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L^1-ESTIMATES FOR THE CUBIC TERM OF THE BACKSCATTERING TRANSFORM

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Abstract. An analysis of the backscattering data for the Schrödinger operator in odd dimensions $n \geq 3$ motivates the introduction of the backscattering transform $B : C_0^\infty(\mathbb{R}^n;\mathbb{C}) \to C^\infty(\mathbb{R}^n;\mathbb{C})$. This is an entire analytic mapping and we write $Bv = \sum_1^\infty B_N v$ where $B_N v$ is the N:th order term in the power series expansion at $v = 0$. In this paper we obtain L^1-estimates for $\nabla^{n-2} B_3 v$ in terms of the L^1-norm of $\nabla^{n-2} v$.

1. Introduction

The present paper is a step in the analysis of continuity properties of the backscattering transform for the Schrödinger operator in odd dimensions $n > 1$, in L^1-Sobolev spaces.

In order to state the main result a brief description of the mathematical objects involved is necessary.

Consider $v \in C_0^\infty(\mathbf{R}^n)$ where $n \geq 3$ is odd. This means that v is smooth, compactly supported and complex-valued. We denote by $K_v(t)$, $t \geq 0$, the wave group associated to the operator

$$\Box_v = \partial_t^2 - \Delta_x - M_v,$$

where M_v is multiplication by $v(x)$. Thus $u(x,t) = (K_v(t)u_0)(x)$ is for every $u_0 \in C_0^\infty(\mathbf{R}^n)$ the unique solution in $C^1([0,\infty), L^2(\mathbf{R}^n))$ to the problem

$$\Box_v u(x,t) = 0, \quad u(x,0) = 0, \ (\partial_t u)(x,0) = u_0(x). \tag{1.1}$$

We have

$$K_v(t) = \sum_0^\infty K_N(t), \tag{1.2}$$

where the K_N are defined inductively by

$$K_0(t) = \sin(t|D|)/|D|,$$

and

$$K_N(t) = \int_0^t K_{N-1}(t-s) M_v K_0(s)\, ds, \quad t \geq 0, \tag{1.3}$$

where, as before, M_v is multiplication by v. Then $K_N(t)$ is a strongly continuous function of t with values in the space of continuous linear operators on $L^2(\mathbf{R}^n)$ and one has the estimate

$$\|K_N(t)\| \leq \|v\|_{L^\infty}^N t^{2N+1}/(2N+1)! \ .$$

The convergence of (1.2) follows from this estimate, and (1.1) is easily verified. It follows also that $K_v(t)$ is strongly continuous in t. We have $|x-y| \leq t$ in the support of $K_v(x,y,t)$. We use the notation $A(x,y)$ for the distribution kernel of an operator A continuous from $C_0^\infty(\mathbf{R}^l)$ to $\mathcal{D}'(\mathbf{R}^m)$ and we always identify such an operator A with its distribution kernel $A(x,y)$. Since n is odd it is true also that $|x-y|=t$ in the support of $K_0(x,y,t)$. This allows us to define the operator

$$G = G_v = \int_0^\infty K_v(t) M_v \dot{K}_0(t)\, dt, \tag{1.4}$$

where $\dot{K}_0(t)$ is the time derivative of $K_0(t)$. Then G is a continuous linear operator on $L^2_{\mathrm{cpt}}(\mathbf{R}^n)$ and we make the following definition.

Definition 1.1. Assume $v \in C_0^\infty(\mathbf{R}^n)$. Then the backscattering transform Bv of v is defined by

$$(Bv)(x) = v(x) + 2^n \int v(x-y) G_v(x-y, x+y)\, dy \tag{1.5}$$

where the integral is interpreted in the distribution sense.

Since v is compactly supported there is a bound for y in the support of $v(x-y)G_v(x-y,x+y)$ when x stays in a compact set. This implies that Bv is defined at least as a distribution. As we shall see soon Bv is a smooth function for any $v \in C_0^\infty(\mathbf{R}^n)$.

It is possible to extend Bv by continuity to a much larger class of potentials than $C_0^\infty(\mathbf{R}^n)$. In particular Bv may be defined when $v \in L^q_{\mathrm{cpt}}$ and q is sufficiently large.

To motivate our definition we consider the situation when $v \in C_0^\infty(\mathbf{R}^n)$ is real-valued so that the Schrödinger operator

$$H_v = H_0 - M_v, \quad H_0 = -\Delta$$

is a self-adjoint operator in $L^2(\mathbf{R}^n)$ with domain the Sobolev space $\mathcal{H}_{(2)}(\mathbf{R}^n)$. We recall that in this case the wave operators defined by

$$W_\pm = \lim_{t\to\pm\infty} e^{itH_v} e^{-itH_0}$$

exist and are complete. Since $H_v u$ and $H_0 u$ are real-valued when u is real-valued the complex conjugate of the distribution kernel of $e^{itH_v}e^{-itH_0}$ is obtained by replacing t by $-t$ in this expression. It follows that the distribution kernel of W_- is the complex conjugate of the distribution kernel of W_+, and hence

$$W = (W_+ + W_-)/2$$

has a real-valued distribution kernel. The operators W and G are connected through the formula

$$W = I + G + \sum_1^\mu f_j \otimes g_j \tag{1.6}$$

Here the $(f_j)_{j=1}^\mu$ are real-valued and form an orthonormal basis for the space spanned by the eigenvectors of H_v. The functions g_j are smooth and real-valued. (See Theorem 7.1 of [5].) When v is sufficiently small there are no eigenvalues and

$$(Bv)(x) = 2^n \int v(x-y) W(x-y, x+y)\, dy. \tag{1.7}$$

The right-hand side, which is interpreted in the distribution sense, is the real part of

$$2^n \int v(x-y)W_+(x-y,x+y)\,dy \tag{1.8}$$

This expression represents (after Fourier transformation) the backscattering part of the scattering matrix. (We refer to [5] for more details.)

There are several reasons why we want to work with Bv instead of the transform given by (1.8). The first reason is that we want to represent backscattering data in terms of functions that may be viewed as perturbations of the original potential v. Hence it is natural to require these functions to be real-valued. The second reason is the Bv turns out to be entire analytic in v. Thus we have a convergent expansion of Bv into an infinite sum $\sum_1^\infty B_N v$. It was proved in [6] and [1] that the degree of smoothness of $B_N v$ increases with N for potential in $L^q_{\mathrm{cpt}}(\mathbf{R}^n)$ as soon as q is sufficiently large. Thus information about the first terms in the power series expansion of Bv may carry important information about the smoothness properties of v. L^2-Sobolev estimates were given in [6], showing in particular that if v is assumed to be in an L^2-Sobolev space, the regularity of $B_N v$ increases with $N \geq 2$. Finally, the functions g_j in (1.6) are smooth regardless of the smoothness of v. Hence $R(x,y) = \sum_j f_j(x)g_j(y)$ is smooth in the second argument, and since

$$\int v(x-y)R(x-y,x+y)\,dy = \int v(y)R(y,2x-y)\,dy$$

we see that the right-hand side of (1.5) only differs from the real part of the expression (1.7) by a smooth term.

Estimates for B_N, $N \geq 2$, in L^1-Sobolev in the case when $n = 3$ were obtained by R. Lagergren in his thesis (see [4]). He proved that

$$\|\nabla B_N v\|_{L^1(\mathbf{R}^3)} \leq CN^3(8\pi)^{-N}\|\nabla v\|^N_{L^1(\mathbf{R}^3)},$$

where C is a constant that does not depend on N. We aim to understand if such an estimate could be valid for larger dimensions, starting here with the cubic term. (For the second order term see Section 2.1 below and the references there.) For reasons of homogeneity, the ∇v should be replaced by $\nabla^{n-2}v$. The main result of the present paper is Theorem 3.5.

In Section 2 we recall formulas from [5] and [1], the analyticity of B when defined on $C_0^\infty(\mathbf{R}^n)$, and discuss the quadratic term. Formulas for the cubic term are derived in Section 3, while Sections 4 and 5 contain all the technical weight of the proof of Theorem 3.5

Although it is not necessary we have found it convenient to allow complex-valued potentials in our analysis of Bv.

2. The power series representation of the backscattering transform

We shall need multi-linear versions of the transforms $B_N v$. Let $v_1, v_2, \ldots$ be a sequence in $C_0^\infty(\mathbf{R}^n)$ and define $K_N(t,\vec{v})$ inductively by

$$\begin{aligned} K_N(t,\vec{v}) &= K_N(t,v_1,\ldots,v_N) \\ &= \int_0^t K_{N-1}(t-s,v_1,\ldots,v_{N-1})M_{v_N}K_0(s)\,ds,\ 0 \leq t,\ N = 1,2,\ldots. \end{aligned} \tag{2.1}$$

Then

$$\|K_N(t,\vec{v})\| \le \Big(\prod_1^N \|v_j\|_{L^\infty}\Big) t^{2N+1}/(2N+1)!,$$

which implies that $K_v(t)$ is entire analytic in v when viewed as an element in the Banach space of continuous linear operators in L^2, since $K_N(t) = K_N(t,(v,\dots,v))$. We define

$$G_0 = I$$

and

$$\begin{aligned} G_N(\vec{v}) &= G_N(v_1,\dots,v_N) \\ &= \int_0^\infty K_{N-1}(t,v_1,\dots,v_{N-1}) M_{v_N} \dot{K}_0(t)\,dt, \quad N \ge 1, \end{aligned} \tag{2.2}$$

and set

$$\begin{aligned} B_N(\vec{v}) &= B_N(v_1,\dots,v_N) \\ &= 2^n \int v_N(x-y) G_{N-1}(x-y,x+y,v_1,\dots,v_{N-1})\,dy, \quad N \ge 1, \end{aligned} \tag{2.3}$$

where the integral is interpreted in the distribution sense and $G_N(x,y,\vec{v})$ denotes the distribution kernel of $G_N(\vec{v})$. We notice that $B_1(v) = v$.

Theorem 2.1. *Assume $v \in C_0^\infty(\mathbf{R}^n)$. Then $B(\lambda v)$ is an entire function of λ with values in $C^\infty(\mathbf{R}^n)$ with power series expansion $\sum_1^\infty \lambda^N B_N v$, where $B_N v = B_N(v,\dots,v)$.*

Proof. Let m be an arbitrary nonnegative integer. It follows from Theorem 11.1 in [5] that one may write, with an $M = M(m) \ge 0$,

$$K_{\lambda v}(x,y,t) = \sum_{|\alpha|\le M} (\partial_x - \partial_y)^\alpha U_{\alpha,\lambda}(x,y,t), \tag{2.4}$$

where the $U_{\alpha,\lambda}$ are entire analytic functions of λ with values in the space $C^m(\mathbf{R}^n \times \mathbf{R}^n \times \overline{\mathbf{R}_+})$. It follows from Leibniz' formula that the assertion about $K_v(t)$ is also true for $K_v(t)M_v$, and then it is easily seen that it is true also for $K_v(t)M_v\dot{K}_0(t)$. This is because $K_0(t)$ is a convolution operator with distribution kernel

$$k_0(x,t) = tp_n(t\partial_t) \int_{S^{n-1}} \delta(x - t\omega)\,d\omega$$

with p_n polynomial of degree $(n-3)/2$ (see formula (5.7) in [5]). If $T > 0$ and

$$G_{T,\lambda} = \lambda \int_0^T K_{\lambda v}(t) M_v \dot{K}_0(t)\,dt$$

it follows that

$$G_{T,\lambda}(x,y) = \sum_{|\alpha|\le M} (\partial_x - \partial_y)^\alpha g_{\alpha,T,\lambda}(x,y),$$

where the $g_{\alpha,T,\lambda}$ are entire analytic function of λ with values in $C^\mu(\mathbf{R}^n \times \mathbf{R}^n)$. Since v is compactly supported it follows from the support properties of the distribution kernels of $K_v(t)$ and $K_0(t)$ that for every open set Ω in $\mathbf{R}^n$ there is a positive constant $T = T(\Omega)$ such that

$$v(x-y)G_{\lambda v}(x-y,x+y) = v(x-y)G_{T,\lambda}(x-y,x+y) \quad \text{in } \Omega \times \mathbf{R}^n.$$

Hence, when $x \in \Omega$ and $T = T(\Omega)$, we have

$$\begin{aligned} B(\lambda v)(x) &= \lambda \int v(x-y) G_{T,\lambda}(x-y, x+y)\, dy \\ &= \lambda \int v(y) G_{T,\lambda}(y, 2x-y)\, dy \\ &= \lambda \sum_{|\alpha| \le M} \int v(y) \partial_y^\alpha g_{\alpha,T,\lambda}(y, 2x-y)\, dy \\ &= \lambda \sum_{|\alpha| \le M} (-1)^{|\alpha|} \int v^{(\alpha)}(y) g_{\alpha,T,\lambda}(y, 2x-y)\, dy. \end{aligned}$$

Since the right-hand side is an entire analytic function of λ with values in $C^m(\mathbf{R}^n)$ and m was arbitrary we have proved that $B(\lambda v)$ is an entire function of λ with values in $C^\infty(\mathbf{R}^n)$.

To finish the proof we recall that $K_{\lambda v}(t)$, considered as a strongly continuous function of λ with values in the Banach space of bounded operators in L^2, is entire analytic in λ with the power series expansion

$$K_{\lambda v}(t) = \sum_0^\infty \lambda^N K_N(t).$$

It follows that the distribution kernel of $G_{\lambda v}$ is entire analytic with the expansion

$$\sum_0^\infty \lambda^{N+1} G_{N+1},$$

where G_{N+1}, considered as an operator, is given by

$$G_{N+1} = \int_0^\infty K_N(t) M_v \dot{K}_0(t)\, dt.$$

This in turn implies that $B(\lambda v)$ has the expansion

$$B(\lambda v) = \lambda v + \sum_0^\infty \lambda^{N+2} B_{N+2} v,$$

which proves the last assertion of the theorem.

□

2.1. **The quadratic term of the backscattering transform.** The first glimpse into the nature of $B_N v$ is offered by some results, which we are going to recall now, concerning the quadratic term of the backscattering transform.

Let

$$E(x,y) = E_2(x,y) = 4^{-1}(i\pi)^{1-n}\delta^{(n-2)}(x^2-y^2) \tag{2.5}$$

be the unique solution of the ultra-hyperbolic operator $\Delta_x - \Delta_y$ satisfying the properties

$$E(x,y) = -E(y,x)$$

$$E(x,y) \quad \text{is separately rotation invariant in } x \text{ and } y.$$

We recall that

$$E(x,y) = -\int_0^\infty k_0(x,t)\dot{k}_0(y,t)\, dt, \tag{2.6}$$

where $k_0(x,t)$ is the convolution kernel of the operator $K_0(t)$. (See Corollary 10.2 and Theorem 10.4 in [5].) Then

$$\begin{aligned}(B_2v)(x) &= 2^n \iint v_x(y)v_x(z)E_2(y-z,y+z)\,dy\,dz \\ &= -2^{-1}(2\pi i)^{1-n}\iint \delta^{(n-2)}(y\cdot z)v_x(y)v_x(z)\,dy\,dz,\end{aligned} \tag{2.7}$$

when $v\in C_0^\infty(\mathbf{R}^n)$ and $v_x(y)=v(x+y)$. (See Corollary 10.7 and Theorem 10.8 in [5].) Using the latter representation it was proved (see Theorem 10.10 [5]) that

$$\|\nabla^{n-2}B_2v\|_{L^1}\le C_n\|\nabla^{n-2}v\|_{L^1}^2$$

when $v\in C_0^\infty(\mathbf{R}^n)$, where C_n is a constant that only depends on the dimension n ($n\ge 3$ and odd).

For L^2-Sobolev global estimates see [2], and see [3] for precise formulas in the rotation invariant case.

3. The cubic term of the backscattering transform

We shall first derive a formula for $B_3(\vec v)$, where $\vec v=(v_1,v_2,v_3)$, $v_j\in C_0^\infty(\mathbf{R}^n)$. For this we need some preparation.

We consider the space $(\mathbf{R}^n)^3$, with coordinates denoted (x_1,x_2,x_3) where $x_j\in\mathbf{R}^n$, and we write Δ_j for the Laplacian in the variables x_j.

The distribution kernel of

$$K_1(t,v_1)=\int_0^t K_0(t-s)M_{v_1}K_0(s)\,ds,\quad t\ge 0,$$

has the form

$$K_1(x,y,t,v_1)=\int_0^t\Big(\int_{\mathbf{R}^n}k_0(x-z_1,t-s)v_1(z_1)k_0(z_1-y,s)\,dz_1\Big)\,ds. \tag{3.1}$$

(Here and henceforth the integrals are considered in the distribution sense. One can easily see that the integrand above is smooth in z_1, s and t when viewed as a distribution in (x,y), and similar arguments will hold in all the considered cases.) We introduce

$$Q_2(x_1,x_2,t)=\int_0^t k_0(x_1,t-s)k_0(x_2,s)\,ds \tag{3.2}$$

and

$$E_3(x_1,x_2,x_3)=\int_0^\infty Q_2(x_1,x_2,t)\dot k_0(x_3,t)\,dt. \tag{3.3}$$

We notice that $|x_1|+|x_2|=t$ on the support of Q_2 and $|x_3|=t$ on the support of $\dot k_0$, hence

$$|x_1|+|x_2|=|x_3| \tag{3.4}$$

on the support of E_3. Also

$$E_3(x_1, x_2, x_3) \quad \text{is rotation invariant in each of } x_1,\ x_2,\ x_3, \tag{3.5}$$

$$E_3(x_1, x_2, x_3) \quad \text{is symmetric in } x_1,\ x_2. \tag{3.6}$$

Remark. The distribution E_3 is a fundamental solution of the constant coefficient differential operator $P_3 = (\Delta_3 - \Delta_1)(\Delta_3 - \Delta_2)$. Indeed, we see that

$$\partial_t^2 Q_2 - \Delta_1 Q_2 = k_0(x_2, t)\delta(x_1)$$

and using (2.6) we get

$$\begin{aligned}
\Delta_1 E_3(x_1, x_2, x_3) &= \int_0^\infty \Delta_1 Q_2(x_1, x_2, t)\dot{k}_0(x_3, t), dt \\
&= -\int_0^\infty \delta(x_1) k_0(x_2, t)\dot{k}_0(x_3, t)\, dt + \int_0^\infty \partial_t^2 Q_2(x_1, x_2, t)\dot{k}_0(x_3, t)\, dt \\
&= \delta(x_1)E(x_2, x_3) + \int_0^\infty Q_2(x_1, x_2, t)\partial_t^3 k_0(x_3, t)\, dt \\
&= \delta(x_1)E(x_2, x_3) + \Delta_3\Big(\int_0^\infty Q_2(x_1, x_2, t)\dot{k}_0(x_3, t)\, dt\Big).
\end{aligned}$$

It follows that

$$(\Delta_1 - \Delta_3)E_3(x_1, x_2, x_3) = \delta(x_1)E(x_2, x_3),$$

anh hence

$$P_3 E_3(x_1, x_2, x_3) = \delta(x_1)(\Delta_2 - \Delta_3)E(x_2, x_3) = \delta(x_1, x_2, x_3).$$

We can now state the first result.

Lemma 3.1. *For every v_1, v_2, $v_3 \in C_0^\infty(\mathbf{R}^n)$, $B_3(\vec{v})$ is a smooth compactly supported function and*
(3.7)

$$B_3(\vec{v})(x) = \iiint E_3(x_1, x_2, x_3)$$
$$(v_1)_x\Big(\frac{x_1 + x_2 + x_3}{2}\Big)(v_2)_x\Big(\frac{x_1 - x_2 + x_3}{2}\Big)(v_3)_x\Big(\frac{-x_1 + x_2 + x_3}{2}\Big)\, dx_1\, dx_2\, dx_3,$$

where $(v_j)_x(y) = v_j(x + y)$ and $\vec{v} = (v_1, v_2, v_3)$.

Proof. It is clearly enough to establish (3.7); the fact that $B_3(\vec{v})$ is smooth and compactly supported follows from this formula and (3.4). With the notation above we see that (3.1) becomes

$$K_1(x, y, t, v_1) = \int_{\mathbf{R}^n} v_1(z_1) Q_2(x - z_1, z_1 - y, t)\, dz_1.$$

Using this in (2.2) for $N = 2$ we get

$$G_2(x, y, (v_1, v_2))$$
$$= \int\int\int_0^\infty v_2(z_2)v_1(z_1)Q_2(x - z_1, z_1 - z_2, t)\dot{k}_0(z_2 - y, t)\, dt\, dz_1\, dz_2$$
$$= \int\int v_2(z_2)v_1(z_1)E_3(x - z_1, z_2 - z_1, z_2 - y)\, dz_2\, dz_1,$$

where we have used (3.3) in the last equality.

The expression of $B_3(\vec{v})$ is then (see (2.3))

$$B_3(\vec{v})(x) = 2^n \int v_3(z_3)G_2(z_3, 2x - z_3, (v_1, v_2))\, dz_3$$
$$= 2^n \int\int\int v_1(z_1)v_2(z_2)v_3(z_3)E_3(z_3 - z_1, z_1 - z_2, z_2 + z_3 - 2x)\, dz_1\, dz_2\, dz_3.$$

The relation (3.7) follows by changing variables $x_1 = z_1 - z_3$, $x_2 = z_1 - z_2$, $x_3 = z_2 + z_3 - 2x$. □

To go further we need an explicit expression for $E_3(x_1, x_2, x_3)$. We write

$$E_{j,k}(x_1, x_2, x_3) = E(x_j, x_k) \prod_{l \neq j,k} \delta(x_l), \qquad j, k = 1, 2, 3, \tag{3.8}$$

where $E(x, y)$ is the fundamental solution given in (2.5).

The next technical lemma will be useful also when obtaining a more explicit expression for $B_3 v$.

Lemma 3.2. *Assume* $\varphi \in C_0^\infty((\mathbf{R}^n)^3)$ *and set* $\tilde{\varphi}(x_1, x_2, x_3) = \varphi(x_2, x_1, x_3)$. *Then*

$$\langle \varphi, E_{1,2} * E_{2,3} \rangle = -\langle \tilde{\varphi}, E_{1,2} * E_{1,3} \rangle$$
$$= -4^{-1}(2\pi)^{2(1-n)} \int \cdots \int \delta^{(n-2)}(y_1 \cdot z_1)\delta^{(n-2)}(y_2 \cdot z_2)$$
$$\tilde{\varphi}(y_1 + y_2 - z_1 - z_2, y_1 + z_1, y_2 + z_2)\, dy_1\, dy_2\, dz_1\, dz_2.$$

Remark. It is an immediate consequence of the support properties of E that the convolutions $E_{1,2} * E_{2,3}$ and $E_{1,2} * E_{1,3}$ are defined.

Proof. The first equation follows since

$$(E_{1,2} * E_{2,3})(x_1, x_2, x_3)$$
$$= (E_{2,1} * E_{1,3})(x_2, x_1, x_3) = -(E_{1,2} * E_{1,3})(x_2, x_1, x_3).$$

When proving the second equation we notice that

$$\langle \tilde{\varphi}, E_{2,1} * E_{1,3} \rangle = \int \cdots \int E(x_2, x_1 - w)E(w, x_3)\tilde{\varphi}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3\, dw.$$

We change variables and write

$$x_1 = y_1 + y_2 - z_1 - z_2;$$
$$x_2 = y_1 + z_1;$$
$$x_3 = y_2 + z_2;$$
$$w = y_2 - z_2.$$

Then $dx_1\,dx_2\,dx_3\,dw = 2^{2n} dy_1\,dy_2\,dz_1\,dz_2$ and

$$\begin{aligned} E(x_2, x_1 - w)E(w, x_3) &= E(y_1 + z_1, y_1 - z_1)E(y_2 - z_2, y_2 + z_2) \\ &= (4^{-1}(i\pi)^{1-n})^2 \delta^{(n-2)}((y_1 + z_1)^2 - (y_1 - z_1)^2) \\ &\qquad \delta^{(n-2)}((y_2 - z_2)^2 - (y_2 + z_2)^2) \\ &= -4^{-1}(2\pi)^{2(1-n)}\delta^{(n-2)}(y_1 \cdot z_1)\delta^{(n-2)}(y_2 \cdot z_2). \end{aligned}$$

This gives the remaining part of the lemma after a simple computation. □

Proposition 3.3. *The following equality holds*

$$E_3 = E_{1,2} * E_{2,3} - E_{1,2} * E_{1,3}, \tag{3.9}$$

where $E_{j,k}$ are given in (3.8).

We give two proofs for this proposition, each of them pointing out suggestive properties. The second proof, though longer then the first, gives also a direct argument to the fact that (3.4) holds on the support $E_{1,2} * E_{2,3} - E_{1,2} * E_{1,3}$, which is not immediate without having the equality (3.9).

First proof of Proposition 3.3. We notice that

$$(\Delta_1 - \Delta_2)E_3 = E_{2,3} - E_{1,3}$$

Since (x_1, x_2) belongs to a compact set provided x_3 stays in a compact set when (x_1, x_2, x_3) is in the support of E_3 or in the support of $E_{2,3} - E_{1,3}$ it follows that

$$E_3 = E_{1,2} * (\Delta_1 - \Delta_2)E_3 = E_{1,2} * (E_{2,3} - E_{1,3}) = E_{1,2} * E_{2,3} - E_{1,2} * E_{1,3}.$$

□

Second proof of Proposition 3.3. We set

$$U = E_{1,2} * E_{2,3}.$$

The first equality in Lemma 3.2 says that

$$U(x_1, x_2, x_3) + U(x_2, x_1, x_3) = (E_{1,2} * E_{2,3} - E_{1,2} * E_{1,3})(x_1, x_2, x_3). \tag{3.10}$$

When computing the distribution U we recall (3.8) and use formula (2.6) to express the fundamental solution E. We get

$$U(x_1, x_2, x_3) = \int_0^\infty \int_0^\infty \int k_0(x_1, s)\dot{k}_0(x_2 - w, s)k_0(w, t)\dot{k}_0(x_3, t)\, dw\, ds\, dt$$

When integrating with respect to w we notice that

$$\int \dot{k}_0(x_2 - w, s)k_0(w, t)\, dw$$

is the convolution kernel of the operator

$$\begin{aligned} \dot{K}_0(s)K_0(t) &= \cos((s|D|)\sin(t|D|)/|D| \\ &= \frac{1}{2|D|}\left[\sin((t + s)|D|) + \sin((t - s)|D|)\right] \\ &= \frac{1}{2}K_0(t + s) + \frac{1}{2}K_0(t - s) \quad \text{if } t \geq s, \end{aligned}$$

and

$$\dot{K}_0(s)K_0(t) = \frac{1}{2}K_0(t + s) - \frac{1}{2}K_0(s - t) \quad \text{if } t \leq s.$$

It follows that

$$\int \dot{k}_0(x_2-w,s)k_0(w,t)\,dw = \frac{1}{2}(k_0(x_2,t+s)+k_0(x_2,t-s)) \quad \text{if } t\ge s,$$

and

$$\int \dot{k}_0(x_2-w,s)k_0(w,t)\,dw = \frac{1}{2}(k_0(x_2,t+s)-k_0(x_2,s-t)) \quad \text{if } t\le s.$$

We have proved therefore that

$$U(x_1,x_2,x_3) = \frac{1}{2}[U_1(x_1,x_2,x_3)+U_2^+(x_1,x_2,x_3)+U_2^-(x_1,x_2,x_3)]$$

where

$$U_1(x_1,x_2,x_3) = \int_0^\infty\int_0^\infty k_0(x_1,s)k_0(x_2,t+s)\dot{k}_0(x_3,t)\,ds\,dt$$

$$U_2^+(x_1,x_2,x_3) = \int_0^\infty\int_s^\infty k_0(x_1,s)k_0(x_2,t-s)\dot{k}_0(x_3,t)\,dt\,ds.$$

and

$$U_2^-(x_1,x_2,x_3) = -\int_0^\infty\int_t^\infty k_0(x_1,s)k_0(x_2,s-t)\dot{k}_0(x_3,t)\,ds\,dt.$$

We notice that if in U_2^- we change $s_1=s-t$, we get

$$\begin{aligned} U_2^-(x_1,x_2,x_3) &= -\int_0^\infty\int_0^\infty k_0(x_1,s_1+t)k_0(x_2,s_1)\dot{k}_0(x_3,t)\,ds\,dt \\ &= -U_1(x_2,x_1,x_3). \end{aligned}$$

We also have

$$\begin{aligned} U_2^+(x_1,x_2,x_3) &= U_2^+(x_2,x_1,x_3) \\ &= \int_0^\infty\int_0^\infty k_0(x_1,s)k_0(x_2,t)\dot{k}_0(x_3,t+s)\,dt\,ds. \end{aligned}$$

It follows that

$$\begin{aligned} U(x_1,x_2,x_3)+U(x_2,x_1,x_3) &= U_2^+(x_1,x_2,x_3) \\ &= \int_0^\infty\int_s^\infty k_0(x_1,s)k_0(x_2,t-s)\dot{k}_0(x_3,t)\,dt\,ds. \end{aligned}$$

We change the order of integration and use (3.10), (3.2) and (2.6) to obtain

$$\begin{aligned} &(E_{1,2}*E_{2,3}-E_{1,2}*E_{1,3})(x_1,x_2,x_3) \\ &= \int_0^\infty\Big(\int_0^t k_0(x_1,s)k_0(x_2,t-s)\,ds\Big)\dot{k}_0(x_3,t)\,dt \\ &= \int_0^\infty Q_2(x_1,x_2,t)\dot{k}_0(x_3,t)\,dt = E_3(x_1,x_2,x_3), \end{aligned}$$

which completes the proof. □

The next theorem gives the form of the cubic term of the backscattering transform that we use and which might be seen as a trilinear counterpart of (2.7).

Theorem 3.4. *Suppose $n \geq 3$ is odd. The cubic term of the backscattering transform of $v \in C_0^\infty(\mathbf{R}^n)$ can be written*

$$(B_3 v)(x) = -2^{-1}(2\pi i)^{2(1-n)} \int \cdots \int \delta^{(n-2)}(y_1 \cdot z_1)\delta^{(n-2)}(y_2 \cdot z_2) \, v_x(y_1+y_2)v_x(y_2-z_1)v_x(z_1+z_2)\, dy_1\, dy_2\, dz_1\, dz_2, \tag{3.11}$$

where $v_x(y) = v(x+y)$.

Proof. The first equality in Lemma 3.2 shows that

$$\langle \varphi, E_{1,2} * E_{2,3} - E_{1,2} * E_{1,3}\rangle = \langle \varphi, E_{1,2} * E_{2,3}\rangle + \langle \tilde{\varphi}, E_{1,2} * E_{2,3}\rangle.$$

Then the formula (3.11) is a direct consequence of (3.7), Proposition 3.3 and Lemma 3.2 applied to the functions φ and $\tilde{\varphi}$, where

$$\varphi(x_1,x_2,x_3) = (v_1)_x\Big(\frac{x_1+x_2+x_3}{2}\Big)(v_2)_x\Big(\frac{x_1-x_2+x_3}{2}\Big)(v_3)_x\Big(\frac{-x_1+x_2+x_3}{2}\Big),$$

when $x \in \mathbf{R}^n$. □

The aim of this paper is to prove the following theorem.

Theorem 3.5. *Suppose $n \geq 3$ is odd. Then*

$$\|\nabla^{n-2} B_3 v\|_{L^1(\mathbf{R}^n)} \leq C_n \|\nabla^{n-2} v\|_{L^1(\mathbf{R}^n)}, \quad \text{when } v \in C_0^\infty(\mathbf{R}^n), \tag{3.12}$$

where C_n is a constant that only depends on the dimension n.

4. Trilinear singular integral operators

The proof of Theorem 3.5 involves studying singular integrals like the one in the right hand side of (3.11). We use partial integrations in these integrals, by vector fields with non-smooth coefficients. The necessary technical machinery is developed in the present section.

4.1. **Spaces of functions and vector fields.** Define

$$W(Y) = \frac{|y_2|^2 y_1 - \langle y_1, y_2\rangle y_2}{|y_1|^2|y_2|^2 - \langle y_1, y_2\rangle^2} \tag{4.1}$$

when $Y = (y_1, y_2) \in \mathbf{R}^n \times \mathbf{R}^n$, $n \geq 3$, and y_1, y_2 are linearly independent. Then

$$\langle W(Y), y_1\rangle = 1, \quad \langle W(Y), y_2\rangle = 0. \tag{4.2}$$

Using the notation $\hat{y} = y/|y|$, we define

$$w(y_1,y_2) = |W(\hat{y}_1, \hat{y}_2)|. \tag{4.3}$$

This function is separately homogeneous of degree 0 in y_1 and y_2, and we define $w(Y) = +\infty$ when y_1 and y_2 are linearly dependent. A simple computation yields

$$w(\omega,\theta) = (1 - \langle \omega, \theta\rangle^2)^{-1/2}, \quad \omega,\, \theta \in S^{n-1}. \tag{4.4}$$

We notice also that $w(\omega,\theta) = w(\theta,\omega)$ and

$$\sup_{\omega \in S^{n-1}} \int_{S^{n-1}} w(\omega,\theta)^p \, d\theta < \infty \quad \text{when } p < n-1. \tag{4.5}$$

Thus $w(Y)$ is a Borel-measurable function and $w \in L^p_{\rm loc}(\mathbf{R}^n \times \mathbf{R}^n)$ when $p < n-1$ in view of (4.5) and the homogeneity properties of w.

We denote the elements in $(\mathbf{R}^n)^4$ by (Y, Z) where $Y = (y_1, y_2)$ and $Z = (z_1, z_2)$. The set $\Omega \subset \mathbf{R}^n \times \mathbf{R}^n$ is defined by

$$\Omega = \{Y \in \mathbf{R}^n \times \mathbf{R}^n; 1/2 < |y_1|, |y_2| < 2\}.$$

Definition 4.1. Let b and c be nonnegative integers satisfying

$$b + c \leq n - 2.$$

Then $\Phi_{b,c}$ is the space of functions $\varphi = \varphi(Y, Z)$ in $\Omega \times \Omega$ which satisfy the following conditions:

(i) $\varphi \in C^\infty$ in the open set Σ where $w(Y)$ and $w(Z)$ are finite, and φ is a compactly supported in $\Omega \times \Omega$;

(ii) $\varphi \in C_0^\infty(\Omega \times \Omega)$ if $b = c = 0$;

(iii) if $c = 0$ then

$$|\partial_Y^\beta \partial_Z^\gamma \varphi(Y, Z)| \leq C_{\beta,\gamma} w(Y)^{b+|\beta|}, \quad (Y, Z) \in \Sigma,$$

when $b + |\beta| \leq n - 2$ and γ is arbitrary;

(iv) if $b = 0$ then

$$|\partial_Y^\beta \partial_Z^\gamma \varphi(Y, Z)| \leq C_{\beta,\gamma} w(Z)^{c+|\gamma|}, \quad (Y, Z) \in \Sigma,$$

when $c + |\gamma| \leq n - 2$ and β is arbitrary;

(v) if b and c are both positive then

$$|\partial_Y^\beta \partial_Z^\gamma \varphi(Y, Z)| \leq C_{\beta,\gamma} |w(Y)|^{b+|\beta|} w(Z)^{c+|\gamma|}, \quad (Y, Z) \in \Sigma,$$

when $b + c + |\beta| + |\gamma| \leq n - 2$.

It will be convenient also to let $\Phi_{b,c} = \{0\}$ in case one of the indices b and c are negative. We notice that the right-hand sides of the inequalities above are integrable in $\Omega \times \Omega$ as a consequence of (4.5).

Lemma 4.2. *Assume $\varphi \in \Phi_{b,c}$ and that $b + c + |\beta| + |\gamma| \leq n - 2$. Then*

$$\partial_Y^\beta \partial_Z^\gamma \varphi \in \Phi_{b+|\beta|, c+|\gamma|},$$

where the derivatives are taken in the distribution sense. Also, with derivatives taking in the distribution sense, one has the inclusions

$$\partial_Z^\gamma \Phi_{b,0} \subset \Phi_{b,0}, \quad \partial_Y^\beta \Phi_{0,c} \subset \Phi_{0,c}$$

for arbitrary β and γ.

Proof. A simple argument of induction allows us to assume that $|\beta| + |\gamma| = 1$ when proving the first part of the lemma. It suffices then to prove that the derivatives of $L\varphi$ can be computed pointwise, i.e. that $L\varphi = (L\varphi)_0$ (with equality in the distribution sense) whenever L is a first-order vector field with smooth coefficients and $(L\varphi)_0$ is the unique L^1-function which agrees with $L\varphi$ in the set where φ is smooth. The problem is local and independent of the choice of coordinates, and we consider the situation around some point κ in $\Omega \times \Omega$. We express the components y_1, y_2 of Y and z_1, z_2 of Z in polar coordinates and write $y_j = r_j \omega_j$, $z_k = \rho_k \theta_k$. After a rotation of the angular variables we may assume that $\langle \omega_1, \omega_2 \rangle \geq 0$, $\langle \theta_1, \theta_2 \rangle \geq 0$ at κ so that $w(Y)$ is of the same order of magnitude as $|\omega_1 - \omega_2|$ and $w(Z)$ of the same order of magnitude as $|\theta_1 - \theta_2|$. If $w(Y)$ is large we may assume that both ω_1

and ω_2 are close to e_n. Then we may replace these variables by their projections ω_1' and ω_2' in $\mathbf{R}^{n-1}$ along e_n and we take $u_1 = \omega_1' - \omega_2'$, $\omega_1' + \omega_2'$, r_1, and r_2 as coordinates instead of Y. By introducing a similar system of coordinates instead of Z when $w(Z)$ is large we find that there are local coordinates (z, u_1, u_2) at κ, with $z \in \mathbf{R}^{2n+2}$ and $u_1, u_2 \in \mathbf{R}^{n-1}$ such that

$$|\psi| + |\nabla\psi| \le C(|u_1|^{2-n} + |u_2|^{2-n}) \quad \text{when } |u_1|\,|u_2| \neq 0, \tag{4.6}$$

where ψ is the expression for φ in the new coordinates. Thus we have reduced our problem to showing that the distribution derivative of ψ can be computed pointwise when $\psi = \psi(z, u_1, u_2) \in L^1((\mathbf{R}^n)^4)$ satisfies (4.6) and is smooth when $|u_1|$ and $|u_2|$ are not zero. But if g is smooth and compactly supported and L is a constant coefficient vector field, then an integration first along lines parallel to L, and not intersecting the set where $|u_1|\,|u_2| = 0$, together with an application of Fubini's theorem, shows that

$$\int \cdots \int (Lg)\psi \, dz\, du_1\, du_2 = -\int \cdots \int gL\psi \, dz\, du_1\, du_2,$$

where the derivatives in the right-hand side are computed pointwise. Hence $L\psi$ computed pointwise agrees with $L\psi$ computed in the distribution sense. This proves the first part of the lemma and the second part follows from similar arguments. □

Definition 4.3. If b and c are nonnegative integers satisfying $b + c \le n - 2$ then $\mathcal{X}_{b,c}$ is the space of all first order differential operators

$$L = \langle g, \partial_Y \rangle + \langle h, \partial_Z \rangle + q \tag{4.7}$$

in $\Omega \times \Omega$ such that

$$\eta g_\nu \in \Phi_{b-1,c},\ \eta h_\nu \in \Phi_{b,c-1},\ \eta q \in \Phi_{b,c}$$

when $\eta \in C_0^\infty(\Omega \times \Omega)$, where g_ν and h_ν denote the components of g and h. (Recall that $\Phi_{b,c} = \{0\}$ if one of the indices b or c is negative so that $g = 0$ if $b = 0$ and $h = 0$ if $c = 0$.) The subspace $\mathcal{X}_1 \subset \mathcal{X}_{1,1}$ is the set of all $L = \langle h, \partial_Z \rangle + q$ where $\eta h_\nu \in \Phi_{1,0}$ and $\eta q \in \Phi_{1,0}$ with η as above. Finally, $\mathcal{X}_2 \subset \mathcal{X}_{1,1}$ consists of all $L = \langle g, \partial_Y \rangle + q$ with ηg_ν and ηq in $\Phi_{0,1}$ instead.

It is easily seen the the formal transpose L^t of L belongs to $\mathcal{X}_{\mu,\nu}$ if L is in that space.

The following lemma is immediate from the definitions above together with the fact that $\Phi_{b_1,c_1} \cdot \Phi_{b_2,c_2} \subseteq \Phi_{b_1+b_2,c_1+c_2}$ when $b_1 + b_2 + c_1 + c_2 \le n - 2$.

Lemma 4.4. *Assume $\varphi \in \Phi_{b_1,c_1}$ and $L \in \mathcal{X}_{b_2,c_2}$ where $b_1 + b_2 + c_1 + c_2 \le n - 2$. Then $L\varphi \in \Phi_{b_1+b_2,c_1+c_2}$. Moreover, $\mathcal{X}_1\Phi_{b,0} \subset \Phi_{b+1,0}$ and $\mathcal{X}_2\Phi_{0,c} \subset \Phi_{0,c+1}$ when $b, c < n - 2$.*

Lemma 4.5. *Assume $L \in \mathcal{X}_{b,c}$, and that $\varphi_j \in \Phi_{b_j,c_j}$, $j = 1, 2$, where*

$$b + b_1 + b_2 + c + c_1 + c_2 \le n - 2.$$

Then

$$\iint (L\varphi_1)\varphi_2 \, dY\, dZ = \iint \varphi_1 L^t \varphi_2 \, dY\, dZ.$$

This equation is also fulfilled if

$$L \in \mathcal{X}_1,\ \varphi_j \in \Phi_{b_j,0},\ b_1 + b_2 < n - 2$$

or

$$L \in \mathcal{X}_2,\ \varphi_j \in \Phi_{0,c_j},\ c_1 + c_2 < n - 2.$$

Proof. Let L be as in (4.7). Pointwise computations give

$$\begin{aligned}&(L\varphi_1)\varphi_2 - \varphi_1(L^t\varphi_2)\\ &= \operatorname{div}_Y(\varphi_1\varphi_2 g) + \operatorname{div}_Z(\varphi_1\varphi_2 h).\end{aligned}$$

Therefore $(L\varphi_1)\varphi_2 - \varphi_1(L^t\varphi_2)$ is a compactly supported L^1-function which, by Lemma 4.2, at the same time is the sum of first order derivatives of compactly supported L^1-functions, where the derivatives are taken in the distribution sense. Hence

$$\iint ((L\varphi_1)\varphi_2 - \varphi_1(L^t\varphi_2))\, dY\, dZ = 0$$

and the lemma follows. □

Lemma 4.6. *One has the estimates*

$$|\partial_x^\alpha \partial_y^\beta W(x,y)| \le C_{\alpha,\beta} w(x,y)^{1+|\alpha|+|\beta|}, \quad \text{when } x, y \in \Omega.$$

Proof. Let P_μ be the space of all vector valued functions in Ω of the form

$$\sum_{|\alpha+\beta|\le\mu} a_{\alpha,\beta}(x,y) W(x,y)^\alpha W(y,x)^\beta,$$

where $a_{\alpha,\beta}$ smooth vector-valued functions on Ω. Then $W \in P_1$, and a simple computation shows that

$$\begin{aligned}\langle z, \partial_x\rangle W(x,y) &= |W(x,y)|^2(z - \langle z, \widehat{y}\rangle\widehat{y})\\ &\quad + 2\frac{\langle x,y\rangle\langle z,y\rangle - \langle x,z\rangle |y|^2}{|x|^2|y|^2 - \langle x,y\rangle^2} W(x,y)\\ &= |W(x,y)|^2(z - \langle z,\widehat{y}\rangle\widehat{y}) - 2\langle W(x,y), z\rangle W(x,y),\end{aligned}$$

where $\hat{y} = y/|y|$, and

$$\begin{aligned}&\langle z, \partial_y\rangle W(x,y)\\ &= -2\langle W(y,x), z\rangle W(x,y) + |y|^{-2}|W(x,y)|^2(2\langle y,z\rangle x - \langle x,z\rangle y - \langle x,y\rangle z).\end{aligned}$$

It follows from this that

$$\langle z, \partial_x\rangle W(x,y),\ \langle z, \partial_y\rangle W(x,y) \in P_2.$$

A simple argument of induction shows then that

$$\partial_x^\alpha \partial_y^\beta W(x,y) \in P_{1+|\alpha|+|\beta|}.$$

This completes the proof. □

We write $Y' = (y_2, y_1)$ when $Y = (y_1, y_2)$. The next lemma is an imediate consequence of the previous one since $w(Y') = w(Y)$.

Lemma 4.7. *We have*

$$W(Y)^{\beta_1} W(Y')^{\beta_2} W(Z)^{\gamma_1} W(Z')^{\gamma_2} \varphi \in \Phi_{b+|\beta_1|+|\beta_2|, c+|\gamma_1|+|\gamma_2|}$$

if $\Phi \in \Phi_{b,c}$ *and*

$$b + c + |\beta_1| + |\beta_2| + |\gamma_1| + |\gamma_2| \le n - 2.$$

4.2. **Singular integrals.** Next we define the singular integrals

$$\int \cdots \int \delta^{(\mu)}(y_1 \cdot z_1)\delta^{(\nu)}(y_2 \cdot z_2)\varphi(Y,Z)\,dY\,dZ \tag{4.8}$$

when $\varphi \in \Phi_{b,c}$ and $\mu + \nu + b + c \le n-2$.

We start with some general remarks. Let ρ be the measure $\delta(y_1 \cdot z_1)\delta(y_2 \cdot z_2)$ in $\Omega \times \Omega$ and consider $w_1 = w(Y)$ and $w_2 = w(Z)$ as functions in $\Omega \times \Omega$. We notice that y_1 is orthogonal to z_2 at the points in the support of ρ where $w_1 = +\infty$. Hence $y_1 \cdot z_2$ vanishes at such points, and since the differential of $y_1 \cdot z_2$ is linearly independent from the differentials of $y_1 \cdot z_1$ and $y_2 \cdot z_2$ in that set, it follows that the set where $w_1 = +\infty$ is of ρ-measure 0, and for similar reasons this is also true for the set where $w_2 = +\infty$. It follows that w_1 and w_2 are ρ-measurable. By introducing polar coordinates and writing $y_1 = r_1\omega_1$, $y_2 = r_2\omega_2$ one finds that

$$\int \cdots \int_{\Omega\times\Omega} w_1(Y)^{n-2}\,d\rho(Y,Z)$$

$$= \int_{Z\in\Omega} \cdots \int_{r_j\in(1/2,2)} \delta(\omega_1 \cdot z_1)\delta(\omega_2 \cdot z_2)w(\omega_1,\omega_2)^{n-2}r_1^{n-2}r_2^{n-2}\,dr_1\,dr_2\,dZ\,d\omega_1\,d\omega_2$$

$$= C_n \iint w(\omega_1,\omega_2)^{n-2}\,d\omega_1\,d\omega_2 < \infty,$$

where we have used (4.5). Hence $w_1^{n-2} \in L^1_\rho(\Omega\times\Omega) = L^1(\Omega\times\Omega, d\rho)$ and the same is true for w_2. It follows that

$$w_1^\mu w_2^\nu \in L^1_\rho(\Omega\times\Omega) \quad \text{when } \mu,\nu \ge 0,\ \mu+\nu \le n-2.$$

This in turn implies that the spaces $\Phi_{b,c}$ are contained in $L^1_\rho(\Omega\times\Omega)$ when $b+c \le n-2$.

We want to integrate by parts in (4.8) and therefore we consider the vector fields

$$L_1 = \langle z_1, \partial_{y_1}\rangle/|z_1|^2, \quad L_2 = \langle y_2, \partial_{z_2}\rangle/|y_2|^2 \tag{4.9}$$

in Ω. We notice that $L_1 \in \mathcal{X}_{1,0}$, $L_2 \in \mathcal{X}_{0,1}$, and

$$L_1 g(y_1 \cdot z_1)h(y_2 \cdot z_2) = g'(y_1 \cdot z_1)h(y_2 \cdot z_2)$$

while

$$L_2 g(y_1 \cdot z_1)h(y_2 \cdot z_2) = g(y_1 \cdot z_1)h'(y_2 \cdot z_2)$$

when g and h are distributions in $\mathbf{R}$. This leads to

$$\delta^{(\mu)}(y_1 \cdot z_1)\delta^{(\nu)}(y_2 \cdot z_2) = L_1^\mu L_2^\nu \delta(y_1 \cdot z_1)\delta(y_2 \cdot z_2) = L_1^\mu L_2^\nu \rho$$

in $\Omega \times \Omega$. Hence

$$\int \cdots \int \delta^{(\mu)}(y_1 \cdot z_1)\delta^{(\nu)}(y_2 \cdot z_2)\varphi(Y,Z)\,dY\,dZ$$

$$= (-1)^{\mu+\nu} \int \cdots \int \delta(y_1 \cdot z_1)\delta(y_2 \cdot z_2)L_1^\mu L_2^\nu \varphi(Y,Z)\,dY\,dZ$$

$$= (-1)^{\mu+\nu} \int L_1^\mu L_2^\nu \varphi\,d\rho$$

when $\varphi \in \Phi_{0,0} = C_0^\infty(\Omega\times\Omega)$. If $b+c+\mu+\nu \le n-2$ it follows from Lemma 4.4 that the right-hand side is defined also when $\varphi \in \Phi_{b,c}$, and this motivates the following definition:

Definition 4.8. Assume $\varphi \in \Phi_{b,c}$ and that $\mu+\nu+b+c \le n-2$. Then we define

$$\int\cdots\int \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ = (-1)^{\mu+\nu}\int L_1^{\mu}L_2^{\nu}\varphi\,d\rho.$$

We could also have used, for example, the vector fields

$$M_1 = \langle y_1, \partial_{z_1}\rangle/|y_1|^2 \in \mathcal{X}_{0,1},\ \ M_2 = \langle z_2, \partial_{y_2}\rangle/|z_2|^2 \in \mathcal{X}_{1,0}, \tag{4.10}$$

since

$$\begin{aligned}&\int\cdots\int \delta^{(\mu)}(x_1\cdot y_1)\delta^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ\\ &= (-1)^{\mu+\nu}\int\cdots\int \delta(y_1\cdot z_1)\delta(y_2\cdot z_2)M_1^{\mu}M_2^{\nu}\varphi(Y,Z)\,dY\,dZ\\ &= (-1)^{\mu+\nu}\int M_1^{\mu}M_2^{\nu}\varphi\,d\rho\end{aligned}$$

when $\varphi \in \Phi_{0,0}$. This would give an alternative definition. Its equivalence with the first definition will follow from the lemma below.

Choose some $\eta \in C_0^\infty(\mathbf{R})$ such that $0 \le \eta$ and $\int \eta(s)\,ds = 1$. Set $\eta_\varepsilon(s) = \varepsilon^{-1}\eta(s/\varepsilon)$ when $0<\varepsilon<1$ so that $\eta_\varepsilon(s) \to \delta(s)$ as $\varepsilon \to 0$.

Lemma 4.9. *Assume $\varphi \in \Phi_{b,c}$ and that $\mu+\nu+b+c \le n-2$. Then*

$$\begin{aligned}&\int\cdots\int \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ\\ &= \lim_{\varepsilon\to 0}\int\cdots\int \eta_\varepsilon^{(\mu)}(y_1\cdot z_1)\eta_\varepsilon^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ.\end{aligned} \tag{4.11}$$

Proof. Since $L_1 \in \mathcal{X}_{1,0}$, $L_2 \in \mathcal{X}_{0,1}$ and $L_j^t = -L_j$ it follows from Lemma 4.5 that the right-hand side of (4.11) is equal to

$$(-1)^{\mu+\nu}\int\cdots\int \eta_\varepsilon(y_1\cdot z_1)\eta_\varepsilon(y_2\cdot z_2)L_1^{\mu}L_2^{\nu}\varphi(Y,Z)\,dY\,dZ.$$

To finish the proof it suffices to show that

$$\int\cdots\int \eta_\varepsilon(y_1\cdot z_1)\eta_\varepsilon(y_2\cdot z_2)\psi(Y,Z)\,dY\,dZ \to \int \psi\,d\rho \tag{4.12}$$

when $\psi \in \Phi_{b,c}$ and $b+c \le n-2$. To prove this we let $0<\delta<1$ and set

$$\psi_\delta(Y,Z) = (1+\delta w_1^2 w_2^2)^{-n}\psi(Y,Z),$$

where $w_1 = w(Y)$ and $w_2 = w(Z)$. Then $\psi_\delta \in C_0(\Omega\times\Omega)$ and

$$\int\cdots\int \eta_\varepsilon(y_1\cdot z_1)\eta_\varepsilon(y_2\cdot z_2)\psi_\delta(Y,Z)\,dY\,dZ \to \int \psi_\delta\,d\rho \tag{4.13}$$

as $\varepsilon \to +0$. We also notice that

$$\sup_{0<\varepsilon<1}\int\cdots\int \eta_\varepsilon(y_1\cdot z_1)\eta_\varepsilon(y_2\cdot z_2)|\psi_\delta(Y,Z)-\psi(Y,Z)|\,dY\,dZ \to 0 \tag{4.14}$$

as $\delta \to 0$. In fact, we have the estimate

$$
\begin{aligned}
&|\psi_\delta(Y,Z) - \psi(Y,Z)| \\
&\le C\Big(1 - \Big(\frac{1}{1+\delta w_1^2 w_2^2}\Big)^n\Big)(w_1^{n-2} + w_2^{n-2}) \\
&\le C'(1 - \frac{1}{1+\delta w_1^2 w_2^2})(w_1^{n-2} + w_2^{n-2}) \\
&= C' \frac{\delta w_1^2 w_2^2}{1+\delta w_1^2 w_2^2}(w_1^{n-2} + w_2^{n-2}) \\
&\le C'' \Big(\frac{\delta w_1^2 w_2^2}{1+\delta w_1^2 w_2^2}\Big)^{1/8}(w_1^{n-2} + w_2^{n-2}) \\
&\le C''' \delta^{1/8}(w_1^{n-3/2} + w_2^{n-3/2}).
\end{aligned} \tag{4.15}
$$

After introducing polar coordinates for y_1 and y_2 we see that

$$
\begin{aligned}
&\int \cdots \int_{\Omega\times\Omega} \eta_\varepsilon(y_1 \cdot z_1)\eta_\varepsilon(y_2 \cdot z_2) w(Y)^{n-3/2}\, dY\, dZ \\
&\le C \int \cdots \int_{1/2<r_j<2,\, 1/2<|z_1|,\, |z_2|<2} \eta_\varepsilon(r_1\langle\omega_1, z_1\rangle)\eta_\varepsilon(r_2\langle\omega_2, z_2\rangle) \\
&\qquad |w(\omega_1,\omega_2)|^{n-3/2}\, dr_1\, dr_2\, d\omega_1\, d\omega_2\, dz_1\, dz_2.
\end{aligned}
$$

Since there is a constant C such that

$$
\int_{|z|<2} \eta_\varepsilon(r\langle\omega, z\rangle)\, dz \le C, \quad \omega \in S^{n-1},\ 1/2 < r < 2,
$$

it follows from (4.5) and (4.15) that (4.14) holds after we have carried out a similar argument when $w(Y)$ has been replaced by $w(Z)$. Since $|\psi_\delta(Y,Z)| \le |\psi(Y,Z)|$ and $\psi_\delta(Y,Z) \to \psi(Y,Z)$ a.e. with respect to the measure ρ when $\delta \to 0$, a combination of (4.13) and (4.14) gives (4.12) and thus the proof is complete. □

We shall next consider a family of vector fields depending on a real parameter. We define

$$
\begin{aligned}
N_{11}(r) &= \langle W(Y), \partial_{z_1} - r\partial_{z_2}\rangle \in \mathcal{X}_1, \\
N_{12}(r) &= \langle W(Z), \partial_{y_1} - r\partial_{y_2}\rangle \in \mathcal{X}_2, \\
N_{21}(r) &= \langle W(Y'), \partial_{z_2} - r\partial_{z_1}\rangle \in \mathcal{X}_1, \\
N_{22}(r) &= \langle W(Z'), \partial_{y_2} - r\partial_{y_1}\rangle \in \mathcal{X}_2.
\end{aligned}
$$

Recall that $Y' = (y_2, y_1)$ if $Y = (y_1, y_2)$. We notice that $N_{jk}^t(r) = -N_{jk}(r)$ and that

$$
N_{1k}(r) g(y_1 \cdot z_1) h(y_2 \cdot z_2) = g'(y_1 \cdot z_1) h(y_2 \cdot z_2) \tag{4.16}
$$

while

$$
N_{2k}(r) g(y_1 \cdot z_1) h(y_2 \cdot z_2) = g(y_1 \cdot z_1) h'(y_2 \cdot z_2) \tag{4.17}
$$

when $g, h \in C^\infty(\mathbf{R})$. This follows from (4.2).

Theorem 4.10. *Assume $\varphi \in \Phi_{b,c}$. If $\mu > 0$ then*

$$\begin{aligned}&\int\cdots\int \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ\\ &= -\int\cdots\int \delta^{(\mu-1)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)T\varphi(Y,Z)\,dY\,dZ\end{aligned} \tag{4.18}$$

when one of the following conditions is fulfilled:

$$\mu+\nu+b+c \le n-3,\ T = N_{11}(r)\ \text{or}\ T = N_{12}(r); \tag{4.19}$$

$$c = 0,\ \mu+\nu+b \le n-2,\ T = N_{11}(r); \tag{4.20}$$

$$b = 0,\ \mu+\nu+c \le n-2,\ T = N_{12}(r); \tag{4.21}$$

$$\mu+\nu+b+c \le n-2,\ T = L_1\ \text{or}\ T = M_1. \tag{4.22}$$

If instead $\nu > 0$ then

$$\begin{aligned}&\int\cdots\int \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ\\ &= -\int\cdots\int \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu-1)}(y_2\cdot z_2)T\varphi(Y,Z)\,dY\,dZ,\end{aligned} \tag{4.23}$$

provided one of the following conditions holds:

$$\mu+\nu+b+c \le n-3,\ T = N_{21}(r)\ \text{or}\ T = N_{22}(r); \tag{4.24}$$

$$c = 0,\ \mu+\nu+b \le n-2,\ T = N_{21}(r); \tag{4.25}$$

$$b = 0,\ \mu+\nu+b \le n-2,\ T = N_{22}(r); \tag{4.26}$$

$$\mu+\nu+b+c \le n-2,\ T = L_2\ \text{or}\ T = M_2. \tag{4.27}$$

Proof. It follows from Lemma 4.9 that the left-hand side of (4.18) is the limit as $\varepsilon \to 0$ of

$$I_\varepsilon = \int\cdots\int \eta_\varepsilon^{(\mu)}(y_1\cdot z_1)\eta_\varepsilon^{(\nu)}(y_2\cdot z_2)\varphi(Y,Z)\,dY\,dZ \tag{4.28}$$

with η_ε as in that lemma. Assume $\mu > 0$. If (4.19) holds we write

$$\eta_\varepsilon^{(\mu)}(y_1\cdot z_1)\eta_\varepsilon^{(\nu)}(y_2\cdot z_2) = T\eta_\varepsilon^{(\mu-1)}(y_1\cdot z_1)\eta_\varepsilon^{(\nu)}(y_2\cdot z_2).$$

Since $T \in \mathcal{X}_{1,1}$, $T = -T^t$ and

$$1+1+b+c \le 2+n-3-\mu \le n-2$$

we may apply Lemma 4.5 to conclude that

$$I_\varepsilon = -\int\cdots\int \eta_\varepsilon^{(\mu-1)}(y_1\cdot z_1)\eta_\varepsilon^{(\nu)}(y_2\cdot z_2)T\varphi(Y,Z)\,dY\,dZ. \tag{4.29}$$

Since $T\varphi \in \Phi_{b+1,c+1}$ and

$$\mu-1+\nu+b+1+c+1 \le n-2.$$

an application of Lemma 4.9 shows that I_ε tends to the right-hand side of (4.18) as $\varepsilon \to 0$, which proves the validity of that formula.

Next assume that $\mu > 0$ and that (4.20) holds. Then $T \in \mathcal{X}_{1,0}$ and since $b < n-2$ we can apply Lemma 4.5 to conclude that (4.29) holds. Then, by Lemma 4.4, $T\varphi \in \Phi_{b+1,0}$ and an application of Lemma 4.9 shows again that (4.18) is valid.

If $b = 0$ and (4.21) holds then by repeating the previous arguments with the roles of the Y and Z-variables interchanged we obtain (4.18) again. If (4.22) holds then $T \in \mathcal{X}_{1,0}$ or $T \in \mathcal{X}_{0,1}$ and we may apply Lemma 4.5 since $b + c < n - 2$ to get (4.29), and then (4.18) follows by another application of Lemma 4.9.

Since the same arguments can be applied with the roles of the variables (y_1, z_1) and (y_2, z_2) interchanged we also see that the second part of the theorem is true. ☐

4.3. **Spaces of distribution-valued functions in $\mathbf{R}^4_+$.** If $\mu + \nu + b + c \le n - 2$ and $\varphi \in \Phi_{b,c}$ then we define $\rho_{\mu,\nu,\varphi} \in \mathcal{E}'((\mathbf{R}^n)^4)$ by

$$\langle f, \rho_{\mu,\nu,\varphi}\rangle = \iint \delta^{(\mu)}(y_1 \cdot z_1)\delta^{(\nu)}(y_2 \cdot z_2)\varphi(Y, Z)f(Y, Z)\, dY\, dZ \tag{4.30}$$

when $f \in C^\infty((\mathbf{R}^n)^4)$ (see Definition 4.8).

When $(\vec{s}, \vec{t}) = (s_1, s_2, t_1, t_2) \in \mathbf{R}^4_+$, let $R(\vec{s}, \vec{t}) : (\mathbf{R}^n)^4 \to (\mathbf{R}^n)^3$ be the linear mapping

$$R(\vec{s}, \vec{t})(Y, Z) = (s_1 y_1 + s_2 y_2, s_2 y_2 - t_1 z_1, t_1 z_1 + t_2 z_2). \tag{4.31}$$

We denote by $R_* : \mathcal{E}'((\mathbf{R}^n)^4) \to \mathcal{D}'((\mathbf{R}^n)^3)$ the corresponding push-forward map between spaces of distributions.

Definition 4.11. If $\Gamma \subset \mathbf{R}^4_+$ is a measurable set and $\mu + \nu + b + c \le n - 2$ then $X(b, c, \mu, \nu, \Gamma)$ is the linear space generated by all mappings

$$\Gamma \ni (\vec{s}, \vec{t}) \mapsto \gamma(\vec{s}, \vec{t}) R_*(\vec{s}, \vec{t})\rho_{\mu,\nu,\varphi} \in \mathcal{D}'((\mathbf{R}^n)^3),$$

where γ is a bounded measurable function in Γ and $\varphi \in \Phi_{b,c}$.

We shall denote the variables in $(\mathbf{R}^n)^3$ by $u = (u_1, u_2, u_3)$ and let ∂_k be the gradient vector corresponding to the variable u_k. We denote by S^m_{jk} the space of polynomials of degree $\le m$ in the variable $s_j \partial_k$ when $1 \le j \le 2$, $1 \le k \le 2$, and T^m_{jk} is the space of polynomials of degree $\le m$ in $t_j \partial_k$ when $1 \le j \le 2$, $2 \le k \le 3$. If $m = 1$ we use the notation S_{jk} instead of S^1_{jk}, and similarly for T_{jk}.

Theorem 4.12. *We have the following inclusions of spaces when $b + \mu + \nu \le n - 2$ and $c + \mu + \nu \le n - 2$:*

$$X(0, c, \mu, \nu, \Gamma) \subset S_{11} X(0, c, \mu - 1, \nu, \Gamma), \quad \mu > 0; \tag{4.32}$$

$$X(b, 0, \mu, \nu, \Gamma) \subset T_{23} X(b, 0, \mu, \nu - 1, \Gamma), \quad \nu > 0; \tag{4.33}$$

$$X(b, 0, \mu, \nu, \Gamma) \subset (T_{12} + T_{13}) X(b, 0, \mu - 1, \nu, \Gamma), \quad \mu > 0; \tag{4.34}$$

$$X(0, c, \mu, \nu, \Gamma) \subset (S_{21} + S_{22}) X(0, c, \mu, \nu - 1, \Gamma), \quad \nu > 0; \tag{4.35}$$

$$\begin{gathered} X(b, 0, \mu, \nu, \Gamma) \subset T_{12} X(b + 1, 0, \mu - 1, \nu, \Gamma), \\ \textit{if } \mu > 0 \textit{ and } t_1/t_2 \textit{ is bounded in } \Gamma; \end{gathered} \tag{4.36}$$

$$\begin{gathered} X(0, c, \mu, \nu, \Gamma) \subset S_{22} X(0, c + 1, \mu, \nu - 1, \Gamma), \\ \textit{if } \nu > 0 \textit{ and } s_2/s_1 \textit{ is bounded in } \Gamma; \end{gathered} \tag{4.37}$$

$$\begin{gathered} X(0, c, \mu, \nu, \Gamma) \subset S_{12} X(0, c + 1, \mu - 1, \nu, \Gamma), \\ \textit{if } \mu > 0 \textit{ and } s_1/s_2 \textit{ is bounded in } \Gamma; \end{gathered} \tag{4.38}$$

$$X(b,0,\mu,\nu,\Gamma)\subset T_{22}X(b+1,0,\mu,\nu-1,\Gamma),\quad \textit{if } \nu>0 \textit{ and } t_2/t_1 \textit{ is bounded in } \Gamma. \tag{4.39}$$

Proof. An element of $X(0,c,\mu,\nu,\Gamma)$ is a finite linear combination of bounded functions of $(\vec{s},\vec{t})\in\Gamma$ times $R_*(\vec{s},\vec{t})\rho_{\mu,\nu,\varphi}$, where $\varphi\in\Phi_{0,c}$. If $\mu>0$ it follows from Theorem 4.10 that

$$\begin{aligned}\langle f,R_*(\vec{s},\vec{t})\rho_{\mu,\nu,\varphi}\rangle &= \langle R^*(\vec{s},\vec{t})f,\rho_{\mu,\nu,\varphi}\rangle\\ &= -\iint \delta^{(\mu-1)}(y_1\cdot z_1)\delta^{(\nu)}(y_2\cdot z_2)(T\varphi R^*(\vec{s},\vec{t})f)(Y,Z)\,dY\,dZ,\end{aligned} \tag{4.40}$$

where $T=L_1$ or $T=N_{12}(r)$. If $T=L_1$ then $L_1\varphi\in\Phi_{0,c}$ and it is easily seen that

$$T\varphi R^*(\vec{s},\vec{t})f\in\Phi_{0,c}R^*(\vec{s},\vec{t})S_{11}f.$$

Hence the right-hand side of (4.40), considered as a distribution valued function of $(\vec{s},\vec{t})\in\Gamma$ acting on f, is an element of $S_{11}X(0,c,\mu-1,\nu,\Gamma)$. This proves (4.32). If instead $T=N_{12}(r)$ we choose $r=s_1/s_2$ and assume s_1/s_2 is bounded in Γ. We notice that

$$(\partial_{y_1}-(s_1/s_2)\partial_{y_2})R^*(\vec{s},\vec{t})f\in R^*(\vec{s},\vec{t})S_{12}f$$

and that $T\varphi$ is a linear combination of elements in $\Phi_{0,c+1}$ with coefficients that are bounded functions of $(\vec{s},\vec{t})\in\Gamma$. It follows that

$$N_{12}(s_1/s_2)\varphi R^*(\vec{s},\vec{t})f\in L^\infty(\Gamma)\Phi_{0,c+1}R^*(\vec{s},\vec{t})S_{12}f,$$

from which (4.38) follows.

If $\nu>0$ an application of Theorem 4.10 shows that

$$\begin{aligned}\langle f,R_*(\vec{s},\vec{t})\rho_{\mu,\nu,\varphi}\rangle &= \langle R^*(\vec{s},\vec{t})f,\rho_{\mu,\nu,\varphi}\rangle\\ &= -\iint \delta^{(\mu)}(y_1\cdot z_1)\delta^{(\nu-1)}(y_2\cdot z_2)(T\varphi R^*(\vec{s},\vec{t})f)(Y,Z)\,dY\,dZ,\end{aligned} \tag{4.41}$$

where $T=M_2$ or $T=N_{22}(r)$. If $T=M_2$, then $T\varphi\in\Phi_{0,c}$ and

$$T\varphi R^*(\vec{s},\vec{t})f\in\Phi_{0,c}R^*(\vec{s},\vec{t})(S_{21}+S_{22})f.$$

This proves (4.35). If $T=N_{22}(r)$ we choose $r=s_2/s_1$ and assume that s_2/s_1 is bounded in Γ. Then one finds that

$$T\varphi R^*(\vec{s},\vec{t})f\in L^\infty(\Gamma)\Phi_{0,c+1}R^*(\vec{s},\vec{t})S_{22}f$$

and (4.37) follows.

Next we consider elements of $X(b,0,\mu,\nu,\Gamma)$. We may assume these are of the form $R_*(\vec{s},\vec{t})\rho_{\mu,\nu,\varphi}$ where $\varphi\in\Phi_{b,0}$. Assume first that $\mu>0$. Then an application of Theorem 4.10 gives us (4.40) with $T=M_1$ or $T=N_{11}(r)$. If $T=M_1$ then $T\varphi\in\Phi_{b,0}$ and

$$T\varphi R^*(\vec{s},\vec{t})f\in\Phi_{b,0}R^*(\vec{s},\vec{t})(T_{12}+T_{13})f.$$

This gives us (4.34). If instead $T=N_{12}(r)$ we choose $r=t_1/t_2$ and assume t_1/t_2 is bounded in Γ. Then

$$T\varphi R^*(\vec{s},\vec{t})f\in L^\infty(\Gamma)\Phi_{b+1,0}R^*(\vec{s},\vec{t})T_{12}f$$

and (4.36) follows.

Finally, if $\nu>0$ then it follows from Theorem 4.10 that (4.41) holds with $T=L_2$ or $T=N_{21}(r)$. If $T=L_2$ then $T\varphi\in\Phi_{b,0}$ and

$$T\varphi R^*(\vec{s},\vec{t})f\in\Phi_{b,0}R^*(\vec{s},\vec{t})T_{23}f,$$

which proves (4.33). If instead $T = N_{21}(r)$ we choose $r = t_2/t_1$ and assume t_2/t_1 is bounded in Γ. Then

$$T\varphi R^*(\vec{s},\vec{t})f \in L^\infty(\Gamma)\Phi_{b+1,0}R^*(\vec{s},\vec{t})T_{22}f$$

which proves (4.39) and completes the proof of the theorem. □

5. Estimates of singular integrals

5.1. **An equivalent formulation of the problem.** We consider the distribution

$$F(Y,Z) = \delta^{(n-2)}(y_1\cdot z_1)\delta^{(n-2)}(y_2\cdot z_2)$$

in $(\mathbf{R}^n)^4$ together with the mapping

$$R : (\mathbf{R}^n)^4 \to (\mathbf{R}^n)^3, \quad R(Y,Z) = (y_1+y_2, y_2-z_1, z_1+z_2).$$

If $R(Y,Z) = (u_1,u_2,u_3)$ and $(Y,Z)\in \operatorname{supp}(F)$ then

$$|u_1-u_2|^2 = |y_1|^2+|z_1|^2,\ |u_2+u_3|^2 = |y_2|^2+|z_2|^2.$$

From this observation it follows that the push-forward

$$U = R_*F$$

of F under R is defined. With this notation, the formula (3.11) for B_3v becomes

$$(B_3v)(x) = -2^{-1}(2\pi)^{2(1-n)}\langle v_x\otimes v_x\otimes v_x, U\rangle$$

when $v\in C_0^\infty(\mathbf{R}^n)$. We have already seen that B_3v is a smooth function, and

$$(\partial_x^\alpha B_3v)(x)$$
$$= -2^{-1}(2\pi)^{2(1-n)}\sum_{\gamma_1+\gamma_2+\gamma_3=\alpha}\frac{\alpha!}{\gamma_1!\gamma_2!\gamma_3!}\langle\partial^{\gamma_1}v_x\otimes\partial^{\gamma_2}v_x\otimes\partial^{\gamma_3}v_x, U\rangle,$$

when $\alpha\in\mathbf{N}^n$ with $|\alpha| = n-2$. Therefore Theorem 3.5 is proved when we have shown that if $\vec{k} = (k_1,k_2,k_3)$, where the integers k_j satisfy

$$0\le k_j\le n-2,\ k_1+k_2+k_3 = 2(n-2), \tag{5.1}$$

then

$$\int_{\mathbf{R}^n} |\langle f_x, U\rangle|\,dx \le C\|\nabla_1^{k_1}\nabla_2^{k_2}\nabla_3^{k_3}f\|_{L^1} \tag{5.2}$$

when $f\in C_0^\infty((\mathbf{R}^n)^3)$ and $f_x(u_1,u_2,u_3) = f(x+u_1,x+u_2,x+u_3)$.

Since $F(Y,Z)$ is homogeneous of degree $1-n$ in each of the variables y_1,y_2,z_1,z_2 we can find $\chi\in C_0^\infty(\Omega\times\Omega)$ such that

$$F = \iint F(\vec{s},\vec{t})\,d\vec{s}\,d\vec{t}$$

where the integration is over $\mathbf{R}_+^4$ and $F(\vec{s},\vec{t})\in\mathcal{E}'((\mathbf{R}^n)^4)$ is defined by

$$\langle\psi, F(\vec{s},\vec{t})\rangle$$
$$= \iint \delta^{(n-2)}(y_1\cdot z_1)\delta^{(n-2)}(y_2\cdot z_2)\chi(Y,Z)\psi(s_1y_1,s_2y_2,t_1z_1,t_2z_2)\,dY\,dZ.$$

It follows that

$$U = \iint U(\vec{s},\vec{t})\,d\vec{s}\,d\vec{t}$$

where

$$\langle f, U(\vec{s},\vec{t})\rangle = \iint \delta^{(n-2)}(y_1\cdot z_1)\delta^{(n-2)}(y_2\cdot z_2)\chi(Y,Z)$$
$$f(s_1y_1+s_2y_2, s_2y_2-t_1z_1, t_1z_1+t_2z_2)\,dY\,dZ.$$

Recalling (4.30) and (4.31) we see that

$$U(\vec{s},\vec{t}) = R_*(\vec{s},\vec{t})\rho_{n-2,n-2,\chi}$$

Hence $U(\vec{s},\vec{t})$, considered as a distribution valued function of $(\vec{s},\vec{t})$, is an element of $X(0,0,n-2,n-2,\mathbf{R}^4_+)$.

Therefore (5.2), and hence Theorem 3.5, follows if we prove the following theorem.

Theorem 5.1. *Assume* $V = V(\vec{s},\vec{t}) \in X(0,0,n-2,n-2,\mathbf{R}^4_+)$. *Then*

$$\iint\limits_{\mathbf{R}^4_+\times\mathbf{R}^n} |\langle f_x, V(\vec{s},\vec{t})\rangle|\,d\vec{s}\,d\vec{t}\,dx \le C\|\nabla_1^{k_1}\nabla_2^{k_2}\nabla_3^{k_3}f\|_{L^1} \tag{5.3}$$

when $f\in C_0^\infty((\mathbf{R}^n)^3)$, *where* k_1, k_2, k_3 *are integers,* $0\le k_j\le n-2$, $k_1+k_2+k_3 = 2(n-2)$.

We shall write $\mathbf{R}^4_+$ as the union of eight conic sets. Define $\Gamma_j\subset\mathbf{R}^2_+$ by

$$\Gamma_1 = \{s_1\le s_2\},\ \Gamma_2 = \{s_1\ge s_2\},$$

and define when $j=1,2$

$$\Gamma_{jk1} = (\Gamma_j\times\Gamma_k)\cap\{s_2\le t_2\}$$

while

$$\Gamma_{jk2} = (\Gamma_j\times\Gamma_k)\cap\{s_2\ge t_2\}$$

Then $\mathbf{R}^4_+$ is the union of the sets Γ_{jkl} where $1\le j,k,l\le 2$.

Then Theorem 5.1 follows if we prove that

$$\begin{aligned}&\iiint\limits_{\Gamma_{jkl}\times\mathbf{R}^n} |\langle f_x, V(\vec{s},\vec{t})\rangle|\,d\vec{s}\,d\vec{t}\,dx\\ &\le C_V\|\nabla_1^{k_1}\nabla_2^{k_2}\nabla_3^{k_3}f\|_{L^1},\quad f\in C_0^\infty((\mathbf{R}^n)^3)\end{aligned} \tag{5.4}$$

when $V\in X(0,0,n-2,n-2,\Gamma_{jkl})$ and for all combinations of j,k,l.

5.2. **Cones and corresponding admissible partial integrations.** Let $\vec{k} = (k_1,k_2,k_3)$ be a vector of nonnegative integers as in (5.1). We consider sequences

$$\vec{\sigma} = (\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22}),\ \vec{\tau} = (\tau_{12},\tau_{13},\tau_{22},\tau_{23})$$

where the σ_{jk} and τ_{jk} are nonnegative integers, together with nonnegative integers b,c such that

$$\begin{gathered}\sigma_{11}+\sigma_{21}\le k_1,\ \sigma_{12}+\sigma_{22}+\tau_{12}+\tau_{22}\le k_2,\ \tau_{13}+\tau_{23}\le k_3,\\ \sigma_{11}+\sigma_{12}\le n-2,\ \sigma_{21}+\sigma_{22}\le n-2,\ \tau_{12}+\tau_{13}\le n-2, \tau_{22}+\tau_{23}\le n-2\end{gathered} \tag{5.5}$$

and

$$b+c\le n-2. \tag{5.6}$$

The set of all sequences $(b,c,\vec{\sigma},\vec{\tau})$ satisfying the above conditions is denoted $\mathcal{A}(\vec{k})$.

If a_1, a_2, b_1, b_2 are nonnegative integers, b, c are as above and $\Gamma \subset \mathbf{R}^4_+$ is a closed conic set we define the measure $Q_{b,c}(a_1, a_2, b_1, b_2, \Gamma)$ in $(\mathbf{R}^n)^3$ by

$$\begin{aligned} &\langle g, Q_{b,c}(a_1, a_2, b_1, b_2, \Gamma)\rangle \\ &= \iint\limits_{\widetilde{\Gamma}} \delta(y_1 \cdot z_1)\delta(y_2 \cdot z_2) w(Y)^b w(Z)^c |y_1|^{a_1-(n-2)} |y_2|^{a_2-(n-2)} \\ &\quad |z_1|^{b_1-(n-2)} |z_2|^{b_2-(n-2)} g(y_1 + y_2, y_2 - z_1, z_1 + z_2)\, dY\, dZ \end{aligned} \tag{5.7}$$

when $g \in C_0((\mathbf{R}^n)^3)$, where $\widetilde{\Gamma}$ is the set of all (Y, Z) such that

$$(y_1/s_1, y_2/s_2, z_1/t_1, z_2/t_2) \in \Omega \times \Omega$$

for some $(s_1, s_2, t_1, t_2) \in \Gamma$.

Definition 5.2. We say that $(b, c, \vec{\sigma}, \vec{\tau}) \in \mathcal{A}(\vec{k})$ is $(\Gamma, \vec{k})$-admissible if

$$\begin{aligned} &\int\limits_{\mathbf{R}^n} |\langle g_x, Q_{b,c}(a_{11} + a_{12}, a_{21} + a_{22}, b_{12} + b_{13}, b_{22} + b_{23}, \Gamma)\rangle|\, dx \\ &\quad \le C \|\nabla_1^{k_1 - a_{11} - a_{21}} \nabla_2^{k_2 - a_{12} - a_{22} - b_{12} - b_{22}} \nabla_3^{k_3 - b_{13} - b_{23}} g\|_{L^1} \end{aligned} \tag{5.8}$$

when $g \in C_0^\infty((\mathbf{R}^n)^3)$, whenever the a_{jk}, b_{jk} are nonnegative integers satisfying

$$\begin{aligned} &a_{11} \le \sigma_{11},\ a_{12} \le \sigma_{12},\ a_{21} \le \sigma_{21},\ a_{22} \le \sigma_{22}, \\ &b_{12} \le \tau_{12},\ b_{13} \le \tau_{13},\ b_{22} \le \tau_{22},\ b_{23} \le \tau_{23}. \end{aligned} \tag{5.9}$$

The motivation of this definition is made apparent by the following lemma.

Lemma 5.3. *Assume that*

$$V \in S_{11}^{\sigma_{11}} S_{12}^{\sigma_{12}} S_{21}^{\sigma_{21}} S_{22}^{\sigma_{22}} T_{12}^{\tau_{12}} T_{13}^{\tau_{13}} T_{22}^{\tau_{22}} T_{23}^{\tau_{23}} X(b, c, 0, 0, \Gamma) \tag{5.10}$$

where $(b, c, \vec{\sigma}, \vec{\tau}) \in \mathcal{A}(\vec{k})$ *is* $(\Gamma, \vec{k})$*-admissible. Then*

$$\iiint\limits_{\Gamma \times \mathbf{R}^n} |\langle f_x, V(\vec{s}, \vec{t})\rangle|\, d\vec{s}\, d\vec{t}\, dx \le C \|\nabla_1^{k_1} \nabla_2^{k_2} \nabla_3^{k_3} f\|_{L^1} \tag{5.11}$$

when $f \in C_0^\infty((\mathbf{R}^n)^3)$.

Proof. We may write

$$\begin{aligned} V(\vec{s}, \vec{t}) = \sum s_1^{|\alpha_{11}|+|\alpha_{12}|} s_2^{|\alpha_{21}|+|\alpha_{22}|} t_1^{|\beta_{12}|+|\beta_{13}|} t_2^{|\beta_{22}|+|\beta_{23}|} \\ \partial_1^{\alpha_{11}+\alpha_{21}} \partial_2^{\alpha_{12}+\alpha_{22}+\beta_{12}+\beta_{22}} \partial_3^{\beta_{13}+\beta_{23}} V_{\alpha,\beta}(\vec{s}, \vec{t}) \end{aligned}$$

where

$$V_{\alpha,\beta} \in X(b, c, 0, 0, \Gamma)$$

and (5.9) is fulfilled with $a_{jk} = |\alpha_{jk}|$, $b_{jk} = |\beta_{jk}|$. Hence

$$\begin{aligned} &|\langle f_x, V(\vec{s}, \vec{t})\rangle| \\ &\le C \sum |\langle (g_{\alpha,\beta})_x, V_{\alpha,\beta}(\vec{s}, \vec{t})\rangle|\, s_1^{|\alpha_{11}+\alpha_{12}|} s_2^{|\alpha_{21}+\alpha_{22}|} t_1^{|\beta_{12}+\beta_{13}|} t_2^{|\beta_{22}|+|\beta_{23}|}, \end{aligned}$$

where

$$g_{\alpha,\beta} = \partial_1^{\alpha_{11}+\alpha_{21}} \partial_2^{\alpha_{12}+\alpha_{22}+\beta_{12}+\beta_{22}} \partial_3^{\beta_{13}+\beta_{23}} f.$$

Then (5.11) follows if we prove that

$$\iiiint |\langle g_x, V_{\alpha,\beta}(\vec{s},\vec{t})\rangle|\, s_1^{|\alpha_{11}+\alpha_{12}|} s_2^{|\alpha_{21}+\alpha_{22}|} t_1^{|\beta_{12}+\beta_{13}|} t_2^{|\beta_{22}+\beta_{23}|}\, d\vec{s}\, d\vec{t}\, dx \\ \le C\|\nabla_1^{k_1-|\alpha_{11}+\alpha_{21}|}\nabla_2^{k_2-|\alpha_{12}+\alpha_{22}+\beta_{12}+\beta_{22}|}\nabla_3^{k_3-|\beta_{13}+\beta_{23}|} g\|_{L^1} \tag{5.12}$$

when $g \in C_0^\infty((\mathbf{R}^n)^3)$. This inequality holds since the left-hand side can be estimated from above by a constant times

$$\iint \delta(y_1\cdot z_1)\delta(y_2\cdot z_2) w(Y)^b w(Z)^c |y_1|^{|\alpha_{11}+\alpha_{12}|-(n-2)}|y_2|^{|\alpha_{21}+\alpha_{22}|-(n-2)} \\ |z_1|^{|\beta_{12}+\beta_{13}|-(n-2)}|z_2|^{|\beta_{22}+\beta_{23}|-(n-2)} \\ |g(x+y_1+y_2, x+y_2-z_1, x+z_1+z_2)|\, dY\, dZ,$$

and the lemma follows. □

5.3. **Proof of Theorem 5.1.** As indicated above, to prove Theorem 5.1 it suffices to show that (5.4) holds. To this end we shall show that for each of the cones $\Gamma = \Gamma_{jkl}$, $j,k,l = 1,2$, and $\vec{k}$ satisfying (5.1) there exists a sequence $(b,c,\vec{\sigma},\vec{\tau}) \in \mathcal{A}(\vec{k})$ that is admissible with respect to the considered cone Γ and such that (5.10) holds for every $V \in X(0,0,n-2,n-2,\Gamma)$.

We first give a preparatory lemma.

Lemma 5.4. *Assume that q_1, q_2, q_3, a, b are nonnegative integers such that*

$$q_j \le n-2,\ a,b \le n-1,\ a+b = q_1+q_2+q_3, \\ q_1 \le a,\ q_3 \le b. \tag{5.13}$$

Set

$$R_{a,b}(Y,Z) = (w(Y)^{n-2}|y_2|^{2-n} + w(Z)^{n-2}|z_2|^{2-n}) \\ (|y_1|^{2-n}+|z_1|^{2-n})|y_1+z_1|^{-a}|y_2+z_2|^{-b}. \tag{5.14}$$

Then the integral

$$I(g) = \int\cdots\int \delta(y_1\cdot z_1)\delta(y_2\cdot z_2) R_{a,b}(Y,Z) \\ |g(x+y_1+y_2, x+y_2-z_1, x+z_1+z_2)|\, dY\, dZ\, dx \tag{5.15}$$

is convergent and

$$I(g) \le C\|\nabla_1^{q_1}\nabla_2^{q_2}\nabla_3^{q_3} g\|_{L^1} \tag{5.16}$$

when $g \in C_0^\infty((\mathbf{R}^n)^3)$.

Proof. We replace x by $x-y_2$ in the right-hand side of (5.15) and write

$$I(g) = \int\cdots\int \delta(y_1\cdot z_1)\delta(y_2\cdot z_2) R_{a,b}(Y,Z) \\ |g(x+y_1, x-z_1, x+z_1+z_2-y_2)|\, dY\, dZ\, dx.$$

By introducing polar coordinates r_2, ω_2 for y_2 one finds that

$$\begin{aligned}
&\iint \delta(y_2 \cdot z_2) w(Y)^{n-2} |y_2|^{2-n} |f(y_2 - z_2)| \, dy_2 \, dz_2 \\
&= \iiint \delta(\omega_2 \cdot z_2) w(y_1, \omega_2)^{n-2} |f(r_2\omega_2 - z_2)| \, dr_2 \, dz_2 \, d\omega_2 \\
&\leq \iint_{\mathbf{R}^n \times S^{n-1}} w(y_1, \omega_2)^{n-2} |f(u_2)| \, du_2 \, d\omega_2 \\
&\leq C\|f\|_{L^1}
\end{aligned}$$

when $f \in C_0(\mathbf{R}^n)$. We have the same estimate for

$$\iint \delta(y_2 \cdot z_2) w(Z)^{n-2} |z_2|^{2-n} |f(y_2 - z_2)| \, dy_2 \, dz_2.$$

Since $|y_2 + z_2| = |y_2 - z_2|$ on the support of $\delta(y_2 \cdot z_2)$, this shows that

$$\begin{aligned}
I(g) \leq C \int \cdots \int \delta(y_1 \cdot z_1)(|y_1|^{2-n} + |z_1|^{2-n}) |y_1 + z_1|^{-a} \\
|u_2|^{-b} |g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx.
\end{aligned} \tag{5.17}$$

We have proved therefore that

$$I(g) \leq C(I_1(g) + I_2(g)),$$

where

$$\begin{aligned}
I_1(g) = \int \cdots \int \delta(y_1 \cdot z_1) |y_1|^{2-n} |y_1 + z_1|^{-a} \\
|u_2|^{-b} |g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
I_2(g) = \int \cdots \int \delta(y_1 \cdot z_1) |z_1|^{2-n} |y_1 + z_1|^{-a} \\
|u_2|^{-b} |g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx.
\end{aligned} \tag{5.19}$$

We first prove the statement concerning the convergence of the integral in (5.15). We write

$$\begin{aligned}
I_2(g) = \int \cdots \int \delta(y_1 \cdot z_1) |z_1|^{2-n} |y_1 + z_1|^{-a} |u_2|^{-b} \\
|g(x, x - z_1 - y_1, x - (-z_1 + y_1) + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx
\end{aligned}$$

and take polar coordinates r_1, ω_1 for z_1 noticing that

$$-z_1 + y_1 = T_{\omega_1} u_1$$

if $u_1 = z_1 + y_1 = r_1\omega_1 + y_1$, $y_1 \cdot z_1 = 0$ and $T_\omega z = z - 2\langle z, \omega \rangle \omega$. This proves that

$$\begin{aligned}
I_2(g) \leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b} \\
|g(x, x - u_1, x - T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, dw_1 \, dx.
\end{aligned} \tag{5.20}$$

We notice that the integral in the right-hand side is convergent when $g \in C_0$.

We write

$$\begin{aligned}
I_1(g) = \int \cdots \int \delta(y_1 \cdot z_1) |y_1|^{2-n} |y_1 + z_1|^{-a} |u_2|^{-b} \\
|g(x, x - z_1 - y_1, x + (-y_1 + z_1) + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx
\end{aligned}$$

and take polar coordinates (r_1, ω_1) for y_1. This leads to the estimates

$$\begin{aligned} I_1(g) \le C \int \cdots \int |u_1|^{-a} |u_2|^{-b} \\ |g(x, x-u_1, x+T_{\omega_1} u_1 + u_2)|\, du_1\, du_2\, dx\, d\omega_1, \end{aligned} \tag{5.21}$$

and since this integral is also convergent we have proved now that $I(g) < \infty$.

We return now to the proof of (5.16). It is enough to show that

$$\begin{aligned} I_1(g) &\le C \|\nabla_1^{q_1} \nabla_2^{q_2} \nabla_3^{q_3} g\|_{L^1}, \\ I_2(g) &\le C \|\nabla_1^{q_1} \nabla_2^{q_2} \nabla_3^{q_3} g\|_{L^1}. \end{aligned} \tag{5.22}$$

Suppose first that $q_2 = 0$. Then since $q_1 + q_3 = a + b$ and $q_1 \le a$, $q_3 \le b$, it follows that $q_1 = a$ and $q_3 = b$. We have then, applying the Hardy's inequality in the integration with respect to u_2:

$$\begin{aligned} I_1(g) &= \int \cdots \int \delta(y_1 \cdot z_1) |y_1|^{2-n} |y_1 + z_1|^{-q_1} |u_2|^{-q_3} \\ &\quad |g(x+y_1, x-z_1, x+z_1+u_2)|\, dy_1\, dz_1\, du_2\, dx \\ &\le C \int \cdots \int \delta(y_1 \cdot z_1) |y_1|^{2-n} |y_1 + z_1|^{-q_1} \\ &\quad |\nabla_3^{q_3} g(x+y_1, x-z_1, u_2)|\, dy_1\, dz_1\, du_2\, dx. \end{aligned}$$

We change variables $x \to x - z_1$. We get thus

$$\begin{aligned} I_1(g) &\le C \int \cdots \int \delta(y_1 \cdot z_1) |y_1|^{2-n} |y_1 + z_1|^{-q_1} \\ &\quad |\nabla_3^{q_3} g(x+y_1+z_1, x, u_2)|\, dy_1\, dz_1\, du_2\, dx \\ &= \int\int\int |u_1|^{-q_1} |\nabla_3^{q_3} g(x+u_1, x, u_2)|\, du_1\, du_2\, dx \\ &\le C \int\int\int |\nabla_1^{q_1} \nabla_3^{q_3} g(x+u_1, x, u_2)|\, du_1\, du_2\, dx = C \|\nabla_1^{q_1} \nabla_3^{q_3} g\|_{L^1}. \end{aligned} \tag{5.23}$$

In a similar way

$$\begin{aligned} I_2(g) &= \int \cdots \int \delta(y_1 \cdot z_1) |z_1|^{2-n} |y_1 + z_1|^{-q_1} |u_2|^{-q_3} \\ &\quad |g(x+y_1, x-z_1, x+z_1+u_2)|\, dy_1\, dz_1\, du_2\, dx \\ &\le C \int \cdots \int \delta(y_1 \cdot z_1) |z_1|^{2-n} |y_1 + z_1|^{-q_1} \\ &\quad |\nabla_3^{q_3} g(x+y_1, x-z_1, u_2)|\, dy_1\, dz_1\, du_2\, dx \\ &\le C \int\int\int |u_1|^{-q_1} |\nabla_3^{q_3} g(x+u_1, x, u_2)|\, du_1\, du_2\, dx \end{aligned}$$

and

$$\begin{aligned} I_2(g) &\le C \int\int\int |\nabla_1^{q_1} \nabla_3^{q_3} g(x+u_1, x, u_2)|\, du_1\, du_2\, dx \\ &= C \|\nabla_1^{q_1} \nabla_3^{q_3} g\|_{L^1}. \end{aligned}$$

We suppose now that $q_1 = 0$ and that $q_2 + q_3 = a + b$, $q_3 \leq b$, $q_2, q_3 \leq n - 2$. We use the inequality (5.21) to write

$$I_1(g) \leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b} |g(x, x - u_1, x + T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$\leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b+q_3} |\nabla_3^{q_3} g(x, x - u_1, x + T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$\leq C(I_{11}(g) + I_{12}(g)),$$

where

$$I_{11}(g) = \int \cdots \int |u_1|^{-q_2} |\nabla_3^{q_3} g(x, x - u_1, x + T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx$$

and

$$I_{12}(g) = \int \cdots \int |u_2|^{-q_2} |\nabla_3^{q_3} g(x, x - u_1, x + T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

In $I_{11}(g)$ we change variables $u_2 \to u_2 - T_{\omega_1} u_1$ and get

$$I_{11}(g) = C \int \int \int |u_1|^{-q_2} |\nabla_3^{q_3} g(x, x - u_1, x + u_2)| \, du_1 \, du_2 \, dx$$
$$\leq C \int \int \int |\nabla_2^{q_2} \nabla_3^{q_3} g(x, x - u_1, x + u_2)| \, du_1 \, du_2 \, dx$$
$$= C \|\nabla_2^{q_2} \nabla_3^{q_3} g\|_{L^1}.$$

In $I_{12}(g)$ we change variables $u_2 \to T_{\omega_1} u_2$, and then $u_1 \to u_1 - u_2$ and get

$$I_{12}(g) = C \int \cdots \int |u_2|^{-q_2} |\nabla_3^{q_3} g(x, x - u_1 + u_2, x + T_{\omega_1} u_1)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$\leq C \int \cdots \int |\nabla_2^{q_2} \nabla_3^{q_3} g(x, x - u_1 + u_2, x + T_{\omega_1} u_1)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$= C \|\nabla_2^{q_2} \nabla_3^{q_3} g\|_{L^1}.$$

We now consider $I_2(g)$ using (5.20). We can write

$$I_2(g) \leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b}$$
$$|g(x, x - u_1, x - T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$\leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b+q_3}$$
$$|\nabla_3^{q_3} g(x, x - u_1, x - T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx$$
$$\leq C \int \cdots \int (|u_1|^{-q_2} + |u_2|^{-q_2})$$
$$|\nabla_3^{q_3} g(x, x - u_1, x - T_{\omega_1} u_1 + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

Repeating the argument above we get

$$I_2(g) \leq C \|\nabla_2^{q_2} \nabla_3^{q_3} g\|_{L^1}.$$

We consider now the case $q_1 \geq 1$, $q_2 \geq 1$ and q_1, q_2, q_3, a, b satisfying (5.13). We prove (5.22) by induction over $q_1 + q_2$, assuming that $q_1 + q_2 > 1$ and that the inequality (5.16) is already proved for lower values on that number. (Here q_3 is fixed but a and b may vary.)

Suppose now that $q_1 \geq 1$, $q_2 \geq 1$. Using (5.21) we can write

$$I_1(g) \leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b}$$
$$|g(x - T_{\omega_1} u_1, x - u_1 - T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

We apply Hardy's inequality in the integration with respect to u_1 and find that

$$I_1(g) \leq C(I_1'(g) + I_1''(g)),$$

where

$$I_1'(g) = \int \cdots \int |u_1|^{-a+1} |u_2|^{-b}$$
$$|\nabla_1 g(x - T_{\omega_1} u_1, x - u_1 - T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$
$$I_1''(g) = \int \cdots \int |u_1|^{-a+1} |u_2|^{-b}$$
$$|\nabla_2 g(x - T_{\omega_1} u_1, x - u_1 - T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

In the integrals above we write $u_1 = z_1 + r_1 \omega_1$ and integrate over the set where $z_1 \cdot \omega_1 = 0$ and $-\infty \leq r_1 \leq \infty$. Since r_1 only occurs as a factor in front of ω_1 we may restrict the integration with respect to r_1 to an integration over $\mathbf{R}_+$ if we multiply the integral by 2. Viewing (r_1, ω_1) as polar coordinates for $y_1 \in \mathbf{R}^n$ we find that

$$\begin{aligned} I_1'(g) &\leq C \int \cdots \int \delta(z_1 \cdot y_1) |z_1 + y_1|^{-a+1} |y_1|^{2-n} |u_2|^{-b} \\ &\quad |\nabla_1 g(x + y_1 - z_1, x - 2z_1, x + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \\ &\leq C \int \cdots \int \delta(z_1 \cdot y_1) |z_1 + y_1|^{-a+1} |y_1|^{2-n} |u_2|^{-b} \\ &\quad |\nabla_1 g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} I_1''(g) &\leq C \int \cdots \int \delta(z_1 \cdot y_1) |z_1 + y_1|^{-a+1} |y_1|^{2-n} |u_2|^{-b} \\ &\quad |\nabla_2 g(x + y_1 - z_1, x - 2z_1, x + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \\ &\leq C \int \cdots \int \delta(z_1 \cdot y_1) |z_1 + y_1|^{-a+1} |y_1|^{2-n} |u_2|^{-b} \\ &\quad |\nabla_2 g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx. \end{aligned} \tag{5.25}$$

In the last integrals in (5.24) and (5.25) we apply the induction hypothesis for $q_1' = q_1 - 1$, $q_2' = q_2$, $q_3' = q_3$, $a' = a - 1$ and $b' = b$, and $q_1' = q_1$, $q_2' = q_2 - 1$, $q_3' = q_3$, $a' = a - 1$ and $b' = b$ respectively, and get thus

$$I_1'(g) \leq C \| \nabla_1^{q_1} \nabla_2^{q_2} \nabla_3^{q_3} g \|_{L^1},$$
$$I_1''(g) \leq C \| \nabla_1^{q_1} \nabla_2^{q_2} \nabla_3^{q_3} g \|_{L^1}.$$

Using (5.20) we can write

$$I_2(g) \leq C \int \cdots \int |u_1|^{-a} |u_2|^{-b}$$
$$|g(x + T_{\omega_1} u_1, x - u_1 + T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

We apply Hardy's inequality in the integration with respect to u_1 and find that

$$I_2(g) \leq C(I_2'(g) + I_2''(g)),$$

where

$$I_2'(g) = \int \cdots \int |u_1|^{-a+1}|u_2|^{-b}$$
$$|\nabla_1 g(x + T_{\omega_1} u_1, x - u_1 + T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$
$$I_2''(g) = \int \cdots \int |u_1|^{-a+1}|u_2|^{-b}$$
$$|\nabla_2 g(x + T_{\omega_1} u_1, x - u_1 + T_{\omega_1} u_1, x + u_2)| \, du_1 \, du_2 \, d\omega_1 \, dx.$$

In the integrals above we write $u_1 = y_1 + r_1\omega_1$ and integrate over the set where $y_1 \cdot \omega_1 = 0$ and $-\infty \le r_1 \le \infty$. As before, viewing (r_1, ω_1) as polar coordinates for $z_1 \in \mathbf{R}^n$ we find that

$$\begin{aligned} I_2'(g) &\le C \int \cdots \int \delta(z_1 \cdot y_1)|z_1 + y_1|^{-a+1}|y_1|^{2-n}|u_2|^{-b} \\ &\quad |\nabla_1 g(x + y_1 - z_1, x - 2z_1, x + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \\ &\le C \int \cdots \int \delta(z_1 \cdot y_1)|z_1 + y_1|^{-a+1}|y_1|^{2-n}|u_2|^{-b} \\ &\quad |\nabla_1 g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} I_2''(g) &\le C \int \cdots \int \delta(z_1 \cdot y_1)|z_1 + y_1|^{-a+1}|y_1|^{2-n}|u_2|^{-b} \\ &\quad |\nabla_2 g(x + y_1 - z_1, x - 2z_1, x + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx \\ &\le C \int \cdots \int \delta(z_1 \cdot y_1)|z_1 + y_1|^{-a+1}|y_1|^{2-n}|u_2|^{-b} \\ &\quad |\nabla_2 g(x + y_1, x - z_1, x + z_1 + u_2)| \, dy_1 \, dz_1 \, du_2 \, dx. \end{aligned} \tag{5.27}$$

In (5.26) and (5.27) we apply the induction hypothesis for $q_1' = q_1 - 1$, $q_2' = q_2$, $q_3' = q_3$, $a' = a - 1$ and $b' = b$, and $q_1' = q_1$, $q_2' = q_2 - 1$, $q_3' = q_3$, $a' = a - 1$ and $b' = b$, respectively, and get thus

$$I_2'(g) \le C\|\nabla_1^{q_1}\nabla_2^{q_2}\nabla_3^{q_3} g\|_{L^1},$$
$$I_2''(g) \le C\|\nabla_1^{q_1}\nabla_2^{q_2}\nabla_3^{q_3} g\|_{L^1}.$$

This completes the proof. □

Assume now that q_1, q_2, q_3 are nonnegative integers satisfying

$$q_j \le n - 2, \; q_1 + q_2 + q_3 \le 2(n - 2). \tag{5.28}$$

When (5.28) is fulfilled we define

$$m = m(\vec{q}) = \max(q_1, q_1 + q_2 + q_3 - n + 1), \; M = M(\vec{q}) = \min(q_1 + q_2, n - 1). \tag{5.29}$$

Then $m(\vec{q}) \le M(\vec{q})$ and we define

$$\begin{aligned} R(\vec{q}, Y, Z) &= (w(Y)^{n-2}|y_2|^{2-n} + w(Z)^{n-2}|z_2|^{2-n})(|y_1|^{2-n} + |z_1|^{2-n}) \\ &\quad |y_2 + z_2|^{-(q_1+q_2+q_3)}\Big(\big(\frac{|y_2 + z_2|}{|y_1 + z_1|}\big)^m + \big(\big(\frac{|y_2 + z_2|}{|y_1 + z_1|}\big)^M\Big). \end{aligned} \tag{5.30}$$

We notice that $(a, b) = (m, q_1 + q_2 + q_3 - m)$ and $(a, b) = (M, q_1 + q_2 + q_3 - M)$ satisfy the conditions in the previous lemma. Hence we have

Corollary 5.5. *Assume that* (5.28) *is fulfilled. Then*

$$I(\vec{q},g)\le C\|\nabla_1^{q_1}\nabla_2^{q_2}\nabla_3^{q_3}g\|_{L^1},\quad g\in C_0^\infty((\mathbf{R}^n)^3),$$

where

$$\begin{aligned}I(\vec{q},g)=\int\cdots\int\delta(y_1\cdot z_1)\delta(y_2\cdot z_2)R(\vec{q},Y,Z)\\ |g(x+y_1+y_2,x+y_2-z_1,x+z_1+z_2)|\,dY\,dZ\,dx.\end{aligned}\tag{5.31}$$

In what follows we assume that $\vec{k}=(k_1,k_2,k_3)$ satisfies (5.1) and we ask for conditions that imply that the inequality (5.8) holds.

We assume first that $c=0$ and that $s_2\le t_2$ in the conic set Γ. Then it follows from the corollary above and (5.7) that (5.8) holds provided

$$\begin{aligned}&s_1^{a_{11}+a_{12}-n+2}s_2^{a_{21}+a_{22}-n+2}t_1^{b_{12}+b_{13}-n+2}t_2^{b_{22}+b_{23}-n+2}\\ &\le Cs_2^{2-n}(s_1^{2-n}+t_1^{2-n})t_2^{-(q_1+q_2+q_3)}\Big(\big(\frac{t_2}{s_1+t_1}\big)^m+\big(\frac{t_2}{s_1+t_1}\big)^M\Big)\end{aligned}\tag{5.32}$$

when $(s_1,s_2,t_1,t_2)\in\Gamma$. Here $m=m(\vec{q})$, $M=M(\vec{q})$ and

$$q_1=k_1-a_{11}-a_{21},\ q_2=k_2-a_{12}-a_{22}-b_{12}-b_{22},\ q_3=k_3-b_{13}-b_{23}.\tag{5.33}$$

We notice that both sides of (5.32) are homogeneous of degree

$$\sum_{j,k}a_{jk}+\sum_{j,k}b_{jk}-4(n-2).$$

It suffices therefore to check condition (5.32) in the intersection Γ' of Γ with the set where $t_2=1$. After multiplying both sides of that inequality by $(s_1s_2t_1)^{n-2}$ we get the condition

$$\begin{aligned}&s_1^{a_{11}+a_{12}}s_2^{a_{21}+a_{22}}t_1^{b_{12}+b_{13}}\\ &\le C(s_1^{n-2}+t_1^{n-2})\Big((s_1+t_1)^{-m}+(s_1+t_1)^{-M}\Big).\end{aligned}$$

Since $s_1^{n-2}+t_1^{n-2}$ is of the same order of magnitude as $(s_1+t_1)^{n-2}$ we have proved that (5.8) holds provided

$$\begin{aligned}&s_1^{a_{11}+a_{12}}s_2^{a_{21}+a_{22}}t_1^{b_{12}+b_{13}}\\ &\le C\Big((s_1+t_1)^{n-m-2}+(s_1+t_1)^{n-M-2}\Big),\quad (s_1,s_2,t_1,1)\in\Gamma'.\end{aligned}\tag{5.34}$$

Lemma 5.6. *The sequence* $(n-2,0,\vec{\sigma},\vec{\tau})\in\mathcal{A}(\vec{k})$ *is* $(\Gamma,\vec{k})$*-admissible in each of the following cases:*

$$\Gamma=\Gamma_{111}\qquad k_3+\sigma_{21}+\sigma_{22}+\tau_{22}\le n-2;\tag{5.35}$$

$$\Gamma=\Gamma_{121}\qquad k_1+\tau_{12}+\tau_{13}\le n-2;\tag{5.36}$$

$$\Gamma=\Gamma_{211}\qquad k_3+\tau_{22}\le n-2,\ k_1+\sigma_{12}\le n-2;\tag{5.37}$$

$$\Gamma=\Gamma_{221}\qquad k_1+\sigma_{12}+\tau_{12}+\tau_{13}\le n-2.\tag{5.38}$$

Proof. In view of the discussions above it suffices to verify that (5.34) holds in each of the four cases when (5.9) is fulfilled.

We first verify (5.34) when (5.35) holds. Then $s_1\le s_2$, $t_1\le 1$ and $s_2\le 1$ in Γ'. Set

$$a_j=a_{j1}+a_{j2},\ b_j=b_{j2}+b_{j3}.$$

We estimate the left-hand side of (5.34) from above by $(s_1+t_1)^{a_1+b_1}$ and find that (5.34) holds if the inequality $a_1+b_1 \geq n-M-2$ is fulfilled. Recalling (5.29) and (5.33) we see that M may be replaced by q_1+q_2 and that the inequality is equivalent to

$$\begin{aligned} a_1+b_1 &\geq n-2-q_1-q_2 = n-2+a_1+a_2+b_{12}+b_{22}-k_1-k_2 \\ &= a_1+a_2+b_{12}+b_{22}+k_3-(n-2). \end{aligned}$$

We write this as

$$k_3+a_2+b_{22} \leq n-2+b_{13}$$

and find that (5.34) follows from (5.35).

Assume next that (5.36) holds. Then $s_1 \leq s_2 \leq 1$ and $t_1 \geq 1$ in Γ' and (5.34) holds if $b_1 \leq n-m-2$. Since

$$n-2-(q_1+q_2+q_3-n-1) = 1+a_1+a_2+b_1+b_2$$

we may replace m by q_1 and get the inequality

$$k_1+b_1 \leq n-2+a_{11}+a_{21},$$

which follows from (5.36).

If (5.37) holds we have $s_2 \leq 1, s_2 \leq s_1, t_1 \leq 1$ in Γ' and the left-hand side of (5.34) may be estimated from above by $(s_1+t_1)^{a_1+b_1}(\min(s_1,1))^{a_2}$. Hence (5.34) holds in $\Gamma' \cap \{s_1 \leq 1\}$ provided

$$a_1+a_2+b_1 \geq n-M-2.$$

A simple computation shows that this inequality holds since $k_3+\tau_{22} \leq n-2$. When $s_1 \geq 1$ we estimate the left-hand side of (5.34) from above by a constant times $s_1^{a_1}$ and the right-hand side from below by s_1^{n-m-2} if C is large. This gives us the condition $a_1 \leq n-m-2$ which again is a consequence of the assumptions.

Finally, when $\Gamma = \Gamma_{221}$ we have $s_1 \geq s_2$, $s_2 \leq 1$, $t_1 \geq 1$ in Γ'. When $s_1 \leq 1$ then (5.34) holds if $b_1 \leq n-m-2$ and if $s_1 \geq 1$ it holds if $a_1+b_1 \leq n-m-2$. We may replace m by q_1 and need that

$$a_1+b_1 \leq n-2-q_1 = n-2-k_1+a_{11}+a_{21}$$

which may be written

$$k_1+a_{12}+b_{12}+b_{13} \leq n-2+a_{21}.$$

The last condition of the lemma guarantees this inequality. □

We assume next that $b=0$ and that $t_2 \leq s_2$ in the conic set Γ. Then it follows from the Corollary 5.5 and (5.7) that (5.8) holds provided

$$\begin{aligned} &s_1^{a_{11}+a_{12}-n+2} s_2^{a_{21}+a_{22}-n+2} t_1^{b_{12}+b_{13}-n+2} t_2^{b_{22}+b_{23}-n+2} \\ &\leq C t_2^{2-n}(s_1^{2-n}+t_1^{2-n}) s_2^{-(q_1+q_2+q_3)} \Big(\big(\frac{s_2}{s_1+t_1}\big)^m + \big(\frac{s_2}{s_1+t_1}\big)^M \Big) \end{aligned} \tag{5.39}$$

when $(s_1,s_2,t_1,t_2) \in \Gamma$. Here $m=m(\vec{q})$, $M=M(\vec{q})$ and

$$q_1 = k_1-a_{11}-a_{21},\ q_2 = k_2-a_{12}-a_{22}-b_{12}-b_{22},\ q_3 = k_3-b_{13}-b_{23}. \tag{5.40}$$

Both sides of (5.39) are homogeneous of degree $\sum_{j,k} a_{jk} + \sum_{j,k} b_{jk} - 4(n-2)$. It suffices therefore to check condition (5.39) in the intersection Γ'' of Γ with the set where $s_2 = 1$. After multiplying both sides of that inequality by $(s_1 t_1 t_2)^{n-2}$ we get the condition

$$s_1^{a_{11}+a_{12}} t_1^{b_{12}+b_{13}} t_2^{b_{22}+b_{23}} \le C(s_1^{n-2} + t_1^{n-2})\Big((s_1+t_1)^{-m} + (s_1+t_1)^{-M}\Big).$$

Since $s_1^{n-2} + t_1^{n-2}$ is of the same order of magnitude as $(s_1+t_1)^{n-2}$ we have proved that (5.8) holds provided

$$s_1^{a_{11}+a_{12}} t_1^{b_{12}+b_{13}} t_2^{b_{22}+b_{23}} \le C\Big((s_1+t_1)^{n-m-2} + (s_1+t_1)^{n-M-2}\Big), \quad (s_1,s_2,t_1,1) \in \Gamma''. \tag{5.41}$$

Lemma 5.7. *The sequence* $(0, n-2, \vec{\sigma}, \vec{\tau}) \in \mathcal{A}(\vec{k})$ *is* $(\Gamma, \vec{k})$*-admissible in each of the following cases:*

(5.42) $\Gamma = \Gamma_{112}$ $\quad k_3 + \sigma_{21} + \sigma_{22} + \tau_{22} \le n-2;$

(5.43) $\Gamma = \Gamma_{122}$ $\quad k_1 + \tau_{12} + \tau_{13} \le n-2;\ k_3 + \sigma_{21} + \sigma_{22} \le n-2;$

(5.44) $\Gamma = \Gamma_{212}$ $\quad k_1 + \sigma_{12} \le n-2;$

(5.45) $\Gamma = \Gamma_{222}$ $\quad k_1 + \sigma_{12} + \tau_{12} + \tau_{13} \le n-2.$

Proof. In view of the discussions above it suffices to verify that (5.41) holds in each of the four cases when (5.9) is fulfilled. We use the notation in Lemma 5.6:

$$a_j = a_{j1} + a_{j2},\ b_j = b_{j2} + b_{j3}.$$

We suppose first that $\Gamma = \Gamma_{112}$ and (5.42) holds. Then $s_1 \le 1$ and $t_1 \le t_2 \le 1$, $t_2 \le 1$ on Γ''. To check (5.41) it is enough to see that

$$(s_1+t_1)^{a_1+b_1} \le C(s_1+t_1)^{n-M-2}$$

when $s_1 + t_1 \le 2$; therefore it suffices to have $a_1 + b_1 \ge n - M - 2$. Here M can be replaced by $q_1 + q_2$ (the inequality being obvious for $M = n-1$). So we get the condition $k_3 + a_{21} + a_{22} + b_{22} \le n - 2 + b_{13}$ which is true if (5.42) holds.

We assume now that $\Gamma = \Gamma_{122}$ and (5.43) holds. We have $s_1 \le 1$, $t_2 \le t_1$ and $t_2 \le 1$ on Γ''. If $t_1 \ge 1$, it is enough to see that $b_1 \le n - m - 2$. If $m = q_1 + q_2 + q_3 - n + 1$ this inequality is obviously true. We suppose that $m = q_1$. The inequality to check becomes

$$k_1 + b_{12} + b_{13} \le n - 2 + a_{11} + a_{21}$$

which is true when $k_1 + \tau_{12} + \tau_{13} \le n-2$. If $t_1 \le 1$ we use that

$$s_1^{a_1} t_2^{b_2} t_1^{b_1} \le C(s_1+t_1)^{a_1+b_1+b_2},$$

and then the inequality (5.41) is true provided $a_1 + b_1 + b_2 \ge n - 2 - (q_1 + q_2)$. The later inequality holds when $k_3 + \sigma_{21} + \sigma_{22} \le n-2$.

Assume next that $\Gamma = \Gamma_{212}$ and that (5.44) holds. Then $s_1 \ge 1$, $t_1 \le t_2$ and $t_2 \le 1$ on Γ''. The inequality (5.41) is true if $s_1^{a_1} \le C(s_1+t_1)^{n-m-2}$. It is therefore sufficient to have $a_1 \le n - m - 2$, which is true if (5.44) holds.

Finally, we suppose $\Gamma = \Gamma_{222}$ and (5.45) holds. In this case $s_1 \ge 1$, $t_1 \ge t_2$ and $t_2 \le 1$ on Γ''. The left-hand side of (5.41) is estimated from above by $(s_1+t_1)^{a_1+b_1}$. To have (5.41) it suffices to check that $a_1 + b_1 \le n - m - 2$. If $m = q_1 + q_2 + q_3 - n + 1$ this inequality is obviously true. When $m = q_1$, it becomes

$$k_1 + a_{12} + b_{12} + b_{13} \le n - 2 + a_{21}$$

which is true when (5.45) holds. □

Lemma 5.8. *Assume* $V \in X(0,0,n-2,n-2,\Gamma)$, *where* $\Gamma = \Gamma_{jk1}$ *and* $1 \le j,k \le 2$. *Assume* $k_1 + k_2 + k_3 = 2(n-2)$, $0 \le k_j \le n-2$. *Then*

$$\int_{\Gamma\times\mathbf{R}^n}\!\!\cdots\!\int |\langle f_x, V(\vec{s},\vec{t})\rangle|\, d\vec{s}\, d\vec{t}\, dx \le C\|\nabla_1^{k_1}\nabla_2^{k_2}\nabla_3^{k_3} f\|_{L^1} \tag{5.46}$$

when $f \in C_0^\infty((\mathbf{R}^n)^3)$.

Proof. We shall apply Theorem 4.12, Lemma 5.3 and Lemma 5.6. Using Theorem 4.12 we shall replace V successively by elements in the spaces $QX(b,c,\mu,\nu,\Gamma)$ where the Q are polynomials in $S_{11}, S_{12}, \dots, T_{23}$ and $c = 0$.

Consider first the case when $\Gamma = \Gamma_{j11}$. Since we want c to be equal to 0 we can only apply (4.32)–(4.36) (Notice that t_1/t_2 is bounded in Γ_{j11}.) The best strategy turns out to be bring down ν to 0 through applications of (4.32)–(4.35) before applying (4.36).

We first apply (4.35) which gives (where $\Gamma = \Gamma_{j11}$)

$$\begin{aligned} V \in X(0,0,n-2,n-2,\Gamma) &\subset (S_{21}+S_{22})^{n-2-k_3} X(0,0,n-2,k_3,\Gamma) \\ &\subset \sum_{0\le l\le n-2-k_3} S_{21}^{n-2-k_3-l} S_{22}^{l} X(0,0,n-2,k_3,\Gamma). \end{aligned}$$

Then (4.33) applied to the l:th term gives

$$\begin{aligned} &S_{21}^{n-2-k_3-l} S_{22}^{l} X(0,0,n-2,k_3,\Gamma) \\ &\subset S_{21}^{n-2-k_3-l} S_{22}^{l} T_{23}^{k_3} X(0,0,n-2,0,\Gamma). \end{aligned} \tag{5.47}$$

We notice that $k_2 \ge l$ since

$$k_2 - l \ge k_2 - (n-2-k_3) = k_2 + k_3 - n + 2.$$

An application of (4.32) shows that right-hand side of (5.47) is contained in

$$S_{11}^{n-2-k_2+l} S_{21}^{n-2-k_3-l} S_{22}^{l} T_{23}^{k_3} X(0,0,k_2-l,0,\Gamma).$$

We then apply (4.36) to the right-hand side above $k_2 - l$ times to conclude that

$$V \in \sum_{0\le l\le n-2-k_3} S_{11}^{n-2-k_2+l} S_{21}^{n-2-k_3-l} S_{22}^{l} T_{12}^{k_2-l} T_{23}^{k_3} X(k_2-l,0,0,0,\Gamma).$$

Hence V is in the sum of the spaces

$$S_{11}^{\sigma_{11}} S_{12}^{\sigma_{12}} S_{21}^{\sigma_{21}} S_{22}^{\sigma_{22}} T_{12}^{\tau_{12}} T_{13}^{\tau_{13}} T_{22}^{\tau_{22}} T_{23}^{\tau_{23}} X(n-2,0,0,0,\Gamma),$$

where $(n-2,0,\vec{\sigma},\vec{\tau}) \in \mathcal{A}(\vec{k})$ and

$$\vec{\sigma} = (n-2-k_2+l, 0, n-2-k_3-l, l),\ \ \vec{\tau} = (k_2-l, 0, 0, k_3).$$

In view of Lemma 5.3 and Lemma 5.6 the inequality (5.46) holds since (5.35) and (5.37) are fulfilled. In fact,

$$k_3 + \sigma_{21} + \sigma_{22} + \tau_{22} = k_3 + (n-2-k_3-l) + l = n-2$$

and

$$k_3 + \tau_{22} = k_3 \le n-2,\ \ k_1 + \sigma_{12} = k_1 \le n-2.$$

Next we consider the case where $\Gamma = \Gamma_{j21}$. Since we want c to be equal to 0 we can only apply (4.32)–(4.35) together with (4.39). (Notice that t_2/t_1 is bounded in

Γ_{j21}.) The best strategy turns out to be bring down μ to 0 through applications of (4.32)–(4.35) before using (4.39). To start with we apply (4.34) to get

$$V \in X(0,0,n-2,n-2,\Gamma) \subset (T_{12}+T_{13})^{n-2-k_1} X(0,0,k_1,n-2,\Gamma)$$
$$\subset \sum_{0\le l\le n-2-k_1} T_{12}^l T_{13}^{n-2-k_1-l} X(0,0,k_1,n-2,\Gamma).$$

We apply (4.32) to the l:th term and get

$$T_{12}^l T_{13}^{n-2-k_1-l} X(0,0,k_1,n-2,\Gamma) \subset S_{11}^{k_1} T_{12}^l T_{13}^{n-2-k_1-l} X(0,0,0,n-2,\Gamma).$$

Since $l \le k_2$ an application of (4.33) shows that the right-hand side above is contained in

$$S_{11}^{k_1} T_{12}^l T_{13}^{n-2-k_1-l} T_{23}^{n-2-k_2+l} X(0,0,0,k_2-l,\Gamma)$$

and by (4.39) this space is in turn contained in

$$S_{11}^{k_1} T_{12}^l T_{13}^{n-2-k_1-l} T_{22}^{k_2-l} T_{23}^{n-2-k_2+l} X(k_2-l,0,0,0,\Gamma).$$

Hence V is in the sum of the spaces

$$S_{11}^{\sigma_{11}} S_{12}^{\sigma_{12}} S_{21}^{\sigma_{21}} S_{22}^{\sigma_{22}} T_{12}^{\tau_{12}} T_{13}^{\tau_{13}} T_{22}^{\tau_{22}} T_{23}^{\tau_{23}} X(n-2,0,0,0,\Gamma),$$

where $(n-2,0,\vec{\sigma},\vec{\tau}) \in \mathcal{A}(\vec{k})$ and

$$\vec{\sigma} = (k_1,0,0,0), \ \ \vec{\tau} = (l, n-2-k_1-l, k_2-l, n-2-k_2+l),$$

where $0 \le l \le n-2-k_1$. In view of Lemma 5.3 and Lemma 5.6 the inequality (5.46) holds since (5.36) and (5.38) are fulfilled. In fact,

$$k_1 + \sigma_{12} + \sigma_{22} + \tau_{12} + \tau_{13} = k_1 + \tau_{12} + \tau_{13} = k_1 + l + n - 2 - k_1 - l = n-2.$$

This completes the proof of the lemma. □

Lemma 5.9. *Assume* $V \in X(0,0,n-2,n-2,\Gamma)$, *where* $\Gamma = \Gamma_{jk2}$ *and* $1 \le j,k \le 2$. *Assume* $k_1+k_2+k_3 = 2(n-2)$, $0 \le k_j \le n-2$. *Then*

$$\int_{\Gamma\times\mathbf{R}^n}\!\!\cdots\!\int |\langle f_x, V(\vec{s},\vec{t})\rangle| \, d\vec{s}\, d\vec{t}\, dx \le C \|\nabla_1^{k_1}\nabla_2^{k_2}\nabla_3^{k_3} f\|_{L^1} \tag{5.48}$$

when $f \in C_0^\infty((\mathbf{R}^n)^3)$.

Proof. We shall apply Theorem 4.12, Lemma 5.3 and Lemma 5.7. Using Theorem 4.12 we shall replace V successively by elements in the spaces $QX(b,c,\mu,\nu,\Gamma)$ where the Q are polynomials in $S_{11}, S_{12}, \dots, T_{23}$ and $b=0$.

Consider first the case when $\Gamma = \Gamma_{1j2}$. Since we want b to be equal to 0 we can only apply (4.32)–(4.35) together with (4.38) (Notice that s_1/s_2 is bounded in Γ_{1j2}.) The best strategy turns out to be bringing down ν to 0 through applications of (4.32)–(4.35) before applying (4.38). Thus we can start as in the first part of the proof of the previous lemma and use that V is contained in the sum of the spaces

$$S_{11}^{n-2-k_2+l} S_{21}^{n-2-k_3-l} S_{22}^l T_{23}^{k_3} X(0,0,k_2-l,0,\Gamma),$$

where $0 \le l \le n-2-k_3$. An application of (4.38) shows that the spaces occurring here are contained in

$$S_{11}^{n-2-k_2+l} S_{12}^{k_2-l} S_{21}^{n-2-k_3-l} S_{22}^l T_{23}^{k_3} X(0,k_2-l,0,0,\Gamma).$$

Thus we find that V is in the sum of spaces

$$S_{11}^{n-2-k_2+l} S_{12}^{k_2-l} S_{21}^{n-2-k_3-l} S_{22}^l T_{23}^{k_3} X(0,n-2,0,0,\Gamma)$$

and hence in the sum of

$$S_{11}^{\sigma_{11}} S_{12}^{\sigma_{12}} S_{21}^{\sigma_{21}} S_{22}^{\sigma_{22}} T_{12}^{\tau_{12}} T_{13}^{\tau_{13}} T_{22}^{\tau_{22}} T_{23}^{\tau_{23}} X(0, n-2, 0, 0, \Gamma),$$

where

$$\vec{\sigma} = (n-2-k_2+l, k_2-l, n-2-k_3-l, l), \ \vec{\tau} = (0,0,0,k_3), \quad 0 \le l \le n-2-k_3.$$

We note that $(0, n-2, \vec{\sigma}, \vec{\tau}) \in \mathcal{A}(\vec{k})$ and

$$k_3 + \sigma_{21} + \sigma_{22} + \tau_{22} = n-2$$
$$k_1 + \tau_{12} + \tau_{13} = k_1 \le n-2, \ k_3 + \sigma_{21} + \sigma_{22} = n-2,$$

hence (5.42) and (5.43) holds. Consequently, in this case the inequality (5.48) follows from Lemma 5.7 and Lemma 5.3.

Consider now the case when $\Gamma = \Gamma_{2j2}$. Since we want b to be equal to 0 we can only apply (4.32)–(4.35) together with (4.37), since s_2/s_1 is bounded in Γ_{2j2}. Here the best strategy is to bring down μ to 0 through applications of (4.32)–(4.35) before applying (4.37). Starting as in the second part of the proof of the previous lemma, we find that V is contained in the sum of the spaces

$$S_{11}^{k_1} T_{12}^{l} T_{13}^{n-2-k_1-l} T_{23}^{n-2-k_2+l} X(0,0,0,k_2-l,\Gamma)$$

where $l \le n-2-k_1$. An application of (4.37) shows that these spaces are contained in

$$S_{11}^{k_1} S_{22}^{k_2-l} T_{12}^{l} T_{13}^{n-2-k_1-l} T_{23}^{n-2-k_2+l} X(0,k_2-l,0,0,\Gamma).$$

Hence V belongs to the sum of spaces

$$S_{11}^{k_1} S_{22}^{k_2-l} T_{12}^{l} T_{13}^{n-2-k_1-l} T_{23}^{n-2-k_2+l} X(0,n-2,0,0,\Gamma)$$

and hence to the sum of

$$S_{11}^{\sigma_{11}} S_{12}^{\sigma_{12}} S_{21}^{\sigma_{21}} S_{22}^{\sigma_{22}} T_{12}^{\tau_{12}} T_{13}^{\tau_{13}} T_{22}^{\tau_{22}} T_{23}^{\tau_{23}} X(0, n-2, 0, 0, \Gamma),$$

where

$$\vec{\sigma} = (k_1, 0, 0, k_2-l), \ \vec{\tau} = (l, n-2-k_1-l, 0, n-2-k_2+l), \quad 0 \le l \le n-2-k_1.$$

We find that $(0, n-2, \vec{\sigma}, \vec{\tau}) \in \mathcal{A}(\vec{k})$ and notice that

$$k_1 + \sigma_{12} = k_1 \le n-2, \ k_1 + \sigma_{12} + \tau_{12} + \tau_{13} = n-2,$$

hence (5.44) and (5.45) are fulfilled. Therefore the inequality (5.48) is a consequence of Lemma 5.7 and Lemma 5.3. □

Proof of Theorem 5.1. Let $V \in X(0,0,n-2,n-2,\mathbf{R}_+^4)$ and $\vec{k} = (k_1,k_2,k_3)$ be a sequence of integers as in (5.1). The inequality (5.4), and hence (5.3), is now a consequence of Lemma 5.8 and 5.9. □

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