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Local Near Field Refractors and Reflectors

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LOCAL NEAR FIELD REFRACTORS AND REFLECTORS

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ABSTRACT. The problem of designing an interface optical surface separating two media with different refractive indices that focuses monochromatic collimated radiation, from a plane domain Ω into a target destination Ω' lying in 3d-space, is locally solved. Similar questions are also considered for reflection. The surface solutions satisfy first order pdes.

1. INTRODUCTION

This paper considers the problem of designing a surface separating two homogenous media I and II that refracts a collimated beam of rays from a domain $\Omega \subset \mathbb{R}^n$ into a target destination $\mathcal{T}(\Omega) \subset \mathbb{R}^{n+1}$. We find conditions on the transformation $\mathcal{T}$ such that a refracting surface exists locally. For example, given a plane image F surrounded by medium I and an image F' in 3d-space surrounded by medium II, we find conditions so that there is a surface separating the media that refracts F into F' locally. We also consider in Section 3 a similar question when the rays emanate from a point source. The main result is Theorem 2.1 and examples of illustration are given in Subsection 2.2. Our method also gives similar results for reflection. With the same method we also show in Subsection 2.3 that given a plane curve γ and a space curve Γ satisfying (2.17), there exists a surface that refracts γ into Γ. All problems in the paper are solved by studying the systems of first order pdes (2.2) and (3.10).

We remark that the paper focuses only in the destination of the points after refraction regardless of the input and output intensities of radiation. Some results in this direction were obtained in [GT11] and [Gut11] where magnification of images are considered. See also [HP05] for reflectors in the far field case. If one is interested in the problem of constructing surfaces that after refraction only preserve energy with given intensities, then one is lead to second order fully nonlinear equations of Monge-Ampère type. These kind of problems have been recently considered in several papers: [GH13] in the case of one source; [GT13] in the case of collimated beams. In these cases the destination of the points is not specifically given.

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The results are organized as follows. Section 2 contains a derivation of the pde for the collimated beam case and the proof of the main result, the local existence Theorem 2.1. In Subsection 2.2 we show examples of application, and in Remark 2.3 we discuss the case when the given mapping does not satisfy the conditions of Theorem 2.1. Subsection 2.3 discuss the refractor mapping one curve into another. Section 3 discusses the case when the rays emanate from one point source, where the main result is Theorem 3.1, which is illustrated by Example 3.2 and Remark 3.3. We work most of the time in 3d-space which is the relevant case for the physical problem.

2. Case when the rays emanate in a collimated beam

2.1. **Derivation of the equation.** We have two homogenous and isotropic media I_1, I_2 with refractive indices n_1, n_2 respectively; and we set $\kappa = \dfrac{n_2}{n_1}$. We consider light rays that emanate from a domain $\Omega \subset \mathbb{R}^n$ with direction e_{n+1}, and we are given a C^1 transformation $\mathcal{T} := (T, S) : \Omega \to \mathbb{R}^{n+1}$, where $Tx \in \mathbb{R}^n$, and $Sx \in \mathbb{R}$. Ω is surrounded by medium I_1, and $\mathcal{T}(\Omega)$ is surrounded by medium I_2.

We seek a surface $z = u(x)$, interface of the media I and II, that refracts the rays emanating from Ω into the destination $\mathcal{T}(\Omega)$, in other words, each point $x \in \Omega$ is refracted into the point $\mathcal{T}x$. The question we answer here is for which transformations $\mathcal{T}$ is there a refracting surface u doing the described job locally.

Let us first derive the differential equation for the problem. The outer unit normal to the surface $z = u(x)$ is $N = \dfrac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}}$; $x = (x_1, \cdots, x_n)$. From the Snell law in vector form we have

$$m = \frac{\mathcal{T}x - (x, u(x))}{|\mathcal{T}x - (x, u(x))|} = \frac{1}{\kappa}\left(e_{n+1} - \Phi(e_{n+1} \cdot N)\, N\right) \tag{2.1}$$

where $\Phi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1 - t^2)}$, see [GH09, formula (2.1)]. Of course, a physical condition on refraction must be satisfied to avoid internal reflection. That is, $m \cdot e_{n+1} \geq \kappa$ when $\kappa < 1$ and $m \cdot e_{n+1} \geq 1/\kappa$ if $\kappa > 1$, see [GH09, Lemma 2.1]. We then obtain from (2.1) the following two equations:

$$\frac{Tx - x}{|\mathcal{T}x - (x, u(x))|} = \frac{1}{\kappa}\,\Phi\left(\frac{1}{\sqrt{1 + |Du(x)|^2}}\right)\frac{Du(x)}{\sqrt{1 + |Du(x)|^2}}$$

and

$$\frac{Sx - u(x)}{|\mathcal{T}x - (x, u(x))|} = \frac{1}{\kappa}\left(1 - \Phi\left(\frac{1}{\sqrt{1 + |Du(x)|^2}}\right)\frac{1}{\sqrt{1 + |Du(x)|^2}}\right).$$

Solving for Du we obtain the first order system of pdes

$$Du(x) = \frac{Tx - x}{\dfrac{1}{\kappa}\,|\mathcal{T}x - (x, u(x))| + u(x) - Sx}. \tag{2.2}$$

Then our problem is reduced to see under what conditions on the transformation $\mathcal{T} = (T, S)$ the system (2.2) has solutions. We notice that for reflection, (2.1) is replaced by $m = e_{n+1} - 2(e_{n+1} \cdot N)N$, and the resulting equation for u is (2.2) with $\kappa = 1$.

2.2. **Local existence of solutions passing through a point.** Let us set

$$F(x,z) := \frac{Tx - x}{\frac{1}{\kappa}|\mathcal{T}x - (x,z))| + z - Sx} = (F_1(x,z), \cdots, F_n(x,z)),$$

so we have $Du(x) = F(x, u(x))$. If u solves (2.2) and is C^2, then $u_{ij} = u_{ji}$, we therefore obtain

$$\frac{\partial F_i}{\partial x_j}(x,u(x)) + \frac{\partial F_i}{\partial z}(x,u(x))\, F_j(x,u(x)) = \frac{\partial F_j}{\partial x_i}(x,u(x)) + \frac{\partial F_j}{\partial z}(x,u(x))\, F_i(x,u(x)). \tag{2.3}$$

From [Har02, p. 117-118], the condition

$$\frac{\partial F_i}{\partial x_j}(x,z) + \frac{\partial F_i}{\partial z}(x,z)\, F_j(x,z) = \frac{\partial F_j}{\partial x_i}(x,z) + \frac{\partial F_j}{\partial z}(x,z)\, F_i(x,z) \tag{2.4}$$

for all x, z in an open set for $1 \le i, j \le n$, implies that *a solution u to* (2.2) *passing through a fixed point exists.* We will find conditions on the mapping $\mathcal{T}$ that are equivalent to (2.4) when $n = 2$ which is the significant dimension for the physical problem. More precisely we have the following theorem.

Theorem 2.1. *Let $x_0 \in \Omega \subset \mathbb{R}^2$, $y_0 \in \mathbb{R}$ with $(Tx_0, Sx_0) \neq (x_0, y_0)$ and such that the physical condition for refraction is satisfied:*

$$\frac{(Tx_0, Sx_0) - (x_0, y_0)}{|(Tx_0, Sx_0) - (x_0, y_0)|} \cdot e_{n+1} > \kappa \tag{2.5}$$

when $\kappa < 1$, and

$$\frac{(Tx_0, Sx_0) - (x_0, y_0)}{|(Tx_0, Sx_0) - (x_0, y_0)|} \cdot e_{n+1} > 1/\kappa \tag{2.6}$$

if $\kappa > 1$. Given the mapping $\mathcal{T} = (T_1, T_2, S)$ we define

$$\begin{aligned} F &= \frac{\partial T_1}{\partial x_2} - \frac{\partial T_2}{\partial x_1} \\ G &= (T_1x - x_1)\frac{\partial S}{\partial x_2} - (T_2x - x_2)\frac{\partial S}{\partial x_1} \\ H &= (T_1x - x_1)^2\frac{\partial T_1}{\partial x_2} - (T_2x - x_2)^2\frac{\partial T_2}{\partial x_1} + (T_1x - x_1)(T_2x - x_2)\left(\frac{\partial T_2}{\partial x_2} - \frac{\partial T_1}{\partial x_1}\right). \end{aligned} \tag{2.7}$$

If $F = G = H = 0$ for all x in a neighborhood of x_0, then there is a neighborhood U of x_0, and a unique solution $u(x)$ to (2.2) *defined for $x \in U$ with $u(x_0) = y_0$, such that each ray emanating from $x \in U$ with direction e_{n+1} is refracted by the surface $z = u(x)$ into the destination $\mathcal{T}x$.*

For reflexion, if $F = G = H = 0$ for x in a neighborhood of x_0 and if $|(Tx_0, Sx_0) - (x_0, y_0)| + y_0 - Sx_0 \neq 0$, then there exists a neighborhood U of x_0, and a unique solution $u(x)$ to (2.2) *with $\kappa = 1$ defined for $x \in U$ with $u(x_0) = y_0$, such that each ray emanating from $x \in U$ with direction e_{n+1} is reflected by the surface $z = u(x)$ into the destination $\mathcal{T}x$.*

Proof. Notice that by continuity, $\mathcal{T}x \neq (x, u(x))$ for x in a neighborhood of x_0. And therefore the physical condition for refraction given in the statement of the theorem holds also in a neighborhood of x_0.

Set $Tx = (T_1x, \cdots, T_nx)$; $x = (x_1, \cdots, x_n)$. We then have $F_i(x,z) = \dfrac{T_ix - x_i}{\Delta(x,z)}$, $1 \leq i \leq n$, where

$$\Delta(x,z) = \frac{1}{\kappa}\,|\mathcal{T}x - (x,z)| + z - Sx.$$

Notice that if $\kappa < 1$, then $\Delta(x_0, y_0) > 0$. And if $\kappa > 1$, then from (2.6) we have $\Delta(x_0, y_0) < 0$. Therefore, there is a neighborhood V of (x_0, y_0) where $\Delta \neq 0$. And similarly when $\kappa = 1$ by the assumption.

We have for $i \neq j$

$$\frac{\partial F_i}{\partial x_j}(x,z) = \frac{\dfrac{\partial T_i}{\partial x_j}(x)\,\Delta(x,z) - (T_ix - x_i)\,\dfrac{\partial \Delta}{\partial x_j}(x,z)}{\Delta(x,z)^2},$$

and

$$\frac{\partial F_i}{\partial z}(x,z) = -\frac{(T_ix - x_i)\,\dfrac{\partial \Delta}{\partial z}(x,z)}{\Delta(x,z)^2}.$$

Substituting these into (2.4) and simplifying yields

$$\left(\frac{\partial T_i}{\partial x_j} - \frac{\partial T_j}{\partial x_i}\right)\Delta = (T_ix - x_i)\,\frac{\partial \Delta}{\partial x_j} - (T_jx - x_j)\,\frac{\partial \Delta}{\partial x_i}, \tag{2.8}$$

for $1 \leq i, j \leq n$. We have

$$\Delta(x,z) = \frac{1}{\kappa}\left(\sqrt{|Tx - x|^2 + (z - Sx)^2}\right) + z - Sx,$$

and

$$\frac{\partial \Delta}{\partial x_j} = \frac{1}{\kappa}\left[\frac{\sum_{k=1}^{n}\left(\dfrac{\partial T_k}{\partial x_j} - \delta_{kj}\right)(T_kx - x_k) + (Sx - z)\dfrac{\partial S}{\partial x_j}}{\sqrt{|Tx - x|^2 + (Sx - z)^2}}\right] - \frac{\partial S}{\partial x_j}$$

Let us assume from now on that $n = 2$, which is the relevant case for the physical application. Then (2.8) becomes

$$\left(\frac{\partial T_1}{\partial x_2} - \frac{\partial T_2}{\partial x_1}\right)\Delta = (T_1x - x_1)\,\frac{\partial \Delta}{\partial x_2} - (T_2x - x_2)\,\frac{\partial \Delta}{\partial x_1}, \tag{2.9}$$

valid for all $x = (x_1, x_2)$ in a neighborhood of a point $x_0 = (x_1^0, x_2^0)$ and all z in a neighborhood of a point y_0.

Inserting the derivatives of Δ in (2.9) and simplifying yields

$$\left(\frac{1}{\kappa}\sqrt{|Tx-x|^2+(Sx-z)^2}+z-Sx\right)\sqrt{|Tx-x|^2+(Sx-z)^2}\,F + \left(\sqrt{|Tx-x|^2+(Sx-z)^2}+\frac{1}{\kappa}(z-Sx)\right)G=\frac{1}{\kappa}H. \tag{2.10}$$

From (2.10) we can write

$$\frac{1}{\kappa}\left(|Tx-x|^2+(Sx-z)^2\right)F + \sqrt{|Tx-x|^2+(Sx-z)^2}\,((z-Sx)F+G)+\frac{1}{\kappa}(z-Sx)\,G=\frac{1}{\kappa}H$$

which implies that

$$\left(|Tx-x|^2+(Sx-z)^2\right)((z-Sx)F+G)^2 = \left(\frac{1}{\kappa}H-\frac{1}{\kappa}|Tx-x|^2F-\frac{1}{\kappa}(z-Sx)G-\frac{1}{\kappa}(z-Sx)^2F\right)^2.$$

Expanding this equation, we then get a fourth order polynomial equation in the variable $z - Sx$, with coefficients depending only on x, and satisfying

$$\begin{aligned}&\left(1-\frac{1}{\kappa}\right)F^2\,(z-Sx)^4 \\ &+2\left(1-\frac{1}{\kappa}\right)FG\,(z-Sx)^3 \\ &+\left(\left(1-\frac{2}{\kappa^2}\right)|Tx-x|^2F^2+\frac{2}{\kappa^2}HF+\left(1-\frac{1}{\kappa^2}\right)G^2\right)(z-Sx)^2 \\ &+\left(2\left(1-\frac{1}{\kappa^2}\right)|Tx-x|^2F+2\frac{1}{\kappa^2}H\right)G\,(z-Sx) \\ &+|Tx-x|^2G^2-\frac{1}{\kappa^2}\left(H-|Tx-x|^2F\right)^2=0,\end{aligned} \tag{2.11}$$

notice that when $\kappa = 1$ this is a quadratic polynomial in $z - Sx$. In order for this equation to hold for all x_1, x_2, z on an open set we must have that all four coefficients of the powers of $z - Sx$ are equal 0. In fact, given (x_1, x_2) if we evaluate the above expression at $z = S(x_1, x_2)$, then we get that the constant coefficient must be 0. We next take the derivative with respect to z of the fourth order polynomial and evaluate again at $z = S(x_1, x_2)$, yields that the quantity multiplying $z - Sx$ must be 0. Taking more derivatives with respect to z we obtain the desired result. Notice that $F = G = H = 0$ is a necessary and sufficient condition for the fourth order polynomial to be zero in a neighborhood of a point.

Thus, we have proved that (2.9) implies that $F = G = H = 0$. And reciprocally, if $F = G = H = 0$, then (2.9) holds.

Let us then assume that $F = G = H = 0$ and proceed to find the solution u to (2.2).

Since $F = 0$, it follows that there exists a potential function $w = w(x_1, x_2)$ satisfying $w_{x_1}(x_1, x_2) = T_1x - x_1$ and $w_{x_2}(x_1, x_2) = T_2x - x_2$. Inserting this in $H = 0$, we get that w must solve the quasilinear pde

$$(w_{x_1}^2 - w_{x_2}^2)w_{x_1x_2} + w_{x_1}w_{x_2}(w_{x_2x_2} - w_{x_1x_1}) = 0. \tag{2.12}$$

The condition $G = 0$ yields a linear pde for $S = S(x, y)$ given by

$$w_{x_1}S_{x_2} - w_{x_2}S_{x_1} = 0.$$

At this point we have proved that given a mapping $(T_1(x_1, x_2), T_2(x_1, x_2), S(x_1, x_2))$ satisfying the three conditions $F = G = H = 0$ and setting

(2.13)

$$F_1(x_1, x_2, z) = \frac{T_1(x_1, x_2) - x_1}{\frac{1}{\kappa}\sqrt{(T_1(x_1, x_2) - x_1)^2 + (T_2(x_1, x_2) - x_2)^2 + (S(x_1, x_2) - z)^2} + z - S(x_1, x_2)}$$

$$F_2(x_1, x_2, z) = \frac{T_2(x_1, x_2) - x_2}{\frac{1}{\kappa}\sqrt{(T_1(x_1, x_2) - x_1)^2 + (T_2(x_1, x_2) - x_2)^2 + (S(x_1, x_2) - z)^2} + z - S(x_1, x_2)},$$

we have that $(F_1)_{x_2} + (F_1)_zF_2 = (F_2)_{x_1} + (F_2)_zF_1$.

We now prove that there exists a function $u = u(x_1, x_2)$ that locally solves the system $u_{x_1}(x_1, x_2) = F_1(x_1, x_2, u(x_1, x_2))$ and $u_{x_2}(x_1, x_2) = F_2(x_1, x_2, u(x_1, x_2))$ and passes through a given point, i.e., $u(x_1^0, x_2^0) = y_0$. This is proved in general in [Har02, Chapter VI], but give a proof in this simpler case for convenience of the reader. Indeed, since we assume that $\mathcal{T}$ is C^1, then F_1 and F_2 are Lipschitz and so the existence and uniqueness theorems for odes apply. To prove this, let ϕ solve the ode $\phi'(x_2) = F_2(x_0^1, x_2, \phi(x_2))$ with $\phi(x_2^0) = y_0$, for x_2 in some open interval $I_{(x_1^0, x_2^0)}$ containing x_2^0. Next, for each fixed $x_2 \in I_{(x_1^0, x_2^0)}$, let $\psi(x_1, x_2)$ solve the ode $\dfrac{d\psi(x_1, x_2)}{dx_1} = F_1(x_1, x_2, \psi(x_1, x_2))$ with $\psi(x_1^0, x_2) = \phi(x_2)$ for $x_1 \in J$ an interval around x_1^0 whose size depends on x_2.

Define $u(x_1, x_2) = \psi(x_1, x_2)$. It follows that $u(x_1^0, x_2^0) = y_0$ and $u_{x_1}(x_1, x_2) = F_1(x_1, x_2, u(x_1, x_2))$.

It remains to show that $u_{x_2}(x_1, x_2) = F_2(x_1, x_2, u(x_1, x_2))$.

Since $u_{x_1}(x_1, x_2) = F_1(x_1, x_2, u(x_1, x_2))$, we have

$$u_{x_1x_2}(x_1, x_2) = \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial x_2} + \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial z}u_{x_2}(x_1, x_2).$$

This means that for each fixed x_2 the function $u_{x_2}(x_1, x_2)$ solves the ODE

$$\frac{\partial}{\partial x_1}(u_{x_2}(x_1, x_2)) = \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial x_2} + \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial z}u_{x_2}(x_1, x_2).$$

For fixed x_2 consider the function $F_2(x_1, x_2, u(x_1, x_2))$. We have that

$$\begin{aligned}
&\frac{\partial}{\partial x}(F_2(x_1, x_2, u(x_1, x_2)))\\
&= \frac{\partial F_2(x_1, x_2, u(x_1, x_2))}{\partial x_1} + \frac{\partial F_2(x_1, x_2, u(x_1, x_2))}{\partial z} u_{x_1}(x_1, x_2)\\
&= \frac{\partial F_2(x_1, x_2, u(x_1, x_2))}{\partial x_1} + \frac{\partial F_2(x_1, x_2, u(x_1, x_2))}{\partial z} F_1(x_1, x_2, u(x_1, x_2))\\
&= \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial x_2} + \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial z} F_2(x_1, x_2, u(x_1, x_2)),
\end{aligned}$$

where in the last equality we have used the condition satisfied by F_1 and F_2.

We have shown that, for each fixed x_2, both functions $u_{x_2}(x_1, x_2)$ and $F_2(x_1, x_2, u(x_1, x_2))$ solve the linear ODE

$$p'(x_1) = \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial x_2} + \frac{\partial F_1(x_1, x_2, u(x_1, x_2))}{\partial z} p(x_1),$$

and satisfy $u_{x_2}(x_1^0, x_2) = F_2(x_1^0, x_2, u(x_0, x_2))$. Therefore $u_{x_2}(x_1, x_2) = F_2(x_1, x_2, u(x_1, x_2))$ for all (x_1, x_2) near (x_1^0, x_2^0).

Therefore we have proved that the system (2.2) has a solution under the assumption (2.9). This completes the proof of the theorem.

□

Example 2.2 (Examples)**.** We give examples of mappings that verify $F = H = G = 0$, and so (2.9).

Let $w = w(x_1, x_2)$ solve the pde

$$\sqrt{w_{x_1}^2 + w_{x_2}^2} = g(w), \tag{2.14}$$

for some function g, then w satisfies (2.12). Conversely we notice in passing that, if w satisfies (2.12), then $w_{x_1}^2 + w_{x_2}^2$ equals a function of w*.

Let $T_1(x_1, x_2) = x_1 + w_{x_1}(x_1, x_2)$, $T_2(x_1, x_2) = x_2 + w_{x_2}(x_1, x_2)$, and let $S(x_1, x_2) = h(w(x_1, x_2))$ for some function h, where w is a solution to (2.14) for some g. Then, it is easy to check that the mapping (T_1, T_2, S) verifies the conditions $F = G = H = 0$. Therefore the system (2.2) has a solution given by the function $u(x_1, x_2) = f(w(x_1, x_2))$ where f is a solution of the ODE

$$f'(t) = \frac{1}{\frac{1}{\kappa}\sqrt{g(t)^2 + (f(t) - h(t))^2} + f(t) - h(t)}. \tag{2.15}$$

If we choose w and h such that $T_1 = 0, T_2 = 0$ and $S = M > 0$ (that is, u reflects/refracts all points to $(0, 0, M)$), then u is a paraboloid when $\kappa = 1$, and it is

*Let $u = f(w)$. We have that u verifies $u_x w_y - u_y w_x = 0$. From (2.12), $v = |Dw|^2$ also verifies the same equation. If we set $u(x, 0) = f(w(x, 0)) = v(x, 0)$, then $z = w(x, 0)$ for a unique x if w is increasing in x. So we choose x such that $f(w(x, 0)) = v(x, 0)$.

ellipsoid when $\kappa > 1$. We have $w(x_1, x_2) = -\frac{1}{2}(x_1^2 + x_2^2) + c$, $g(w) = (2(c - w))^{\frac{1}{2}}$ and $h = M$. So f solves the ODE

$$f'(t) = \frac{1}{\frac{1}{\kappa}\sqrt{2(c-t) + (f(t) - M)^2} + f(t) - M}.$$

This means $\frac{1}{\kappa}\sqrt{(f(t) - M)^2 + 2(c-t)} + f(t) - M = b$. If $\kappa = 1$, then $f(t) = \frac{b^2 + 2t - 2c}{2b} + M$ with $b > 0$ arbitrary. Hence $u(x_1, x_2) = \frac{b^2 - x_1^2 - x_2^2}{2b} + M$, a paraboloid with focus at $(0, 0, M)$. If $\kappa > 1$, then we get $f(t) = \frac{-\frac{1}{\kappa}b - \sqrt{b^2 + (1 - \frac{1}{\kappa^2})2(t - c)}}{1 - \frac{1}{\kappa^2}} + M$. Hence $u(x_1, x_2) = \frac{-\frac{1}{\kappa}b - \sqrt{b^2 - (1 - \frac{1}{\kappa^2})(x_1^2 + x_2^2)}}{1 - \frac{1}{\kappa^2}} + M$, an ellipsoid with upper focus at $(0, 0, M)$. The case $\kappa < 1$ is similar.

The case when all rays are reflected/refracted into a vertical segment, is when $T_1 = x_1 + w_{x_1} = 0$, $T_2 = x_2 + w_{x_2} = 0$, and $S = h(w(x_1, x_2))$ where h is an arbitrary positive function. Again we have $w(x, y) = -\frac{1}{2}(x^2 + y^2)$, $g(w) = (-2w)^{\frac{1}{2}}$ and we get $u(x, y) = f(w(x, y))$ where f solves the ODE (2.15) with $g(t) = (-2t)^{1/2}$.

If instead we let $w(x, y) = \sqrt{x^2 + y^2} - \frac{1}{2}(x^2 + y^2)$ then $T_1 = \frac{x}{\sqrt{x^2 + y^2}}$ and $T_2 = \frac{y}{\sqrt{x^2 + y^2}}$. Let $S = 0$ and $u(x, y) = f(w(x, y))$ where f solves the ODE

$$f'(t) = \frac{1}{\frac{1}{\kappa}\sqrt{1 - 2t + f(t)^2} + f(t)}.$$

then all points are refracted(reflected) to the unit circle on the xy plane.

The last example is to find a reflector such that all points emanating from Ω given are reflected to points of a given curve $y = f(x)$ on the xy-plane (without prescribing the exact destination of each point). This means that given (x, y) near the origin we must find T_1 and T_2 such that $T_2(x, y) = f(T_1(x, y))$. We claim that we can find T_1 and T_2 satisfying $F = H = 0$ as defined above.

We define T_1 implicitly by the formula $T_1(x, y) - x + (f(T_1(x, y)) - y)f'(T_1(x, y)) = 0$ and $T_2(x, y) = f(T_1(x, y))$. From implicit function theorem it is enough to assume that f satisfies the condition $1 + (f'(z))^2 + (f(z) - y)f''(z) \neq 0$ for y near zero. This means given (x, y) near $(0, 0)$ there is a unique z such that $z - x + (f(z) - y)f'(z) = 0$.

We now show that T_1, T_2 satisfy $F = 0$. First, differentiating implicitly, one can check that $\frac{\partial T_1(x, y)}{\partial y} = f'(T_1(x, y))\frac{\partial T_1(x, y)}{\partial x}$ and since $T_2(x, y) = f(T_1(x, y))$, we also have $\frac{\partial T_2(x, y)}{\partial x} = f'(T_1(x, y))\frac{\partial T_1(x, y)}{\partial x}$. Therefore, $F = 0$.

We check $H = 0$. Notice that $\dfrac{\partial T_2(x,y)}{\partial y} = f'(T_1(x,y))\dfrac{\partial T_1(x,y)}{\partial y} = (f'(T_1(x,y))^2\dfrac{\partial T_1(x,y)}{\partial x}$.
We have

$$H = (T_1(x,y)-x)^2 f'(T_1(x,y))\frac{\partial T_1(x,y)}{\partial x} - (f(T_1(x,y))-y)^2 f'(T_1(x,y))\frac{\partial T_1(x,y)}{\partial x} + (T_1(x,y)-x)(f(T_1(x,y))-y)\left((f'(T_1(x,y)))^2\frac{\partial T_1(x,y)}{\partial x} - \frac{\partial T_1(x,y)}{\partial x}\right).$$

Inserting $T_1(x,y) - x = -(f(T_1(x,y)) - y)f'(T_1(x,y))$, we get $H = 0$.

Therefore, there exists a solution u that reflects according with the mapping $(T_1, f(T_1), 0)$ near a given point.

An equivalent way of solving this problem is as follows. Let u be defined by the formula

$$u(x,y) = \sup\left\{\frac{1}{2}(1-(x-t)^2-(y-f(t))^2) : t \in [a,b]\right\}.$$

One can check that each point (x,y) is reflected by u to the point $(T_1(x,y), f(T_1(x,y)), 0)$, where $(T_1(x,y), f(T_1(x,y)))$ is the point on the curve $C = \{y = f(x)\}$ where the distance to (x,y) is attained.

For refraction we define

$$u(x,y) = \inf\left\{\frac{-\frac{1}{\kappa} - \sqrt{1-(1-\frac{1}{\kappa^2})((x-t)^2+(y-f(t))^2)}}{1-\frac{1}{\kappa^2}} + M : t\in[a,b]\right\},$$

and each point (x,y) is refracted by u to the point $(T_1(x,y), f(T_1(x,y)), 0)$ as before.

Remark 2.3. We analyze here the case when the given mapping (T_1, T_2, S) is such that the condition $F = G = H = 0$ in the neighborhood of a point is not satisfied. We notice that if the system (2.2) would have a C^2 solution u, then following the calculations in Subsection 2.2, this time using (2.3), we would get that u verifies the polynomial equation (2.11) with $z = u(x_1, x_2)$, and therefore u could be determined by solving such equation.

If for example $F \neq 0$ and $G = 0$, then it follows that u is determined up to a sign from the equation

(2.16)
$$\left(1-\frac{1}{\kappa}\right)F^2(u-Sx)^4+\left(\left(1-\frac{2}{\kappa^2}\right)|Tx-x|^2F^2+\frac{2}{\kappa^2}HF\right)(u-Sx)^2-\frac{1}{\kappa^2}\left(H-|Tx-x|^2F\right)^2 = 0.$$

If for instance we consider the magnification mapping $T_1(x_1,x_2) = a+\delta x_1$, $T_2(x_1,x_2) = b+\beta x_2$, and $S(x_1,x_2) = M$, then $F = G = 0$, and $H = (a+(\delta-1)x_1)(b+(\beta-1)x_2)(\beta-\delta)$. If for this map there would be a solution u to the system (2.2), then from (2.11) we would get $H = 0$ and therefore for this mapping there are no solutions u unless $\beta = \delta$. A similar situation occurs in the far field case

Similarly, if we consider the mapping $T_1(x_1,x_2) = a+\delta x_2$, $T_2(x_1,x_2) = b+\beta x_1$ and $S(x_1,x_2) = M$, then $F = \delta - \beta$, $G = 0$, and $H = \delta(a+\delta x_2 - x_1)^2 - \beta(b+\beta x_1 - x_2)^2$. One can check that the function u obtained from (2.16) does not solve the problem and hence there is no solution.

As an example for reflection, i.e. $\kappa = 1$, we consider the mapping $T_1 = x_2$, $T_2 = 0$, and $S = 0$. In this case, $F = 1$, $H = (x_2 - x_1)^2$ and $G = 0$. If the system (2.2) would have a solution u, then (2.16) would have solutions $u = \dfrac{x_2^2}{((x_2 - x_1)^2 - x_2^2)^{1/2}}$ or $u = -\dfrac{x_2^2}{((x_2 - x_1)^2 - x_2^2)^{1/2}}$. A calculation shows that none of these functions solve (2.2) and hence this system has no solution.

Once again, in the case of reflection, $\kappa = 1$, we consider mappings of the form $T_1 = 0$, $T_2 = f(y)$ and $S = g(x, y)$. In this case, $F = 0$, $H = -(f(y) - y)xf'(y)$ and $G = -x\dfrac{\partial g(y)}{\partial y} - (f(y) - y)\dfrac{\partial g(y)}{\partial x}$. The formula for the possible solution is obtained from $2GH(u - g(x, y)) + (x^2 + (f(y) - y)^2)G^2 - H = 0$. One can check from this that mappings of the form $T_1 = 0$,$T_2 = \delta y + a$ and $S = \beta x + b$ have no solutions.

2.3. **Existence of solutions refracting a given curve into another.** Suppose we have a curve $\gamma(s) = (x(s), y(s))$ on the xy-plane, and a curve in 3-d given by $\Gamma(s) = (\phi(s), \psi(s), h(s))$. We consider the problem of finding a refractor (reflector) surface $z = u(x, y)$ such that the ray starting from the point $\gamma(s)$ and moving upwards, is refracted (or reflected) by graph of u into the point $\Gamma(s)$, at least locally near a given point on γ.

We show using Cauchy-Kovalevsky's theorem that this problem is solvable. Let $z(s)$ satisfy the equation

$$z'(s) = (\phi(s) - x(s))x'(s) + (\psi(s) - y(s))y'(s).$$

Next, let w solve the Cauchy problem,

$$(w_x^2 - w_y^2)w_{xy} + w_x w_y(w_{yy} - w_{xx}) = 0,$$

with

$$\begin{aligned} w(x(s), y(s)) &= z(s) \\ w_x(x(s), y(s)) &= \phi(s) - x(s) \\ w_y(x(s), y(s)) &= \psi(s) - y(s). \end{aligned}$$

This problem is solvable provided the initial curves are non-characteristic [Joh82, Chapter 2, pp. 33-34] and the data are analytic. In particular, to have a solution w

the initial curves must satisfy

(2.17)

$$\det\begin{bmatrix} x'(s) & y'(s) & 0\\ 0 & x'(s) & y'(s)\\ -(\phi(s)-x(s))(\psi(s)-y(s)) & (\phi(s)-x(s))^2-(\psi(s)-y(s))^2 & (\phi(s)-x(s))(\psi(s)-y(s))\end{bmatrix}$$
$$= -(\phi(s)-x(s))(\psi(s)-y(s))(y'(s))^2 - ((\phi(s)-x(s))^2-(\psi(s)-y(s))^2)x'(s)y'(s)$$
$$+ (\phi(s)-x(s))(\psi(s)-y(s))(x'(s))^2 \neq 0.$$

We now define $T_1(x,y) = x + w_x(x,y)$, $T_2(x,y) = y + w_y(x,y)$, and notice that $T_1(x(s),y(s)) = \phi(s)$ and $T_2(x(s),y(s)) = \psi(s)$. Next, let $S(x,y)$ solve the linear pde

$$(T_2(x,y)-y)S_x - (T_1(x,y)-x)S_y = 0 \tag{2.18}$$

with initial condition $S(x(s),y(s)) = h(s)$.

At this point we have constructed a mapping $(T_1(x,y),T_2(x,y),S(x,y))$ that satisfies the conditions $F = G = H = 0$, with F,G,H defined in (2.7), and maps the curve γ into the 3-d curve $(\phi(s),\psi(s),h(s))$. From the existence result in Subsection 2.2, we can find a function u passing through a given point that solves

$$u_x(x,y) = F_1(x,y,u(x,y)),\ \text{and}\ u_y(x,y) = F_2(x,y,u(x,y))$$

with F_1 and F_2 as defined by (2.13).

The graph of u is the desired refracting surface.

The solutions u and w can be related as follows. Notice that both u and w solve the linear pde (2.18). If $u(x(s),y(s)) = f(z(s))$ for some function f, then $u(x,y) = f(w(x,y))$ for all (x,y). Because $u(x,y)$ and $f(w(x,y))$ solve (2.18) and coincide on the curve γ.

3. Case when the rays emanate from a point source

We consider the refractor problem for rays emanating from the origin in $\mathbb{R}^3$. We use spherical coordinates: $r(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$.

Consider a surface $\mathcal{S}$ given parametrically by

$$\mathcal{S} = \{S(\theta,\phi) = s(\theta,\phi)r(\theta,\phi) : \theta\in[0,2\pi], \phi\in[0,\pi]\}$$

where s is a positive scalar function.

We are given a transformation $Y(\theta,\phi) = (y_1(\theta,\phi), y_2(\theta,\phi), y_3(\theta,\phi))$ and we seek for a surface $\mathcal{S}$ so that each ray emanating from the origin in the direction $r(\theta,\phi)$ is refracted off by the surface $\mathcal{S}$ into the point $Y(\theta,\phi) = (y_1(\theta,\phi), y_2(\theta,\phi), y_3(\theta,\phi))$. We are going to determine conditions on the mapping Y so that the refracting surface $\mathcal{S}$ exists. In fact, we will derive a system of first order pdes for the scalar function $s(\theta,\phi)$, that will be solvable under certain assumptions on Y.

Let us use the notation $Y_\phi = \left(\dfrac{\partial y_1(\theta,\phi)}{\partial\phi}, \dfrac{\partial y_2(\theta,\phi)}{\partial\phi}, \dfrac{\partial y_3(\theta,\phi)}{\partial\phi}\right)$, and similarly Y_θ, r_ϕ, etc. Let $N(\theta,\phi)$ be the unit normal to $\mathcal{S}$ such that $\langle r,N\rangle > 0$.

From the law of refraction we have the equation

$$\frac{Y - sr}{|Y - sr|} = \frac{1}{\kappa} r + \beta N \tag{3.1}$$

with $\beta = -\frac{1}{\kappa}\Phi(r \cdot N)$ where Φ is from (2.1); however, the explicit form of β is not used in the following calculations.

We will first derive an expression for N. We have that $N = \dfrac{S_\phi \times S_\theta}{|S_\phi \times S_\theta|}$, and $S_\theta = s_\theta r + s r_\theta$ and $S_\phi = s_\phi r + s r_\phi$. Using that r, r_θ, and r_ϕ are mutually perpendicular, $|r| = |r_\phi| = 1$, and $|r_\theta| = \sin\phi$, we get

$$\begin{aligned} S_\phi \times S_\theta &= s_\phi s (r \times r_\theta) + s_\theta s (r_\phi \times r) + s^2 (r_\phi \times r_\theta) \\ &= -\sin(\phi) s_\phi s r_\phi - \frac{s s_\theta}{\sin(\phi)} r_\theta + s^2 \sin(\phi) r. \end{aligned}$$

Therefore, we obtain

$$N = \frac{s\sin^2(\phi) r - s_\theta r_\theta - \sin^2(\phi) s_\phi r_\phi}{\sin(\phi)\left[\sin^2(\phi) s_\phi^2 + s_\theta^2 + s^2\sin^2(\phi)\right]^{1/2}} := \frac{s\sin^2(\phi) r - s_\theta r_\theta - \sin^2(\phi) s_\phi r_\phi}{\gamma}.$$

Inserting in (3.1) the formula for N, we get

$$\frac{Y - sr}{|Y - sr|} = \frac{1}{\kappa} r + \beta \frac{s\sin^2(\phi) r - s_\theta r_\theta - \sin^2(\phi) s_\phi r_\phi}{\gamma}. \tag{3.2}$$

We write $\dfrac{Y - sr}{|Y - sr|} = ar + br_\theta + cr_\phi$ with

$$a = \frac{\langle Y, r\rangle - s}{|Y - sr|}; \quad b\sin^2(\phi) = \frac{\langle Y, r_\theta\rangle}{|Y - sr|}; \quad c = \frac{\langle Y, r_\phi\rangle}{|Y - sr|}.$$

From (3.2) we get that

$$\begin{aligned} \frac{\langle Y, r\rangle - s}{|Y - sr|} &= \frac{1}{\kappa} + \frac{\beta}{\gamma} s \sin^2(\phi) \\ \frac{\langle Y, r_\theta\rangle}{|Y - sr|} &= -\frac{\beta}{\gamma}\sin^2(\phi) s_\theta \\ \frac{\langle Y, r_\phi\rangle}{|Y - sr|} &= -\frac{\beta}{\gamma}\sin^2(\phi) s_\phi. \end{aligned}$$

From the first equality we get that $-\dfrac{\beta}{\gamma} = \dfrac{\frac{1}{\kappa}|Y - sr| + s - \langle Y, r\rangle}{s\sin^2(\phi)|Y - sr|}$ and inserting this in the last two equations we get that the scalar function s satisfies the system

$$s_\theta = \frac{s\langle Y, r_\theta\rangle}{\frac{1}{\kappa}|Y - sr| + s - \langle Y, r\rangle} := F_1(\theta, \phi, s) \tag{3.3}$$

and

$$s_\phi = \frac{s\langle Y, r_\phi\rangle}{\frac{1}{\kappa}|Y - sr| + s - \langle Y, r\rangle} := F_2(\theta, \phi, s). \tag{3.4}$$

Now, we will find conditions on the mapping Y so that the system (3.3) and (3.4) can be solved. We will require that

$$\frac{\partial F_1}{\partial \phi} + \frac{\partial F_1}{\partial s} F_2 = \frac{\partial F_2}{\partial \theta} + \frac{\partial F_2}{\partial s} F_1 \tag{3.5}$$

for all $s > 0$ and for all θ, ϕ near a fixed value θ_0, ϕ_0. We will now derive conditions on the mapping Y so that (3.5) holds.

If we set $Q = \dfrac{1}{\kappa} R^{1/2} + s - \langle Y, r\rangle$ and $R = |Y|^2 - 2s\langle Y, r\rangle + s^2$, then we can write

$$F_1 = sQ^{-1}\langle Y, r_\theta\rangle, \quad \text{and} \quad F_2 = sQ^{-1}\langle Y, r_\phi\rangle.$$

Then

$$\frac{\partial F_1}{\partial \phi} + \frac{\partial F_1}{\partial s} F_2 = sQ^{-1}\langle Y, r_\phi\rangle\langle Y, r_\theta\rangle(Q^{-1} - sQ^{-2}Q_s) - sQ^{-2}\langle Y, r_\theta\rangle Q_\phi + sQ^{-1}\frac{\partial}{\partial \phi}\langle Y, r_\theta\rangle,$$

and

$$\frac{\partial F_2}{\partial \theta} + \frac{\partial F_2}{\partial s} F_1 = sQ^{-1}\langle Y, r_\phi\rangle\langle Y, r_\theta\rangle(Q^{-1} - sQ^{-2}Q_s) - sQ^{-2}\langle Y, r_\phi\rangle Q_\theta + sQ^{-1}\frac{\partial}{\partial \theta}\langle Y, r_\phi\rangle.$$

Therefore, (3.5) is equivalent to

$$\langle Y, r_\theta\rangle Q_\phi - Q\frac{\partial}{\partial \phi}\langle Y, r_\theta\rangle = \langle Y, r_\phi\rangle Q_\theta - Q\frac{\partial}{\partial \theta}\langle Y, r_\phi\rangle. \tag{3.6}$$

By computation

$$Q_\theta = \frac{\langle Y, Y_\theta\rangle - (s + \kappa R^{1/2})\frac{\partial}{\partial \theta}\langle Y, r\rangle}{\kappa R^{1/2}},$$

and

$$Q_\phi = \frac{\langle Y, Y_\phi\rangle - (s + \kappa R^{1/2})\frac{\partial}{\partial \phi}\langle Y, r\rangle}{\kappa R^{1/2}}.$$

Inserting these in (3.6) and multiplying the resulting expression by $\kappa R^{1/2}$ yields

$$\langle Y, r_\theta\rangle \left\{\langle Y, Y_\phi\rangle - (s + \kappa R^{1/2})\frac{\partial}{\partial \phi}\langle Y, r\rangle\right\} - \left(R + \kappa R^{1/2}(s - \langle Y, r\rangle)\right)\frac{\partial}{\partial \phi}\langle Y, r_\theta\rangle$$
$$= \langle Y, r_\phi\rangle \left\{\langle Y, Y_\theta\rangle - (s + \kappa R^{1/2})\frac{\partial}{\partial \theta}\langle Y, r\rangle\right\} - \left(R + \kappa R^{1/2}(s - \langle Y, r\rangle)\right)\frac{\partial}{\partial \theta}\langle Y, r_\phi\rangle.$$

This is equivalent to

$$\kappa R^{1/2}\left\{\langle Y,r_\phi\rangle\frac{\partial}{\partial\theta}\langle Y,r\rangle+(s-\langle Y,r\rangle)\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle-\langle Y,r_\theta\rangle\frac{\partial}{\partial\phi}\langle Y,r\rangle-(s-\langle Y,r\rangle)\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle\right\}$$
$$=\langle Y,r_\phi\rangle\langle Y,Y_\theta\rangle-s\langle Y,r_\phi\rangle\frac{\partial}{\partial\theta}\langle Y,r\rangle-R\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle-\langle Y,r_\theta\rangle\langle Y,Y_\phi\rangle$$
$$+s\langle Y,r_\theta\rangle\frac{\partial}{\partial\phi}\langle Y,r\rangle+R\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle.$$

If we let

$$\begin{aligned}
A&=\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle-\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle\\
B&=\langle Y,r_\phi\rangle\frac{\partial}{\partial\theta}\langle Y,r\rangle-\langle Y,r\rangle\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle-\langle Y,r_\theta\rangle\frac{\partial}{\partial\phi}\langle Y,r\rangle+\langle Y,r\rangle\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle\\
C&=\langle Y,r_\theta\rangle\frac{\partial}{\partial\phi}\langle Y,r\rangle-2\langle Y,r\rangle\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle-\langle Y,r_\phi\rangle\frac{\partial}{\partial\theta}\langle Y,r\rangle+2\langle Y,r\rangle\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle\\
D&=\langle Y,r_\phi\rangle\langle Y,Y_\theta\rangle-\langle Y,r_\theta\rangle\langle Y,Y_\phi\rangle+|Y|^2\left(\frac{\partial}{\partial\phi}\langle Y,r_\theta\rangle-\frac{\partial}{\partial\theta}\langle Y,r_\phi\rangle\right),
\end{aligned}$$

then

$$\kappa(s^2-2s\langle Y,r\rangle+|Y|^2)^{1/2}(As+B)=-As^2+Cs+D.$$

Squaring both sides, we get

$$\kappa^2(s^2-2s\langle Y,r\rangle+|Y|^2)(As+B)^2=(-As^2+Cs+D)^2. \tag{3.7}$$

Expanding the powers in (3.7) yields

(3.8)
$$(\kappa^2-1)A^2s^4+2As^3\left\{\kappa^2B-\kappa^2\langle Y,r\rangle A+C\right\}+s^2\left\{\kappa^2B^2-4\kappa^2\langle Y,r\rangle AB+\kappa^2A^2|Y|^2+2AD-C^2\right\}$$
$$+s\left\{2\kappa^2AB|Y|^2-2\kappa^2\langle Y,r\rangle B^2-2CD\right\}+\kappa^2|Y|^2B^2-D^2=0,$$

for all $s>0$ and (θ,ϕ) close to (θ_0,ϕ_0). Setting $E=\langle Y,r_\phi\rangle\langle Y_\theta,r\rangle-\langle Y,r_\theta\rangle\langle Y_\phi,r\rangle$, we get after a short calculation that

$$\begin{aligned}
A&=\langle Y_\theta,r_\phi\rangle-\langle Y_\phi,r_\theta\rangle\\
B&=-\langle Y,r\rangle A+E\\
C&=2\langle Y,r\rangle A-E\\
D&=\langle Y,r_\phi\rangle\langle Y,Y_\theta\rangle-\langle Y,r_\theta\rangle\langle Y,Y_\phi\rangle+|Y|^2A.
\end{aligned}$$

We can also re-write

$$\begin{aligned}
E&=\langle\langle Y,r_\phi\rangle Y_\theta-\langle Y,r_\theta\rangle Y_\phi,r\rangle\\
D&=\langle\langle Y,r_\phi\rangle Y_\theta-\langle Y,r_\theta\rangle Y_\phi,Y\rangle+|Y|^2A.
\end{aligned}$$

Notice that sufficient conditions for (3.8) to hold are that

$$\begin{aligned}\langle Y_\theta, r_\phi\rangle - \langle Y_\phi, r_\theta\rangle &= 0\\ \langle\langle Y, r_\phi\rangle Y_\theta - \langle Y, r_\theta\rangle Y_\phi, r\rangle &= 0\\ \langle\langle Y, r_\phi\rangle Y_\theta - \langle Y, r_\theta\rangle Y_\phi, Y\rangle &= 0.\end{aligned} \tag{3.9}$$

We therefore have the following theorem.

Theorem 3.1. *Let Y be a C^1 mapping on a neighborhood U of (θ_0, ϕ_0) such that*

$$\begin{aligned}\langle Y_\theta, r_\phi\rangle - \langle Y_\phi, r_\theta\rangle &= 0\\ \langle\langle Y, r_\phi\rangle Y_\theta - \langle Y, r_\theta\rangle Y_\phi, r\rangle &= 0\\ \langle\langle Y, r_\phi\rangle Y_\theta - \langle Y, r_\theta\rangle Y_\phi, Y\rangle &= 0,\end{aligned} \tag{3.10}$$

for all $(\theta, \phi) \in U$. Let $X_0 = s_0 r(\theta_0, \phi_0)$ and $Y_0 = Y(\theta_0, \phi_0)$. Assume that $\left(\frac{Y_0 - X_0}{|Y_0 - X_0|}\right) \cdot \frac{X_0}{|X_0|} > \frac{1}{\kappa}$ if $\kappa > 1$, $\left(\frac{Y_0 - X_0}{|Y_0 - X_0|}\right) \cdot \frac{X_0}{|X_0|} > \kappa$ if $\kappa < 1$, and $\left(\frac{Y_0 - X_0}{|Y_0 - X_0|}\right) \cdot \frac{X_0}{|X_0|}\rangle \neq 1$ if $\kappa = 1$.

Then there exists a unique surface S defined parametrically for (θ, ϕ) in a neighborhood V of (θ_0, ϕ_0) passing through $X_0 = s_0\, r(\theta_0, \phi_0)$ such that each ray emanating from the origin with direction $r(\theta, \phi)$ with $(\theta, \phi) \in V$ is refracted by $\mathcal{S}$ into the destination $Y(\theta, \phi)$.

When $\kappa = 1$ the conclusion is similar but the surface $\mathcal{S}$ reflects.

Example 3.2. We give some examples of mappings that satisfy the conditions.

If $\kappa = 1$ and Y is the constant mapping $Y = Y_0 = (x_0, y_0, z_0)$, then the solutions of the system (3.3), (3.4) are given by $s(\theta, \phi) = \dfrac{b^2 - |Y_0|^2}{2(b - \langle r, Y_0\rangle)}$, and the corresponding surfaces $S = \{s(\theta, \phi) r(\theta, \phi) : \theta \in [0, 2\pi], \phi \in [0, \pi]\}$ are ellipsoids of revolution with foci 0 and Y_0. For $\kappa \neq 1$ we get that the solutions to (3.3), (3.4) satisfy the equation $|sr - Y_0| + \frac{1}{\kappa} s = b$, that is, the corresponding surfaces are Descartes ovals.

If $Y = (\cos\theta, \sin\theta, 0)$, then Y satisfies the conditions (3.10), and the solution s to (3.3), (3.4) depends only on ϕ.

Remark 3.3. When at least one of the coefficients in the polynomial (3.8) is different from zero then the only possible solution of the problem is obtained by solving this polynomial equation in s.

For example if M is a 3×3 constant symmetric matrix (not a multiple of the identity) and $Y(\theta, \phi) = Mr(\theta, \phi)$, then we have $A = B = C = 0$ and $D \neq 0$ and hence the problem has no solution.

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