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POSITIVITY OF $|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a$

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ABSTRACT. We show that

$$\mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a)$$

is positive under suitable conditions on the exponents a and b and the underlying dimension n. (Here $\mathfrak{q}$ is the multiplication by x and $\mathfrak{p} := \mathrm{i}^{-1}\nabla$.) Furthermore we show a generalization of the generalized Hardy inequalities for the fractional Laplacians.

1. Introduction

The classical Hardy inequality ([3, Formula (4)]))

$$\int_a^\infty \left(\frac{F}{x}\right)^\kappa \mathrm{d}x \leq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa \mathrm{d}x$$

with $F(x) = \int_a^x f(t)\mathrm{d}t$ and $\kappa > 1$ is one of the longest known inequalities allowing to bound the weighted L^κ-norm of a decaying function by the L^κ-norm of its gradient (Hardy [3]). In modern textbooks, see, e.g., Reed and Simon [7, p. 169], this occurs ($\kappa = 2$) as the quantum mechanical uncertainty principle lemma and is written in three dimensions as

$$\int_{\mathbb{R}^3} |\nabla\psi|^2 \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\mathrm{d}x. \tag{1}$$

This, in turn was generalized by Herbst [4] (see also Yafaev [8] and Frank et al [1]) to fractional Laplacians (see (19)).

In a seemingly different context, the excess charge problem of atoms, Lieb[5] needed

$$|\mathfrak{q}||\mathfrak{p}|^2 + |\mathfrak{p}|^2|\mathfrak{q}| > 0 \tag{2}$$

which, however, turned out to be equivalent to the quantum mechanical uncertainty principle. Here $\mathfrak{p} = -\mathrm{i}\nabla$ is the momentum operator and $\mathfrak{q}$ (multiplication by x) is the position operator. Lieb [5] showed in fact, that also

$$|\mathfrak{q}||\mathfrak{p}| + |\mathfrak{p}||\mathfrak{q}| > 0 \tag{3}$$

in three dimensions by reducing it to (2).

With the advent of graphene physics, two-dimensional versions of Lieb's inequality became of physical interest which, however, could not simply be reduced to (1). – Instead, (3) was directly proven [2].

The purpose of this article is to show, that the positivity of the Jordan product $\mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a)$ is in fact a generalization which reduces, for $b = n - a$ to Hardy inequalities for fractional Laplacians. Here a and b are positive constants and n is the underlying dimension of the appropriate function space.

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2. Positivity and Relation to Generalized Hardy Inequalities

Our basic result is the following operator inequality on $L^2(\mathbb{R}^n)$ for the momentum operator $\mathfrak{p} = -\mathrm{i}\nabla$ and the position operator $\mathfrak{q}$ (multiplication by x).

Theorem 1. *Assume $n \geq a+b$ and $\min\{a,b\} \in (0,2]$. Then on $C_0^\infty(\mathbb{R}^n)$*

$$0 < \mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a). \tag{4}$$

In fact, our proof shows more, namely

$$|\mathfrak{q}|^{b/2}\mathcal{H}_{a,n}|\mathfrak{q}|^{b/2} \leq \mathcal{J}_{a,b,n} \tag{5}$$

where $\mathcal{H}_{a,n}$ is the Hardy operator of (19).

As indicated in the introduction, the case $n=3$, $a=2$, and $b=1$ has an important consequence in atomic physics: it is an essential ingredient in bounding the total number of electrons that atoms can bind: the number of electrons that an atom can bind can never exceed twice its nuclear charge. This special case was proven and applied in this context by Lieb [5]. The case $n=2$ and $a=b=1$ plays a similar role in investigating how many electrons a magnetic quantum dot in a graphene layer can bound and was proved and applied in that context (Handrek and Siedentop [2]).

Proof. For the proof we can assume that $a \leq b$, since, if not, we use the Fourier transform to exchange the role of $\mathfrak{p}$ and $\mathfrak{q}$.

We first treat the case, that $a<2$. In this case we follow the strategy of [2] and use the identity (20). Thus, by polarization

$$\begin{aligned} t :=&\frac{1}{2}(\psi, (|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a)\psi)\\ =&\alpha_{a,n}\Re\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{(\overline{\psi(x)}-\overline{\psi(y)})(|x|^b\psi(x)-|y|^b\psi(y))}{|x-y|^{n+a}}\\ =&\alpha_{a,n}\Re\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{2|x|^b|\psi(x)|^2-(|x|^b+|y|^b)\overline{\psi(x)}\psi(y)}{|x-y|^{n+a}}. \end{aligned} \tag{6}$$

Now, setting $\psi = g/|.|^{(n+b-a)/2}$ we get

$$\begin{aligned} \frac{t}{\alpha_{a,n}} =& \int\mathrm{d}x\int\mathrm{d}y\frac{\frac{|x|^b|g(x)|^2}{|x|^{n+b-a}}+\frac{|y|^b|g(y)|^2}{|y|^{n+b-a}}-\frac{2\Re\overline{g(x)}g(y)|y|^b}{|x|^{(n+b-a)/2}|y|^{(n+b-a)/2}}}{|x-y|^{n+a}}\\ =&\int_{\mathbb{R}^n}\frac{\mathrm{d}x}{|x|^n}|g(x)|^2\int_{\mathbb{R}^n}\mathrm{d}y\frac{2|x|^a-|x|^{\frac{n+a-b}{2}}|y|^{\frac{-n+a+b}{2}}-|x|^{\frac{n+a+b}{2}}|y|^{\frac{-n+a-b}{2}}}{|x-y|^{n+a}}\\ &+\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{(|x|^b+|y|^b)|g(x)-g(y)|^2}{2|x|^{(n+b-a)/2}|x-y|^{n+a}|y|^{(n+b-a)/2}}. \end{aligned} \tag{7}$$

At this point we could simply drop the last term, since it is positive. However, with minimal extra effort we estimate the last term using $|x|^b+|y|^b \geq 2|x|^{b/2}|y|^{b/2}$ and obtain using (21)

$$
\begin{aligned}
t \geq & \alpha_{a,n}\int_{\mathbb{R}^n}\mathrm{d}x\frac{|g(x)|^2}{|x|^n}\int_{\mathbb{R}^n}\mathrm{d}y\frac{2-|y|^{\frac{a+b-n}{2}}-|y|^{\frac{-n+a-b}{2}}}{(2|y|)^{\frac{n+a}{2}}\left(\frac{|y|+|y|^{-1}}{2}-\omega\cdot\mathfrak{e}\right)^{\frac{n+a}{2}}}+(\psi,\mathcal{H}_{a,n}\psi)\\
(8)\quad = & \frac{\alpha_{a,n}}{2^{\frac{n+a}{2}}}\int_{\mathbb{R}^n}\mathrm{d}x\frac{|g(x)|^2}{|x|^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}y}{|y|^n}\frac{2|y|^{\frac{n-a}{2}}-|y|^{b/2}-|y|^{-b/2}}{(\frac{|y|+|y|^{-1}}{2}-\omega\cdot\mathfrak{e})^{\frac{n+a}{2}}}+(\psi,\mathcal{H}_{a,n}\psi)\\
= & \frac{\alpha_{a,n}}{2^{\frac{n+a}{2}}}\int_{\mathbb{R}^n}\mathrm{d}x\frac{|g(x)|^2}{|x|^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}y}{|y|^n}\frac{|y|^{\frac{n-a}{2}}+|y|^{-\frac{n-a}{2}}-|y|^{\frac{b}{2}}-|y|^{-\frac{b}{2}}}{(\frac{|y|+|y|^{-1}}{2}-\omega\cdot\mathfrak{e})^{\frac{n+a}{2}}}+(\psi,\mathcal{H}_{a,n}\psi)\\
> & 0
\end{aligned}
$$

assuming – in the last line – that ψ is not identical zero. The positivity, i.e., the last inequality, follows from the positivity of the numerator of the last integral which is a consequence of the fact that the function $f(\alpha) := r^\alpha + r^{-\alpha}$ is strictly monotone increasing for positive r and $n - a \geq b$.

We now supply the missing case that $\min\{a, b\} = 2$. Again we may assume that $a \leq b$ without loss of generality. An easy calculation shows

$$
\begin{aligned}
& \frac{1}{2}|\mathfrak{p}|^2|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^2\\
(9)\quad = & |\mathfrak{q}|^{\frac{b}{2}}\left(|\mathfrak{p}|^2+\frac{1}{2}|\mathfrak{q}|^{-\frac{b}{2}}[[|\mathfrak{p}|^2,|\mathfrak{q}|^{\frac{b}{2}}],|\mathfrak{q}|^{\frac{b}{2}}]|\mathfrak{q}|^{-\frac{b}{2}}\right)|\mathfrak{q}|^{\frac{b}{2}}\\
= & |\mathfrak{q}|^{\frac{b}{2}}\left(|\mathfrak{p}|^2-\frac{b^2}{4}|\mathfrak{q}|^{-2}\right)|\mathfrak{q}|^{\frac{b}{2}}\\
\geq & |\mathfrak{q}|^{\frac{b}{2}}\mathcal{H}_{2,n}|\mathfrak{q}|^{\frac{b}{2}} > 0,
\end{aligned}
$$

because $b \leq n - 2$. Since the first inequality is actually an equality in the case $b = n - 2$, it shows that our assumption $a + b \leq n$ is critical, since Herbst's inequalities are sharp. □

3. Ground State Representation

The result of the previous section can be viewed as a warmup for the following result

Theorem 2. *Assume $a, b \in (0,\infty)$ with $a + b \leq n$, $\min\{a, b\} \in (0, 2)$, and $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Then*

$$
\begin{aligned}
(10)\quad (\psi,(\mathcal{J}_{a,b,n}-L_{a,b,n}|\mathfrak{q}|^{b-a})\psi) = & (\psi,|\mathfrak{q}|^{\frac{b}{2}}\mathcal{H}_{a,n}|\mathfrak{q}|^{\frac{b}{2}}\psi)\\
& +\alpha_{a,n}\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{(|x|^{\frac{b}{2}}-|y|^{\frac{b}{2}})^2\,|\psi(x)|x|^\gamma-\psi(y)|y|^\gamma|^2}{2|x|^\gamma|x-y|^{n+a}|y|^\gamma}
\end{aligned}
$$

where $\gamma = (n + b - a)/2$ and

$$
(11)\qquad L_{a,b,n} = 2^a\frac{\Gamma(\frac{n-b+a}{4})\Gamma(\frac{n+b+a}{4})}{\Gamma(\frac{n+b-a}{4})\Gamma(\frac{n-b-a}{4})}.
$$

Monotonicity of $L_{a,b,n}$: Note that $L_{a,b,n}$ is a strictly montone decreasing function in b on the interval $[0, n - a]$ and vanishes at $n - a$. The second claim is obvious, since $\lim_{x\to 0_+}\Gamma(x) = 0$. For the first claim we use the log convexity of the Gamma function (Bohr and Mollerup theorem).

Sharpness: Formula (10) implies the inequality

$$
(12)\qquad \mathcal{J}_{a,b,n} > L_{a,b,n}|\mathfrak{q}|^{b-a} + \mathcal{H}_{a,n}
$$

is strict under the assumptions of the theorem, since the remainder term in (10) vanishes, if and only if $\psi(x) = c|x|^{-\gamma}$ which is only in L^2 when $c = 0$.

However, the remainder can be made arbitrarily small by a smooth cut-off tending to infinity.

If $a=2$ equality holds in (12) because of the calculation (9).

Proof. Pick $\gamma := \frac{n+b-a}{2}$. By Fourier transform of $|\cdot|^{-\alpha}$ (see (18)), we know that

$$
\begin{aligned}
&(|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma}) \\
=\ & \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{d}\xi |\xi|^a \Big(\overline{(|\psi(x)|^2|x|^\gamma)^\wedge(\xi)}(|x|^{b-\gamma})^\wedge(\xi) \\
& \quad + \overline{(|\psi(x)|^2|x|^{\gamma+b})^\wedge(\xi)}(|x|^{-\gamma})^\wedge(\xi)\Big) \\
=\ & \frac{1}{2}\Big(\frac{B_{n-(\gamma-b)}}{B_{\gamma-b}}\int_{\mathbb{R}^n}\mathrm{d}\xi|\xi|^{a-n+(\gamma-b)}(|\psi(x)|^2|x|^\gamma)^\wedge(\xi) \\
& \quad + \frac{B_{n-\gamma}}{B_\gamma}\int_{\mathbb{R}^n}\mathrm{d}\xi|\xi|^{a-n+\gamma}(|\psi(x)|^2|x|^{\gamma+b})^\wedge(\xi)\Big) \\
=\ & \frac{1}{2}\Big(\frac{B_{a+\gamma-b}B_{n-(\gamma-b)}}{B_{n-a-(\gamma-b)}B_{\gamma-b}} + \frac{B_{a+\gamma}B_{n-\gamma}}{B_{n-a-\gamma}B_\gamma}\Big)\int_{\mathbb{R}^n}\mathrm{d}x|\psi(x)|^2|x|^{b-a} \\
=\ & 2^a\frac{\Gamma(\frac{n-b+a}{4})\Gamma(\frac{n+b+a)}{4})}{\Gamma(\frac{n+b-a}{4})\Gamma(\frac{n-b-a}{4})}\int_{\mathbb{R}^n}\mathrm{d}x|\psi(x)|^2|x|^{b-a} \\
=\ & L_{a,b,n}\int_{\mathbb{R}^n}\mathrm{d}x|\psi(x)|^2|x|^{b-a}.
\end{aligned}
\tag{13}
$$

We have a similar computation for the operator $|\mathfrak{q}|^{\frac{b}{2}}|\mathfrak{p}|^a|\mathfrak{q}|^{\frac{b}{2}}$,

$$
\begin{aligned}
&(|\psi|^2|x|^\gamma, |\mathfrak{q}|^{\frac{b}{2}}|\mathfrak{p}|^a|\mathfrak{q}|^{\frac{b}{2}}|x|^{-\gamma}) \\
=\ & \int_{\mathbb{R}^n}\mathrm{d}\xi|\xi|^a\overline{(|\psi(x)|^2|x|^{\frac{b}{2}+\gamma})^\wedge(\xi)}(|x|^{\frac{b}{2}-\gamma})^\wedge(\xi) \\
=\ & \frac{B_{n-(\gamma-\frac{b}{2})}}{B_{\gamma-\frac{b}{2}}}\int_{\mathbb{R}^n}\mathrm{d}\xi|\xi|^{a-n+(\gamma-\frac{b}{2})}(|\psi(x)|^2|x|^{\frac{b}{2}+\gamma})^\wedge(\xi) \\
=\ & \frac{B_{a+\gamma-\frac{b}{2}}B_{n-(\gamma-\frac{b}{2})}}{B_{n-a-(\gamma-\frac{b}{2})}B_{\gamma-\frac{b}{2}}}\int_{rz^n}\mathrm{d}x|\psi(x)|^2|x|^{b-a} \\
=\ & 2^a\left(\frac{\Gamma(\frac{n+a}{4})}{\Gamma(\frac{n-a}{4})}\right)^2\int_{rz^n}\mathrm{d}x|\psi(x)|^2|x|^{b-a}.
\end{aligned}
\tag{14}
$$

On the other hand, by using (20) and polarization we can compute the above quantities again in the following,

$$
\begin{aligned}
&(|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma}) \\
=\ & \frac{1}{2}\alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}\Big\{(|\psi(x)|^2|x|^\gamma - |\psi(y)|^2|y|^\gamma)(|x|^{b-\gamma}-|y|^{b-\gamma}) \\
& \quad + (|\psi(x)|^2|x|^{b+\gamma}-|\psi(y)|^2|y|^{b+\gamma})(|x|^{-\gamma}-|y|^{-\gamma})\Big\} \\
=\ & \frac{1}{2}\alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}\Big\{2|\psi(x)|^2|x|^b + 2|\psi(y)|^2|y|^b \\
& \quad -|\psi(x)|^2|x|^\gamma|y|^{b-\gamma} - |\psi(y)|^2|x|^{b-\gamma}|y|^\gamma \\
& \quad -|\psi(x)|^2|x|^{b+\gamma}|y|^{-\gamma} - |\psi(y)|^2|x|^{-\gamma}|y|^{b+\gamma}\Big\}.
\end{aligned}
\tag{15}
$$

By (6) and subtraction and addition of $2\Re\overline{\psi(x)}\psi(y)|y|^b + 2\Re\psi(x)\overline{\psi(y)}|x|^b$ in the above braces we get

$$
\begin{aligned}
(16)\qquad & (|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma})\\
= \quad & \alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}(\overline{\psi(x)}-\overline{\psi(y)})(|x|^b\psi(x)-|y|^b\psi(y))\\
& +\frac{1}{2}\alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}\Big\{2\Re\overline{\psi(x)}\psi(y)|y|^b + 2\Re\psi(x)\overline{\psi(y)}|x|^b\\
& \quad -|\psi(x)|^2|x|^\gamma|y|^{b-\gamma} - |\psi(y)|^2|x|^{b-\gamma}|y|^\gamma\\
& \quad -|\psi(x)|^2|x|^{b+\gamma}|y|^{-\gamma} - |\psi(y)|^2|x|^{-\gamma}|y|^{b+\gamma}\Big\}\\
= \quad & (\psi, \mathcal{J}_{a,b,n}\psi) + \alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\frac{1}{2}(|x|^b+|y|^b)\mathrm{d}x\mathrm{d}y}{|x|^\gamma|x-y|^{n+a}|y|^\gamma}\left|\psi(x)|x|^\gamma - \psi(y)|y|^\gamma\right|^2.
\end{aligned}
$$

The sesquilinear form of $|\mathfrak{q}|^{\frac{b}{2}}|\mathfrak{p}|^a|\mathfrak{q}|^{\frac{b}{2}}$ is

$$
\begin{aligned}
(17)\qquad & (|\psi|^2|x|^\gamma, |\mathfrak{q}|^{\frac{b}{2}}|\mathfrak{p}|^a|\mathfrak{q}|^{\frac{b}{2}}|x|^{-\gamma})\\
= \quad & \alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}(|\psi(x)|^2|x|^{\frac{b}{2}+\gamma} - |\psi(y)|^2|y|^{\frac{b}{2}+\gamma})(|x|^{\frac{b}{2}-\gamma} - |y|^{\frac{b}{2}-\gamma})\\
= \quad & \alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+a}}\Big\{|\psi(x)|^2|x|^b + |\psi(y)|^2|y|^b\\
& \quad -2\Re\overline{\psi(x)}\psi(y)|x|^{\frac{b}{2}}|y|^{\frac{b}{2}} + 2\Re\overline{\psi(x)}\psi(y)|x|^{\frac{b}{2}}|y|^{\frac{b}{2}}\\
& \quad -|\psi(x)|^2|x|^{\frac{b}{2}+\gamma}|y|^{\frac{b}{2}-\gamma} - |\psi(y)|^2|x|^{\frac{b}{2}-\gamma}|y|^{\frac{b}{2}+\gamma}\Big\}\\
= \quad & (\psi, |\mathfrak{q}|^{\frac{b}{2}}|\mathfrak{p}|^a|\mathfrak{q}|^{\frac{b}{2}}\psi) - \alpha_{a,n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathrm{d}x\mathrm{d}y\frac{\left|\psi(x)|x|^\gamma - \psi(y)|y|^\gamma\right|^2}{|x|^{\gamma-\frac{b}{2}}|x-y|^{n+a}|y|^{\gamma-\frac{b}{2}}}.
\end{aligned}
$$

A combination of the computions (13) to (17) and the ground state representation (21)gives us the desired result. □

Appendix A. Auxiliary Facts

For the reader's convenience we collect some helpful known facts:

Fourier transforms of powers: For $\alpha \in (0, n)$

$$B_\alpha \mathcal{F}(|\cdot|^{-\alpha}) = B_{n-\alpha}|\cdot|^{-n+\alpha} \tag{18}$$

with $B_\alpha := 2^{\frac{\alpha}{2}}\Gamma(\alpha/2)$ (see, e.g., Lieb and Loss [6, Theorem 5.9])

Generalized Hardy Inequalities (Herbst [4]): Assume $a \in (0, n)$. Then, on $H^{a/2}(\mathbb{R}^n)$

$$\mathcal{H}_{a,n} := |\mathfrak{p}|^a - 2^a\left[\frac{\Gamma\left(\frac{n+a}{4}\right)}{\Gamma\left(\frac{n-a}{4}\right)}\right]^2|\mathfrak{q}|^{-a} > 0. \tag{19}$$

The inequality is sharp in the sense that there is no smaller constant in front of $|\mathfrak{q}|^{-a}$ which allows this inequality on $C_0^\infty(\mathbb{R}^n)$.

Hardy's classical inequality is obtained for $a = 2$, Kato's inequality is the case $a = 1$.

Fractional Laplacian: For $\psi \in H^{a/2}(\mathbb{R}^n)$

$$(\psi, |\mathfrak{p}|^a\psi) = \alpha_{a,n}\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+a}} \tag{20}$$

with

$$\alpha_{a,n} = \frac{2^{a-1}}{\pi^{n/2}}\frac{\Gamma(\frac{n+a}{2})}{|\Gamma(-\frac{a}{2})|}$$

(Frank et al [1, Formula (3.2)]).

Ground State transformed Hardy inequality (Frank et al. [1]**):** For all $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$(\psi, \mathcal{H}_{a,n}\psi) = \alpha_{a,n} \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y \frac{\left|\psi(x)|x|^{\frac{n-a}{2}} - \psi(y)|y|^{\frac{n-a}{2}}\right|^2}{|x|^{\frac{n-a}{2}}|x-y|^{n+a}|y|^{\frac{n-a}{2}}}. \tag{21}$$

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