to X$ *is* ψ*-condensing (condensing with respect to* ψ*) iff* $\psi(S(A)) \leq \psi(A)$ *for any bounded* A*, and* $\psi(S(A)) < \psi(A)$ *if* $\psi(A) > 0$ *(i.e., if* $\overline{A}$ *is not compact).*

The compact maps $X \to X$ are ψ-condensing, as well as strict contractions. Also, in a Banach space, any map of the form *contraction + compact* is ψ-condensing for any ψ.

Now we introduce two additional assumptions. The following assumption is always true if the number of regimes (the set $\mathcal{J}$) and the number of dwell times are finite.

(A3) *For any bounded set* $A \subset X$*, the maps* S_j^τ*, restricted to* A*, depend uniformly continuously on* j *and* τ*. More precisely, given a bounded* A*, for every* $\epsilon > 0$ *there exist* $\delta_{\mathcal{J}} > 0$ *and* $\delta_{\mathcal{I}} > 0$ *such that*

$$\sup_{x \in A} d_X(S_{j_1}^{\tau_1}(x), S_{j_2}^{\tau_2}(x)) \leq \epsilon$$

provided $d_{\mathcal{J}}(j_1, j_2) \leq \delta_{\mathcal{J}}$ *and* $|\tau_1 - \tau_2| = d_{\mathcal{I}}(\tau_1, \tau_2) \leq \delta_{\mathcal{I}}$*.*

The fact that $\mathcal{J}$ and $\mathcal{I}$ are compact and property **(A3)** have the following consequence that we include for further reference.

Lemma 9.

[1] *For every bounded set A and for every $\epsilon > 0$ there are finite subsets $\mathcal{J}(A,\epsilon)$ and $\mathcal{I}(A,\epsilon)$ such that for any choice of $j \in \mathcal{J}$ and $\tau \in \mathcal{I}$*

$$\inf_{j' \in \mathcal{J}(A,\epsilon),\ \tau' \in \mathcal{I}(A,\epsilon)} d_X(S_j^\tau(x), S_{j'}^{\tau'}(x)) \le \epsilon\,, \quad \forall x \in A\,.$$

Also, for any $j \in \mathcal{J}$ and $\tau \in \mathcal{I}$, there are $j' \in \mathcal{J}_\epsilon$ and $\tau' \in \mathcal{I}_\epsilon$ such that for any $C \subset A$,

$$d^\sharp\left(S_j^\tau(C), S_{j'}^{\tau'}(C)\right) \le \epsilon\,.$$

[2] *If $\tau_k \to \tau_*$ and $j_k \to j_*$, and if C is a bounded set, then*

$$d^\sharp\left(S_{j_k}^{\tau_k}(C), S_j^\tau(C)\right) \to 0\,.$$

Proof. For the first part observe that

$$e\left(S_j^\tau(C), S_{j'}^{\tau'}(C)\right) = \sup_{x\in C}\inf_{y\in C} d\left(S_j^\tau(x), S_{j'}^{\tau'}(y)\right) \le \sup_{x\in C} d\left(S_j^\tau(x), S_{j'}^{\tau'}(x)\right) \le \epsilon$$

and the same for $e\left(S_{j'}^{\tau'}(C), S_j^\tau(C)\right)$. For the second part,

$$\sup_{x\in C}\inf_{y\in C} d\left(S_{j_k}^{\tau_k}(x), S_{j_*}^{\tau_*}(y)\right) \le \sup_{x\in C} d\left(S_{j_k}^{\tau_k}(x), S_{j_*}^{\tau_*}(x)\right) \to 0,$$

and the same with x and y swapped. □

A simple consequence of the continuity of the maps S_j^τ and assumption **(A3)** is the following fact.

Lemma 10. *If $x_n \to x$ in X, $j_n \to j$ in $\mathcal{J}$, and $\tau_n \to \tau$ in $\mathcal{I}$, then $S_{j_n}^{\tau_n}(x_n) \to S_j^\tau(x)$ in X.*

The proof is the inequality $d(S_{j_n}^{\tau_n}(x_n), S_j^\tau(x)) \le d(S_{j_n}^{\tau_n}(x_n), S_j^\tau(x_n)) + d(S_j^\tau(x_n), S_j^\tau(x))$.

• In the next assumption we use the following notation. Given a one-sided infinite string $h \in \Sigma_{\mathcal{I}}$, $h[n]$ denotes the finite word made by the first n symbols in h, i.e., $h[n] = h(0)h(1)\cdots h(n-1)$, and $\|h[n]\|$ stands for $h(0) + h(1) + \cdots + h(n-1)$. If $w \in \Sigma_{\mathcal{J}}$, we denote $w[n] = w(0)w(1)\cdots w(n-1)$. Also, we denote by $S_{w[n]}^{h[n]}$ the composition

$$S_{w[n]}^{h[n]} = S_{w(n-1)}^{h(n-1)} \circ \cdots \circ S_{w(1)}^{h(1)} \circ S_{w(0)}^{h(0)}\,.$$

(A4) *Assume there is a closed, bounded set $\mathbf{B} \subset X$ such that for every bounded $A \subset X$ there exists $T(A) > 0$ such that $S_{w[n]}^{h[n]}(A) \subset \mathbf{B}$, for any word $h \in \Sigma_{\mathcal{I}}$ such that $\|h[n]\| > T(A)$ and any word $w \in \Sigma_{\mathcal{J}}$.*

This assumption requiring the existence of an absorbing set is necessary for the existence of a global attractor.

Next, we fix notation for everything related to the hyperspace $X^\sharp$.

• $X^\sharp$ is the space of all nonempty closed, bounded subsets of X equipped with the Hausdorff distance, $d^\sharp$. It is well known, [23], that $X^\sharp$ is a complete metric space. The points of $X^\sharp$ will be denoted A, B, etc. The sets in $X^\sharp$ will be denoted $\mathscr{A}$, $\mathscr{B}$, etc. The set-theoretic operations in $X^\sharp$ will have the superscript $^\sharp$.

• The union of the sets $\mathscr{A}$ and $\mathscr{B}$ we write as $\mathscr{A} \biguplus^{\sharp} \mathscr{B}$, and their intersection is $\mathscr{A} \bigcap^{\sharp} \mathscr{B}$. To indicate that $\mathcal{L} \subset^\sharp X^\sharp$ is composed of elements $A \subset X$ we can write $\mathcal{L} = \bigcup^{\sharp}_{A\in\mathcal{L}} A$.

• We denote by $\mathcal{N}^\flat$ the union in X of the sets comprising $\mathcal{N} \subset^\sharp X^\sharp$, i.e.,

$$\mathscr{N}^\flat = \bigcup_{A\in\mathscr{N}} A\,.$$

• For $A \in X^\sharp$, we denote by $A^\sharp$ the set of nonempty closed subsets of A, i.e.,

$$A^\sharp = \bigcup^{\sharp}_{C\in X^\sharp, C\subset A} C\,.$$

Clearly, $\left(A^\sharp\right)^\flat = A$. However, $\mathscr{N} \subset^\sharp \left(\mathscr{N}^\flat\right)^\sharp$, in general.

• On a few occasions we use the Hausdorff distance between the sets of $X^\sharp$ for which we use the symbol $d^{\sharp\sharp}$.

4 Global attractors

The assumptions on X, $\mathcal{J}$, $\mathcal{I}$, and S_j^τ are the same as in the previous section.

Theorem 11. *Consider the variable time step dynamics with choice, $(\mathfrak{S}, \mathfrak{X})$, where $\mathfrak{X} = X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$ and, as in (6),*

$$\mathfrak{S} : (x, w, h) \mapsto (S_{w(0)}^{h(0)}(x), \sigma(w), \sigma(h))\,.$$

The system $(\mathfrak{S}, \mathfrak{X})$ possesses a global compact attractor. This attractor, $\mathfrak{A}$, is a subset of the product space $\mathfrak{X}$, and is invariant under the map $\mathfrak{S}$.

Proof. From the general theory of global attractors (see [40, 16] and, for more details, – [19]), we know that a necessary and sufficient condition for system $(\mathfrak{S}, \mathfrak{X})$ to possess a global attractor has two parts: First, the system should have a global absorbing set, i.e., a set $\mathfrak{B}$ such that for every bounded set $\mathfrak{C} \subset \mathfrak{X}$, $\mathfrak{S}^n(\mathfrak{C}) \subset \mathfrak{B}$ for all sufficiently large n, and second, for every bounded sequence $(x_k, w_k, h_k) \in \mathfrak{X}$, and every increasing sequence of integers n_k, the sequence $\mathfrak{S}^n(x_k, w_k, h_k) \in \mathfrak{X}$ should have a convergent subsequence. The first part is obviously satisfied with $\mathfrak{B} = \mathbf{B} \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$. Since $\mathfrak{S}$ is a single-valued, continuous map on $\mathfrak{X}$, the second part (about convergent subsequence) will be verified (see [19]) if we could find an mnc on $\mathfrak{X}$ with respect to which the map $\mathfrak{S}$ is condensing. We show next that $\mathfrak{S}$ is condensing with respect to the following mnc on $\mathfrak{X}$:

$$\psi_{\mathfrak{X}}(\mathfrak{C}) = \psi_X(\mathfrak{C}_X)\,,$$

where $\mathfrak{C}_X$ is the projection of the set $\mathfrak{C} \subset X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$ onto X, i.e,, $\mathfrak{C}_X = \{x \in X \,:\, \exists u \in \Sigma_{\mathcal{J}}\ \exists h \in \Sigma_{\mathcal{I}}$ such that $(x, u, h) \in \mathfrak{C}\}$. It is not hard to show in general that if $Y_1, Y_2, \ldots, Y_M$ are metric spaces with the mnc's ψ_{Y_j}, then

$$\psi_{Y_1\times\cdots\times Y_M}(A) = \max_n\ \psi_{Y_n}(A_{Y_n})$$

is an mnc on the product space $Y_1 \times \cdots \times Y_M$. In our case, it does not matter what mnc's we choose on the spaces $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{I}}$, because the spaces are compact.

Take a bounded set $\mathfrak{C} \subset \mathfrak{X}$. We have

$$\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) = \psi_X\left(\mathfrak{S}(\mathfrak{C})_X\right) \le \psi_X\left(\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(\mathfrak{C}_X)\right) = \psi_X(\mathcal{F}(\mathfrak{C}_X).$$

As we prove in Lemma 12[2], the map $\mathcal{F}$ is condensing. Hence,

$$\psi_X\left(\mathfrak{S}(A)_X\right) \le \psi_X\left(\mathcal{F}(\mathfrak{C}_X)\right) \le \psi_X(\mathfrak{C}_X) = \psi_{\mathfrak{X}}(\mathfrak{C})$$

with a strict inequality if $\mathfrak{C}$ is not relatively compact in $\mathfrak{X}$.

Invariance is one of the general properties of global attractors in semi-dynamical systems with continuous maps; so, $\mathfrak{S}(\mathfrak{A}) = \mathfrak{A}$. □

Turning to iterated function systems, we start by establishing some properties of the multi-valued maps F^τ, F_j, and $\mathcal{F}$.

Lemma 12.

[1] *For any compact set $C \subset X$ we have*

$$F^\tau(C) = \overline{\bigcup_{j\in\mathcal{J}} S_j^\tau(C)} = \bigcup_{j\in\mathcal{J}} S_j^\tau(C) \tag{18}$$

$$F_j(C) = \overline{\bigcup_{\tau\in\mathcal{I}} S_j^\tau(C)} = \bigcup_{\tau\in\mathcal{I}} S_j^\tau(C) \tag{19}$$

$$\mathcal{F}(C) = \overline{\bigcup_{\tau\in\mathcal{I}} F^\tau(C)} = \bigcup_{\tau\in\mathcal{I}} F^\tau(C) = \bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(C) \tag{20}$$

[2] *The maps F^τ,F_j, and $\mathcal{F}$, viewed as maps on subsets of X, are ψ_X-condensing.*

[3] *Each of the maps F^τ, F_j, and $\mathcal{F}$ has the following property. If A_n are closed, bounded sets, if $A_n \to A$ in the Hausdorff metric, and if A is compact, then $\mathcal{F}(A_n) \to \mathcal{F}(A)$ in the Hausdorff metric (the same for F^τ and F_j).*

Proof. To prove the first part, pick any $x \in F^\tau(C)$. Then $x = \lim x_n$ where $x_n = S^\tau_{j_n}(y_n)$ for some $j_n \in \mathcal{J}$ and $y_n \in C$. Because of compactness of the sets $\mathcal{J}$ and C, there is a subsequence n_k such that $j_{n_k} \to j^* \in \mathcal{J}$ and $y_{n_k} \to y^* \in C$. By Lemma 10, along the subsequence, we have $\lim S^{\tau_n}_{j_n}(y_n) = S^{\tau^*}_{j^*}(y^*)$. Thus, $x = S^{\tau^*}_{j^*}(y^*) \in S^{\tau^*}_{j^*}(C)$. Similar argument works for F_j.

To prove the second equality in (20), take any point $x \in \overline{\bigcup_{\tau\in\mathcal{I}} F^\tau(C)}$. Then $x = \lim_n S^{\tau_n}_{j_n}(y_n)$ for some $y_n \in C$, and some $j_n \in \mathcal{J}$ and $\tau_n \in \mathcal{I}$. Because of compactness, we may assume that $y_n \to y$, $j_n \to j$ and $\tau_n \to \tau$. Then, by Lemma 10, $x = S^\tau_j(y) \in S^\tau_j(C)$, and we are done.

For the second part, let us prove that $\mathcal{F}$ is ψ_X-condensing. That F^τ and F_j are condensing is proved similarly. Let $B \subset X$ be a bounded set. We will show that $\psi_X(\mathcal{F}(B)) \le \psi_X(B)$ and, if $\overline{B}$ is not compact, $\psi_X(\mathcal{F}(B)) < \psi_X(B)$. Pick an $\epsilon_0 > 0$ and let $\mathcal{J}_0 = \mathcal{J}(\overline{B}, \epsilon_0)$ and $\mathcal{I}_0 = \mathcal{I}(\overline{B}, \epsilon_0)$ be the finite sets described in Lemma 9. Using property mnc(iv) and Lemma 9 we obtain

$$\begin{aligned}&\left|\psi_X\left(\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S^\tau_j(B)\right) - \psi_X\left(\bigcup_{\tau'\in\mathcal{I}_0}\bigcup_{j'\in\mathcal{J}_0} S^{\tau'}_{j'}(B)\right)\right| \\ &\le c(\psi_X)\, d^\sharp\left(\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S^\tau_j(B), \bigcup_{\tau'\in\mathcal{I}_0}\bigcup_{j'\in\mathcal{J}_0} S^{\tau'}_{j'}(B)\right) \le c(\psi_X)\,\epsilon_0.\end{aligned} \tag{21}$$

Here we have used that

$$e\left(\bigcup_{\tau'\in\mathcal{I}_0}\bigcup_{j'\in\mathcal{J}_0} S^{\tau'}_{j'}(B), \bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S^\tau_j(B)\right) = 0$$

and

$$e\left(\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S^\tau_j(B), \bigcup_{\tau'\in\mathcal{I}_0}\bigcup_{j'\in\mathcal{J}_0} S^{\tau'}_{j'}(B)\right) = \sup_{j\in\mathcal{J},\tau\in\mathcal{I},x\in B} d\left(S^\tau_j(x), \bigcup_{\tau'\in\mathcal{I}_0}\bigcup_{j'\in\mathcal{J}_0} S^{\tau'}_{j'}(B)\right) \le \epsilon_0\,.$$

Thus,

$$\begin{aligned}&\psi_X\left(\mathcal{F}(B)\right) = \psi_X\left(\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S^\tau_j(B)\right) \le \psi_X\left(\bigcup_{\tau'\mathcal{I}(\delta^0_{\mathcal{I}})}\bigcup_{j'\in\mathcal{J}(\delta^0_{\mathcal{J}})} S^{\tau'}_{j'}(B)\right) + c(\psi)\,\epsilon_0 \\ &= \psi_X(S^{\tau^0}_{j^0}(B)) + c(\psi_X)\,\epsilon^0\,,\end{aligned} \tag{22}$$

for some $j^0 \in \mathcal{J}(B, \epsilon_0)$ and $\tau^0 \in \mathcal{I}(B, \epsilon_0)$ such that

$$\psi_X(S^{\tau^0}_{j^0}(B)) = \max_{\tau'\in\mathcal{I}(B,\epsilon_0),\, j'\in\mathcal{J}(B,\epsilon_0)} \psi_X(S^{\tau'}_{j'}(B))\,.$$

The second equality in (22) is due to the properties mnc(ii) and mnc(iii). Similarly, we can choose $\epsilon_1 < \epsilon_0$ and find $j^1 \in \mathcal{J}(B, \epsilon_1) \subset \mathcal{J}$ and $\tau^1 \in \mathcal{I}(B, \epsilon_1) \subset \mathcal{I}$ such that

$$\psi_X\left(\mathcal{F}(B)\right) \le \psi_X(S^{\tau^1}_{j^1}(B)) + c(\psi_X)\,\epsilon_1\,.$$

Proceeding like this, let $\epsilon_k \searrow 0$, and let $j^k \in \mathcal{J}$ and $\tau^k \in \mathcal{I}$ be the corresponding sequences such that for every $k \geq 0$, we have

$$\psi_X(\mathcal{F}(B)) \leq \psi_X(S_{j^k}^{\tau^k}(B)) + c(\psi_X)\,\epsilon_k\,.$$

Since $\mathcal{J}$ and $\mathcal{I}$ are compact sets, we can find convergent subsequences $j^{k_l} \to j^* \in \mathcal{J}$ and $\tau^{k_l} \to \tau^* \in \mathcal{I}$. Again, due to **(A3)** and mnc(iv), we have

$$\psi_X(S_{j^k}^{\tau^k}(B)) + c(\psi_X)\,\epsilon^k \to \psi_X(S_{j^*}^{\tau^*}(B))$$

as $k \to \infty$. Therefore,

$$\psi_X(\mathcal{F}(B)) \leq \psi_X(S_{j^*}^{\tau^*}(B)) \leq \psi_X(B).$$

The second inequality follows since the maps S_j^τ are condensing for all $j \in \mathcal{J}$ and all $\tau \in \mathcal{I}$, and it is a strict inequality if B is not relatively compact.

We turn to the proof of the third statement. At some point we use the following simple inequality:

$$d^\sharp\left(\mathscr{N}_1^\flat, \mathscr{N}_2^\flat\right) \;\leq\; d^{\sharp\sharp}(\mathscr{N}_1, \mathscr{N}_2)\,, \tag{23}$$

for any two subsets $\mathscr{N}_1$ and $\mathscr{N}_2$ of $X^\sharp$.

Let A_n be a sequence in $X^\sharp$ converging to A, and A is compact. Our goal is to show that $\mathcal{F}(A_n) \to \mathcal{F}(A)$. We have

$$d^\sharp(\mathcal{F}(A_n), \mathcal{F}(A)) = d^\sharp\Big(\bigcup_{j\in\mathcal{J},\tau\in\mathcal{I}} S_j^\tau(A_n), \bigcup_{k\in\mathcal{J},\lambda\in\mathcal{I}} S_k^\lambda(A)\Big)\,. \tag{24}$$

Fix a bounded set B such that all A_n and A are its subsets. Pick an $\epsilon > 0$ and use Lemma 9 to choose finite sets $\mathcal{J}_\epsilon \subset \mathcal{J}$ and $\mathcal{I}_\epsilon \subset \mathcal{I}$ so that

$$\inf_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} d^\sharp(S_j^\tau(C), S_{j'}^{\tau'}(C)) \leq \epsilon$$

for every $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, and for any set $C \subset B$. Applying (23), we obtain

$$d^\sharp\left(\bigcup_{j\in\mathcal{J},\tau\in\mathcal{I}} S_j^\tau(C), \bigcup_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} S_{j'}^{\tau'}(C)\right) \leq d^{\sharp\sharp}\left(\biguplus_{j\in\mathcal{J},\tau\in\mathcal{I}}^{\sharp} S_j^\tau(C), \biguplus_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon}^{\sharp} S_{j'}^{\tau'}(C)\right)\,.$$

Since

$$\sup_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} \inf_{j\in\mathcal{J},\tau\in\mathcal{I}} d^\sharp\left(S_j^\tau(C), S_{j'}^{\tau'}(C)\right) = 0$$

and

$$\sup_{j\in\mathcal{J},\tau\in\mathcal{I}} \inf_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} d^\sharp\left(S_j^\tau(C), S_{j'}^{\tau'}(C)\right) \leq \epsilon\,,$$

we have

$$d^\sharp\left(\bigcup_{j\in\mathcal{J},\tau\in\mathcal{I}} S_j^\tau(C), \bigcup_{j'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} S_{j'}^{\tau'}(C)\right) \leq \epsilon\,.$$

Thus,

$$d^\sharp\left(\bigcup_{k\in\mathcal{J},\tau\in\mathcal{I}} S_k^\tau(A_n), \bigcup_{k'\in\mathcal{J},\tau'\in\mathcal{I}} S_{k'}^{\tau'}(A)\right) \leq d^\sharp\left(\bigcup_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon} S_k^\tau(A_n), \bigcup_{k'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} S_{k'}^{\tau'}(A)\right) + 2\epsilon\,. \quad (25)$$

In view of (23),

$$d^\sharp\left(\bigcup_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon} S_k^\tau(A_n), \bigcup_{k'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon} S_{k'}^{\tau'}(A)\right) \leq d^{\sharp\sharp}\left(\biguplus_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon}^{\sharp} S_k^\tau(A_n), \biguplus_{k'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon}^{\sharp} S_{k'}^{\tau'}(A)\right),$$

and

$$d^\sharp\left(\biguplus_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon}^{\sharp} S_k^\tau(A_n), \biguplus_{k'\in\mathcal{J}_\epsilon,\tau'\in\mathcal{I}_\epsilon}^{\sharp} S_{k'}^{\tau'}(A)\right) \leq \sup_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon} d^\sharp\left(S_k^\tau(A_n), S_k^\tau(A)\right).$$

Since the sets $\mathcal{J}_\epsilon$ and $\mathcal{I}_\epsilon$ are finite, there exist $k_* \in \mathcal{J}_\epsilon$ and $\tau_* \in \mathcal{I}_\epsilon$ such that

$$\sup_{k\in\mathcal{J}_\epsilon,\tau\in\mathcal{I}_\epsilon} d^\sharp\left(S_k^\tau(A_n), S_k^\tau(A)\right) = d^\sharp(S_{k_*}^{\tau_*}(A_n), S_{k_*}^{\tau_*}(A)) \quad (26)$$

for infinitely many n. The desired conclusion follows from the fact that $d^\sharp\left(S_j^\tau(A_n), S_j^\tau(A)\right) \to 0$ for any choice of j and τ, provided A is compact. This fact is easy to prove. Indeed, suppose $\lim e\left(S_j^\tau(A_n), S_j^\tau(A)\right) \geq \delta > 0$ for $n = n_k \nearrow \infty$. Then there exist $x_n \in A_n$ such that $d\left(S_j^\tau(x_n), S_j^\tau(A)\right) \geq \delta/2$ for $n = n_k$. To each x_n there is a point $y_n \in A$ such that $d(x_n, y_n) \to 0$. The sequence y_n has a subsequence converging to some $y_* \in A$. The corresponding subsequence of x_n converges to the same y_* and, therefore, $d\left(S_j^\tau(x_n), S_j^\tau(A)\right) \to 0$, a contradiction. Similarly, if $\lim e\left(S_j^\tau(A), S_j^\tau(A_n)\right) \geq \delta > 0$ for $n = n_k \nearrow \infty$, then there exist $y_n \in A$ such that $d\left(S_j^\tau(y_n), S_j^\tau(A_n)\right) \geq \delta/2$ for $n = n_k$. Again, because $\lim d^\sharp(A, A_n) = 0$, there is a sequence $x_n \in A_n$ such that $d(x_n, y_n) \to 0$, and the fact that y_n must have a convergent subsequence in A leads to contradiction. □

Recall that we consider three types of IFSs.

- The VTSIFS-1 is generated by the map

$$\mathfrak{F}^{\mathcal{I}} : (A, h) \mapsto (F^{h(0)}(A), \sigma(h))$$

on the space $X^\sharp \times \Sigma_{\mathcal{I}}$.

- The VTSIFS-2 is generated by the map

$$\mathfrak{F}_{\mathcal{J}} : (A, u) \mapsto (F_{u(0)}(A), \sigma(u))$$

on the space $X^\sharp \times \Sigma_{calJ}$.

- The VTSIFS-3 is generated by the map

$$\mathcal{F} : A \mapsto \overline{\bigcup_{\tau\in\mathcal{I}} F^\tau(A)} = \overline{\bigcup_{j\in\mathcal{J}} F_j(A)}$$

on the space $X^\sharp$.

The results on the global attractors for these systems will rely on the theory of global attractors for multi-valued maps which we explain in detail in our paper [19] and which we summarize in the Appendix.

Theorem 13. *Each of the semi-dynamical systems $\left(\mathfrak{F}^{\mathcal{I}}, X^\sharp \times \Sigma_{\mathcal{I}}\right)$, $\left(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Sigma_{\mathcal{J}}\right)$, and $\left(\mathcal{F}, X^\sharp\right)$ possesses a global attractor. Denote by $\mathfrak{N}^{\mathcal{I}}$ the global attractor of $\left(\mathfrak{F}, X^\sharp \times \Sigma_{\mathcal{I}}\right)$, by $\mathfrak{N}_{\mathcal{J}}$ the global attractor of $\left(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Sigma_{\mathcal{J}}\right)$, and by $\mathcal{K}$ the global attractor of $\left(\mathcal{F}, X^\sharp\right)$. The attractors are invariant under the corresponding maps, i.e., $\mathfrak{F}^{\mathcal{I}}(\mathfrak{N}^{\mathcal{I}}) = \mathfrak{N}^{\mathcal{I}}$, $\mathfrak{F}_{\mathcal{J}}(\mathfrak{N}_{\mathcal{J}}) = \mathfrak{N}_{\mathcal{J}}$, and $\mathcal{F}(\mathcal{K}) = \mathcal{K}$. The attractors are the maximal invariant compact sets in their respective spaces.*

Proof. Consider the system $\left(\mathfrak{F}^{\mathcal{I}}, X^\sharp \times \Sigma_{\mathcal{I}}\right)$. We will check certain conditions sufficient for the existence of the global attractor. First we check that there is a bounded absorbing set. This is easy: the set $\mathfrak{B}^{\mathcal{I}} = \mathbf{B}^\sharp \times \Sigma_{\mathcal{I}}$, where $\mathbf{B}$ is given by property **(A4)**, is absorbing for $\mathfrak{F}^{\mathcal{I}}$. Indeed, let $\mathfrak{M}$ be a bounded subset of $X^\sharp \times \Sigma_{\mathcal{I}}$. Replace $\mathfrak{M}$ by a larger subset $\mathscr{M} \times \Sigma_{\mathcal{I}}$, where $\mathscr{M}$ is the projection of $\mathfrak{M}$ onto $X^\sharp$. We have

$$\left(\mathfrak{F}^{\mathcal{I}}\right)^n (\mathscr{M} \times \Sigma_{\mathcal{I}}) = \biguplus_{A \in \mathscr{M},\, h \in \Sigma_{\mathcal{I}}}^{\sharp} \left(F^{h[n]}(A), \sigma^n(h)\right) \, .$$

It is straightforward to show, using continuity of the maps S_j^τ, that

$$F^{h[n]}(A) = \overline{\bigcup_{u \in \Sigma_{\mathcal{J}}} S_{u[n]}^{h[n]}(A)} \, .$$

Thanks to **(A4)**, $F^{h[n]}(A) \in \mathbf{B}^\sharp$ for all sufficiently large n. Similarly, the set $\mathbf{B}^\sharp \times \Sigma_{\mathcal{J}}$ is absorbing for $\mathfrak{F}_{\mathcal{J}}$ and $\mathbf{B}^\sharp$ is absorbing for $\mathcal{F}$.

The second and last condition to check is the following: For any bounded sequence $(A_k, h_k) \in X^\sharp \times \Sigma_{\mathcal{I}}$ and for any increasing sequence of integers $n_k > 0$, the sequence

$$\mathfrak{F}^{n_k}(A_k, h_k)$$

has a convergent subsequence. Since the space $\Sigma_{\mathcal{I}}$ is compact, it suffices to check that every sequence of the form

$$F^{h_k[n_k]}(A_k) \tag{27}$$

has a convergent subsequence in $X^\sharp$. A similar argument for the system $\left(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Sigma_{\mathcal{J}}\right)$ leads to the question of relative compactness of the sequences

$$F_{u_k[n_k]}(A_k) \, . \tag{28}$$

For the system $\left(\mathcal{F}, X^\sharp\right)$ the question is about the relative compactness of the sequences

$$\mathcal{F}^{n_k}(A_k) \, . \tag{29}$$

We prove that (27), (28), and (29) have convergent subsequences by showing that each of the terms is a closed subset of a nested sequence of closed subsets in X with a nonempty, compact intersection and using Corollary 29.

Denote $C = \bigcup_n A_n$. This set is bounded in X. Denote $D_n = \bigcup_{m \geq n} \mathcal{F}^m(C)$ and consider the set $\Omega = \bigcap_n \overline{D_n}$. We have

$$\overline{D_n} = \mathcal{F}^n \left(\overline{\bigcup_{m=0}^{\infty} \mathcal{F}^m(C)} \right) .$$

[Check for $n = 1$: Since $\overline{\bigcup_m A_m} = \overline{\bigcup_m \overline{A_m}}$,

$$\mathcal{F} \left(\overline{\bigcup_{m=0}^{\infty} \mathcal{F}^m(C)} \right) = \overline{\bigcup_{j,\tau} S_j^\tau \left(\overline{\bigcup_{m=0}^{\infty} \mathcal{F}^m(C)} \right)} = \overline{\bigcup_{m=0}^{\infty} \bigcup_{j,\tau} S_j^\tau \left(\mathcal{F}^m(C) \right)} = \overline{\bigcup_{m=0}^{\infty} \overline{\bigcup_{j,\tau} S_j^\tau \left(\mathcal{F}^m(C) \right)}}$$

Hence, $\mathcal{F}\left(\overline{\bigcup_{m=0}^{\infty} \mathcal{F}^m(C)}\right) = \overline{\bigcup_{m=0}^{\infty} \mathcal{F}^{m+1}(C)}$.]

The set $B = \bigcup_{m=0}^{\infty} \mathcal{F}^m(C)$ is bounded due to **(A4)**. The map $\mathcal{F}$ is ψ-condensing by Lemma 12[2]. In addition, $\mathcal{F}$ has the following property: for any set A, every finite subset L of the union $\bigcup_{j,\tau} S_j^\tau(A)$ lies in the image, $\mathcal{F}(L')$, of a finite set $L' \subset A$. These properties, and **(A4)**, allow us to use Proposition 33 and conclude that $\mathcal{F}$ is asymptotically ψ-condensing, i.e., $\psi\left(\mathcal{F}^n(B)\right) \to 0$ for any bounded set (in particular, our set) B. Now, we have a nested sequence $\overline{D_1} \supset \overline{D_2} \supset \cdots$ of closed, bounded sets such that $\psi(\overline{D_n}) = \psi(D_n) \to 0$. By the generalized Cantor-Kuratowski Theorem 28), the intersection, Ω, is a nonempty compact set and $\lim_n d^\sharp(\overline{D_n}, \Omega) = 0$. Then Corollary 29 implies that each of the sequences (27), (28), and (29), has convergent subsequences.

The two properties we have just checked lead to the existence of the global attractors $\mathfrak{N}^{\mathcal{I}}$, $\mathfrak{N}_{\mathcal{J}}$, and $\mathcal{K}$. Their invariance follows from the continuity property described in Lemma 12[3]. □

Next we discuss the relationship between the attractor $\mathfrak{A}$ of $(\mathfrak{S}, X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}})$, the attractor $\mathfrak{N}^{\mathcal{I}}$ of $(\mathfrak{F}^{\mathcal{I}}, X^\sharp \times \Sigma_{\mathcal{I}})$, the attractor $\mathfrak{N}_{\mathcal{J}}$ of $(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Sigma_{\mathcal{J}})$, and the attractor $\mathcal{K}$ of $(\mathcal{F}, X^\sharp)$.

Theorem 14 (The product structure of attractors).

[1] *There exist a compact set $K_0 \subset X$ and compact sets $\mathcal{K}_1 \subset X^\sharp$ and $\mathcal{K}_2 \subset X^\sharp$ such that*

$$\mathfrak{A} = K_0 \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}} \,, \qquad \mathfrak{N}^{\mathcal{I}} = \mathcal{K}_1 \times \Sigma_{\mathcal{I}} \,, \quad \mathfrak{N}_{\mathcal{J}} = \mathcal{K}_2 \times \Sigma_{\mathcal{J}} \,. \tag{30}$$

[2] *The sets K_0, $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}$ are related as follows.*

• *LK: Stuff should be added here.*

$$K_0 = \bigcup_{C \in \mathcal{K}} \bigcup_{\tau \in \mathcal{I}} F^\tau(C) = \bigcup_{C \in \mathcal{K}} \bigcup_{\tau \in \mathcal{I}} \bigcup_{j \in \mathcal{J}} S_j^\tau(C) = \bigcup_{\tau \in \mathcal{I}} \bigcup_{j \in \mathcal{J}} S_j^\tau(\mathcal{K}^\flat) = \mathcal{K}^\flat$$

$$K_0 = \bigcup_{A\in\mathcal{K}_1}\bigcup_{\tau\in\mathcal{I}} F^\tau(A) = \bigcup_{A\in\mathcal{K}_1}\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(A) = \bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(\mathcal{K}_1^\flat) = \mathcal{K}_1^\flat$$

$$K_0 = \bigcup_{A\in\mathcal{K}_2}\bigcup_{j\in\mathcal{J}} F_j(A) = \bigcup_{A\in\mathcal{K}_2}\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(A) = \bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(\mathcal{K}_2^\flat) = \mathcal{K}_2^\flat$$

$$\mathcal{K}_1 = \biguplus_{\tau\in\mathcal{I}}^{\sharp} F^\tau(\mathcal{K}_1) = \biguplus_{C\in\mathcal{K}_1}^{\sharp}\biguplus_{\tau\in\mathcal{I}}^{\sharp} F^\tau(C) = \biguplus_{C\in\mathcal{K}_1}^{\sharp}\biguplus_{\tau\in\mathcal{I}}^{\sharp}\bigcup_{j\in\mathcal{J}} S_j^\tau(C)$$

$$\mathcal{K}_2 = \biguplus_{j\in\mathcal{J}}^{\sharp} F_j(\mathcal{K}_2) = \biguplus_{C\in\mathcal{K}_2}^{\sharp}\biguplus_{j\in\mathcal{J}}^{\sharp} F_j(C) = \biguplus_{C\in\mathcal{K}_2}^{\sharp}\biguplus_{j\in\mathcal{J}}^{\sharp}\bigcup_{\tau\in\mathcal{I}} S_j^\tau(C)$$

$$\mathcal{K} = \biguplus_{C\in\mathcal{K}}^{\sharp}\bigcup_{\tau\in\mathcal{I}} F^\tau(C) = \biguplus_{C\in\mathcal{K}}^{\sharp}\bigcup_{\tau\in\mathcal{I}}\bigcup_{j\in\mathcal{J}} S_j^\tau(C)$$

[3] *Also,*

(a) *For every $A \in \mathcal{K}_1$ (or $A \in \mathcal{K}_2$) there exists a $C \in \mathcal{K}$ such that $A \subset C$.*

(b) *If $C \in \mathcal{K}$ then $C \subset K_0$. Moreover, for every $x \in C$ there exist $A_1 \in \mathcal{K}_1$ and $A_2 \in \mathcal{K}_2$ such that $x \in A_1 \subset C$ and $x \in A_2 \subset C$.*

(c) *For every $x \in K_0$ there are $A_1 \in \mathcal{K}_1$ and $A_2 \in \mathcal{K}_2$, and there are $C_1 \in \mathcal{K}$ and $C_2 \in \mathcal{K}$ such that $x \in A_1 \subset C_1$ and $x \in A_2 \subset C_2$.*

(d) *$\mathcal{K}$ is the global attractor of the induced semi-dynamical system $(\mathcal{F}, (K_0)^\sharp)$.*

Proof. As follows from the general theory (see e.g. [40, Theorem 23.16]), the attractor $\mathfrak{A}$ consists of all points $(x, u, h) \in \mathfrak{X}$ which are limits of the form $(x, u, h) = \lim \mathfrak{S}^{n_k}(x_k, u_k, h_k)$ for some bounded in $\mathfrak{X}$ sequence (x_k, u_k, h_k) and some increasing sequence of integers $n_k \nearrow +\infty$. In our situation, $x = \lim_{k\to\infty} S_{u_k[n_k]}^{h_k[n_k]}(x_k)$, for some bounded sequence $\{x_k\} \in X$, while $\sigma^{n_k}(u_k) \to u$, and $\sigma^{n_k}(h_k) \to h$.

Pick any strings $w \in \Sigma_{\mathcal{J}}$ and $s \in \Sigma_{\mathcal{I}}$. Denote by $u_k[n_k].w$ (respectively $h_k[n_k].s$) the concatenation of the (finite) word $u_k[n_k]$ (resp. $h_k[n_k]$) with the string w (resp. s). Since $\mathfrak{S}^{n_k}(x_k, u_k[n_k].w, h_k[n_k].s) \to (x, w, s)$ as $k \to \infty$, the point (x, w, s) also belongs to the attractor $\mathfrak{A}$. Hence, $\mathfrak{A} = K_0 \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$, where

$$\begin{aligned} K_0 = \{x \in X : x = \lim_{k\to\infty} S_{u_k[n_k]}^{h_k[n_k]}(x_k),\ &\text{for some bounded } \{x_k\} \subset X, \\ \text{some } \{u_k\} \subset \Sigma_{\mathcal{J}},\ \text{some } \{h_k\} \subset \Sigma_{\mathcal{I}},\ &\text{and some } n_k \nearrow +\infty\}\,. \end{aligned} \tag{31}$$

Because $\mathfrak{A}$ is compact in the product $X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$, and both $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{I}}$ are compact spaces, K_0 must be compact. The invariance property, $\mathfrak{S}(\mathfrak{A}) = \mathfrak{A}$, implies that for every point $(x, u, h) \in \mathfrak{A}$ there exist an $x' \in K_0$, a $j \in \mathcal{J}$, and a $\tau \in \mathcal{I}$ such that $\mathfrak{S}(x', j.u, \tau.h) = (x, u, h)$. This shows that $K_0 = \bigcup_j \bigcup_\tau S_j^\tau(K_0)$.

Next, consider the dynamics generated by the map $\mathfrak{F}^{\mathcal{I}}$ on the product space $X^\sharp \times \Sigma_{\mathcal{I}}$. The global attractor, $\mathfrak{N}^{\mathcal{I}}$, is the collection of all limits of the form $(C, h) = \lim_k (F^{h_k[n_k]}(A_k), \sigma^{n_k}(h_k))$, where A_k is a bounded sequence in $X^\sharp$, h_k is a sequence in $\Sigma_\mathcal{I}$, and $n_k \nearrow +\infty$. Pick any string $s \in \Sigma_\mathcal{I}$ and notice that $\left(\mathfrak{F}^\mathcal{I}\right)^{n_k}(A_k, h_k) = (F^{h_k[n_k]}(A_k), \sigma^{n_k}(h_k[n_k].s)) \to (C, s)$. Thus, $\mathfrak{N}^\mathcal{I} = \mathcal{K}_1 \times \Sigma_\mathcal{I}$, where

$$\mathcal{K}_1 = \{C \in X^\sharp : C = \lim_k F^{h_k[n_k]}(A_k), \quad \text{for some bounded } \{A_k\} \subset X^\sharp, \quad \text{some } \{h_k\} \subset \Sigma_\mathcal{I}, \quad \text{and some } n_k \nearrow +\infty\}\,. \tag{32}$$

The set $\mathcal{K}_1$ is compact in $X^\sharp$. This implies, in particular, that every point of $\mathcal{K}_1$, when viewed as a subset of X, is compact. Again, due to the invariance of the attractor, $\mathcal{K}_1 = \biguplus_\tau F^\tau(\mathcal{K}_1)$. This can be extended as

$$\mathcal{K}_1 = \biguplus_{\tau\in\mathcal{I}} F^\tau(\mathcal{K}_1) = \biguplus_{C\in\mathcal{K}_1} \biguplus_{\tau\in\mathcal{I}} F^\tau(C) = \biguplus_{C\in\mathcal{K}_1} \biguplus_{\tau\in\mathcal{I}} \bigcup_{j\in\mathcal{J}} S_j^\tau(C)\,, \tag{33}$$

where the third equality follows from Lemma 12[1]. Similarly, $\mathfrak{N}_\mathcal{J}$, the attractor of $(\mathfrak{F}_\mathcal{J}, X^\sharp \times \Sigma_\mathcal{I})$, is a product of the form $\mathfrak{N}_\mathcal{J} = \mathcal{K}_2 \times \Sigma_\mathcal{J}$, where

$$\mathcal{K}_2 = \{C \in X^\sharp : C = \lim_k F_{u_k[n_k]}(A_k), \quad \text{for some bounded } \{A_k\} \subset X^\sharp, \quad \text{some } \{u_k\} \subset \Sigma_\mathcal{J}, \quad \text{and some } n_k \nearrow +\infty\}\,, \tag{34}$$

and $\mathcal{K}_2 = \biguplus_j F_j(\mathcal{K}_2)$, which expands as follows:

$$\mathcal{K}_2 = \biguplus_{j\in\mathcal{J}} F_j(\mathcal{K}_2) = \biguplus_{C\in\mathcal{K}_2} \biguplus_{j\in\mathcal{J}} F_j(C) = \biguplus_{C\in\mathcal{K}_2} \biguplus_{j\in\mathcal{J}} \bigcup_{\tau\in\mathcal{I}} S_j^\tau(C)\,. \tag{35}$$

Turn now to the attractor $\mathcal{K}$ of the iterated function system inside X. We have

$$\mathcal{K} = \{C \in X^\sharp : C = \lim_k \mathcal{F}^{n_k}(C_k), \quad \text{for some bounded } \{C_k\} \subset X^\sharp, \quad \text{and some } n_k \nearrow +\infty\}\,. \tag{36}$$

The set $\mathcal{K}$ is compact in $X^\sharp$. Note that the points of $\mathcal{K}$ viewed as subsets of X are compact. From the general theory, $\mathcal{K}$ is the maximal compact subset of $X^\sharp$ invariant under $\mathcal{F}$. This means that $\mathcal{K} = \biguplus_{C\in\mathcal{K}} \mathcal{F}(C) = \biguplus_{C\in\mathcal{K}} \overline{\bigcup_{\tau\in\mathcal{I}} F^\tau(C)}$. Using Lemma 12[1], we write

$$\mathcal{K} = \biguplus_{C\in\mathcal{K}} \bigcup_{\tau\in\mathcal{I}} F^\tau(C) = \biguplus_{C\in\mathcal{K}} \bigcup_{\tau\in\mathcal{I}} \bigcup_{j\in\mathcal{J}} S_j^\tau(C)\,. \tag{37}$$

Now consider the assertions of the third part. The argument will be presented for $\mathcal{K}_1$, because for $\mathcal{K}_2$ it is similar. Start with (a). Let $A \in \mathcal{K}_1$ so that $A = \lim_{k\to\infty} F^{h_k[n_k]}(A_k)$ as described in (32). Consider the sequence $B_k = \mathcal{F}^{n_k}(A_k) = \overline{\bigcup_{h\in\Sigma_\mathcal{I}} F^{h[n_k]}(A_k)}$ in $X^\sharp$. It must have a subsequence, B_{k_m}, converging to some point $C \in \mathcal{K}$. Since $F^{h_k[n_k]}(A_k) \subset B_k$, we must have $A \subset C$, which proves (a).

For (b), suppose $C \in \mathcal{K}$ and pick a point $x \in C$. Because $\mathcal{K}$ is invariant under $\mathcal{F}$, $C = \lim_k \mathcal{F}^{n_k}(C_k)$, where C_k is a sequence of points of $\mathcal{K}$. Then, for $x \in C$, there exists a sequence $y_k \in \bigcup_{h \in \Sigma_{\mathcal{J}}} F^{h[n_k]}(C_k)$ that converges to x. We choose a sequence $\{h_k\} \subset \Sigma_{\mathcal{I}}$ such that $y_k \in F^{h_k[n_k]}(C_k)$. As in the proof of Theorem 13, the sequence $A_k = F^{h_k[n_k]}(C_k)$ has a convergent subsequence, A_{k_m}, converging to some $A \in \mathcal{K}_1$. This yields $x \in A \subset C$. In view of Lemma 12[1], we can find $u_k \in \Sigma_{\mathcal{J}}$ so that $y_k \in S^{h_k[n_k]}_{u_k[n_k]}(C_k)$. Moreover, there are $z_k \in C_k$ such that $y_k = S^{h_k[n_k]}_{u_k[n_k]}(z_k)$. The sequence z_k is obviously bounded, and $\lim_k y_k = x$, hence, $x \in K_0$.

Finally, pick an $x \in K_0$. Then $x = \lim S^{h_k[n_k]}_{u_k[n_k]}(x_k)$. Denote by A_k the set of the first k terms of the sequence $\{x_k\}$. We have

$$S^{h_k[n_k]}_{u_k[n_k]}(A_k) \subset F^{h_k[n_k]}(A_k) \subset \mathcal{F}^{n_k}(A_k)\,.$$

The second and the third sequences have convergent subsequences. We choose a common subsequence k_m so that $F^{h_k[n_k]}(A_k) \to A$ and $\mathcal{F}^{n_k}(A_k) \to C$ along $k = k_m$.

Now, by (a) and (b) we have $\mathcal{K}_1^\flat \subset \mathcal{K}^\flat \subset K_0$. On the other hand, (c) shows that $K_0 \subset \mathcal{K}_1^\flat \subset \mathcal{K}^\flat$, hence

$$K_0 = \mathcal{K}^\flat = \mathcal{K}_1^\flat\,. \tag{38}$$

Applying the operation $^\flat$ to (37) and to (33) and comparing the results with (38), we see that

$$K_0 = \bigcup_{C \in \mathcal{K}} \bigcup_{\tau \in \mathcal{I}} F^\tau(C) = \bigcup_{A \in \mathcal{K}_1} \bigcup_{\tau \in \mathcal{I}} F^\tau(A)$$

Thanks to the product structure of $\mathfrak{A}$, for every closed (and hence compact) subset $A \subset K_0$, $S_j^\tau(A) \subset K_0$. Thus, $\mathcal{F}\left((K_0)^\sharp\right)$ is a subset of $(K_0)^\sharp$ in $X^\sharp$. By Theorem 13 the semi-dynamical system $(\mathcal{F}, (K_0)^\sharp)$ has a global attractor, and this attractor must be the maximal invariant compact subset of $(K_0)^\sharp$. Since $\mathcal{K}$ is invariant and maximal in the whole $X^\sharp$, it is the attractor. This completes the proof of the theorem. □

Suppose all the maps S_j^τ are strict contractions on K_0, i.e., there exist numbers $\gamma(\tau, j) \in (0,1)$ such that

$$d(S_j^\tau(x), S_j^\tau(y)) \le \gamma(\tau, j)\, d(x,y)\,, \quad \forall x, y \in K_0\,.$$

Then the maps S_j^τ are strict contractions on K_0 uniformly with respect to τ and j, i.e., there exists a $\gamma \in (0,1)$ such that $\gamma \ge \gamma(\tau, j)$ for all τ and j.

Corollary 15. *If all the maps S_j^τ are strict contractions on K_0, then $\mathcal{K}$ is a singleton, $\mathcal{K} = \{K_0\}$.*

Proof. The map $\mathcal{F}$ on $(K_0)^\sharp$ is a strict contraction. Indeed, for $A, B \in (K_0)^\sharp$,

$$d^\sharp\left(\mathcal{F}(A), \mathcal{F}(B)\right) = d^\sharp\left(\bigcup_{\tau,j} S_j^\tau(A), \bigcup_{\tau',j'} S_{j'}^{\tau'}(B)\right) \le d^{\sharp\sharp}\left(\biguplus_{\tau,j} S_j^\tau(A), \biguplus_{\tau',j'} S_{j'}^{\tau'}(B)\right)$$

$$\leq \sup_{\tau,j} d^\sharp\left(S_j^\tau(A), S_j^\tau(B)\right) \leq \gamma\, d^\sharp(A,B)\,.$$

Let $C \in (K_0)^\sharp$ be the unique fixed point. Clearly, $\{C\}$ is the global attractor of $(\mathcal{F}, (K_0)^\sharp)$, hence $\{C\} = \mathcal{K}$. Since $\mathcal{K}^\flat = K_0$, $C = K_0$. □

When the maps S_j^τ are not contractions, the structure of the attractor $\mathcal{K}$ may be quite complicated.

5 Dynamics with restricted choice

Let $\Lambda_{\mathcal{J}}$ be a subshift of $\Sigma_{\mathcal{J}}$ and let $\Lambda_{\mathcal{I}}$ be a subshift of $\Sigma_{\mathcal{I}}$.

Theorem 16. *Under the assumptions of Section 3, the discrete semi-dynamical system generated by the map $\mathfrak{S}$ on $X \times \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}}$ has a global attractor. The global attractor, $\mathfrak{A}_\Lambda$, is invariant under $\mathfrak{S}$. Also, $\mathfrak{A}_\Lambda \subset \mathfrak{A}$, where $\mathfrak{A}$ is the global attractor of the full system $(\mathfrak{S}, X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}})$.*

The proof is the same as the proof of Theorem 11. That $\mathfrak{A}_\Lambda$ is a subset of $\mathfrak{A}$ follows from the fact that $\mathfrak{A}$ is the maximal $\mathfrak{S}$-invariant compact in $X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$. □

The existence of global attractors of the systems $(\mathfrak{F}^{\mathcal{I}}, X^\sharp \times \Lambda_{\mathcal{I}})$ and $(\mathfrak{F}_{\mathcal{J}}, X^\sharp \times \Lambda_{\mathcal{J}})$ is proven as in the case without restrictions in Theorem 13. We therefore move to the trajectories generating systems $(\{\Theta_\Lambda^{[n]}\}, X)$ and $(\{\Theta_\Lambda^{[n]}\}, X^\sharp)$, with $\Theta_\Lambda^{[n]}$ as in (16). We start with a theoretical aside.

5.1 Global attractors of set-trajectories generated systems

Definition 17. *Let $\Theta^{[1]}, \Theta^{[2]}, \dots$ be a sequence of maps defined on the hyperspace of nonempty, bounded subsets of X. We assume that each $\Theta^{[n]}$ satisfies the following conditions.*

(1) *If A is bounded, then $\Theta^{[n]}(A)$ is bounded.*

(2) *If $A \in X^\sharp$, then $\Theta^{[n]}(A) \in X^\sharp$.*

(3) *If $A \subset B$, then $\Theta^{[n]}(A) \subset \Theta^{[n]}(B)$.*

(4) $\Theta^{[m+n]}(C) \subset \Theta^{[m]}\left(\Theta^{[n]}(C)\right)$, *for any bounded C and for all $m, n \geq 1$. This is a version of the semi-group property.*

We call $\{\Theta^{[n]}\}$ a set-trajectories generating system on X.

The equation $A_n = \Theta^{[n]}(A_0)$ defines the trajectory of a bounded set A_0.

We write $\Theta^{[n]}(x)$ for the image of the singleton $\{x\}$ under the map $\Theta^{[n]}$.

In this setting we say that $B \subset X$ is a global absorbing set if, for every bounded set C, the sets $\Theta^{[n]}(C)$ are subsets of B for all sufficiently large n. Also, we say that a set A

attracts the set C, if for any $\epsilon > 0$ there is an $N = N(\epsilon, C)$ such that $\Theta^{[n]}(C) \subset \mathcal{O}_\epsilon(A)$ for all $n \geq N$. The attractor is a set that attracts all bounded sets. The global attractor of the set-trajectories generating system $\left(\{\Theta^{[n]}\}_{n=1}^\infty, X\right)$ is the minimal compact set $M \subset X$ that attracts all bounded sets.

Theorem 18. *The set-trajectories generating system* $\left(\{\Theta^{[n]}\}_{n=1}^\infty, X\right)$ *has a global attractor iff the following two conditions are satisfied:*

1) *there is a bounded global absorbing set;*

2) *for every (nonempty) bounded set C and for every increasing sequence of integers n_k, any sequence $x_k \in \Theta^{[n_k]}(C)$ has a convergent subsequence.*

Proof. Suppose M is the global attractor. Then, for any $\epsilon > 0$, the ϵ-neighborhood of M is a global absorbing set. If C is a bounded set and $n_k \nearrow \infty$, then the sets $\Theta^{[n_k]}(C)$ will fit into smaller and smaller neighborhoods of M. This implies that for any sequence $x_k \in \Theta^{[n_k]}(C)$ the distance from x_k to M goes to 0. The closest to x_k points of M must have a convergent subsequence because M is compact. The corresponding subsequence of x_k will have the same limit.

In the opposite direction, suppose that the two conditions are satisfied. Then for every bounded set C its "ω-limit set" defined as

$$\Omega(C) = \bigcap_n \overline{\bigcup_{m\geq n} \Theta^{[m]}(C)} \tag{39}$$

is a nonempty compact. It is nonempty because it is built of all points $x \in X$ which can be represented as the limits of sequences of the form $x_k \in \Theta^{[n_k]}(C)$, and such limits exist due to condition 2). Clearly, $\Omega(C)$ is closed. It is compact because for any sequence $y_k \in \Omega(C)$ one can find $n_k \nearrow \infty$ and $x_k \in \Theta^{[n_k]}(C)$ such that $d(x_k, y_k) < 2^{-k}$. Next, $\Omega(C)$ attracts C (if not, there are points $x_k \in \Theta^{[n_k]}(C)$ that stay some distance $\geq \epsilon_0 > 0$ away from $\Omega(C)$, which is not possible).

Let B be a bounded global absorbing set. Consider the set $M = \Omega(B)$. Let us show that M contains the ω-limit sets of all bounded sets. This will imply that B attracts every bounded set. Pick a bounded set C. There is an $N > 0$ such that $\Theta^{[n]}(C) \subset B$ for all $n \geq N$. Using the fourth condition in Definition 17, we obtain

$$\Omega(C) = \textstyle\bigcap_{n=N+1}^\infty \overline{\bigcup_{m\geq n} \Theta^{[m]}(C)} = \bigcap_{N+k=N+1}^\infty \overline{\bigcup_{N+\ell\geq N+k} \Theta^{[N+\ell]}(C)}$$

$$= \textstyle\bigcap_{k=1}^\infty \overline{\bigcup_{\ell\geq k} \Theta^{[N+\ell]}(C)} \subset \bigcap_{k=1}^\infty \overline{\bigcup_{\ell\geq k} \Theta^{[\ell]}(B)} = M\,.$$

A priori, M may not be minimal, but the minimal attractor, K, which then will be the global attractor in our terminology, does exist, because any two nonempty compact attractors have a nonempty compact intersection, which must be an attractor as well. □

Corollary 19. *Assume the system* $\left(\{\Theta^{[n]}\}, X\right)$ *has a global attractor.*

[1] *For every bounded set C, the set $\Omega(C)$ is a nonempty compact.*

[2] *If A and C are bounded and $A \subset C$, then $\Omega(A) \subset \Omega(C)$.*

[3] *If A and C are bounded and $\Theta^{[n]}(A) \subset C$ for some $n \geq 1$, then $\Omega(A) \subset \Omega(C)$.*

[4] *If A and C are bounded and A attracts C, then $\overline{A} \supset \Omega(C)$.*

[5] *For a bounded C, the set $\Omega(C)$ is the minimal compact set that attracts C.*

[6] *Let K be the global attractor of $\big(\{\Theta^{[n]}\}, X\big)$, and let B be a bounded global absorbing set. Then*

$$K = \Omega(B) = \Omega(\overline{B}) = \bigcup_{\text{all bounded } C} \Omega(C) \;=\; \bigcup_{C \in X^\sharp} \Omega(C)\,.$$

[7] *For every $\epsilon > 0$,*

$$K = \Omega(\mathcal{O}_\epsilon(K))\,.$$

Proof. The first three properties follow from the proof of Theorem 18 and the definition (39), and the properties of the maps $\Theta^{[n]}$. If A attracts C, then, given $\epsilon > 0$, $\Theta^{[m]}(C) \subset \mathcal{O}_\epsilon(A)$ for all sufficiently large m. Then

$$\overline{\bigcup_{m \geq n} \Theta^{[m]}(C)} \subset \mathcal{O}_{2\epsilon}(A)$$

for all sufficiently large n. Then $\Omega(C) \subset \mathcal{O}_{2\epsilon}(A)$, and this implies $\Omega(C) \subset \overline{A}$. To prove the fifth item, assume that A is a compact such that $A \subsetneqq \Omega(C)$ and that A attracts C. The fact that A attracts C, as we have just shown, implies $\Omega(C) \subset A$, a contradiction.

It follows from part [4] that $\Omega(C) \subset K$ for every bounded C. Let B be a bounded global absorbing set. The union of all $\Omega(C)$ lies inside of $\Omega(B)$ as part [3] shows, and since B is itself bounded, it must coincide with $\Omega(B)$. If we take only closed bounded C, then the union of $\Omega(C)$ is not smaller than the union over not necessarily closed C, as property [2] shows, but it cannot be larger either. Since $\Theta^{[n]}(\overline{B}) \subset B$ for all large n, we have $\Omega(\overline{B}) \subset \Omega(B)$ by part [3], while the opposite inclusion follows from part [2].

Since $\mathcal{O}_\epsilon(K)$ is a global absorbing set, part [7] follows from part [6]. □

5.2 Global attractors of point-trajectories generating systems in the hyperspace

We now consider the maps $\{\Theta^{[n]}\}$ with the properties listed in Definition 17 as a trajectories generating dynamics on the hyperspace $X^\sharp$. The points of $X^\sharp$ are nonempty, closed, bounded subsets of X. We extend the maps $\Theta^{[n]}$ to subsets of $X^\sharp$: if $\mathscr{A}$ is a subset of $X^\sharp$, then

$$\Theta^{[n]}\,(\mathscr{A}) = \biguplus_{C \in \mathscr{A}} \Theta^{[n]}(C)\,.$$

In this setting, $\mathscr{B}$ is a global absorbing set if, for every bounded set $\mathscr{A}$, $\Theta^{[n]}(\mathscr{A}) \subset^\sharp \mathscr{B}$ for all sufficiently large n. A set $\mathscr{A}$ attracts the set $\mathscr{C}$ if, for every $\epsilon > 0$, $\Theta^{[n]}(\mathscr{C}) \subset^\sharp \mathcal{O}^\sharp_\epsilon(\mathscr{A})$ for all sufficiently large n. The global attractor of the system $\left(\{\Theta^{[n]}\}, X^\sharp\right)$ is the minimal compact set $\mathscr{A}$ that attracts every bounded set.

Theorem 20. *If the set-trajectories generating system $\left(\{\Theta^{[n]}\}, X\right)$ has a global attractor, then the system $\left(\{\Theta^{[n]}\}, X^\sharp\right)$ has a global attractor as well. If K is the global attractor of $\left(\{\Theta^{[n]}\}, X\right)$ and $\mathscr{K}$ is the global attractor of $\left(\{\Theta^{[n]}\}, X^\sharp\right)$, then $\mathscr{K}^\flat = K$. At the same time, $\mathscr{K} \subset^\sharp K^\sharp$.*

Proof. Assume $\left(\{\Theta^{[n]}\}, X\right)$ has a global attractor and let $K \subset X$ be that attractor. The neighborhood $B = \overline{\mathcal{O}_1(K)}$ is a closed and bounded global absorbing set for $\left(\{\Theta^{[n]}\}, X\right)$. Define $\mathscr{B} = B^\sharp$, the set of all closed subsets of B. This subset of $X^\sharp$ is bounded. It is global absorbing for $\left(\{\Theta^{[n]}\}, X^\sharp\right)$. Indeed, if $\mathscr{A}$ is a bounded set, then $\mathscr{A}^\flat = \bigcup_{C \in \mathscr{A}} C$ is bounded in X, and therefore, $\Theta^{[n]}\left(\mathscr{A}^\flat\right) \subset B$ for all sufficiently large n. If $C \in \mathscr{A}$, then $C \subset \mathscr{A}^\flat$, and hence $\Theta^{[n]}(C) \subset \Theta^{[n]}\left(\mathscr{A}^\flat\right) \subset B$.

Next we show that $\mathscr{M} = K^\sharp$ attracts all bounded sets in $X^\sharp$. Let $\mathscr{C}$ be bounded. We check that for every $\epsilon > 0$ there is an N such that

$$\sup_{C \in \mathscr{C}} \inf_{A \in \mathscr{M}} d^\sharp(\Theta^{[n]}(C), A) < \epsilon$$

for all $n \geq N$. Suppose this is not true. Then there are an $\epsilon_0 > 0$, a sequence $m_n \nearrow \infty$, and a sequence $C_n \in \mathscr{C}$ such that

$$\inf_{A \in \mathscr{M}} d^\sharp(\Theta^{[m_n]}(C_n), A) > \epsilon_0 \,. \tag{40}$$

Define

$$\hat{C} = \overline{\mathscr{C}^\flat} = \overline{\bigcup_{C \in \mathscr{C}} C} \,.$$

This is a closed bounded set. Since K is the global attractor of $\left(\{\Theta^{[n]}\}, X\right)$, K attracts $\hat{C}$, i.e., $e(\Theta^{[n]}(\hat{C}), K) \to 0$. By Lemma 27, there is a subsequence of m_n for which $\Theta^{[m_n]}(C_n)$ converges in the Hausdorf metric to a compact subset of K. This contradicts (40).

Since K is compact in X the set $K^\sharp$ is compact in $X^\sharp$. Once a compact set attracting all bounded sets (an attractor) is found, we notice that any two such sets must have a nonempty intersection which is an attractor as well. Hence, there exists the minimal attractor, i.e., the global attractor $\mathscr{K}$. By construction, $\mathscr{K} \subset^\sharp \mathscr{M} = K^\sharp$. Pick any point $x \in K$. Then, $x = \lim_n x_n$, where $x_n \in \Theta^{[m_n]}(C)$, $m_n \nearrow \infty$, and C is some closed, bounded set (according to Corollary 19[6], we can take $C = \overline{B}$, where B is a bounded global absorbing set). Since $e(\Theta^{[m_n]}(C), K) \to 0$, by Lemma 27 the sequence $\Theta^{[m_n]}(C)$ has a convergent in the Hausdorff metric subsequence, and the limit of this subsequence, C_*, must be a compact subset of K. Clearly, $C_* \in \mathscr{K}$. On the other hand, $x \in C_*$, and hence, $x \in \mathscr{K}^\flat$. □

Remark 21. *The system* $\big(\{\Theta^{[n]}\}, X^\sharp\big)$ *having a global attractor does not imply that the system* $\big(\{\Theta^{[n]}\}, X\big)$ *has one. A trivial example: X is an infinite dimensional Banach space, $\Theta^{[n]} = \Phi^n$ where Φ maps every bounded set into the closed unit ball B_1. Then $\mathscr{K} = \{B_1\}$, but* $\big(\{\Theta^{[n]}\}, X\big)$ *does not have a global attractor (as we defined it).*

5.3 Attractors of IFSs with restricted choice

Return to the setting of the maps S_j^τ and two subshifts, $\Lambda_{\mathcal{J}} \subset \Sigma_{\mathcal{J}}$ and $\Lambda_{\mathcal{I}} \subset \Sigma_{\mathcal{I}}$. Define the maps

$$\Theta_\Lambda^{[n]} : A \mapsto \overline{\bigcup_{h\in\Lambda_{\mathcal{I}}} \bigcup_{w\in\Lambda_{\mathcal{J}}} S_{w[n]}^{h[n]}(A)}\,. \tag{41}$$

Under the assumptions of Section 3, these maps satisfy the conditions of Definition 17. Our assumption **(A4)** implies that $\mathbf{B}$ is a global absorbing set for the set-trajectories generating system $\{\Theta_\Lambda^{[n]}\}$ on X. Let us show that for every (nonempty) bounded set C and for every increasing sequence of integers n_k, any sequence $x_k \in \Theta_\Lambda^{[n_k]}(C)$ has a convergent subsequence. It is clear that $\Theta_\Lambda^{[n]}(C) \subset \mathcal{F}^n(C)$, where $\mathcal{F}$ is the map defined in (13), $\mathcal{F}(C) = \overline{\bigcup_{\tau\in\mathcal{I}} \bigcup_{j\in\mathcal{J}} S_j^\tau(C)}$. Also,

$$\mathcal{F}^n(C) = \overline{\bigcup_{h\in\Sigma_{\mathcal{I}}} \bigcup_{w\in\Sigma_{\mathcal{J}}} S_{w[n]}^{h[n]}(C)}\,.$$

It is easy to check that $\mathcal{F}^n$, $n = 1, 2, \ldots$, satisfy the conditions of Definition 17. In the proof of Theorem 13 we have shown that, for any mnc ψ, the map $\mathcal{F}$ is asymptotically ψ-condensing. In addition, we have shown that $\lim_n \psi(\bigcup_{m\geq n} \mathcal{F}^m(C)) = 0$. Since the sequence $\{x_k\}$ is a subset of $\bigcup_{m\geq n} \mathcal{F}^m(C)$ except for a finite number of points, $\psi(\{x_k\}) = 0$. Hence, $\{x_k\}$ is relatively compact. Thus, we can apply Theorems 18 and 20 to obtain most of the proof of the following result.

Theorem 22. *The IFS with restricted choice represented by the set-trajectories generating system* $\Big(\{\Theta_\Lambda^{[n]}\}, X\Big)$ *has a global attractor. When viewed in the hyperspcae $X^\sharp$, the corresponding system* $\Big(\{\Theta_\Lambda^{[n]}\}, X^\sharp\Big)$ *has a global attractor as well. Let K_Λ be the global attractor of* $\Big(\{\Theta_\Lambda^{[n]}\}, X\Big)$ *and let $\mathscr{K}_\Lambda$ be the global attractor of* $\Big(\{\Theta_\Lambda^{[n]}\}, X^\sharp\Big)$. *The attractor K_Λ is the projection on X of the attractor $\mathfrak{A}_\Lambda$ of the semi-dynamical system $(\mathfrak{S}, X \times \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}})$. The attractor $\mathscr{K}_\Lambda$ in $X^\sharp$ is a subset of $(K_\Lambda)^\sharp$. At the same time, the merger of the sets comprising $\mathscr{K}_\Lambda$ is K_Λ, i.e., $(\mathscr{K}_\Lambda)^\flat = K_\Lambda$.*

Proof. The fact that K_Λ is the projection on X of the attractor $\mathfrak{A}_\Lambda$ of the semi-dynamical system $(\mathfrak{S}, X \times \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}})$ follows from the comparison of the descriptions of the attractors. Every point $x \in K_\Lambda$ is a limit of the form $x = \lim_k S_{w_k[n_k]}^{h_k[n_k]}(y_k)$ for some bounded sequence y_k, a sequence $n_k \nearrow \infty$, and sequences $h_k \in \Lambda_{\mathcal{I}}$ and $w_k \in \Lambda_{\mathcal{J}}$. Every point $(x, w, h) \in \mathfrak{A}_\Lambda$ is the limit of the form $(x, w, h) = \lim_k (S_{w_k[n_k]}^{h_k[n_k]}(y_k), \sigma^{n_k}(w_k), \sigma^{n_k}(h_k))$. $\square$

5.4 The case of contractions

Now consider a very special situation. Of the assumptions on the maps from Section 3 we keep **(A3)** and replace other assumptions by requiring that each S_j^τ is a strict contraction and that their Lipschitz constants are bounded from above by some $\gamma < 1$. Then the map $\mathcal{F}$ is a contraction on $X^\sharp$ and

$$d^\sharp(\mathcal{F}(A), \mathcal{F}(B)) \leq \gamma\, d^\sharp(A, B)\,.$$

The map $\mathcal{F}$ has a unique fixed point in $X^\sharp$. This point, K, (better – the singleton $\{K\}$) is the global attractor of the system $(\mathcal{F}, X^\sharp)$. Since the map $\mathcal{F}$ is asymptotically condensing for any mnc ψ, we have $\psi(\mathcal{F}^n(K)) \to 0$, and hence $\psi(K) = 0$, which means K is compact. Also, in the framework of the system $(\{\mathcal{F}^n\}, X)$, K attracts all bounded sets. It is the global attractor of $(\{\mathcal{F}^n\}, X)$ because $\mathcal{F}(K) = K$, and hence K must lie in the global attractor.

Following Hutchinson [15], define the map $p : \Sigma_\mathcal{J} \times \Sigma_\mathcal{I} \to K$ as follows:

$$p(w, h) = \lim_n \overleftarrow{S_{w[n]}^{h[n]}}(x)\,, \tag{42}$$

where x is an arbitrary point of X and

$$\overleftarrow{S_{w[n]}^{h[n]}} = S_{w(0)}^{h(0)} \circ S_{w(1)}^{h(1)} \circ \cdot \circ S_{w(n-1)}^{h(n-1)}\,.$$

The limit in (42) exists because

$$d\left(\overleftarrow{S_{w[n]}^{h[n]}}(x), \overleftarrow{S_{w[n-1]}^{h[n-1]}}(x)\right) \leq \gamma^{n-1}\, d(S_{w(n-1)}^{h(n-1)}(x), x)$$

and $\sup_{j,\tau} d(S_j^\tau(x), x)$ is finite for a fixed x. The limit does not depend on x because all S_j^τ are contractions. The limit is a point of the attractor K because $\overleftarrow{S_{w[n]}^{h[n]}}(x) \in \mathcal{F}^n(x)$ and $\mathcal{F}^n(x) \to K$ in $X^\sharp$. The map $p : \Sigma_\mathcal{J} \times \Sigma_\mathcal{I} \to X$ is continuous because, once we fix x, we have $\mathcal{F}^N(x) \subset \mathcal{O}_1(K)$ for some N, and we can write

$$p_n(w, h) = \overleftarrow{S_{w[n]}^{h[n]}}(x) = \overleftarrow{S_{w[n-N]}^{h[n-N]}}(y(N, w, h))$$

where, regardless of w and h, all $y(N, w, h)$ lie in a fixed bounded set $\mathcal{F}^N(x)$, and we can use property **(A3)** to show that $d(p_n(w, h), p_n(u, s))$ is small if the distances between w and u in $\Sigma_\mathcal{J}$ and between h and s in $\Sigma_\mathcal{I}$ are small.

The image of $\Sigma_\mathcal{J} \times \Sigma_\mathcal{I}$ under p is the whole K because every point $z \in K$ is a limit of some points $z_k \in \mathcal{F}^{n_k}(x)$, and each z_k is of the form $z_k = \overleftarrow{S_{w_{n_k}}^{h_{n_k}}}(x)$ for some finite words w_{n_k} and h_{n_k} of length n_k. Since $\Sigma_\mathcal{J}$ and $\Sigma_\mathcal{I}$ are compact, there are subsequences of w_{n_k} and h_{n_k} converging to some $w \in \Sigma_\mathcal{J}$ and $h \in \Sigma_\mathcal{I}$. Hence, $z = p(w, h)$.

For the point-trajectories generating system $\left(\{\mathcal{F}^n\}, X^\sharp\right)$ the attractor is $\mathscr{K} = \{K\}$, because every bounded set converges (along its trajectory) to K.

This completes the description of attractors of IFSs generated by contractions. In a similar fashion we can analyze the restricted choice IFSs. We would like to identify their attractors (as we understand them in connection with the set-trajectories generating systems) with the image of a subshift under the map p. In [7, Definition 4.16.2], Barnsley defines what he calls *the set attractor of the directed IFS* as the image $p(\Lambda)$. However, he does not discuss any properties of $p(\Lambda)$ that would make it a true attractor. What he is intersted instead is the fractal nature of $p(\Lambda)$. We, on the other hand, are interested in attractors, and it turns out that the attractor of the set-trajectories generating system associated with a subshift $\Lambda = \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}}$ is the image under p of the dual subshift, Λ'. We could not find the notion of a dual subshift in the literature, so we proceed with a brief discussion.

Let Σ be the full one-sided shift over the alphabet $\mathcal{A}$ (and $\mathcal{A}$ is a compact metric space with the metric $d_{\mathcal{A}}$), and let Λ be its subshift. Define

$$\Lambda' = \{u \in \Sigma : \forall n \geq 0 \, \exists w_n \in \Lambda \text{ such that } u[n] = \overleftarrow{w_n[n]}\} , \tag{43}$$

where $\overleftarrow{w_n[n]}$ is the word $w_n[n]$ written in the reversed order.

Lemma 23. Λ' *is a subshift.*

Proof. Take any $u \in \Lambda'$ and consider $\sigma(u)$. The first letters of strings in Λ' are the same as the first letters of strings in Λ. Given $n \geq 0$, we find $w_{n+1} \in \Lambda$ such that

$$u(0)u(1)\ldots u(n) = w_{n+1}(n)w_{n+1}(n-1)\ldots w_{n+1}(0) .$$

Then

$$(\sigma(u))\,[n] \equiv u(1)\ldots u(n) = w_{n+1}(n-1)\ldots w_{n+1}(0) ,$$

which shows that $\sigma(u) \in \Lambda'$ because $(\sigma(u))\,[n] = \overleftarrow{w_{n+1}[n]}$. It remains to show that Λ' is closed in Σ. Assume the sequence $u^m \in \Lambda'$ converges to $u \in \Sigma$. This means that

$$\sum_{\ell=0}^{\infty} \frac{d_{\mathcal{A}}(u^m(\ell), u(\ell))}{2^\ell} \underset{m\to\infty}{\to} 0 .$$

Pick any integer $N \geq 0$. For each u^m find an appropriate w^m_{N+1}. Then

$$\sum_{\ell=0}^{N} \frac{d_{\mathcal{A}}(u^m(\ell), u(\ell))}{2^\ell} = \sum_{\ell=0}^{N} \frac{d_{\mathcal{A}}(w^m_{N+1}(N-\ell), u(\ell))}{2^\ell} \underset{m\to\infty}{\to} 0 .$$

Since Λ is a compact space, the sequence w^m has a convergent subsequence. Let $w \in \Lambda$ be its limit. Then $u[N+1] = \overleftarrow{w[N+1]}$. The lemma is proven. □

Definition 24. *The subshift* Λ' *defined in* (43) *is called dual to the subshift* Λ.

It is clear that $(\Lambda')' = \Lambda$. There are self-dual subshifts, the ones that satisfy $\Lambda' = \Lambda$. For example, the full one-sided shift Σ and the golden mean subshift are self-dual. The finite-type subshift of the full one-sided shift over the alphabet $\{1, 2, 3\}$ defined by the forbidden word 123 is not self-dual. The subshift dual to this subshift is defined by the forbidden word 321.

Consider now the restricted choice IFSs.

Theorem 25. *Let the maps S_j^τ, $j \in \mathcal{J}$, $\tau \in \mathcal{I}$, be strict contractions with a common contractivity coefficient $\gamma < 1$. Suppose, in addition, that S_j^τ satisfy the hypothesis* **(A3)***. Let $\Lambda_{\mathcal{J}}$ be a subshift of $\Sigma_{\mathcal{J}}$ and let $\Lambda_{\mathcal{I}}$ be a subshift of $\Sigma_{\mathcal{I}}$. Let K_Λ be the global attractor of $\left(\{\Theta_\Lambda^{[n]}\}, X\right)$ and let $\mathscr{K}_\Lambda$ be the global attractor of $\left(\{\Theta_\Lambda^{[n]}\}, X^\sharp\right)$. Then $\mathscr{K}_\Lambda$ is a singleton, $\mathscr{K}_\Lambda = \{K_\lambda\}$. In addition, K_Λ can be characterized as the image of the subshift Λ' dual to $\Lambda = \Lambda_{\mathcal{J}} \times \Lambda_{\mathcal{I}}$ under the map $p : \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}} \to K$, where K is the global attractor of the unrestricted system $(\mathcal{F}, X)$.*

Proof. The maps $\Theta_\Lambda^{[n]}$ satisfy the conditions of Definition 17. The two conditions of Theorem 18 are satisfied for the same reasons as in the unrestricted case. Thus, the existence of the global attractors K_Λ and $\mathscr{K}_\Lambda$ follows from Theorems 18 and 20. It is clear that $K_\Lambda \subset K$.

For any bounded set A, we have

$$d^\sharp(\Theta_\Lambda^{[n]}(A), \Theta_\Lambda^{[n+1]}(A)) \leq \gamma^n \sup_{j,\tau} d^\sharp(S_j^\tau(A), A)\,.$$

Hence, the sequence $\Theta_\Lambda^{[n]}(A)$ in $X^\sharp$ is Cauchy. Since

$$d^\sharp(\Theta_\Lambda^{[n+1]}(A), \Theta_\Lambda^{[n+1]}(B)) \leq \gamma^n \sup_{j,\tau} d^\sharp(S_j^\tau(A), S_j^\tau(B)) \leq \gamma^{n+1} d^\sharp(A, B)\,,$$

the limit of $\Theta_\Lambda^{[n]}(A)$ does not depend on A. Temporarily, denote this limit by K_*. Since $\Theta_\Lambda^{[n]}(A) \subset \mathcal{F}^n(A)$ and $\mathcal{F}^n(A)$ converges to the compact K, K_* is a compact subset of K. Thus, K_* is the global attractor that we denote K_Λ. Since each of the maps S_j^τ is continuous, we can argue as in the proof of Lemma 12[3] and show that each map $\Theta_\Lambda^{[m]}$ is continuous in the sense that, if $B_n, B \in X^\sharp$, $d^\sharp(B_n, B) \to 0$ and B is compact, then $\Theta_\Lambda^{[m]}(B_n)$ converges to $\Theta_\Lambda^{[m]}(B)$ in $X^\sharp$. Using this fact and the semi-group property, we can pass to the limit in the inclusion $\Theta_\Lambda^{[m]}\left(\Theta_\Lambda^{[n]}(A)\right) \supset \Theta_\Lambda^{[m+n]}(A)$ and obtain

$$K_\Lambda \subset \Theta_\Lambda^{[m]}(K_\Lambda) = \bigcup_{h \in \Lambda_{\mathcal{I}}} \bigcup_{w \in \Lambda_{\mathcal{J}}} S_{w[m]}^{h[m]}(K_\Lambda) \tag{44}$$

for every m. This is the replacement of the fixed point property of K.

That K_Λ is the image of Λ' under the map p follows from the fact that every point $z \in K_\Lambda$ is a limit of the points $x_k = S_{w_k[n_k]}^{h_k[n_k]}(x)$, where $h_k \in \Lambda_{\mathcal{I}}$ and $w_k \in \Lambda_{\mathcal{J}}$. We can write this as

$$z = \lim_{k\to\infty} \overleftarrow{S_{u_k[n_k]}^{s_k[n_k]}}(x)$$

with $s_k[n_k] = \overleftarrow{h_k[n_k]}$ and $u_k[n_k] = \overleftarrow{w_k[n_k]}$. Since $s_k[n_k]$ has a subsequence converging to some $s \in \Lambda'_{\mathcal{I}}$ while $u_k[n_k]$ converges to some $u \in \Lambda'_{\mathcal{J}}$, we obtain $z = p(s, u)$.

It is clear that $\mathscr{K}_\Lambda = \{K_\Lambda\}$. □

6 Example

In many switched systems coming from ordinary or partial differential equations, assumptions **(A1)** and **(A2)** of Section 3 can be verified by establishing *a priori* estimates depending only on the bounds of intervals where switching parameters lie. To illustrate this point we shall discuss the second example mentioned in Introduction. [In the next section we give a deceivingly simple example where such estimates cannot be obtained.]

In a bounded domain Ω in $\mathbb{R}^2$ consider the familiar Navier-Stokes equations with the no-slip boundary condition,

$$\begin{aligned} &\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}(x), \quad \operatorname{div}\mathbf{v} = 0, \ \text{ in } (0,+\infty)\times\Omega\,, \\ &\mathbf{v}(t,x) = 0 \quad \text{for } x \in \partial\Omega\,. \end{aligned} \tag{45}$$

Here $\nu > 0$ is a given coefficient of kinematic viscosity and $\mathbf{f}$ is a given vector-field representing external forces. For a detailed discussion of solutions to this problem we refer to [24, 31]. The functional setting is as follows. Denote by $\mathcal{D}$ the space of C^∞ divergence-free vector-fields with supports in Ω. Denote by H the closure of $\mathcal{D}$ in $L^2(\Omega)$ (strictly speaking, $[L^2(\Omega)]^2$, because of vector-valued functions, but we will omit the power with the understanding that each component lies in L^2). The L^2 inner product $(\mathbf{u},\mathbf{w}) = \int_\Omega u^i\, w^i\, dx$ makes H a Hilbert space. In addition, denote by V the closure of $\mathcal{D}$ in the Sobolev space $W^{1,2}(\Omega)$ with the norm $\|\mathbf{u}\|_V = \left(\int_\Omega |\mathbf{u}_x|^2\, dx\right)^{1/2}$. V is Hilbert with the inner product $((\mathbf{u},\mathbf{w})) = \int_\Omega u^i_{x_k}\, w^i_{x_k}\, dx$ (summation over the repeated indices is assumed throughout). The space H is a convenient state space for system (45). We assume that $\mathbf{f} \in L^2(\Omega)$. For an initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}(0) \in H\,, \tag{46}$$

the corresponding solution of (45) is defined as a vector-function $\mathbf{v}(x,t)$ with the following properties:

1) $\mathbf{v}(\cdot,t)$ is a continuous function of $t \in [0,+\infty)$ with values in H, satisfying (46);

2) In addition, $\mathbf{v}(\cdot,t) \in V$ for all $t > 0$, and moreover, $\mathbf{v} \in L^2([0,T],V)$, for any $T > 0$;

3) The time derivative of $\mathbf{v}(\cdot,t)$ exists for $t > 0$ as an element of the space V' dual to V, and moreover, $\mathbf{v}_t \in L^2([0,T],V')$, for any $T > 0$;

4) The Navier-Stokes equations are satisfied in the sense that, for any $T > 0$, the following equality
$$\int_0^T \left[(\mathbf{v}_t,\mathbf{w}) + \nu\,((\mathbf{v},\mathbf{w})) + (v^i\mathbf{v}_{x_i},\mathbf{w})\right]\, dt = \int_0^T (\mathbf{f},\mathbf{w})\, dt \tag{47}$$
holds for every $\mathbf{w} \in C([0,T],H) \cap L^2([0,T],V)$.

It is well known, [24], that so defined solution exists globally in time and is unique for any initial condition $\mathbf{v}(0) \in H$. Also, the solution map $S^t : \mathbf{v}(0) \mapsto S^t(\mathbf{v}(0)) = \mathbf{v}(t)$ is continuous as a map from H into itself. Also, it satisfies the semi-group property: $S^t \circ S^\tau = S^{t+\tau}$.

In a remarkable early paper [25], Ladyzhenskaya showed that the Navier-Stokes dynamics on the space H possesses a global compact attractor. She established the basic properties of that attractor and showed, in particular, that when restricted to the attractor, the semi-dynamical system S^t, $t \geq 0$, can be uniquely extended to negative t to form a dynamical system. Later developments by Ladyzhenskaya and other researchers have simplified and at the same time generalized the theory. From the modern perspective, once the properties of S^t listed in the previous paragraph are established (in particular, the semi-group property and the fact that S^t is compact because of property 2) and the compactness of the embedding $V \subset H$), the existence of a global compact attractor would follow from the existence of a bounded absorbing set. This in turn is a consequence of the following basic *a priori* estimate:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}(t)\|_H^2 + \nu\,\|\mathbf{v}(t)\|_V^2 \leq \|\mathrm{Proj}_H \mathbf{f}\|_H\,\|\mathbf{v}(t)\|_H \tag{48}$$

Applying the Poincaré inequality $\|\mathbf{w}\|_V^2 \geq \lambda_1\,\|\mathbf{w}\|_H^2$ (where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet Stokes operator in Ω), we obtain

$$\frac{d}{dt}\|\mathbf{v}(t)\|_H + \nu\,\lambda_1\,\|\mathbf{v}(t)\|_H \leq \|\mathrm{Proj}_H \mathbf{f}\|_H \tag{49}$$

Integrating this inequality leads us to the inequality

$$\|\mathbf{v}(t)\|_H \leq e^{-\nu\,\lambda_1\,t}\,\|\mathbf{v}(0)\|_H + \frac{1-e^{-\nu\,\lambda_1\,t}}{\nu\,\lambda_1}\,\|\mathrm{Proj}_H \mathbf{f}\|_H \tag{50}$$

It follows that the ball of radius

$$R = 1/(\nu\,\lambda_1)\,\|\mathrm{Proj}_H \mathbf{f}\|_H + 1 \tag{51}$$

centered at $0 \in H$ can be chosen as the absorbing set: for any bounded in H set of initial conditions there exists a time after which all solutions will stay inside $B_R(0) \subset H$.

Now we use this information to extend Ladyzhenskaya's result to the variable time step dynamics with choice generated by the following system

$$\begin{aligned} &\mathbf{v}_t - \nu_{w(n)}\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \mathbf{f}_{w(n)}(x), \quad \mathrm{div}\,\mathbf{v} = 0, \quad \text{in } [t_n, t_{n+1}) \times \Omega\,, \\ &\mathbf{v}(t,x) = 0 \quad \text{for } x \in \partial\Omega\,. \end{aligned} \tag{52}$$

We assume, that

1. $h(n) = t_{n+1} - t_n \in \mathcal{I} = [a, b]$, where the lower limit on the inter-switching time, a, is strictly positive;

2. $\mathcal{J} = \{0, 1\}$, i.e., $w(n)$ is either 0 or 1;

3. the two values, ν_0 and ν_1, of the viscosity coefficient lie within some finite interval $[\nu_{min}, \nu_{max}]$, and $\nu_{min} > 0$;

4. the two admissible vector-fields of external forces, $\mathbf{f}_0(x)$ and $\mathbf{f}_1(x)$, have components in $L^2(\Omega)$.

We choose H as the state space. In view of the general properties of solutions of (45) discussed above, both maps S_0^t and S_1^t are continuous and compact in H. Assumption **(A1)** of Section 3 follows from the continuity of each map S_0^t and S_1^t in t. To verify assumption **(A2)**, we go back to inequality (49) and write it in the form valid for both $w(n) = 0$ and $w(n) = 1$:

$$\frac{d}{dt}\|\mathbf{v}(t)\|_H + \nu_{min}\,\lambda_1\,\|\mathbf{v}(t)\|_H \leq F\max\{\|\mathrm{Proj}_H\mathbf{f}_0\|_H,\,\|\mathrm{Proj}_H\mathbf{f}_1\|_H\}\,, \tag{53}$$

where

$$F = \max\{\|\mathrm{Proj}_H\mathbf{f}_0\|_H,\,\|\mathrm{Proj}_H\mathbf{f}_1\|_H\}\,.$$

Since (53) is valid on each interval $[t_n, t_{n+1})$, we can integrate it to obtain

$$\|\mathbf{v}(t)\|_H \leq e^{-\nu_{min}\,\lambda_1\,t}\,\|\mathbf{v}(0)\|_H + \frac{1 - e^{-\nu_{\min}\,\lambda_1\,t}}{\nu_{min}\,\lambda_1}\,F \tag{54}$$

for any $t > 0$. Hence, again, the ball of radius $R = 1/(\nu_{min}\,\lambda_1)\,F + 1$ will do as an absorbing set.

Next, we show that the solution maps S_j^τ for system (52) are uniformly continuous on bounded subsets of the state space H; in other words, they satisfy assumption **(A3)**. The needed estimates are essentially the same as in the proof of continuous dependence for (45). We give here a formal derivation, referring the reader to [24] for details. Let $\mathbf{v}(t)$ and $\tilde{\mathbf{v}}(t)$ be two solutions of (45) corresponding to the initial conditions $\mathbf{v}(0)$ and $\tilde{\mathbf{v}}(0)$, both of which are in the ball of radius R centered at $\mathbf{0}$ in H. The difference, $\mathbf{u}(t) = \mathbf{v}(t) - \tilde{\mathbf{v}}(t)$, is divergence free and satisfies the equation

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\tilde{\mathbf{v}}\cdot\nabla)\,\mathbf{u} + \nabla p' = \mathbf{0}\,.$$

Multiply this equation by $\mathbf{u}$ and integrate over Ω to obtain

$$\frac{1}{2}\,\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2}^2 + \nu\,\|\nabla\mathbf{u}(t)\|_{L^2}^2 \;=\; -\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,dx\,. \tag{55}$$

To bound the integral on the right, we first observe that

$$\Big|\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,dx\Big| \leq \|\nabla\mathbf{v}\|_{L^2}\,\|\mathbf{u}\|_{L^4}^2\,.$$

In the two-dimensional case we have the following inequality

$$\|\mathbf{u}\|_{L^4} \leq c\,\|\mathbf{u}\|_{L^2}^{1/2}\,\|\nabla\mathbf{u}\|_{L^2}^{1/2}$$

for all $\mathbf{u} \in V$. Thus,

$$\Big| \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,dx\Big| \le c^2\,\|\nabla\mathbf{v}\|_{L^2}\,\|\nabla\mathbf{u}\|_{L^2}\,\|\mathbf{u}\|_{L^2}\,,$$

and, we continue,

$$\le \frac{\nu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{c^4}{2\nu}\,\|\nabla\mathbf{v}\|_{L^2}^2\,\|\mathbf{u}\|_{L^2}^2\,.$$

Now, as follows from (55),

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2}^2 + \frac{\nu}{2}\,\|\nabla\mathbf{u}(t)\|_{L^2}^2 \le \frac{c^4}{2\nu}\,\|\nabla\mathbf{v}(t)\|_{L^2}^2\,\|\mathbf{u}(t)\|_{L^2}^2\,,$$

but for our purposes a weaker inequality,

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2}^2 \le \frac{c^4}{\nu}\,\|\nabla\mathbf{v}(t)\|_{L^2}^2\,\|\mathbf{u}(t)\|_{L^2}^2\,,$$

will do. Integrating this inequality, we obtain

$$\|\mathbf{u}(t)\|_{L^2}^2 \le \|\mathbf{u}(0)\|_{L^2}^2\,\exp\left(\frac{c^4}{\nu}\int_0^t \|\nabla\mathbf{v}(s)\|_{L^2}^2\,ds\right)\,.$$

It remains to mention, that $\int_0^t \|\nabla\mathbf{v}(s)\|_{L^2}^2\,ds$ is a continuous function, as follows from (48) and (54).

Thus, all conditions of Theorems 11, 13, and 14 are satisfied. Hence, **the variable time step dynamics with choice associated with the switched Navier-Stokes equations (52) has a global attractor. The corresponding variable time step iterated function systems also possess global attractors. All the attractors are related as described in Theorem 14.**

We arrive at the same conclusion if the switching assumptions for (52) are changed to $\mathcal{J} = \{0, 1, \ldots, N-1\}$ or $\mathcal{J} = [0, 1]$, with the understanding that $0 < \nu_{min} \le \nu_{w(n)} \le \nu_{max}$ and $\sup_{\mathcal{J}} \|\mathrm{Proj}_H \mathbf{f}_{w(n)}\|_H < \infty$.

7 A cautionary example

In this section we give an example of a two-dimensional switched system for which the validity of assumption **(A2)** of Section 3 depends on the choice of the interval $\mathcal{I}$ of admissible time steps. The idea comes from a well-known example of stabilization by means of switching of two individually unstable systems, see [9, 29]. We build the example by first patching a couple of systems of ODEs on the plane and then transplanting the result to an infinite cylinder.

The first system is this:

$$\begin{aligned}\dot{x} &= -y + x\left(1 - x^2 - y^2\right)\\ \dot{y} &= x + y\left(1 - x^2 - y^2\right)\end{aligned} \tag{56}$$

The origin is its unstable focus and $x^2+y^2 = 1$ is the stable limit-circle. In the neighborhood of the origin, we shall glue in two linear systems between which the switching will occur. Those systems are

$$\begin{aligned}\dot{x} &= \epsilon x - 4y\\ \dot{y} &= \ \ x + \epsilon y\end{aligned} \quad (57) \qquad\qquad \begin{aligned}\dot{x} &= \epsilon x - \ \ y\\ \dot{y} &= 4x + \epsilon y,\end{aligned} \quad (58)$$

where ϵ is a positive parameter. This type of systems is well known in the stability theory of switched systems, [29]. For each (57) and (58), the origin is an unstable fixed point, but with the right switching between (57) and (58) (e.g., using (57) if $xy > 0$ and using (58) if $xy < 0$), the origin becomes globally asymptotically stable, see [9]. Transfer systems (56), (57), and (58) to a cylinder by changing the coordinates x and y to s and θ, where $x = e^s \cos\theta$, $y = e^s \sin\theta$. In the (s, θ) coordinates system (56) reads

$$\dot{s} = 1 - e^{2s}\,, \quad \dot{\theta} = 1\,, \tag{59}$$

system (57) reads

$$\dot{s} = \epsilon - \frac{3}{2}\sin(2\theta), \quad \dot{\theta} = \frac{1}{2}(5 - 3\cos(2\theta)), \tag{60}$$

and system (58) reads

$$\dot{s} = \epsilon + \frac{3}{2}\sin(2\theta), \quad \dot{\theta} = \frac{1}{2}(5 + 3\cos(2\theta)). \tag{61}$$

The half of the cylinder with $s < 0$ corresponds to the interior of the unit circle, and the half with $s > 0$ - to its exterior. We want system (59) to operate in the region $s \geq -1$ and systems (60) and (61) to operate where $s < -4$. This will be achieved by adding the right sides of (59) with coefficient $\zeta(s)$ to the corresponding right sides of systems (60) or (61) with coefficient $(1 - \zeta(s))$, where $\zeta(s)$ is a smooth, monotone increasing function such that $\zeta(s) = 0$ for $s \leq -4$ and $\zeta(s) = 1$ for $s \geq -1$. The region $-4 < s < -1$, we call transfer region. We write the resulting systems symbolically as

$$\frac{d}{dt}\begin{bmatrix} s \\ \theta \end{bmatrix} = f_{w(n)}(s, \theta)\,, \tag{62}$$

where $w(n) = 1$ corresponds to (60) combined with (59), and $w(n) = 2$ corresponds to (61). Note, that on the cylinder, system (59) and each of the two systems (62) have the circle $s = 0$ as the global compact attractor. However, depending on the strategy of switching between system number 1 and system number 2 of (62), the attractor may survive, or there may be no global attractor. To understand this we only need to understand how switching works for systems (57) and (58). Indeed, consider the corresponding switched system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \epsilon \begin{bmatrix} x \\ y \end{bmatrix} + A_{w(n)}\begin{bmatrix} x \\ y \end{bmatrix} \tag{63}$$

where $w(n) = 1$ or 2 and

$$A_1 = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \tag{64}$$

We will exhibit an interval $[a, b] \subset (0, +\infty)$ such that for any choice of time steps τ_n from this interval and for any sequence of regime switching $w(n)$, all trajectories starting not at the origin go to infinity. This will imply that on the cylinder, all trajectories of system (62) starting at the points with $s < 0$ (in fact, any disc in this region) will move in the direction of the circle $s = 0$. The trajectories starting at the points with $s > 0$ will move to the same circle because in the xy-coordinates $s = 0$ is the limit-circle of system (56). Thus, the variable time step system (62) with $\mathcal{J} = \{1, 2\}$ and $\mathcal{I} = [a, b]$ does have a global attractor. In the setting of Section 4, the global attractor of the variable time step dynamics with choice is $\mathfrak{A} = \{s = 0\} \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$ and the fractal K is the circle $s = 0$.

Choosing the wrong interval for the time steps will destroy this picture. Some trajectories of system (63) will converge to the origin. On the cylinder this means $s \to -\infty$, and there is no global attractor for (62). We will give an example of a bad interval.

To find a "good" interval, it suffices to find those τ for which the solutions

$$e^{\epsilon\tau}\, e^{\tau A_1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \text{ and } e^{\epsilon\tau}\, e^{\tau A_2} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

of (57) and (58) are both farther from the origin than the initial condition $[x_0, y_0]^T$ by some factor. In other words,

$$e^{\epsilon\tau} \, \| e^{\tau A_{1,2}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \| \geq \gamma \, \| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \| \tag{65}$$

for some $\gamma > 1$ and all $[x_0, y_0]^T$ on the unit circle. Just for an example of some "good" interval, we find that

$$\| e^{\tau A_{1,2}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \|^2 \geq \frac{1}{2} \, \| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \|^2$$

when $5/9 < \cos(4\tau) \leq 1$, and take those τ, which in addition guarantee that $e^{2\epsilon\tau}/2 \geq \gamma^2 > 1$, for some $\gamma > 1$. If $\epsilon = 0.3$, we may take $\tau \in \mathcal{I} = [1.326, 1.810]$.

A different argument is needed to find the "bad" intervals. Note, that the eigenvalues of the matrices $e^{\tau A_j}$ are $e^{\pm 2\tau i}$, and the trajectories corresponding to sequences without switching do not go to the origin. However, both products $e^{\tau A_1} e^{\tau A_2}$ and $e^{\tau A_2} e^{\tau A_1}$ have real, negative eigenvalues, and the largest eigenvalue, $\lambda(\tau)$, is such that $|\lambda(\tau)| < 1$, provided τ satisfies $|\cos(2\tau)| < 3/5$. Thus, given an $\epsilon > 0$, we find intervals of τ such that $e^{2\epsilon\tau}|\lambda(\tau)| < \delta < 1$, for some δ. Those intervals are bad. Denote $M_1^\tau = e^{2\epsilon\tau} e^{\tau A_1} e^{\tau A_2}$ and $M_2^\tau = e^{2\epsilon\tau} e^{\tau A_2} e^{\tau A_1}$. We can choose any sequence τ_j from any of the bad intervals and look at the trajectories $[x_{n+1}, y_{n+1}]^T = M_{w(n)}^{\tau_n} [x_n, y_n]^T$, for any infinite string w of 1's and 2's. The distance from the origin to the consecutive images of any circle $(x_0)^2 + (y_0)^2 = r^2$ will go to 0 exponentially fast (although some points in the images will go to infinity). If $\epsilon = 0.3$, and $\mathcal{I} = [0.5, 1.06]$, the dynamics with choice does not have a global compact attractor.

8 Appendix

In this section we collect a few useful facts about the space $X^\sharp$ and convergence of sets in X and in $X^\sharp$. Also, we present some results on the global attractors of systems arising from iterations of a single multi-valued map.

The notation convention is the same as in Section 3. As in the main part of the paper, the space (X, d) is complete. Then $(X^\sharp, d^\sharp)$ is also complete,[23, §33, IV].

Lemma 26.

(1) *$d^\sharp(A, B) = d^\sharp(A, \overline{B})$; if $d^\sharp(A, B) = 0$, then $\overline{A} = \overline{B}$.*

(2) *If B is a nonempty, closed, bounded set in X, then $B^\sharp$, the family of all nonempty, closed subsets of B, is a bounded set in $X^\sharp$ (it need not be closed).*

(3) *If $\mathscr{A}$ is a nonempty, bounded subset of $X^\sharp$, then $\mathscr{A}^\flat = \bigcup_{C\in\mathscr{A}} C$ is a bounded subset of X.*

(4) *If $A \subset X$ is compact, then $A^\sharp$ is compact in $X^\sharp$.*

The proof is an easy exercise. □

• We say that a set A attracts the sets A_n if $\lim_n e(A_n, A) = 0$.

The following observation proves to be quite useful.

Lemma 27. *Let $A \subset X$ be a compact set. Let B_n be a sequence of sets attracted by A. Then the sequence $\overline{B_n}$ has a subsequence converging in the Hausdorff metric. Its limit is a compact subset of A.*

Proof. Choose a sequence $n_k \nearrow +\infty$ such that $\overline{B_{n_k}} \subset \mathcal{O}_{2^{-k}}(A)$. Because A is compact, we can define the sets

$$A_k = \{y \in A \,:\, \exists x \in B_{n_k} \text{ such that } d(x, y) = d(x, A)\}\,.$$

Obviously, $d^\sharp(B_{n_k}, A_k) \le 2^{-k}$. Since $A^\sharp$ is compact in $X^\sharp$, the sequence $\overline{A_k}$ has a convergent subsequence with a limit in $A^\sharp$. The corresponding subsequence of B_{n_k} will have the same limit. □

The limit of a Cauchy sequence, A_n, in the hyperspace $X^\sharp$ can be described with the help of the formula due to Hausdorff, [23, §29, IV(8)]:

$$A_n \to A = \bigcap_n \overline{\bigcup_{m\ge n} A_m} = \{x \in X \,:\, \exists n_k \nearrow \infty\, \exists x_k \in A_{n_k} \text{ such that } x = \lim x_k\}\,. \tag{66}$$

The basic convergence results stem from a generalization of the Cantor nested intervals theorem. The following theorem is a version of the Kuratowski generalization of Cantor's result, [23, §34]. The first part is in [36, Theorem 3.1] and the second follows from [27, Prop. 3(viii)].

Theorem 28. *Assume $A_1 \supset A_2 \supset \dots$ is a nested sequence of nonempty, closed, bounded subsets in X. Assume that $\psi(A_n) \to 0$ for some mnc ψ. Then,*

(1) *the set $A = \bigcap_n A_n$ is nonempty and compact;*

(2) *$A_n \to A$ in $X^\sharp$.*

Combining this with Lemma 27, we obtain the following corollary.

Corollary 29. *Suppose $A_1 \supset A_2 \supset \dots$ is a nested sequence of non-empty, closed, and bounded sets, and assume that $\lim_n \psi(A_n) = 0$. Then any sequence of closed subsets $B_n \subset A_n$ has a convergent in the Hausdorff metric subsequence (with the limit a subset of A).*

The implications of convergence to a compact set are as follows.

Lemma 30. *Assume $A_n \to A$ in $X^\sharp$ and A is compact. Then*

(1) *any sequence C_k of nonempty, closed subsets of A_{n_k}, where $n_k \nearrow \infty$, has a convergent in $X^\sharp$ subsequence, and its limit is a compact subset of A;*

(2) $\psi(A_n) \to 0\,;$

(3) $\overline{\bigcup_{m\geq n} A_m} \to A\,;$

(4) $\psi(\overline{\bigcup_{m\geq n} A_m}) \to 0\,.$

Proof. The first statement follows from Lemma 27. For the second observe that $\psi(A_n) = \psi(A_n) - \psi(A) \leq c_\psi d^\sharp(A_n, A) \to 0$. The third statement is a consequence of the first and formula (66). Finally, the fourth statement follows from the second. □

It is convenient to have a different description of the condition $\lim_n \psi(A_n) = 0$.

Lemma 31. *Let $A_1 \supset A_2 \supset \dots$ be a nested sequence of nonempty, closed, and bounded sets. The following conditions are equivalent:*

1) $\lim_n \psi(A_n) = 0$.

2) *Every sequence x_k, where $x_k \in A_{n_k}$, $n_k \nearrow +\infty$, has a convergent subsequence.*

Proof. Let the first condition be satisfied. For any n, $\{x_k\} \subset A_n$ except for at most a finite number of x_k's. Hence, $\psi(\{x_k\}) \leq \psi(A_n)$, and therefore $\psi(\{x_k\}) = 0$, which implies the second condition.

Now suppose the second condition is satisfied. Note that if a set C has a finite ϵ-net $Q \subset C$, then $d^\sharp(C, Q) \leq \epsilon$, and therefore, by the property mnc(iv),

$$\psi(C) \leq c_\psi\, \epsilon\,.$$

If $\psi(A_n)$ does not converge to 0, then there is subsequence n_k and an $\epsilon > 0$ such that

$$\psi(A_{n_k}) > c_\psi\, \epsilon\,.$$

Hence, the sets A_{n_k} do not have finite ϵ-nets. Take a point $x_1 \in A_{n_1}$. There exists $x_2 \in A_{n_2}$ such that $d(x_1, x_2) > \epsilon/2$. There exists $x_3 \in A_{n_3}$ such that $d(x_3, \{x_1, x_2\}) > \epsilon/2$, and so on. The sequence $\{x_n\}$ is not totally bounded. However, by construction, $x_k \in A_{n_k}$, and by assumption, the sequence $\{x_k\}$ must have a convergent subsequence. A contradiction. □

Consider now a map Φ defined on nonempty bounded subsets of X with values also bounded subsets of X. The first two assumptions on Φ are these:

Φ1: Φ maps $X^\sharp$ into itself.

Φ2: If $A \subset B$, then $\Phi(A) \subset \Phi(B)$.

We do not assume that Φ is a multi-valued map in the traditional sense. In our considerations the set $\Phi(A)$ may be larger than the union of the sets $\Phi(x)$ over all $x \in A$ (where $\Phi(x)$ is $\Phi(\{x\})$, of course).

The iterations Φ^n satisfy the conditions of Definition 17. They generate the dynamics of sets on X and on $X^\sharp$. These are the systems $(\{\Phi^n\}, X)$ and $\big(\{\Phi^n\}, X^\sharp\big)$ in the notation of Section 5. We are interested in the global attractors of these systems. Here is a corollary of Theorems 18, 20, and Lemma 31.

Theorem 32.

(1) *For the system $(\{\Phi^n\}, X)$ to possess a global attractor it is necessary and sufficient that the following two conditions hold.*

  a) There is a bounded global absorbing set;

  b) For some mnc ψ, $\psi(\Phi^n(A)) \to 0$ for every bounded set A.

(2) *If both conditions a) and b) are satisfied, the system $\big(\{\Phi^n\}, X^\sharp\big)$ possesses a global attractor. If K is the global attractor of $(\{\Phi^n\}, X)$ and $\mathscr{K}$ is the global attractor of $\big(\{\Phi^n\}, X^\sharp\big)$, then $\mathscr{K}^\flat = K$.*

(3) *Assume both conditions a) and b) are satisfied. Let B be a bounded global absorbing set. Then*

$$K = \omega(B) = \bigcap_n \overline{\bigcup_{m \ge n} \Phi^m(B)} \quad \text{and} \quad \mathscr{K} = \omega^\sharp(B^\sharp) = \bigcap\nolimits^\sharp_n \overline{\bigcup\nolimits^\sharp_{m \ge n} \Phi^m(B^\sharp)}^\sharp .$$

In the third statement, the formula for K follows from Corollary 19[6], and the formula for $\mathscr{K}$ uses a similar argument.

We call the maps Φ satisfying the condition $b)$ asymptotically ψ-condensing. If Φ originates from a map $\Phi : X \to 2^X$ so that $\Phi(A) = \bigcup_{x \in A} \Phi(x)$, then Φ is asymptotically ψ-condensing if it is ψ-condensing (this requires a proof, see [36]). Recall that Φ is ψ-condensing with respect to some mnc ψ with the properties mnc(i)-(iv) iff $\psi(\Phi(B)) \le \psi(B)$ for any bounded B, and the inequality is strict if $\psi(B) > 0$. The condition $\Phi(A) = \bigcup_{x \in A} \Phi(x)$ can be weakened as follows, see [36, Theorems 4.1 and 4.2]. Consider the following two properties ("compact sets" in **Φ4a** are replaced by "finite sets" in **Φ4b**).

Φ4a(b): For every $D \in X^\sharp$, there is a sequence of sets $B_n \subset \Phi^n(D)$, $n = 1, 2, \ldots$, such that

1) $d^\sharp(B_n, \Phi^n(D)) \to 0$;
2) for $n = 1, 2, \ldots$, for every compact set (**b** : finite set) $L \in B_n$ there is a compact set (**b** : finite set) $L' \subset B_{n-1}$ such that

$$L \subset \Phi(L') \,. \tag{67}$$

[The reason assumption **Φ4** precedes **Φ3** is that we would like to have the same names for the properties of Φ here as in [19].] An example of a map that satisfies **Φ4b** is the Hutchinson-Barnsley map (13). The following result is due essentially to Massatt, [36].

Proposition 33. *Let Φ satisfy the assumptions* **Φ1**, **Φ2**, *and* **Φ4a** *or* **Φ4b**. *In addition, let there be a bounded global absorbing set for the iterations of Φ in X. Then, if Φ is ψ-condensing, it is asymptotically ψ-condensing. As a result, both* $(\{\Phi^n\}, X)$ *and* $\left(\{\Phi^n\}, X^\sharp\right)$ *have global attractors.*

It is easier to check that a map is condensing than that it is asymptotically condensing. For example, Φ is condensing if it is compact (i.e., it maps bounded sets into relatively compact sets). For another example, assume $\Phi(A) = \bigcup_{x \in A} S(x)$ and S is a single-valued map. Then Φ is condensing if S is a contraction, or if S is compact. If X is a Banach space and ψ is the Kuratowski, or the Hausdorff, or any other mnc with the additional property that $\psi(A+B) \le \psi(A) + \psi(B)$ for any pair of bounded sets, then if S is a sum of a finite number of ψ-condensing maps, Φ is ψ-condensing.

The invariance of attractors K and $\mathscr{K}$ does not come free, some form of continuity of Φ is needed. For example, the following assumption will do:

Φ3a : If $A_k, A \in X^\sharp$ and A is compact, then

$$A_k \to A \quad \Longrightarrow \quad \Phi(A_k) \to \Phi(A) \,.$$

Proposition 34. *Assume* $(\{\Phi^n\}, X)$ *has a global attractor. Let K be this attractor and let* $\mathscr{K}$ *be the global attractor of* $\left(\{\Phi^n\}, X^\sharp\right)$. *Assume Φ has the property* **Φ3a**. *Then* $K = \Phi(K)$ *and* $\mathscr{K} = \Phi(\mathscr{K})$. *This, in turn, implies that K is the union in X of bounded two-sided set-trajectories of Φ. [A two-sided trajectory is a sequence of bounded, closed sets* A_n, $n \in \mathbb{Z}$, *such that* $\Phi(A_n) = A_{n+1}$. *It is bounded if* $\{A_n\}$ *is bounded in* $X^\sharp$.*] The attractor* $\mathscr{K}$ *is the union (in* $X^\sharp$*) of bounded two-sided point-trajectories of Φ. Also, K is the maximal compact Φ-invariant subset of X and* $\mathscr{K}$ *is the maximal Φ-invariant compact subset of* $X^\sharp$.

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