THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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A. Pushnitski, G. Raikov and C. Villegas-Blas

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PUSHNITSKI, A. B.
RAIKOV, G. .D.
VILLEGAS-BLAS, C.

A TRACE FORMULA FOR EIGENVALUE CLUSTERS OF THE PERTURBED LANDAU HAMILTONIAN

A. B. PUSHNITSKI

Department of Mathematics, King's College London,
London, WC2R 2LS, United Kingdom
E-mail: alexander.pushnitski@kcl.ac.uk

G. D. RAIKOV

Facultad de Matemáticas, Pontificia Universidad Católica de Chile,
Santiago de Chile
E-mail: graikov@mat.puc.cl

C. VILLEGAS-BLAS

Instituto de Matemáticas, Universidad Nacional Autonóma de México,
Cuernavaca, Mexico
E-mail: villegas@matcuer.unam.mx

We consider the Landau Hamiltonian perturbed by an electric potential V which decays sufficiently fast at infinity. The spectrum of the perturbed Hamiltonian consists of clusters of eigenvalues which accumulate to the Landau levels. Applying a suitable version of the anti-Wick quantization, we investigate the asymptotic distribution of the eigenvalues within a given cluster as the number of the cluster tends to infinity. We obtain an explicit description of the asymptotic density of the eigenvalues in terms of the Radon transform of V.

Keywords: Perturbed Landau Hamiltonian; Asymptotic density for eigenvalue clusters; Anti-Wick quantization.

1. Introduction

Consider the operator

$$H_0 := \left(-i\frac{\partial}{\partial x} + \frac{B}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{B}{2}x\right)^2,$$

self-adjoint in $L^2(\mathbb{R}^2)$. The operator H_0 is the Landau Hamiltonian, i.e. the 2D Schrödinger operator with constant magnetic field $B > 0$. The spectrum of H_0 consists of eigenvalues called Landau levels $\lambda_q := B(2q+1)$, $q \in \mathbb{Z}_+ := 0, 1, 2, \ldots$.

The multiplicity of each of them is infinite, and so

$$\sigma(H_0) = \sigma_{\rm ess}(H_0) = \bigcup_{q=0}^{\infty}\{\lambda_q\}.$$

Next, let $V \in C(\mathbb{R}^2)$ be a real-valued function which satisfies the estimate

$$|V(\mathbf{x})| \leq C\langle \mathbf{x}\rangle^{-\rho}, \quad \mathbf{x} \in \mathbb{R}^2, \quad \rho > 1, \tag{1}$$

where $\langle \mathbf{x}\rangle := (1+|\mathbf{x}|^2)^{1/2}$. Set $H := H_0 + V$. Since V decays at infinity, and hence is a relatively compact perturbation of H_0, we have

$$\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H_0) = \bigcup_{q=0}^{\infty}\{\lambda_q\}.$$

However, while the Landau levels $\{\lambda_q\}_{q\in\mathbb{Z}_+}$ are eigenvalues of infinite multiplicity of the unperturbed operator H_0, typically they play the role of accumulation points of the discrete eigenvalues of H. Thus, the spectrum of H consists of clusters of discrete eigenvalues around the Landau levels.

The purpose of the note is to describe the high-energy behaviour of the eigenvalue clusters of H. First, we estimate the size of the qth cluster and show that it shrinks to the Landau level λ_q as $q \to \infty$ (see Theorem 2.1). Our main result is Theorem 2.2 where we find the asymptotic density of the eigenvalues of H written explicitly in the terms of the Radon transform of V. A detailed description of these results and their proofs are contained in [1].

2. Main results

Theorem 2.1. *Assume* (1)*; then there exists a constant* $C > 0$ *such that for all* $q \in \mathbb{Z}_+$ *one has*

$$\sigma(H) \cap [\lambda_q - B, \lambda_q + B] \subset (\lambda_q - C\lambda_q^{-1/2},\, \lambda_q + C\lambda_q^{-1/2}), \tag{2}$$

i.e. for large q *the eigenvalue clusters shrink to the Landau levels at the rate* $q^{-1/2}$ *as* $q \to \infty$.

Remarks.

(i) The estimate $O(\lambda_q^{-1/2})$ *cannot be improved; this follows from Theorem 2.2 below.*

(ii) Theorem 2.1 was proven first in [2] for $V \in C_0^{\infty}(\mathbb{R}^2)$*. The proof in [1] not only covers the case of more general potentials* V*, but also is based on different ideas than those of [2].*

Fix the eigenvalue cluster q; let us define the rescaled in accordance with (2) eigenvalue counting measure μ_q of the qth cluster as follows. For any bounded interval $[\alpha,\beta] \subset \mathbb{R}\setminus\{0\}$ we set

$$\mu_q([\alpha,\beta]) = \sum_{\lambda_q+\alpha\lambda_q^{-1/2} \leq \lambda \leq \lambda_q+\beta\lambda_q^{-1/2}} \dim \operatorname{Ker}(H - \lambda I);$$

it is easy to see that for any fixed bounded interval $[\alpha,\beta] \subset \mathbb{R}\backslash\{0\}$ and all sufficiently large values of q, the measure $\mu_q([\alpha,\beta])$ is finite. Below we study the asymptotics of the measure μ_q as $q \to \infty$. For $\omega = (\omega_1,\omega_2) \in \mathbb{S}^1$, set $\omega^\perp = (-\omega_2,\omega_1)$, and introduce the Radon transform

$$\widetilde{V}(\omega,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(b\omega + t\omega^\perp)dt, \quad \omega \in \mathbb{S}^1, \quad b \in \mathbb{R},$$

of the potential V. Then

$$|\widetilde{V}|(\omega,b) \le C_\rho \|V\|_{X_\rho} \langle b\rangle^{1-\rho}, \quad b \in \mathbb{R}, \tag{3}$$

where $\|V\|_{X_\rho} = \sup_{\mathbf{x}\in\mathbb{R}^2} \langle \mathbf{x}\rangle^\rho |V(\mathbf{x})|$. Let us define the limiting measure μ as follows: for any interval $[\alpha,\beta] \subset \mathbb{R} \setminus \{0\}$ we set

$$\mu([\alpha,\beta]) = \frac{1}{2\pi}\left|\left\{(\omega,b) \in \mathbb{S}^1 \times \mathbb{R} \,|\, \alpha \le B\widetilde{V}(\omega,b) \le \beta\right\}\right|$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{S}^1 \times \mathbb{R}$. Evidently, for any bounded interval $[\alpha,\beta] \subset \mathbb{R} \setminus \{0\}$ we have $\mu([\alpha,\beta]) < \infty$. Moreover, estimate (3) implies that μ has a bounded support in $\mathbb{R}$. Our main result is:

Theorem 2.2. *Let $V \in C(\mathbb{R}^2)$ be a continuous function that satisfies* (1)*. Then, for any function $\varrho \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have*

$$\lim_{q\to\infty} \lambda_q^{-1/2} \int_{\mathbb{R}} \varrho(\lambda)d\mu_q(\lambda) = \int_{\mathbb{R}} \varrho(\lambda)d\mu(\lambda). \tag{4}$$

Asymptotics (4) can be more explicitly written as

$$\lim_{q\to\infty} \lambda_q^{-1/2}\, \mathrm{Tr}\, \varrho(\sqrt{\lambda}_q(H-\lambda_q)) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \varrho(B\widetilde{V}(\omega,b))\, db\, d\omega. \tag{5}$$

By standard approximation arguments, (5) can be extended to a wider class of continuous functions ϱ. Moreover, it follows from Theorem 2.2 that if $[\alpha,\beta] \subset \mathbb{R}\backslash\{0\}$, and $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$, then

$$\lim_{q\to\infty} \lambda_q^{-1/2}\mu_q([\alpha,\beta]) = \mu([\alpha,\beta]).$$

3. Semiclassical interpretation

Consider the classical Hamiltonian function

$$\mathcal{H}(\boldsymbol{\xi},\mathbf{x}) = (\xi + \tfrac{1}{2}By)^2 + (\eta - \tfrac{1}{2}Bx)^2, \quad \boldsymbol{\xi} := (\xi,\eta) \in \mathbb{R}^2, \quad \mathbf{x} := (x,y) \in \mathbb{R}^2,$$

in the phase space $T^*\mathbb{R}^2 = \mathbb{R}^4$ with the standard symplectic form. The projections onto the configuration space of the orbits of the Hamiltonian flow of $\mathcal{H}$ are circles of radius $\sqrt{E}/B$, where $E > 0$ is the value of the energy corresponding to the orbit. The classical particles move along these circles with period $T_B = \pi/B$. The set of these orbits can be parameterized by the energy $E > 0$ and the center $\mathbf{c} \in \mathbb{R}^2$ of the circle. Denote the path in the configuration space corresponding to such an orbit by $\gamma(\mathbf{c}, E, t)$, $t \in [0, T_B)$, and set

$$\langle V \rangle(\mathbf{c}, E) = \frac{1}{T_B} \int_0^{T_B} V(\gamma(\mathbf{c}, E, t))dt, \quad T_B = \pi/B. \tag{6}$$

For an energy $E > 0$, consider the set M_E of all orbits with this energy. The set M_E is a smooth manifold with coordinates $\mathbf{c} \in \mathbb{R}^2$. It can be considered as the quotient of the constant energy surface

$$\Sigma_E = \{(\boldsymbol{\xi}, \mathbf{x}) \in \mathbb{R}^4 \mid \mathcal{H}(\boldsymbol{\xi}, \mathbf{x}) = E\}$$

with respect to the flow of $\mathcal{H}$. Restricting the standard Lebesgue measure of $\mathbb{R}^4$ to Σ_E and then taking the quotient, we obtain the measure $B\, dc_1\, dc_2$ on M_E. An elementary calculation shows that the r.h.s. of (5) can be rewritten as

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \varrho(B\widetilde{V}(\omega, b))db\, d\omega = \frac{1}{2\pi} \lim_{E\to\infty} \frac{1}{\sqrt{E}} \int_{\mathbb{R}^2} \varrho(\sqrt{E}\, \langle V \rangle(\mathbf{c}, E))\, B\, dc_1\, dc_2. \tag{7}$$

This calculation is based on the fact that as $E \to \infty$, the radius $\sqrt{E}/B$ of the classical orbits tends to infinity and therefore the orbits approximate straight lines on any compact domain of the configuration space. Given (7), we can rewrite (5) as

$$\lim_{q\to\infty} \frac{1}{\sqrt{\lambda_q}} \operatorname{Tr} \varrho(\sqrt{\lambda_q}(H - \lambda_q)) = \frac{1}{2\pi} \lim_{E\to\infty} \frac{1}{\sqrt{E}} \int_{\mathbb{R}^2} \varrho(\sqrt{E}\, \langle V \rangle(\mathbf{c}, E))\, B\, dc_1\, dc_2$$

which agrees with the semiclassical intuition, and more precisely with the "averaging principle" for systems close to integrable ones. This principle states that a good approximation is obtained if one replaces the original perturbation by the one which results by averaging the original perturbation along the orbits of the free dynamics (see [3, Section 52]).

4. Related results

4.1. *Asymptotics of eigenvalue clusters for manifolds with closed geodesics*

In [4] A. Weinstein considered the operator $-\Delta_{\mathcal{M}} + V$, where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on a compact Riemannian manifold $\mathcal{M}$ with periodic bicharacteristic flow (e.g. a sphere), and $V \in C(\mathcal{M}; \mathbb{R})$. In this case, all eigenvalues of $\Delta_{\mathcal{M}}$ have finite multiplicities which however grow with the eigenvalue number. Adding the perturbation V creates clusters of eigenvalues. A. Weinstein proved that the asymptotic density of eigenvalues in these clusters can be described by the density function obtained by averaging V along the closed geodesics on $\mathcal{M}$. Let us illustrate these results with the case $\mathcal{M} = \mathbb{S}^2$. The eigenvalues of $-\Delta_{\mathbb{S}^2}$ are $\lambda_q = q(q + 1)$, $q \in \mathbb{Z}_+$, and their multiplicities are $d_q = 2q + 1$. For $V \in C(\mathbb{S}^2; \mathbb{R})$ set

$$\widetilde{V}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} V(\mathcal{C}_\omega(s))ds, \quad \omega \in \mathbb{S}^2,$$

where $\mathcal{C}_\omega(s) \in \mathbb{S}^2$ is the great circle orthogonal to ω, and s is the arc length on this circle. Then for each $\varrho \in C_0^\infty(\mathbb{S}^2;\mathbb{R})$ we have

$$\lim_{q\to\infty} \frac{\operatorname{Tr} \varrho(-\Delta_{\mathbb{S}^2} + V - \lambda_q)}{d_q} = \int_{\mathbb{S}^2} \varrho(\widetilde{V}(\omega))dS(\omega) \tag{8}$$

where dS is the normalized Lebesgue measure on $\mathbb{S}^2$. Since $\mathbb{S}^2$ can be identified with its set of oriented geodesics $\mathcal{G}$, the r.h.s. of (8) can be interpreted as an integral with respect to the $SO(3)$–invariant normalized measure on $\mathcal{G}$. This result admits extensions to the case $\mathcal{M} = \mathbb{S}^d$ with $d > 2$, and, more generally, to the case where $\mathcal{M}$ is a compact symmetric manifold of rank 1 (see [4, 5]).

4.2. *Strong magnetic field asymptotics*

Let us compare our Theorem 2.2 with the asymptotics of the eigenvalues of H as $B \to \infty$. It was found in [6] that

$$\lim_{B\to\infty} B^{-1} \operatorname{Tr} \varrho(H - \lambda_q) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \varrho(V(\mathbf{x}))d\mathbf{x} = \int_{\mathbb{R}} \varrho(t)dm(t) \tag{9}$$

where $\varrho \in C_0^\infty(\mathbb{R} \setminus \{0\})$, $q \in \mathbb{Z}_+$, $V \in L^p(\mathbb{R}^2)$, $p > 1$, and

$$m([\alpha, \beta]) := \frac{1}{2\pi} \left|\left\{\mathbf{x} \in \mathbb{R}^d \,|\, \alpha \le V(\mathbf{x}) \le \beta\right\}\right|$$

for each interval $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$. Similarly to Theorem 2.2, asymptotic relation (9) admits a semiclassical interpretation which, however, is not identical with that of Theorem 2.2. In particular, as explained in [1], Theorem 2.2 retains "more quantum flavour" than (9), and hence its proof is technically much more involved.

4.3. *The spectral density of the scattering matrix for high energies*

In the recent work [7] inspired by [1], D. Bulger and A. Pushnitski considered the scattering matrix $S(\lambda)$, $\lambda > 0$, for the operator pair $(-\Delta + V, -\Delta)$ where Δ is the standard Laplacian in $L^2(\mathbb{R}^d)$, $d \ge 2$, and $V \in C(\mathbb{R}^d)$ is an electric potential which satisfies an estimate analogous to (1). Then the asymptotics as $\lambda \to \infty$ of the eigenvalues for $S(\lambda)$ can be written in terms of the X-ray transform of V in a manner similar to (4).

5. Sketch of the proofs of Theorems 2.1–2.2

5.1. *General outline*

Let P_q be the orthogonal projection in $L^2(\mathbb{R}^2)$ onto $\operatorname{Ker}(H_0 - \lambda_q)$. For $\ell \ge 1$, let S_ℓ be the Schatten-von Neumann class, with the norm $\|\cdot\|_\ell$; the usual operator norm is denoted by $\|\cdot\|$. Applying some fairly standard analytic tools, we reduce the proofs of Theorems 2.1 – 2.2 to the following statement:

Theorem 5.1. *Let V satisfy* (1) *and let $B_0 > 0$.*

(i) For some $C = C(B_0)$, we have

$$\sup_{q\geq 0}\sup_{B\geq B_0} \lambda_q^{1/2} B^{-1}\|P_q V P_q\| \leq C\|V\|_{X_\rho}. \tag{10}$$

(ii) For any integer $\ell > 1/(\rho-1)$, we have $P_q V P_q \in S_\ell$, and

$$\lim_{q\to\infty} \lambda_q^{(\ell-1)/2}\,\mathrm{Tr}(P_q V P_q)^\ell = \frac{B^\ell}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \widetilde{V}(\omega,b)^\ell\, db\, d\omega. \tag{11}$$

Remark. Estimate (10) could be of independent interest, for instance in the study of the spectral properties of perturbed 2D Schrödinger operators with constant electromagnetic fields (see [8]).

In what follows we describe briefly the main steps in the proof of Theorem 5.1.

5.2. *Unitary equivalence of $P_q V P_q$ and $\mathrm{Op}^w(V_B * \Psi_q)$*

Let $s : \mathbb{R}^{2d} \to \mathbb{C}$, $d \geq 1$, be a symbol from an appropriate class. Then $\mathrm{Op}^w(s) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the Weyl pseudo-differential operator (ΨDO) with symbol s, defined by

$$(\mathrm{Op}^w(s)u)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} s\left(\frac{x+x'}{2},\xi\right) e^{i(x-x')\cdot\xi} u(x')dx'd\xi$$

(see e.g. [9, 10]). Further, let $h := -\frac{d^2}{dx^2} + x^2$ be the harmonic oscillator self-adjoint in $L^2(\mathbb{R})$. We have

$$\sigma(h) = \bigcup_{q=0}^{\infty}\{2q+1\}, \quad \dim\mathrm{Ker}(h-(2q+1)) = 1, \quad q \in \mathbb{Z}_+.$$

Let p_q be the orthogonal projection onto $\mathrm{Ker}(h-(2q+1))$, $q \in \mathbb{Z}_+$. We have $p_q = |\varphi_q\rangle\langle\varphi_q|$ where

$$\varphi_q(x) := \frac{\mathrm{H}_q(x)e^{-x^2/2}}{(\sqrt{\pi}2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

and H_q are the Hermite polynomials (see [13, Chapter 22]). Let $2\pi\Psi_q$ be the Wigner function associated with φ_q, i.e. the Weyl symbol of p_q. We have

$$\Psi_q(x,\xi) = \frac{(-1)^q}{\pi}\,\mathrm{L}_q(2(x^2+\xi^2))e^{-(x^2+\xi^2)}, \quad (x,\xi) \in \mathbb{R}^2, \quad q \in \mathbb{Z}_+,$$

where L_q are the Laguerre polynomials (see [13, Chapter 22]). Finally, set

$$V_B(x,y) = V(-B^{-1/2}y, -B^{-1/2}x), \quad (x,y) \in \mathbb{R}^2.$$

Theorem 5.2. *There exists a unitary operator $\mathcal{U}_B : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ such that for each $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and each $q \in \mathbb{Z}_+$, we have*

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = p_q \otimes \mathrm{Op}^w(V_B * \Psi_q).$$

Remarks.

(i) The operator $\mathcal{U}_B$ is a metaplectic operator (see [9, Theorem 18.5.9]) corresponding to the linear symplectomorphism which maps the symbol of the operator H_0 into the symbol of $(Bh) \otimes I$. An explicit formula for $\mathcal{U}_B$ can be found in [1].

(ii) The operator $\mathrm{Op}^w(V_B * \Psi_0)$ is the standard ΨDO with anti-Wick symbol V_B (see [10, Section 24]). The operators $\mathrm{Op}^w(V_B * \Psi_q)$, $q \geq 1$, fall into the more general class of ΨDOs with contravariant symbols (see [11]).

5.3. *Reduction of* $\mathrm{Op}^w(V_B * \Psi_q)$ *to* $\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})$

For $r > 0$ define the distribution $\delta_r \in \mathcal{S}'(\mathbb{R}^2)$ by

$$\delta_r(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta) d\theta, \quad \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Lemma 5.1. *Let $V \in C_0^\infty(\mathbb{R}^2)$ and let $B_0 > 0$. Then there exists a constant $C = C(V, B_0)$ such that for any $q \in \mathbb{Z}_+$ and $B \geq B_0$ we have*

$$\| \mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}}) \| \leq C B \lambda_q^{-3/4}, \tag{12}$$

$$\| \mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}}) \|_2 \leq C B \lambda_q^{-3/4}. \tag{13}$$

Remark (Remark). Estimates (12)–(13) could be interpreted in the spirit of the equidistribution of the eigenfunctions φ_q of the harmonic oscillator h, i.e. the weak convergence of the Wigner function Ψ_q to the measure invariant with respect to the classical flow (see e.g. [12]). Note also that up to a change of variable, $V_B * \delta_{\sqrt{2q+1}}$ coincides with the classical average $\langle V \rangle(\cdot, \lambda_q)$ defined in (6).

5.4. *Norm estimate of* $\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})$

Lemma 5.2. *Let $V(\mathbf{x}) = \langle \mathbf{x} \rangle^{-\rho}$, $\mathbf{x} \in \mathbb{R}^2$, $\rho > 1$. Then*

$$\| \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}}) \| = O(\lambda_q^{-1/2}), \quad q \to \infty.$$

The proof of Lemma 5.2 is based on well known estimates of the norms of Weyl ΨDOs (see [14]). Theorem 5.1 (i) follows now from Theorem 5.2, estimate (12), and Lemma 5.2.

5.5. *Asymptotics of traces*

Lemma 5.3. *Let $V \in C_0^\infty(\mathbb{R}^2)$. Then for each $\ell \in \mathbb{N}$ we have*

$$\lim_{q \to \infty} \lambda_q^{(\ell-1)/2} \operatorname{Tr} \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})^\ell = \frac{B^\ell}{2\pi} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \widetilde{V}(\omega, b)^\ell \, db \, d\omega.$$

The proof of Lemma 5.3 is based on the usual composition formula for Weyl ΨDOs (see e.g. [10]), and a subsequent application of the stationary phase method (see e.g. [15]). Theorem 5.1 (ii) now follows from Theorem 5.2, estimate (13), Lemma 5.3, and standard interpolation methods.

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