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CONTINUITY ESTIMATES FOR POROUS MEDIUM TYPE EQUATIONS WITH MEASURE DATA

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ABSTRACT. We consider parabolic equations of porous medium type of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = \mu \qquad \text{in } E_T,$$

in some space time cylinder E_T. The most prominent example covered by our assumptions is the classical porous medium equation

$$u_t - \Delta u^m = \mu \qquad \text{in } E_T.$$

We establish a sufficient condition for the continuity of u in terms of a natural Riesz potential of the right-hand side measure μ. As an application we come up with a borderline condition ensuring the continuity of u: more precisely, if $\mu \in L\big(\frac{N+2}{2}, 1\big)$, then u is continuous in E_T.

1. INTRODUCTION

In this paper we establish a characterization ensuring the continuity of solutions of non-homogeneous porous medium type equations whose most prominent example is given by the classical porous medium equation

$$u_t - \operatorname{div}\big(\mathbf{a}(x,t)Du^m\big) = \mu \qquad \text{in } E_T, \tag{1.1}$$

where the matrix $\mathbf{a}$ is only measurable and positive-definite in E_T. Here, E_T stands for the space-time cylinder of height $T > 0$ over a bounded open domain $E \subset \mathbb{R}^N$, $N \geq 2$. The inhomogeneity μ is a non-negative Radon-measure on E_T with finite total mass $\mu(E_T) < \infty$. Without loss of generality, we assume that the measure μ is defined on $\mathbb{R}^{N+1}$ by letting $\mu\lfloor\big(\mathbb{R}^{N+1} \setminus E_T\big) = 0$. More generally, we consider porous medium type equations of the type

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = \mu \qquad \text{in } E_T. \tag{1.2}$$

For the vector-field $\mathbf{A}\colon E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ we assume that it is measurable with respect to $(x,t) \in E_T$ for all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (u,ξ) for a.e. $(x,t) \in E_T$, and moreover satisfies the following growth and ellipticity conditions:

$$\begin{cases} \mathbf{A}(x,t,u,\xi) \cdot \xi \geq mC_o|u|^{m-1}|\xi|^2, \\ |\mathbf{A}(x,t,u,\xi)| \leq mC_1|u|^{m-1}|\xi|, \end{cases} \tag{1.3}$$

whenever $(x,t) \in E_T$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, for some $0 < C_o \leq C_1 < \infty$. Throughout the paper we consider the case $m \geq 1$, i.e. we are concerned with the *degenerate case* in the porous medium equation.

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In [16, 1] we established a sufficient condition ensuring the local boundedness of weak solutions to (1.2). More precisely we proved for given $\lambda \in (0, \frac{1}{N}]$ the following pointwise bound for solutions to (1.2):

$$u(z_o) \le 2\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \boldsymbol{\gamma}\bigg[\frac{1}{r^{n+2}}\iint_{Q_{r,\theta}(z_o)} u^{m+\lambda}\,dxdt\bigg]^{\frac{1}{1+\lambda}} + \boldsymbol{\gamma}\,\mathbf{I}^{\mu}_2(z_o,r,\theta) \tag{1.4}$$

which holds true whenever $Q_{r,\theta}(z_o) \subset E_T$ for a.e. $z_o \in E_T$ with a universal constant $\boldsymbol{\gamma}$ depending only on the data N, m, C_o, C_1, and on λ. Here, the localized (or truncated) parabolic Riesz potential is defined by

$$\mathbf{I}^{\mu}_{\beta}(z_o,r,\theta) := \int_0^r \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2}(z_o))}{\varrho^{N+2-\beta}}\frac{d\varrho}{\varrho}\,, \qquad \beta\in(0,N+2]\,,$$

for $z_o \in E_T$ and $r,\theta > 0$ such that $Q_{r,\theta}(z_o) \subset E_T$. Here, $Q_{\varrho,\varrho^2\theta/r^2}(z_o)$ stands for a general parabolic cylinder in E_T, see § 2.1. In the case that $\theta = r^2$ the potential $\mathbf{I}^{\mu}_{\beta}$ reduces to the standard localized parabolic Riesz potential. The pointwise estimate (1.4) implies the following boundedness criterion: If $I^{\mu}_2(\cdot,r,r^2) \in L^{\infty}_{\rm loc}(E_T)$ for some $r > 0$, then $u \in L^{\infty}_{\rm loc}(E_T)$. In view of this recent result, we deal with *locally bounded* weak energy solutions, of which we now give the precise definition.

Definition 1.1 (locally bounded, weak energy solution)**.** *Consider a non-negative measurable function $u\colon E_T \to \mathbb{R}$ satisfying*

$$u \in C^0_{\rm loc}\big(0,T;L^2_{\rm loc}(E)\big)\cap L^{\infty}_{\rm loc}(E_T), \quad u^{\frac{m+1}{2}} \in L^2_{\rm loc}\big(0,T;W^{1,2}_{\rm loc}(E)\big); \tag{1.5}$$

it is termed locally bounded, weak energy solution of the porous medium type equation (1.2) *if and only if for every subset $U \Subset E$ and every subinterval $[t_1,t_2] \subset (0,T]$ the following equation*

$$\int_U u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_U \big[-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi\big]\,dxdt = \int_{U\times(t_1,t_2)}\varphi\,d\mu \tag{1.6}$$

holds true for any bounded testing function

$$\varphi \in W^{1,2}_{\rm loc}\big(0,T;L^2(U)\big)\cap L^2_{\rm loc}\big(0,T;W^{1,2}_0(U)\big).$$

In (1.6) *the symbol Du has to be understood in the sense of the following definition:*

$$Du := \tfrac{2}{m+1}\mathbf{1}_{\{u>0\}}u^{\frac{1-m}{2}}Du^{\frac{m+1}{2}}\,.$$

The hypothesis that the testing function φ must be bounded has to be imposed, in order to guarantee that the right-hand side of (1.6) is well defined. All other integrals appearing there are finite, due to the other assumptions on u and φ. The above notion of locally bounded weak energy solution can for instance be retrieved from [7, 11] for the homogeneous, respectively inhomogeneous porous medium equation with a right-hand side $\mu \in L^{\infty}(E_T)$. The notion differs from the most common one, where the regularity condition on $u^{\frac{m+1}{2}}$ is replaced by the assumption $u^m \in L^2_{\rm loc}(0,T;W^{1,2}_{\rm loc}(E))$. The requirements in (1.5) allow the testing of the homogeneous equation by the solution u itself (and not by u^m) and lead to natural energy estimates for u in terms of $Du^{\frac{m+1}{2}}$. For the homogeneous, respectively inhomogeneous equation with a bounded right-hand side $\mu \in L^{\infty}(E_T)$, this notion seems to be the weakest one which allows natural energy estimates. In the following, when talking of solutions, we will omit the term *locally bounded* for the sake of simplicity.

As already mentioned at the beginning of the introduction, we are interested in the continuity properties of weak solutions. Our main result is a sufficient criterion guaranteeing the continuity of solutions. The precise statement is as follows:

Theorem 1.1 (Continuity of weak energy solutions via linear potentials)**.** *Let u be a non-negative, locally bounded, weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity conditions* (1.3)*. Furthermore, consider $E_o \Subset E_T$ and assume that*

$$\lim_{r\downarrow 0} \sup_{z_o\in E_o} \mathbf{I}_2^{\mu}(z_o, r, r^2) = 0 \tag{1.7}$$

holds true. Then, u is continuous in E_o.

The subtle point here is, that the local boundedness of $\mathbf{I}_2^{\mu}(\cdot, r, r^2)$ for some $r > 0$, ensures the local boundedness of u, and moreover implies $\lim_{\varrho\downarrow 0} \mathbf{I}_2^{\mu}(z, \varrho, \varrho^2) = 0$ for a.e. $z \in E_T$, while the information of locally uniform convergence to zero of the Riesz potential from (1.7) implies the continuity of the weak energy solution. With that respect, our results, i.e. the local boundedness and the continuity of weak solutions via Riesz potentials, are of borderline type. It is somewhat surprising that the Riesz $\mathbf{I}_2^{\mu}$ potential plays the same role as in the linear setting. At this stage it would be interesting to consider measures μ for which the Riesz potential $\mathbf{I}_2^{\mu}(\cdot, r, r^2)$ is locally bounded, and moreover satisfies

$$\lim_{\varrho\downarrow 0} \frac{\mu\big(Q_{\varrho,\varrho^2}(z)\big)}{\varrho^N} = 0$$

locally uniformly on E_T with respect to z. By our potential estimate (1.4) weak energy solutions would be locally bounded, and one might conjecture that they are also locally VMO on E_T. Such a result would be between local boundedness and continuity. We will not go into this subject here.

Theorem 1.1 is stated as a result for weak energy solutions, but it also applies to very weak solutions u, as introduced in [1, Definition 1.3] and then built in [1, Theorem 1.4]. Indeed, due to the boundedness of the weak energy solutions u_k making up the approximating sequence of the very weak solution u, it is possible to build the starting cylinder (and consequently, the whole approximating sequence of shrinking cylinders $Q_{r_n,\theta_n}(z_o)$) in a way that is independent of k. A similar argument is discussed, for example, in [4, Chapter 6].

As an application of Theorem 1.1, we consider measures given by measurable functions $\mu \in L^1(E_T)$. In Chapter 5 we establish the following important assertion:

$$\mu \in L\big(\tfrac{N+2}{2}, 1\big) \quad \Longrightarrow \quad u \text{ is locally continuous in } E_T.$$

For the definition of the Lorentz space $L\big(\frac{N+2}{2}, 1\big)$ we refer to (5.2). How subtle this result actually is, can be seen by the classical theory for parabolic equations of the form $u_t - \operatorname{div} \mathbf{A}(x, t, Du) = \mu$ in E_T with coefficients satisfying (1.3) with $m = 1$. Here, it is known that the assumption $\mu \in L^{\frac{N+2}{2}+\varepsilon}(E_T)$, for some arbitrary small $\varepsilon > 0$, implies the continuity of u. This can be retrieved for example from [5, Section IV]. For $\mu \in L^{\frac{N+2}{2}}(E_T)$ solutions might be even unbounded. We note that $L^{\frac{N+2}{2}+\varepsilon} \subset L\big(\frac{N+2}{2}, 1\big)$ for any $\varepsilon > 0$. We mention, that the assumption $\mu \in L\big(\frac{N+2}{2}, 1\big)$ is independent of $m \geq 1$. Finally, an assumption of the type $L^{\frac{N+2}{2}+\varepsilon}(E_T)$ falls into the range of applications covered by Corollary 5.1. Hence in the case $m = 1$, we recover the classical result on the continuity of weak solutions.

Before describing the method of proof, a few words concerning the history of the problem are in order. As far as the regularity for equations with the same structure considered here, with $m > 1$ and $\mu = 0$, is concerned, continuity of solutions was and is still a major issue. An important step forward was the proof that locally bounded solutions are locally Hölder continuous, due to DiBenedetto & Friedman [6]. Hölder continuity for solutions of the Cauchy problem for the prototype equations (1.1), with $\mathbf{a}(x,t) = \mathbb{I}_n$ was established before by Caffarelli & Friedman [3]; their approach relies on the special property of *global* solutions. Continuity of solutions of degenerate parabolic equations $u_t = \Delta(|u|^{m-1}u)$ was proved by Caffarelli & Evans [2], but the modulus of continuity implicit in their proof is essentially of logarithmic kind.

Coming to the method of proof, the continuity of u is, heuristically, the consequence of the following fact: there exists a family of nested and shrinking cylinders $Q_{r_n,\theta_n}(z_o)$, all with the same vertex, such that the oscillation of u in $Q_{r_n,\theta_n}(z_o)$ tends to zero as $n \to \infty$ in a way quantitatively determined by the structure conditions (1.3), and by the measure μ. In order to achieve such a kind of controlled decay, one needs to study separately two cases: Either in the cylinder $Q_{r_n,\theta_n}(z_o)$ u is mostly large in a proper measure-theoretical sense (this will be our first alternative), or such a situation does not occur (this represents the second alternative). In either case, the conclusion is that the oscillation of u in a smaller cylinder about z_o decreases in a way that can be quantitatively measured. By the well-known *intrinsic scaling* technique originally introduced by DiBenedetto [5], the cylinders have to be rescaled, in order to reflect the degeneracy, that is, their height has to be suitably stretched to take into account the lack of homogeneity of the equation. If $m = 1$, the cylinders would be the standard parabolic cylinders, reflecting the natural homogeneity of the space and time variables.

There is a further aspect to be taken into account: if at a certain step $\bar{n}$, the solution u is all bounded away from zero in $Q_{r_{\bar{n}},\theta_{\bar{n}}}(z_o)$ in a precisely quantified way, then, u will remain bounded away from zero in all smaller cylinders $Q_{r_n,\theta_n}(z_o)$ for any $n > \bar{n}$; correspondingly, the equation is no longer degenerate, behaves like a second order, quasilinear parabolic equation with growth of order 2, as considered in [14], and our result follows by classical methods. This last possibility is sketched at the end of the proof of Theorem 1.1

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2. Preliminaries

2.1. **Notations.** For a point $z \in \mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ we shall always write $z = (x,t)$. By $B_r(x_o) \equiv \{x \in \mathbb{R}^N : |x - x_o| < r\}$ we denote the open ball in $\mathbb{R}^N$ with center $x_o \in \mathbb{R}^N$ and radius $r > 0$. Moreover, we write

$$Q_{r,\theta}(z_o) := B_r(x_o) \times (t_o - \theta, t_o),$$

where $z_o = (x_o, t_o) \in \mathbb{R}^{N+1}$ and $r, \theta > 0$. Whenever writing $2Q$ for a cylinder $Q \equiv Q_{r,\theta}(z_o)$ we mean $2Q = Q_{2r,4\theta}(z_o)$.

Finally, by $\mathcal{M}(E_T)$ we denote the set of all non-negative Radon-measures on E_T.

2.2. **Auxiliary lemmas.** Throughout the paper we will frequently use the following parabolic Sobolev embedding; see [5, Prop. 3.7, p. 7].

Lemma 2.1. *Let $Q_{\varrho,\theta}(z_o)$ be a parabolic cylinder with $0<\varrho,\theta\le 1$ and $1<p<\infty$, $0<r<\infty$. Then there exists a constant $\boldsymbol{\gamma}$ depending only on N,p,r such that for every*

$$u\in L^\infty\big(t_o-\theta,t_o;L^r(B_\varrho(x_o))\big)\cap L^p\big(t_o-\theta,t_o;W^{1,p}(B_\varrho(x_o))\big)$$

there holds

$$\iint_{Q_{\varrho,\theta}(z_o)}|u|^q\,dxdt$$
$$\le\boldsymbol{\gamma}\bigg(\sup_{t\in(t_o-\theta,t_o)}\int_{B_\varrho(x_o)\times\{t\}}|u|^r\,dx\bigg)^{\frac{p}{n}}\iint_{Q_{\varrho,\theta}(z_o)}\Big|\frac{u}{\varrho}\Big|^p+|Du|^p\,dxdt\,,$$

where

$$q=\frac{p(N+r)}{N}\,.$$

The following elementary result can be retrieved from [11, 12, 13].

Lemma 2.2. *Assume that u is a non-negative function in E_T such that $u^\sigma\in L^2(0,T;W^{1,2}(E_T))$ for some $\sigma\ge 1$. Let $u^{(\varepsilon)}:=\max\{u,\varepsilon\}$ for some $\varepsilon>0$. Then, $u^{(\varepsilon)}$ has a weak derivative $Du^{(\varepsilon)}\in L^2(E_T;\mathbb{R}^N)$ such that*

$$Du^{(\varepsilon)}=\mathbf{1}_{E_T\cap\{u>\varepsilon\}}Du\,,$$

where Du is defined by

$$Du=\tfrac{1}{\sigma}\mathbf{1}_{E_T\cap\{u>0\}}u^{1-\sigma}Du^\sigma\,.$$

Moreover, we have that

$$\lim_{\varepsilon\downarrow 0}\|u^{\sigma-1}Du^{(\varepsilon)}-\tfrac{1}{\sigma}Du^\sigma\|_{L^2(E_T)}=0\,.$$

2.3. **Auxiliary functions.** For $\lambda\in(0,1)$ and $s\ge 0$, we define the following functions which will show up in a natural way in the energy estimates:

$$\begin{aligned}G_\lambda(s)&:=\int_0^s 1-(1+\sigma)^{-\lambda}\,d\sigma\equiv s-\tfrac{1}{1-\lambda}\big((1+s)^{1-\lambda}-1\big),\\ V_\lambda(s)&:=\int_0^s\sigma^{\frac{m-1}{2}}(1+\sigma)^{-\frac{1+\lambda}{2}}\,d\sigma,\\ W_\lambda(s)&:=\int_0^s(1+\sigma)^{-\frac{1+\lambda}{2}}\,d\sigma\equiv\tfrac{2}{1-\lambda}\big((1+s)^{\frac{1-\lambda}{2}}-1\big).\end{aligned}$$

In the following we state some auxiliary estimates which will be used several times in the course of the proof of the main results. The proofs of Lemmas 2.3, 2.4, 2.5 can be found in [1].

Lemma 2.3. *For any $\varepsilon\in(0,1]$ and $s\ge 0$ we have*

$$V_\lambda(s)\le\tfrac{2}{m-\lambda}\,s^{\frac{m-\lambda}{2}}$$

and

$$s^{m+\lambda}\le\varepsilon^{1+\lambda}s^{m-1}+\boldsymbol{\gamma}_\varepsilon V_\lambda(s)^{\frac{2(m+\lambda)}{m-\lambda}}\,,$$

where the constant $\boldsymbol{\gamma}_\varepsilon$ blows up as $\varepsilon^{-(1+\lambda)\frac{m+\lambda}{m-\lambda}}$ in the limit $\varepsilon\downarrow 0$. We note that $\boldsymbol{\gamma}_\varepsilon$ also depends on m and λ.

Lemma 2.4. *For any $\varepsilon \in (0,1]$ and $s \ge 0$ there holds*

$$W_\lambda(s) \le \tfrac{2}{1-\lambda}\, s^{\frac{1-\lambda}{2}}$$

and

$$s^{1+\lambda} \le \varepsilon^{1+\lambda} + \boldsymbol{\gamma}_\varepsilon W_\lambda(s)^{\frac{2(1+\lambda)}{1-\lambda}},$$

where the constant $\boldsymbol{\gamma}_\varepsilon$ blows up as $\varepsilon^{-\frac{(1+\lambda)^2}{1-\lambda}}$ in the limit $\varepsilon \downarrow 0$. We note that $\boldsymbol{\gamma}_\varepsilon$ also depends on λ.

Lemma 2.5. *For any $\varepsilon \in (0,1]$ and $s \ge 0$ there holds*

$$s \le \varepsilon + \boldsymbol{\gamma}_\varepsilon G_\lambda(s)$$

for a constant $\boldsymbol{\gamma}_\varepsilon \equiv \boldsymbol{\gamma}(\lambda)\varepsilon^{-1}$.

2.4. **The logarithmic function.** For later purposes we introduce the *Logarithmic function* ψ as follows: For parameters a, b, c with $0 < c < a$ and $b \ge 0$ we define for $s < a + b + c$ the function

$$\psi_{(a,b,c)}(s) := \ln_+\left(\frac{a}{a-(s-b)_+ + c}\right) := \begin{cases} \ln\left(\dfrac{a}{a-s+b+c}\right), & \text{if } b+c < s < a+b+c, \\ 0, & \text{if } s \le b+c. \end{cases} \tag{2.1}$$

The first and second derivative can be computed easily. For the first derivative we have

$$0 \le \big(\psi_{(a,b,c)}\big)'(s) = \begin{cases} \dfrac{1}{b-s+a+c}, & \text{if } b+c < s < a+b+c, \\ 0, & \text{if } s < b+c. \end{cases}$$

The second derivative away from $s = b + c$ is given by

$$\big(\psi_{(a,b,c)}\big)''(s) = \big[\big(\psi_{(a,b,c)}\big)'(s)\big]^2 \ge 0.$$

3. Energy estimates

Let k be any real number and for a function $v \in L^1(E)$ we consider the truncations of v given by

$$(v-k)_+ \equiv \sup\{v-k; 0\}, \qquad (v-k)_- \equiv \sup\{-(v-k); 0\}.$$

In the following we prove energy estimates for $(u-k)_-$ and $(u-k)_+$.

Proposition 3.1. *There exists a positive constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, C_o, C_1)$, such that there holds: Whenever u is a non-negative weak energy solution to* (1.2) *in E_T, in the sense of Definition 1.1, then for every cylinder $Q_{\varrho,\theta}(z_o) \subset E_T$, every $k \ge 0$, and every cutoff function $\zeta \in W_0^{1,\infty}(B_\varrho(x_o))$, and such that $0 \le \zeta \le 1$, we have*

$$\begin{aligned} \operatorname*{ess\,sup}_{t_o-\theta<t\le t_o} \int_{B_\varrho(x_o)\times\{t\}} (u-k)_-^2\zeta^2 dx - \int_{B_\varrho(x_o)\times\{t_o-\theta\}} (u-k)_-^2\zeta^2 dx& \\ + C_o \iint_{Q_{\varrho,\theta}(z_o)} |u|^{m-1}|D(u-k)_-|^2\zeta^2 dxdt& \\ \le \boldsymbol{\gamma} \iint_{Q_{\varrho,\theta}(z_o)} \big(u-k\big)_-^2 \zeta|\zeta_t| dxdt + \boldsymbol{\gamma} \iint_{Q_{\varrho,\theta}(z_o)} |u|^{m-1}(u-k)_-^2|D\zeta|^2 dxdt& \end{aligned} \tag{3.1}$$

and, moreover

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t_o-\theta<t\le t_o}\int_{B_\varrho(x_o)\times\{t\}}(u-k)_+^2\zeta^2dx-\int_{B_\varrho(x_o)\times\{t_o-\theta\}}(u-k)_+^2\zeta^2dx\\
+C_o\iint_{Q_{\varrho,\theta}(z_o)}|u|^{m-1}|D(u-k)_+|^2\zeta^2dx\,dt\\
\le\gamma\iint_{Q_{\varrho,\theta}(z_o)}\big(u-k\big)_+^2\zeta|\zeta_t|dx\,dt+\gamma\iint_{Q_{\varrho,\theta}(z_o)}|u|^{m-1}(u-k)_+^2|D\zeta|^2dx\,dt\\
+\gamma\int_{Q_{\varrho,\theta}(z_o)}(u-k)_+\,d\mu.
\end{aligned}
\tag{3.2}
$$

Proof. After a translation we may assume $(x_o,t_o)=(0,0)$. We limit ourselves to the proof for $(u-k)_-$, the one for $(u-k)_+$ being completely analogous, except for the extra term coming from the measure μ. In (1.6) take the testing function

$$\varphi_-=-\zeta^2\,(u-k)_-$$

over $B_\varrho\times(-\theta,t]$, where $-\theta<t\le 0$. The use of $-(u-k)_-$ in this testing function is justified, modulus a mollification procedure with respect to t, as explained in detail in [1, Chapter 2, 3]. We omit the details, since the procedure is quite standard. With this respect the following computations are done on a formal basis, when writing u_t. The testing gives

$$
\begin{aligned}
-\iint_{B_\varrho\times(-\theta,t]}u_\tau(u-k)_-\zeta^2dx\,d\tau-\iint_{B_\varrho\times(-\theta,t]}\mathbf{A}(x,\tau,u,Du)\cdot D(u-k)_-\zeta^2dx\,d\tau\\
-2\iint_{B_\varrho\times(-\theta,t]}(u-k)_-\mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\,dx\,d\tau\le 0,
\end{aligned}
$$

where we have directly taken into account that

$$-\int_{B_\varrho\times(-\theta,t]}(u-k)_-\zeta^2\,d\mu\le 0\,.$$

This is precisely the term, which cannot be discarded, when working with $(u-k)_+$. The first term can be transformed as usual, to get

$$
\begin{aligned}
-\iint_{B_\varrho\times(-\theta,t]}u_\tau(u-k)_-\,\zeta^2dx\,d\tau=-\iint_{Q_{\varrho,\theta}}(u-k)_-^2\zeta\zeta_\tau\,dx\,d\tau\\
+\tfrac12\int_{B_\varrho\times\{t\}}(u-k)_-^2\zeta^2dx-\tfrac12\int_{B_\varrho\times\{-\theta\}}(u-k)_-^2\zeta^2dx.
\end{aligned}
$$

From the first structure condition (1.3) it follows that

$$
\begin{aligned}
-\iint_{B_\varrho\times(-\theta,t]}\mathbf{A}(x,\tau,u,Du)\cdot D(u-k)_-\,\zeta^2dx\,d\tau\\
\ge C_om\iint_{B_\varrho\times(-\theta,t]}|u|^{m-1}|D(u-k)_-|^2\zeta^2dx\,d\tau,
\end{aligned}
$$

and from the second condition in (1.3) and Young's inequality it follows that

$$
\begin{aligned}
2\Big|\iint_{B_\varrho\times(-\theta,t]}(u-k)_-\mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\,dx\,d\tau\Big|\\
\le 2C_1m\iint_{B_\varrho\times(-\theta,t]}|u|^{m-1}(u-k)_-|D(u-k)_-|\zeta|D\zeta|dx\,d\tau
\end{aligned}
$$

$$\leq \tfrac{1}{4}C_o m \iint_{B_\varrho\times(-\theta,t]} |u|^{m-1}|D(u-k)_-|^2\zeta^2 dx\, d\tau$$
$$+ \boldsymbol{\gamma}(m, C_o, C_1)\iint_{Q_{\varrho,\theta}} |u|^{m-1}(u-k)_-^2|D\zeta|^2 dx\, d\tau.$$

Combining these estimates, and taking the supremum over $t \in (-\theta, 0]$ proves the proposition. □

In the sequel we need another estimate for $(u-k)_+$. Let $\lambda \in (0,1)$, $a, d > 0$. We consider (1.6) on parabolic cylinders of the form $Q_\varrho^{(a)}(z_o) := B_\varrho(x_o) \times \Lambda_\varrho^{(a)}(t_o) = B_\varrho(x_o) \times \big(t_o - a^{1-m}\varrho^2, t_o\big)$. These cylinders are natural, since they take into account the structure (scaling) of the parabolic equation which arises from the degeneracy in the u-variable. From [1, (3.2)] we recall the following *energy estimate*:

Proposition 3.2. *There exists a positive constant $\gamma = \gamma(m, C_o, C_1, \lambda) \geq 1$, such that there holds: Whenever u is a weak energy solution to* (1.2)*, in E_T, in the sense of Definition 1.1, then for every cylinder $Q_\varrho^{(a)}(z_o) \subset E_T$, and every $a \geq 0$ we have that the energy estimate*

$$\sup_{t\in\Lambda^{(a)}_{\varrho/2}} \int_{B_{\varrho/2}\times\{t\}\cap\{u>a\}} G_\lambda\Big(\frac{u-a}{d}\Big)\, dx$$
$$+ \iint_{Q^{(a)}_{\varrho/2}\cap\{u>a\}} d^{m-1}\Big|DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 + a^{m-1}\Big|DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 dxdt$$
$$\leq \frac{\boldsymbol{\gamma}}{\varrho^2}\iint_{Q^{(a)}_{\varrho}\cap\{u>a\}} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda} dxdt + \frac{\boldsymbol{\gamma}\mu\big(Q^{(a)}_\varrho\big)}{d}$$

holds true. □

Remark 3.1. Proposition 3.2 has been stated and proved for cylinders $Q_\varrho^{(a)}(z_o)$ in [1]. Later on we shall work with slightly different cylinders of the form

$$Q_\varrho^{(\omega/A)}(z_o) = B_\varrho(x_o) \times \Big(t_o - \Big(\frac{\omega}{A}\Big)^{1-m}\varrho^2, t_o\Big],$$

with $A \in (0,1)$ and $\omega > 0$ such that $\frac14\omega \leq a \leq \frac{13}{12}\omega$. Then, a careful reading of the proof shows that the energy estimate continues to hold with a constant $\frac{\gamma}{A^{m-1}\varrho^2}$ instead of $\frac{\gamma}{\varrho^2}$ on the right-hand side. □

In the following Lemma we consider a weak energy solution to (1.2) in E_T and a general cylinder $Q_{\varrho,\theta}(z_o) \Subset E_T$. From our potential estimate we already know that $\sup_{Q_{\varrho,\theta}(z_o)} u < \infty$ and therefore also

$$H := \sup_{Q_{\varrho,\theta}(z_o)} (u-k)_+ < \infty$$

for any $k \geq 0$. Without loss of generality we can assume that $k < \sup_{Q_{\varrho,\theta}(z_o)} u$. Otherwise, we would have $H = 0$. Finally, let $0 < c < \min\{1, H\}$. From (2.1) we recall the definition of the logarithmic function $\psi_{(H,k,c)}$. With that at hand we define for $z \in Q_{\varrho,\theta}(z_o)$ the function

$$\psi(u)(z) := \big(\psi_{(H,k,c)} \circ u\big)(z) = \ln_+\Big[\frac{H}{H-(u(z)-k)_+ + c}\Big],$$

which will be used in the formulation of the following Lemma.

Proposition 3.3. *There exists a constant $\boldsymbol{\gamma}$, depending only on N, m, C_o, C_1, such that for any weak energy solution u to* (1.2)*, in E_T in the sense of Definition 1.1, for every cylinder $Q_{\varrho,\theta}(z_o) \Subset E_T$, and for every level $k \geq 0$, there holds:*

$$
\begin{aligned}
\sup_{t_o-\theta<t<t_o} &\int_{B_\varrho(x_o)\times\{t\}} \psi^2(u)\zeta^2 dx \\
&\leq \int_{B_\varrho(x_o)\times\{t_o-\theta\}} \psi^2(u)\zeta^2 dx + \boldsymbol{\gamma} \iint_{Q_{\varrho,\theta}(z_o)} u^{m-1}\psi(u)|D\zeta|^2 dx\, dt \\
&\quad + \frac{2}{c}\Big(\ln\frac{H}{c}\Big)\int_{Q_{\varrho,\theta}(z_o)} \chi_{\{u>k\}}\, d\mu.
\end{aligned}
$$

Here, $\zeta \in W^{1,\infty}(B_\varrho(x_o))$ is a cutoff function independent of t.

Proof. Take $(x_o,t_o) = (0,0)$ and work within the cylinder $Q^t \equiv B_\varrho \times (-\theta, t)$, with $-\theta < t < 0$. In the weak formulation (1.6) take the testing function

$$\varphi = \zeta^2\big[\psi^2\big]'(u) = 2\psi(u)\psi'(u)\zeta^2.$$

By direct calculation we infer that $[\psi^2]'' = 2(1+\psi)\psi'^2$, and therefore

$$\big[\psi^2\big]''(u) = \big[2(1+\psi)\psi'^2\big](u) \in L^\infty(Q_{\varrho,\theta}(z_o))$$

which implies that such a φ is an admissible testing function, modulo a mollification procedure with respect to time. Note that $\psi(u) \neq 0$ implies that $u > k + c > 0$ and therefore $|D\varphi| \in L^2(Q_{\varrho,\theta}(z_o))$. Now, since $\psi(u)$ vanishes on the set where $(u-k)_+ = 0$, we find

$$\iint_{Q^t} u_\tau[\psi^2]'(u)\zeta^2 dx\, d\tau = \int_{B_\varrho\times\{t\}} \psi^2(u)\zeta^2 dx - \int_{B_\varrho\times\{-\theta\}} \psi^2(u)\zeta^2 dx.$$

The term involving $\mathbf{A}(x,\tau,u,Du)$ is estimated with the help of the lower bound from (1.3) as follows:

$$
\begin{aligned}
\iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D\varphi\, dx\, d\tau \geq 2mC_o &\iint_{Q^t} \big[(1+\psi)\psi'^2\big](u)u^{m-1}|Du|^2\zeta^2 dx\, d\tau \\
&- 4mC_1 \iint_{Q^t} u^{m-1}|Du|\big[\psi\psi'\big](u)\zeta|D\zeta| dx\, d\tau.
\end{aligned}
$$

By an application of Young's inequality we obtain from this

$$
\begin{aligned}
&\iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D\varphi\, dx\, d\tau \\
&\quad\geq mC_o \iint_{Q^t} \big[(1+\psi)\psi'^2\big](u)u^{m-1}|Du|^2\zeta^2 dxd\tau - \boldsymbol{\gamma}\iint_{Q^t} \psi(u)u^{m-1}|D\zeta|^2 dxd\tau,
\end{aligned}
$$

with a constant $\boldsymbol{\gamma}$ depending on m, C_o and C_1. As for the remaining term, since by the definition of $\psi(u)$ we estimate

$$\psi(u) \leq \ln\Big(\frac{H}{c}\Big) \quad\text{and}\quad \psi'(u) \leq \frac{1}{c},$$

we have

$$2\int_{Q^t} \big[\psi\psi'\big](u)\zeta^2\, d\mu \leq \frac{2}{c}\Big(\ln\frac{H}{c}\Big)\int_{Q_{\varrho,\theta}} \chi_{\{u>k\}}\, d\mu.$$

Here, we also used the fact that $\zeta^2 \in [0,1]$. Collecting these estimates, discarding the positive term in the second last inequality, and taking the supremum over $t \in (-\theta, 0)$, establishes the claim of the proposition. □

4. Proof of Theorem 1.1

For a cylinder $Q \Subset E_T$ let

$$\mathbb{I}^{\mu}_{2,Q}(r,\theta) := \sup_{z\in Q} \mathbf{I}^{\mu}_2(z,r,\theta).$$

Theorem 1.1 will be a consequence of the following result.

Proposition 4.1. *There exist constants $C>1$, and $\delta,\nu_*\in(0,1)$, that can be quantitatively determined only in terms of the structural constants m, C_o, C_1, and the dimension N, such that with $\eta := \sqrt{\frac18\nu_*\delta^{m-1}}$ the following assertion holds true: Let u be a weak energy solution to* (1.2) *in E_T and $z_o\in E_T$. There exist $\varrho>0$, and $\omega>0$ such that with $\varrho_o:=\varrho$, $\omega_o:=\omega$, and*

$$\theta_n := \omega_n^{1-m}, \quad \varrho_n := \eta^n\varrho_o, \quad Q_n := Q_{\varrho_n,\theta_n\varrho_n^2}(z_o) \tag{4.1}$$

and

$$\omega_{n+1} := \max\big\{\delta\omega_n, C\,\mathbb{I}^{\mu}_{2,Q_n}(4\varrho_n,16\theta_n\varrho_n^2)\big\}$$

for $n\in\mathbb{N}_0$, the assertions

$$Q_n \subset Q_{n-1}\subset\cdots\subset Q_o\subset E_T, \tag{4.2}$$

and

$$\operatorname*{osc}_{Q_n} u \le \omega_n, \quad \textit{for all } n\in\mathbb{N}_0. \tag{4.3}$$

hold true. □

Assuming Proposition 4.1 to be true, we proceed with the proof of Theorem 1.1. The idea of the proof is to obtain a quantitative decay of the oscillation in terms of the radius from the discrete decay on the cylinders Q_n.

Proof. By the definitions of θ_n and ω_n, we have

$$\begin{aligned}\theta_{n+1} = \omega_{n+1}^{1-m} &= \Big[\max\Big\{\delta\omega_n, C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\varrho_n,16\theta_n\varrho_n^2\big)\Big\}\Big]^{1-m}\\ &= \min\Big\{(\delta\omega_n)^{1-m}, \big(C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\varrho_n,16\theta_n\varrho_n^2\big)\big)^{1-m}\Big\} \le (\delta\omega_n)^{1-m}.\end{aligned}$$

By iteration, we can then conclude that

$$\theta_n \le \big(\delta^{n-k}\omega_k\big)^{1-m}, \quad \text{for all } k=0,1,\dots,n-1.$$

This implies that

$$\begin{aligned}\omega_{n+1} &\le \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\eta^n\varrho, 16\omega_o^{1-m}\delta^{(1-m)n}\eta^{2n}\varrho^2\big)\\ &\le \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_o}\big(4\varrho,16\omega_o^{1-m}\varrho^2\big),\end{aligned} \tag{4.4}$$

where we used in the last line the inclusion $Q_n\subset Q_o$ and the fact that $\delta^{1-m}\eta^2<1$ which is a consequence of the definition of η and $\nu_*<1$. We iterate the preceding inequality for $j=0,\dots,n$ in order to obtain a first rough bound for ω_n. Abbreviating

$$\mathbb{I}_o := \mathbb{I}^{\mu}_{2,Q_o}\big(4\varrho,16\omega_o^{1-m}\varrho^2\big),$$

we have

$$\omega_n \le \delta^n\omega_o + C\,\mathbb{I}_o\sum_{j=0}^{n-1}\delta^{n-1-j} \le \omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o \tag{4.5}$$

for all $n \in \mathbb{N}$. Now, we utilize the inequalities $\eta \leq (\frac{1}{8}\nu_*)^{\frac{1}{2}}$ and $\delta^{1-m}\eta^2 \leq \frac{1}{8}\nu_*$, which also follows from the definition of η. Instead of (4.4) we now get

$$\omega_{n+1} \leq \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\Big(\frac{\nu_*}{8}\Big)^{\frac{n}{2}}\varrho, 16\omega_o^{1-m}\Big(\frac{\nu_*}{8}\Big)^n\varrho^2\Big), \tag{4.6}$$

for any $n \in \mathbb{N}$. Now, for $\bar\varrho \in (0,\varrho]$ there exists $k \in \mathbb{N}$, such that

$$\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho \leq \bar\varrho < \Big(\frac{\nu_*}{8}\Big)^{\frac{k-1}{2}}\varrho.$$

The number k is uniquely determined by the requirement

$$k-1 < \frac{\ln\frac{\bar\varrho}{\varrho}}{\ln\sqrt{\frac{1}{8}\nu_*}} \leq k. \tag{4.7}$$

Iterating (4.6) yields

$$\begin{aligned}
\omega_n &\leq \delta^{n-k}\omega_k + C\sum_{j=k}^{n-1}\mathbb{I}^{\mu}_{2,Q_o}\Big(4\Big(\frac{\nu_*}{8}\Big)^{\frac{j}{2}}\varrho, 16\omega_o^{1-m}\Big(\frac{\nu_*}{8}\Big)^j\varrho^2\Big)\delta^{n-1-j} \\
&\leq \delta^{n-k}\omega_k + C\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho, 16\omega_o^{1-m}\Big(\frac{\nu_*}{8}\Big)^k\varrho^2\Big)\sum_{j=k}^{n-1}\delta^{n-1-j} \\
&\leq \delta^{n-k}\omega_k + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho, 16\omega_o^{1-m}\Big(\frac{\nu_*}{8}\Big)^k\varrho^2\Big) \\
&\leq \delta^{n-k}\omega_k + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\left(4\bar\varrho, 16\omega_o^{1-m}\bar\varrho^2\right).
\end{aligned}$$

Now, let $\bar r \in (0,\bar\varrho)$ be fixed. At this stage we can argue as in [5, Chapter III, § 3]. First, we fix a number $0 < b < \delta$ and choose $\ell \in \mathbb{N}$ such that $b^\ell\bar\varrho < \bar r \leq b^{\ell-1}\bar\varrho$. The number ℓ is uniquely determined by

$$\ell - 1 \leq \frac{\ln\frac{\bar r}{\bar\varrho}}{\ln b} < \ell. \tag{4.8}$$

Then, with $n = k+\ell$ we estimate

$$\delta^{n-k} = \delta^\ell = \exp(\ell\log\delta) = b^{\alpha\ell} < \Big(\frac{\bar r}{\bar\varrho}\Big)^\alpha,$$

where α is defined by

$$\alpha := \frac{\ln\delta}{\ln b}.$$

Note that $\alpha \in (0,1)$, since $0 < b < \delta < 1$. Thus, we have shown that there exist $\alpha \in (0,1)$, and $\boldsymbol{\gamma} > 1$, that depend only on the data, such that

$$\omega_n \leq \Big(\frac{\bar r}{\bar\varrho}\Big)^\alpha\omega_k + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\left(4\bar\varrho, 16\omega_o^{1-m}\bar\varrho^2\right)$$

holds true for any $0 < \bar r < \bar\varrho \leq \varrho$, where $n = k+\ell$ and k,ℓ are defined by (4.7) and (4.8). The strategy now is as follows: We fix $\beta \in (0,1)$ and consider radii $0 < \bar r < \varrho$. We choose $\bar\varrho \in (\bar r, \varrho)$ according to $\bar\varrho = \varrho^{1-\beta}\bar r^\beta$ and determine k,ℓ according to (4.7) and (4.8) which by the choice of $\bar\varrho$ is equivalent to

$$k-1 < \frac{\beta\ln\frac{\bar r}{\varrho}}{\ln\sqrt{\frac{1}{8}\nu_*}} \leq k, \quad \text{and} \quad \ell-1 < \frac{(1-\beta)\ln\frac{\bar r}{\varrho}}{\ln b} \leq \ell. \tag{4.9}$$

With these choices and letting $n=k+\ell$ we infer from the last inequality that

$$\operatorname*{osc}_{Q_n} u \le \omega_n \le \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\omega_k + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\beta}, 16\omega_o^{1-m}\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big).$$

Plugging in the boundedness of ω_k from (4.5) for any $k\in\mathbb{N}$ we obtain that there holds:

$$\begin{aligned}\operatorname*{osc}_{Q_n} u \le \omega_n &\le \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\big[\omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o\big] + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\beta}, 16\omega_o^{1-m}\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big)\\ &=: \mathrm{I}(\bar r) + \mathrm{II}(\bar r).\end{aligned}$$

We note that both terms of the right-hand side vanish in the limit $\bar r\downarrow 0$. Therefore we can choose $\varrho_o\in(0,\varrho]$ such that $\mathrm{I}(\bar r)+\mathrm{II}(\bar r)\le\omega_o$ for any $0<\bar r\le\varrho_o$. Via (4.9) we determine $n_o\in\mathbb{N}$ such that $\omega_n\le\omega_o$ holds true for $n\ge n_o$. Actually, we can take

$$n_o = \left\lceil \frac{\beta\ln\frac{\varrho_o}{\varrho}}{\ln\sqrt{\frac18\nu_*}}\right\rceil + \left\lceil \frac{(1-\beta)\ln\frac{\varrho_o}{\varrho}}{\ln b}\right\rceil.$$

Then, for $n\ge n_o$, we have

$$\widetilde{Q}_n := B_{\varrho_n}\times(-\omega_o^{1-m}\varrho_n^2,0]\subset B_{\varrho_n}\times(-\theta_n\varrho_n^2,0]=Q_n,$$

and therefore

$$\operatorname*{osc}_{\widetilde{Q}_n} u \le \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\big[\omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o\big] + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\beta}, 16\omega_o^{1-m}\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big).$$

From (4.9) we obtain that

$$n-2 < \left(\frac{\beta}{\ln\sqrt{\frac18\nu_*}} + \frac{(1-\beta)}{\ln b}\right)\ln\frac{\bar r}{\varrho}\le n.$$

We define

$$\sigma := \left(\frac{\beta}{\ln\sqrt{\frac18\nu_*}} + \frac{(1-\beta)}{\ln b}\right)\ln\eta,$$

and obtain

$$\varrho_n = \eta^n\varrho = \varrho e^{n\ln\eta} > \varrho\exp\Big(\sigma\ln\frac{\bar r}{\varrho}+2\ln\eta\Big) = \eta^2\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\sigma}.$$

If we finally let

$$r=\eta^2\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\sigma},$$

since $\tilde{Q}_n\supset Q_{r,\omega_o^{1-m}r^2}$, we have

$$\begin{aligned}\operatorname*{osc}_{Q_{r,\omega_o^{1-m}r^2}} u &\le \frac{1}{\eta^{\frac{2\alpha(1-\beta)}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\alpha(1-\beta)}{\sigma}}\big[\omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o\big]\\ &\quad + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}}, \frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\omega_o^{1-m}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right).\end{aligned}$$

Moreover, if $\omega_o\ge1$, then

$$\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}}, \frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\omega_o^{1-m}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right) \le \mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}}, \frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right).$$

On the other hand, if $\omega_o < 1$, then

$$\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\left(\frac{r}{\varrho}\right)^{\frac{\beta}{\sigma}}, \frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\omega_o^{1-m}\left(\frac{r}{\varrho}\right)^{\frac{2\beta}{\sigma}}\right) \leq \mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\omega_o^{\frac{1-m}{2}}\left(\frac{r}{\varrho}\right)^{\frac{\beta}{\sigma}}, \frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\omega_o^{1-m}\left(\frac{r}{\varrho}\right)^{\frac{2\beta}{\sigma}}\right).$$

In both instances, by a proper, possible further redefinition of r, we are led to consider $\sup_{z\in Q_o}\mathbf{I}^{\mu}_2(z,r,r^2)$, that is, the classical parabolic, truncated Riesz potential, with no intrinsic scaling with respect to time. This proves the continuity of u on $Q_{r,\omega_o^{1-m}r^2}$. Statements concerning the continuity over a compact set, now follow by a standard covering argument. □

We now deal with the proof of Proposition 4.1. Having fixed $z_o \in E_T$, we let $\varepsilon, R \in (0,1)$ such that $Q_{2R,4R^{2-\varepsilon}}(z_o) \subset E_T$. Without loss of generality, we may assume that $z_o \equiv (0,0)$, so that $Q_{R,R^{2-\varepsilon}}(z_o) = Q_{R,R^{2-\varepsilon}}$. Now, if

$$\operatorname*{osc}_{Q_{\varrho,\varrho^{2-\varepsilon}}} u \leq \varrho^{\frac{\varepsilon}{m-1}} \qquad \text{for any } \varrho \in (0,R],$$

there is nothing to prove, since the essential oscillation of u has a power-like decay. Otherwise, there exists $\varrho \in (0,R]$ such that

$$\operatorname*{osc}_{Q_{\varrho,\varrho^{2-\varepsilon}}} u > \varrho^{\frac{\varepsilon}{m-1}}.$$

Then, we define

$$\mu_o^+ := \sup_{Q_{\varrho,\varrho^{2-\varepsilon}}} u, \qquad \mu_o^- := \inf_{Q_{\varrho,\varrho^{2-\varepsilon}}} u, \qquad \omega_o := \mu_o^+ - \mu_o^- > 0. \tag{4.10}$$

By the preceding inequality, we have that

$$\omega_o^{m-1} = \Big(\operatorname*{osc}_{Q_{\varrho,\varrho^{2-\varepsilon}}} u\Big)^{m-1} > \varrho^{\varepsilon},$$

which, letting $\theta_o := \omega_o^{1-m}$, guarantees that

$$Q_o := Q_{\varrho,\theta_o\varrho^2} \subset Q_{\varrho,\varrho^{2-\varepsilon}}, \qquad \operatorname*{osc}_{Q_{\varrho,\theta_o\varrho^2}} u \leq \omega_o. \tag{4.11}$$

Thus (4.11) ensures that (4.2)–(4.3) hold for $n = 0$.

Here, we remark that the role of introducing the cylinder $Q_{\varrho,\varrho^{2-\varepsilon}}$, is to guarantee that the upper bound for the oscillation in (4.11) holds true for the constructed cylinder $Q_{\varrho,\theta_o\varrho^2}$. It will be part of the proof of Proposition 4.1 to show that at each step of the induction argument, the cylinders Q_n and the essential oscillation of u within them, satisfy the right geometry. Apart from this, ε plays no other role in this context.

The remaining part of the section will be devoted to the proof of Proposition 4.1: we will determine constants $\delta, \nu_* \in (0,1)$ and $C > 1$, depending only on the set of data m, N, C_o, C_1, and independent of u and z_o for which (4.2)–(4.3) hold inductively for all n.

4.1. **The Induction Argument.** Assuming (4.2)–(4.3) hold for some $n \in \mathbb{N}_0$, we remove the index n by setting

$$\varrho = \varrho_n, \qquad \omega = \omega_n, \qquad \theta = \theta_n = \omega^{1-m} \qquad Q_{\varrho,\theta\varrho^2} = Q_n,$$

and

$$\mu^+ = \sup_{Q_{\varrho,\theta\varrho^2}} u = \mu_n^+, \qquad \mu^- = \inf_{Q_{\varrho,\theta\varrho^2}} u = \mu_n^-.$$

Note that, by (4.3) we have

$$\omega \geq \mu^+ - \mu^-.$$

Denote by a, ξ and A fixed numbers in $(0,1)$, and let

$$\tilde{\theta} := A^{m-1}\theta = \Big(\frac{A}{\omega}\Big)^{m-1}. \tag{4.12}$$

Then, for $r \in (0, \frac{\varrho}{2}]$ we have

$$Q_{2r,4\tilde{\theta}r^2} \subset Q_{\varrho,\theta\varrho^2}, \qquad Q_{4r,16\tilde{\theta}r^2} \subset E_T.$$

We have the following two DeGiorgi-type results.

Lemma 4.1. *Let u be a weak energy solution to* (1.2)*, in E_T. There exists a positive number ν_-, depending on a, ξ, A and the data m, N, C_o, C_1, such that if*

$$\big|\{u \leq \xi\omega\} \cap Q_{2r,4\tilde{\theta}r^2}\big| \leq \nu_- \big|Q_{2r,4\tilde{\theta}r^2}\big|,$$

then

$$u \geq a\xi\omega \qquad \text{a.e. in } Q_{r,\tilde{\theta}r^2}.$$

Proof. The proof follows from the energy estimate (3.1) in Proposition 3.1; see [7, Lemma 7.1, Chapter 3]. Note that in the case considered here the constant C from the structural conditions (5.2) in Chapter 3 of [7] is 0, and therefore, the first alternative $C\varrho > 1$ from Lemma 7.1 will never occur. □

Remark 4.1. The functional dependence of ν_- on the indicated parameters can be retrieved form [7, Chapter 3, (7.9)]. We have

$$\nu_- = \boldsymbol{\gamma}^{-1} a^{(m-1)\frac{N+2}{2}} (1-a)^{N+2} \frac{[(\xi A)^{m-1}]^{\frac{N}{2}}}{[1+(\xi A)^{m-1}]^{\frac{N+2}{2}}} \tag{4.13}$$

for a quantitative constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, N, C_o, C_1) > 1$, independent of a, ξ and A. □

Lemma 4.2. *Let u be a weak energy solution to* (1.2)*, in E_T. Assume that $\xi \in (0, \frac{1}{2}]$, and*

$$\tfrac{1}{2}\omega \leq \mu^+ - \tfrac{1}{4}\omega \leq \tfrac{5}{6}\omega. \tag{4.14}$$

There exist constants $\nu_+ \in (0,1)$, depending on a, A and the data m, N, C_o, C_1, and $B > 1$ that depends on a, A and m, N, C_o, C_1, such that if

$$\big|\{u \geq \mu^+ - \xi\omega\} \cap Q_{2r,4\tilde{\theta}r^2}\big| \leq \nu_+ \big|Q_{2r,4\tilde{\theta}r^2}\big|, \tag{4.15}$$

then either

$$\xi\omega < B\, \mathbb{I}^{\mu}_{2,Q_{2r,4\tilde{\theta}r^2}}\big(4r, 16\tilde{\theta}r^2\big) \tag{4.16}$$

or

$$u \leq \mu^+ - a\xi\omega \qquad \text{a.e. in } Q_{r,\tilde{\theta}r^2}. \tag{4.17}$$

Remark 4.2. The functional dependence of ν_+ and B on the indicated parameters is given by

$$\nu_+ = \Big(\frac{1-a}{\gamma}\Big)^{1+\lambda}, \qquad B = \frac{\gamma}{1-a} \tag{4.18}$$

for an arbitrary parameter $\lambda \in (0, \frac{1}{N}]$, and a quantitative constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, N, C_o, C_1, \lambda, A) > 1$, independent of a and ξ. □

Proof. Due to the presence of the measure μ, the classical DeGiorgi iteration scheme, as adapted to degenerate parabolic equations by DiBenedetto (see [5]), cannot be applied here, and we have to use the Kilpeläinen-Malý approach, as in [15]. We let $B > 1$ to be determined in a universal way in the course of the proof. In the following, we assume

$$\xi\omega \geq B\, \mathbb{I}^{\mu}_{2, Q_{2r,4\tilde{\theta}r^2}}(4r, 16\tilde{\theta}r^2), \tag{4.19}$$

since otherwise the assertion of the Lemma is trivially satisfied. Let $z_1 = (x_1, t_1) \in Q_{r,\tilde{\theta}r^2}$. In the following we will prove that

$$u(z_1) \leq \mu^+ - a\xi\omega, \tag{4.20}$$

and since z_1 is an arbitrary point in $Q_{r,\tilde{\theta}r^2}$, the claim of the Lemma follows. For the proof of (4.20), we sholl proceed in several steps.

Step 1: Setting up an iteration scheme. For $j = -1, 0, 1, \dots$ we define

$$r_j := \frac{r}{2^j}, \qquad B_j := B_{r_j}(x_1), \qquad Q_j := B_j \times \big(t_1 - \tilde{\theta}r_j^2, t_1\big]$$

and

$$\alpha_j := \int_0^{r_j} \frac{\mu(Q_{\varrho,\tilde{\theta}\varrho^2}(z_1))}{\varrho^N}\frac{d\varrho}{\varrho}.$$

Moreover, we let

$$a_o := \mu^+ - \xi\omega.$$

For $j \geq 0$ we now suppose that $a_o, \dots, a_j$ have already been selected. Then, we choose a_{j+1} as follows: We let $\lambda \in (0, \frac{1}{N}]$ and define

$$\mathbf{K}_j(a) := \frac{1}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} u^{m-1}\left(\frac{u-a_j}{a-a_j}\right)^{1+\lambda} dxdt \tag{4.21}$$

for $a > a_j$. Now, if

$$\mathbf{K}_j\big(a_j + \tfrac{1}{4}(\alpha_{j-1} - \alpha_j)\big) < \kappa,$$

we define

$$a_{j+1} := a_j + \tfrac{1}{4}(\alpha_{j-1} - \alpha_j).$$

Otherwise, if

$$\mathbf{K}_j\big(a_j + \tfrac{1}{4}(\alpha_{j-1} - \alpha_j)\big) \geq \kappa,$$

we choose $a_{j+1} > a_j + \frac{1}{4}(\alpha_{j-1} - \alpha_j)$ according to

$$\mathbf{K}_j(a_{j+1}) = \kappa.$$

Such a choice is always possible since $(a_j, \infty) \ni a \mapsto \mathbf{K}_j(a)$ is a monotone decreasing, continuous function, $\lim_{a \downarrow a_j} \mathbf{K}_j(a) = +\infty$ and $\lim_{a\to\infty} \mathbf{K}_j(a) = 0$. In any case, we have

$$\mathbf{K}_j(a_{j+1}) \leq \kappa \quad \text{and} \quad a_{j+1} \geq a_j + \tfrac{1}{4}(\alpha_{j-1} - \alpha_j). \tag{4.22}$$

Step 2: A first bound on a_j. Here, we will prove a first rough bound on a_j of the form

$$a_j < \tfrac{1}{2}(\mu^+ + a_{j-1}) - \tfrac{1}{4}B\alpha_{j-1} \qquad \text{and} \qquad a_j < \mu^+ - \tfrac{1}{2}B\alpha_{j-1}, \tag{4.23}$$

for any $j \in \mathbb{N}$. We start proving (4.23) in the case $j = 1$. First, we observe that

$$a_o = \mu^+ - \xi\omega \le \mu^+ - B\alpha_{-1},$$

by (4.19), so that

$$\bar{a} := \tfrac{1}{2}(\mu^+ + a_o) - \tfrac{1}{4}B\alpha_o > a_o + \tfrac{1}{4}(\alpha_{-1} - \alpha_o).$$

Moreover, by (4.19) and the definition of α_o, we obtain

$$\bar{a} - a_o = \tfrac{1}{2}\xi\omega - \tfrac{1}{4}B\alpha_o \ge \tfrac{1}{2}\xi\omega - \tfrac{1}{4}\xi\omega = \tfrac{1}{4}\xi\omega$$

and therefore

$$\begin{aligned}
\mathbf{K}_o(\bar{a}) &= \frac{1}{r_o^{N+2}} \iint_{\frac{1}{2}Q_o \cap \{u > a_o\}} u^{m-1} \left(\frac{u - a_o}{\bar{a} - a_o}\right)^{1+\lambda} dxdt \\
&\le \frac{(\mu^+)^{m-1}}{r^{N+2}} \left(\frac{\mu^+ - (\mu^+ - \xi\omega)}{\xi\omega/4}\right)^{1+\lambda} \big|\{u > a_o\} \cap Q_o\big| \\
&\le \frac{4^{1+\lambda}(\mu^+)^{m-1}}{r^{N+2}} \big|\{u > a_o\} \cap Q_{2r, 4\tilde{\theta}r^2}\big| \\
&\le \frac{4^{1+\lambda}(\mu^+)^{m-1}\nu_+}{r^{N+2}} \big|Q_{2r, 4\tilde{\theta}r^2}\big| \\
&\le \boldsymbol{\gamma}\, \omega^{m-1}\tilde{\theta}\nu_+ = \boldsymbol{\gamma}(N, m, A)\, \nu_+.
\end{aligned}$$

Here we have used in turn (4.15), (4.14) and (4.12). Now, for fixed $\kappa \in (0, 1)$ we choose ν_+ small enough to satisfy

$$\boldsymbol{\gamma}\nu_+ \le \tfrac{1}{2}\kappa \quad \iff \quad \nu_+ \le \frac{\kappa}{2\boldsymbol{\gamma}}. \tag{4.24}$$

Note, that κ will be chosen later in the course of the proof in a universal way. With such a choice of ν_+, we have

$$\mathbf{K}_o(\bar{a}) \le \tfrac{1}{2}\kappa.$$

Since $\bar{a} > a_o + \frac{1}{4}(\alpha_{-1} - \alpha_o)$, we conclude from the construction of a_1 that $a_o + \frac{1}{4}(\alpha_{-1} - \alpha_o) \le a_1 < \bar{a}$. This proves the first bound in (4.23). The second bound follows from the first one, the definition of a_o and (4.19). This proves (4.23) for $j = 1$.

Now, we let $j \in \mathbb{N}$ and assume that (4.23) holds for $1, \dots, j$. First, we observe that by the definition of α_j and simple computations we have

$$\begin{cases}
\alpha_{j-1} - \alpha_j \ge \dfrac{1}{2N} \dfrac{\mu\big(Q_{r_j, \tilde{\theta}r_j^2}(z_1)\big)}{r_j^N} = \dfrac{1}{2N} \dfrac{\mu(Q_j)}{r_j^N}, \\[2ex]
\alpha_{j-1} - \alpha_j \le 2^N \dfrac{\mu\big(Q_{r_{j-1}, \tilde{\theta}r_{j-1}^2}(z_1)\big)}{r_{j-1}^N} = 2^N \dfrac{\mu(Q_{j-1})}{r_{j-1}^N}.
\end{cases} \tag{4.25}$$

We now let

$$\bar{a} := \tfrac{1}{2}(\mu^+ + a_j) - \tfrac{1}{4}B\alpha_j.$$

Then, by the second inequality in (4.23), we find that

$$\bar{a} - a_j = \tfrac{1}{2}(\mu^+ - a_j) - \tfrac{1}{4}B\alpha_j > \tfrac{1}{4}B(\alpha_{j-1} - \alpha_j). \tag{4.26}$$

Moreover, using the first inequality in (4.23), we obtain

$$\bar{a} - a_j = \tfrac{1}{2}(\mu^+ - a_j) - \tfrac{1}{4}B\alpha_j = \tfrac{1}{2}(\mu^+ + a_j - 2a_j) - \tfrac{1}{4}B\alpha_j \tag{4.27}$$

$$\geq \tfrac{1}{2}(a_j - a_{j-1}) + \tfrac{1}{4}B(\alpha_{j-1} - \alpha_j) \geq \tfrac{1}{2}(a_j - a_{j-1}).$$

From now on we proceed much as we did in [1]. We have

$$
\begin{aligned}
\text{(4.28)} \qquad \mathbf{K}_j(\bar a) &= \frac{1}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \big(u - a_j + a_j\big)^{m-1} \Big(\frac{u - a_j}{\bar a - a_j}\Big)^{1+\lambda} dxdt \\
&\leq \frac{\boldsymbol{\gamma}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} (u - a_j)^{m-1} \Big(\frac{u - a_j}{\bar a - a_j}\Big)^{1+\lambda} dxdt \\
&\quad + \frac{\boldsymbol{\gamma} a_j^{m-1}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \Big(\frac{u - a_j}{\bar a - a_j}\Big)^{1+\lambda} dxdt =: \mathrm{I} + \mathrm{II},
\end{aligned}
$$

where the constant $\boldsymbol{\gamma}$ depends only on m. First, we consider the term I. We have

$$\mathrm{I} = \frac{\boldsymbol{\gamma}(\bar a - a_j)^{m-1}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \Big(\frac{u - a_j}{\bar a - a_j}\Big)^{m+\lambda} dxdt.$$

By Lemma 2.3, for some fixed $\varepsilon \in (0, 1)$ to be chosen later, we conclude that

$$
\begin{aligned}
\mathrm{I} &\leq \frac{\boldsymbol{\gamma}(\bar a - a_j)^{m-1}}{r_j^{N+2}} \Big[\varepsilon^{1+\lambda} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \Big(\frac{u - a_j}{\bar a - a_j}\Big)^{m-1} dxdt \\
&\qquad\qquad + \boldsymbol{\gamma}_\varepsilon \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \Big(V_\lambda\Big(\frac{u - a_j}{\bar a - a_j}\Big)\Big)^{\frac{2(m+\lambda)}{m-\lambda}} dxdt\Big] \\
&\leq \frac{\boldsymbol{\gamma}\varepsilon^{1+\lambda}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} u^{m-1}\, dxdt \\
&\quad + \frac{\boldsymbol{\gamma}_\varepsilon(\bar a - a_j)^{m-1}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j \cap \{u > a_j\}} \Big(V_\lambda\Big(\frac{u - a_j}{\bar a - a_j}\Big)\Big)^{\frac{2(m+\lambda)}{m-\lambda}} dxdt \\
&= \boldsymbol{\gamma}\varepsilon^{1+\lambda}\kappa + \frac{\boldsymbol{\gamma}_\varepsilon(\bar a - a_j)^{m-1}}{r_j^{N+2}} \iint_{\frac{1}{2}Q_j} \bigg(V_\lambda\bigg(\frac{(u - a_j)_+}{\bar a - a_j}\bigg)\bigg)^{\frac{2(m+\lambda)}{m-\lambda}} dxdt\,,
\end{aligned}
$$

where $\boldsymbol{\gamma}$ depends on N, m and $\boldsymbol{\gamma}_\varepsilon$ depends on m, λ and ε. Here, in the last line we have taken into account the fact that

$$1 \leq \frac{u - a_{j-1}}{a_j - a_{j-1}} \quad \text{on } \{u > a_j\},$$

the inclusion $\frac{1}{2}Q_j \subset \frac{1}{2}Q_{j-1}$ and that $\mathbf{K}_{j-1}(a_j) \leq \kappa$, by (4.22). To estimate the integral on the right-hand side, we apply Gagliardo-Nirenberg's inequality from Lemma 2.1 with $p = 2$, $q = \frac{2(m+\lambda)}{m-\lambda}$ and $r = \frac{2\lambda N}{m-\lambda}$ to conclude that

$$
\begin{aligned}
\mathrm{I} &\leq \boldsymbol{\gamma}\varepsilon^{1+\lambda}\kappa + \boldsymbol{\gamma}_\varepsilon \bigg[\sup_{t \in \Lambda_j} \frac{1}{r_j^N} \int_{\frac{1}{2}B_j \times \{t\}} \bigg|V_\lambda\bigg(\frac{(u - a_j)_+}{\bar a - a_j}\bigg)\bigg|^{\frac{2\lambda N}{m-\lambda}} dx\bigg]^{\frac{2}{N}} \\
&\qquad \cdot \frac{(\bar a - a_j)^{m-1}}{r_j^N} \iint_{\frac{1}{2}Q_j} \frac{1}{r_j^2}\bigg|V_\lambda\bigg(\frac{(u - a_j)_+}{\bar a - a_j}\bigg)\bigg|^2 + \bigg|DV_\lambda\bigg(\frac{(u - a_j)_+}{\bar a - a_j}\bigg)\bigg|^2 dxdt \\
&=: \boldsymbol{\gamma}\varepsilon^{1+\lambda}\kappa + \boldsymbol{\gamma}_\varepsilon\, \mathrm{I}_1\big(\mathrm{I}_2 + \mathrm{I}_3\big)\,,
\end{aligned}
$$

with the obvious labeling for I_1, I_2 and I_3, and $\Lambda_j := \big(-\tilde\theta r_j^2, 0\big]$. In turn, we will separately estimate the appearing terms. We start with the *estimate for* I_1.

By (4.14), we have $\frac{1}{2}\omega \leq \mu^+ - \frac{1}{4}\omega$; since $\xi \in (0, \frac{1}{2}]$, $a_o = \mu - \xi\omega$ and $\{a_j\}$ is a monotone increasing sequence, we have $a_j \geq a_o = \mu^+ - \xi\omega \geq \frac{1}{4}\omega$. Moreover, by (4.23) and (4.14), we have $a_j < \mu^+ \le \frac{13}{12}\omega$. Therefore, we can apply Remark 3.1, and rely on the energy estimate from Proposition 3.2. Using in turn Lemma 2.3, Hölder's inequality (note that $\lambda N \le 1$), Lemma 2.5 for some $\varepsilon_1 \in (0,1)$ to be chosen later in the proof, and finally the energy estimate from Proposition 3.2, we deduce

$$
\begin{aligned}
\mathrm{I}_1 &= \bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}\cap\{u>a_j\}} \Big|V_\lambda\Big(\frac{u-a_j}{\bar a - a_j}\Big)\Big|^{\frac{2\lambda N}{m-\lambda}}\,dx\bigg]^{\frac{2}{N}} \\
&\le \boldsymbol{\gamma}\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}\cap\{u>a_j\}} \Big(\frac{u-a_j}{\bar a - a_j}\Big)^{\lambda N}\,dx\bigg]^{\frac{2}{N}} \\
&\le \boldsymbol{\gamma}\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}\cap\{u>a_j\}} \frac{u-a_j}{\bar a - a_j}\,dx\bigg]^{2\lambda} \\
&\le \boldsymbol{\gamma}\varepsilon_1^{2\lambda} + \frac{\boldsymbol{\gamma}}{\varepsilon_1^{2\lambda}}\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}\cap\{u>a_j\}} G_\lambda\Big(\frac{u-a_j}{\bar a - a_j}\Big)\,dx\bigg]^{2\lambda} \\
&\le \boldsymbol{\gamma}\varepsilon_1^{2\lambda} \\
&\quad + \frac{\boldsymbol{\gamma}}{\varepsilon_1^{2\lambda}}\bigg[\frac{1}{A^{m-1}r_j^{N+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\Big(1+\frac{u-a_j}{\bar a - a_j}\Big)^{1+\lambda}\,dxdt + \frac{\mu(Q_j)}{(\bar a - a_j) r_j^N}\bigg]^{2\lambda}
\end{aligned}
$$

with $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda)$. Considering the first term in the brackets, we have

$$
\begin{aligned}
&\frac{1}{r_j^{N+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\Big(1+\frac{u-a_j}{\bar a - a_j}\Big)^{1+\lambda}\,dxdt \\
&\qquad = \frac{1}{r_j^{N+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{a_j-a_{j-1}}{a_j-a_{j-1}}+\frac{u-a_j}{\bar a - a_j}\Big)^{1+\lambda}\,dxdt \\
&\qquad \le \frac{\boldsymbol{\gamma}}{r_j^{N+2}}\iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}} u^{m-1}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}+\frac{u-a_j}{\bar a - a_j}\Big)^{1+\lambda}\,dxdt \\
&\qquad \le \frac{\boldsymbol{\gamma}}{r_j^{N+2}}\iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}} u^{m-1}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}\,dxdt \le \boldsymbol{\gamma}\kappa,
\end{aligned}
\tag{4.29}
$$

where we have taken into account (4.27) and (4.22) and $Q_j = \frac12 Q_{j-1}$. As for the other term in the brackets, by (4.26) and (4.25)

$$
\frac{\mu(Q_j)}{r_j^N(\bar a - a_j)} \le \frac{4\mu(Q_j)}{B r_j^N(\alpha_{j-1}-\alpha_j)} \le \frac{8N}{B}. \tag{4.30}
$$

Therefore, we conclude that

$$
\mathrm{I}_1 \le \boldsymbol{\gamma}\varepsilon_1^{2\lambda} + \frac{\boldsymbol{\gamma}}{\varepsilon_1^{2\lambda}}\Big(\frac{\kappa}{A^{m-1}}+\frac{1}{B}\Big)^{2\lambda}.
$$

Next, we *estimate the term* I_2. Using Lemma 2.3 and (4.27), we arrive at

$$
\begin{aligned}
\mathrm{I}_2 &= \frac{\boldsymbol{\gamma}(\bar a - a_j)^{m-1}}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}} \Big(\frac{u-a_j}{\bar a - a_j}\Big)^{m-\lambda}\,dxdt \\
&\le \frac{\boldsymbol{\gamma}}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_j}{\bar a - a_j}\Big)^{1-\lambda}\,dxdt
\end{aligned}
$$

$$\le \frac{\gamma}{r_j^{N+2}} \iint_{\frac12 Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_j}{a_j-a_{j-1}}\Big)^{1-\lambda}\,dxdt$$
$$\le \frac{\gamma}{r_j^{N+2}} \iint_{\frac12 Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1-\lambda}\,dxdt\,,$$

where $\gamma=\gamma(m,\lambda)$. Since the quantity in brackets on the right-hand side integral is larger than 1, we can enlarge the exponent from $1-\lambda$ to $1+\lambda$, subsequently enlarge the domain of integration to $\frac12 Q_{j-1}\cap\{u>a_{j-1}\}$, and replace r_j by r_{j-1}. This leads us to the estimate

$$\mathrm{I}_2 \le \frac{\gamma}{r_{j-1}^{N+2}} \iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}} u^{m-1}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}\,dxdt \le \gamma\kappa\,,$$

where in the last inequality we used again (4.22). Note that γ depends on N,m,λ. At this point it remains to *estimate* I_3 by the energy estimate from Proposition 3.2, we obtain

$$\begin{aligned}
\mathrm{I}_3 &= \frac{(\bar a-a_j)^{m-1}}{r_j^N}\iint_{\frac12 Q_j\cap\{u>a_j\}} \Big|DV_\lambda\Big(\frac{u-a_j}{\bar a-a_j}\Big)\Big|^2\,dxdt\\
&\le \gamma\bigg[\frac{1}{A^{m-1}r_j^{N+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\bigg(1+\frac{u-a_j}{\bar a-a_j}\bigg)^{1+\lambda}\,dxdt+\frac{\mu(Q_j)}{r_j^N(\bar a-a_j)}\bigg]\\
&\le \gamma\bigg(\frac{\kappa}{A^{m-1}}+\frac{1}{B}\bigg),
\end{aligned}$$

where we used (4.29) and (4.30) to estimate the two terms from the second last line. Inserting the estimates for I_1, I_2 and I_3 in the right-hand side of the inequality for I, we conclude that

$$\mathrm{I} \le \gamma\varepsilon^{1+\lambda}\kappa+\gamma_\varepsilon\Big[\varepsilon_1^{2\lambda}+\varepsilon_1^{-2\lambda}\big(\kappa+B^{-1}\big)^{2\lambda}\Big]\big(\kappa+B^{-1}\big) \tag{4.31}$$

holds true with constants $\gamma=\gamma(N,m)$ and $\gamma_\varepsilon=\gamma_\varepsilon(N,m,C_o,C_1,\lambda,A,\varepsilon)$.

Next, we turn our attention to the term II from the right-hand side of (4.28). Using Lemma 2.4 and (4.22), we find

$$\begin{aligned}
\mathrm{II} &\le \frac{\gamma\varepsilon^{1+\lambda}}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}} a_j^{m-1}\,dxdt\\
&\qquad + \frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{N+2}}\iint_{\frac12 Q_j}\bigg(W_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\bigg)^{\frac{2(1+\lambda)}{1-\lambda}}\,dxdt\\
&\le \frac{\gamma\varepsilon^{1+\lambda}}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}} u^{m-1}\,dxdt\\
&\qquad + \frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{N+2}}\iint_{\frac12 Q_j}\bigg(W_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\bigg)^{\frac{2(1+\lambda)}{1-\lambda}}\,dxdt\\
&\le \gamma\varepsilon^{1+\lambda}\kappa+\frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{N+2}}\iint_{\frac12 Q_j}\bigg(W_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\bigg)^{\frac{2(1+\lambda)}{1-\lambda}}\,dxdt\,,
\end{aligned}$$

where $\gamma=\gamma(N,m)$, and $\gamma_\varepsilon=\gamma_\varepsilon(m,\lambda,\varepsilon)$. To the integral on the right-hand side of the preceding inequality we apply the Gagliardo-Nirenberg inequality from Lemma 2.1 for the choices $p=2$, $q=\frac{2(1+\lambda)}{1-\lambda}$, $r=\frac{2\lambda N}{1-\lambda}$ and $Q=\frac12 Q_j$. This yields

$$\begin{aligned}\mathrm{II}&\le \boldsymbol\gamma\kappa\varepsilon^{1+\lambda}+\boldsymbol\gamma_\varepsilon\bigg[\sup_{t\in\Lambda_j}\frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}}\Big|W_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\Big|^{\frac{2\lambda N}{1-\lambda}}dx\bigg]^{\frac2N}\\&\quad\cdot\frac{a_j^{m-1}}{r_j^N}\iint_{\frac12 Q_j}\frac{1}{r_j^2}\Big|W_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\Big|^2+\Big|DW_\lambda\Big(\frac{(u-a_j)_+}{\bar a-a_j}\Big)\Big|^2dxdt\\&=:\boldsymbol\gamma\kappa\varepsilon^{1+\lambda}+\boldsymbol\gamma_\varepsilon\mathrm{II}_1\big(\mathrm{II}_2+\mathrm{II}_3\big).\end{aligned}$$

As in the case of the term I, we now consecutively estimate the terms II_i for $i=1,2,3$. We start with the *estimate of the sup-term*, that is II_1. In turn, we use Lemma 2.4 and Hölder's inequality (note that $\lambda N\le 1$) to infer that

$$\begin{aligned}\mathrm{II}_1&\le\boldsymbol\gamma\bigg[\sup_{t\in\Lambda_j}\frac{1}{r_j^N}\int_{\frac12 B_j\times\{t\}\cap\{u>a_j\}}\Big(\frac{u-a_j}{\bar a-a_j}\Big)^{\lambda N}dx\bigg]^{\frac2N}\\&\le\boldsymbol\gamma\bigg[\sup_{t\in\Lambda_j}\frac{1}{r_j^N}\int_{\frac12 B_j\cap\{u>a_j\}}\frac{u-a_j}{\bar a-a_j}\,dx\bigg]^{2\lambda}.\end{aligned}$$

Having arrived at this stage, we can further estimate as for the term I_1 from before, and conclude that the same estimate as for I_1 holds true for II_1 as well, that is, we have that

$$\mathrm{II}_1\le\boldsymbol\gamma\varepsilon_1^{2\lambda}+\frac{\boldsymbol\gamma}{\varepsilon_1^{2\lambda}}\Big(\frac{\kappa}{A^{m-1}}+\frac1B\Big)^{2\lambda}$$

holds true. Next, we come to the *estimate of the term* II_2. Using again Lemma 2.4 and following the arguments from the estimation of I_2, we find that there holds:

$$\begin{aligned}\mathrm{II}_2&\le\frac{\boldsymbol\gamma}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}}a_j^{m-1}\Big(\frac{u-a_j}{\bar a-a_j}\Big)^{1-\lambda}dxdt\\&\le\frac{\boldsymbol\gamma}{r_j^{N+2}}\iint_{\frac12 Q_j\cap\{u>a_j\}}u^{m-1}\Big(\frac{u-a_j}{\bar a-a_j}\Big)^{1-\lambda}dxdt\\&\le\frac{\boldsymbol\gamma}{r_j^{n+2}}\iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}}u^{m-1}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}dxdt\le\boldsymbol\gamma\kappa,\end{aligned}$$

for a constant $\boldsymbol\gamma=\boldsymbol\gamma(N,m,C_o,C_1,\lambda)$. Thus, it remains to bound II_3. However, such a bound immediately follows from the energy estimate from Proposition 3.2 and Remark 3.1 (recall also the definition of $\tilde\theta$ in (4.12)):

$$\begin{aligned}\mathrm{II}_3&=\frac{a_j^{m-1}}{r_j^N}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big|DW_\lambda\Big(\frac{u-a_j}{\bar a-a_j}\Big)\Big|^2dxdt\\&\le\boldsymbol\gamma\bigg[\frac{1}{A^{m-1}r_j^{N+2}}\iint_{Q_j\cap\{u>a_j\}}u^{m-1}\Big(1+\frac{u-a_j}{\bar a-a_j}\Big)^{1+\lambda}dxdt+\frac{\mu(Q_j)}{r_j^N(\bar a-a_j)}\bigg]\\&\le\boldsymbol\gamma\Big(\frac{\kappa}{A^{m-1}}+\frac1B\Big).\end{aligned}$$

Here we have also taken into account (4.26). Altogether, we have shown that also the term II can be estimated by the right-hand side of the inequality (4.31). Inserting this into (4.28), we obtain that

$$\mathbf K_j(\bar a)\le\boldsymbol\gamma\varepsilon^{1+\lambda}\kappa+\boldsymbol\gamma_\varepsilon\Big[\varepsilon_1^{2\lambda}+\varepsilon_1^{-2\lambda}\big(\kappa+B^{-1}\big)^{2\lambda}\Big]\big(\kappa+B^{-1}\big)$$

holds true with constants $\gamma = \gamma(N,m)$ and $\gamma_\varepsilon = \gamma_\varepsilon(N,m,C_o,C_1,\lambda,A,\varepsilon)$. Note that $\varepsilon,\varepsilon_1,\kappa \in (0,1)$ and $B>1$ are still at our disposal. We first choose ε to satisfy $\boldsymbol{\gamma}\varepsilon^{1+\lambda} = \frac{1}{6}$. This fixes $\boldsymbol{\gamma}_\varepsilon$ in dependence on N,m,C_o,C_1,λ and A. Next, we choose B so large that

$$B \geq \frac{1}{\kappa}. \tag{4.32}$$

This yields

$$\mathbf{K}_j(\bar{a}) \leq \tfrac{1}{6}\kappa + \boldsymbol{\gamma}\big[\varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda}\kappa^{2\lambda}\big]\kappa$$

with $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,C_o,C_1,A,\lambda)$. Now, choosing ε_1 in dependence on N,m,C_o,C_1,A,λ such that $\boldsymbol{\gamma}\varepsilon_1^{2\lambda} = \frac{1}{6}$, we obtain

$$\mathbf{K}_j(\bar{a}) \leq \tfrac{1}{3}\kappa + \boldsymbol{\gamma}\kappa^{1+2\lambda},$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,C_o,C_1,A,\lambda)$. Finally, we choose κ in dependence on N,m,C_o,C_1,A,λ small enough to satisfy

$$\boldsymbol{\gamma}\kappa^{2\lambda} \leq \tfrac{1}{6}.$$

With this choice the preceding inequality for $\mathbf{K}_j(\bar{a})$ yields that

$$\mathbf{K}_j(\bar{a}) \leq \tfrac{1}{2}\kappa.$$

Now, we recall from (4.26) that $\bar{a} > a_j + \frac{1}{4}B(\alpha_{j-1} - \alpha_j)$. Therefore, due to the construction of a_{j+1} we may conclude that $a_j + \frac{1}{4}B(\alpha_{j-1} - \alpha_j) \leq a_{j+1} < \bar{a}$. This proves the first bound in (4.23). For the second bound we use the fact that $a_j < \mu^+$, which is a consequence of the second inequality in (4.23), to conclude that

$$a_{j+1} < \bar{a} = \tfrac{1}{2}(\mu^+ + a_j) - \tfrac{1}{4}B\alpha_j < \mu^+ - \tfrac{1}{4}B\alpha_j.$$

This proves (4.23) for $j+1$. Hence, we have proved the claim that (4.23) holds true for any $j \in \mathbb{N}$.

Step 3: Improved iterative bound for a_j. Here, we define

$$d_j := a_{j+1} - a_j$$

and prove that there exists a constant $\boldsymbol{\gamma}$ depending only on N,m,C_o,C_1,λ and A such that

$$d_j \leq \tfrac{1}{2}d_{j-1} + \boldsymbol{\gamma}\,\frac{\mu(2Q_j)}{r_j^N} \tag{4.33}$$

holds true for any $j \in \mathbb{N}$.

The proof is similar to the one in Step 2 and therefore, we only sketch it. We fix $j \geq 1$. Without loss of generality, we can assume that

$$d_j \geq \tfrac{1}{2}d_{j-1}, \qquad d_j > \tfrac{1}{4}(\alpha_{j-1} - \alpha_j),$$

because otherwise, there is nothing to prove; cf. (4.25) for the case that $d_j \leq \frac{1}{4}(\alpha_{j-1}-\alpha_j)$. By the construction of a_{j+1}, the second inequality ensures that we have $\mathbf{K}_j(a_{j+1}) = \kappa$. We now work as in the proof of Step 2, but instead of estimating the term involving the measure as in (4.30), we keep the measure and replace $\bar{a} - a_j$ in the denominator by $d_j = a_{j+1} - a_j$. In this way we obtain

$$\kappa \leq \big(\boldsymbol{\gamma}\varepsilon^{1+\lambda} + \boldsymbol{\gamma}_\varepsilon\varepsilon_1^{2\lambda} + \boldsymbol{\gamma}_\varepsilon\varepsilon_1^{-2\lambda}\kappa^{2\lambda}\big)\kappa + \boldsymbol{\gamma}_\varepsilon\varepsilon_1^{-2\lambda}\bigg[\frac{\mu(2Q_j)}{d_j r_j^N} + \Big(\frac{\mu(2Q_j)}{d_j r_j^N}\Big)^{1+2\lambda}\bigg].$$

With the same choices for κ, ε, ε_1 as in the proof of Step 2, and the argument from the end of [1, Chapter 4.3], we conclude that

$$d_j \le \boldsymbol{\gamma}\, \frac{\mu(2Q_j)}{r_j^N}$$

with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda, A)$. This proves the claim.

Step 4: Quantitative bound for u. We let $J > 1$ and sum up (4.33) for $j = 1, \dots, J-1$. Taking into account the definition of d_j, and the fact that the sequence $\{a_j\}$ is a monotone increasing sequence, we deduce that

$$\begin{aligned} a_J - a_1 &= \sum_{j=1}^{J-1}(a_{j+1} - a_j) \\ &\le \tfrac12 \sum_{j=1}^{J-1}(a_j - a_{j-1}) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N} \\ &= \tfrac12 (a_{J-1} - a_1) + \tfrac12 (a_1 - a_o) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N} \\ &\le \tfrac12 (a_J - a_1) + \tfrac12 (a_1 - a_o) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}. \end{aligned}$$

From this we easily obtain that

$$a_J \le 2a_1 - a_o + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}. \tag{4.34}$$

From the construction of a_1 we have two alternatives. Either

$$a_1 = a_o + \tfrac14 (\alpha_{-1} - \alpha_o),$$

or

$$a_1 > a_o + \tfrac14 (\alpha_{-1} - \alpha_o).$$

In the former case, recalling that $a_o = \mu^+ - \xi\omega$, we have

$$a_J \le \mu^+ - \xi\omega + \tfrac12 (\alpha_{-1} - \alpha_o) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N},$$

from which we conclude, using also $(4.25)_2$ for $j = 0$ and the definition of Q_j, that there holds

$$\begin{aligned} a_J &\le \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{J-1} \frac{\mu(2Q_j)}{r_j^N} = \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{J-1} \frac{\mu(Q_{j-1})}{r_j^N} \\ &\le \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{\infty} \frac{\mu(Q_{j-1})}{r_j^N} = \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{\infty} \frac{\mu(Q_{r_{j-1}, \tilde\theta r_{j-1}^2})}{r_j^N} \\ &\le \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{\infty} \int_{r_{j-1}}^{r_{j-2}} \frac{\mu(Q_{\varrho, \tilde\theta \varrho^2})}{r_j^N} \frac{d\varrho}{r_{j-2} - r_{j-1}} \\ &\le \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=0}^{\infty} \int_{r_{j-1}}^{r_{j-2}} \frac{\mu(Q_{\varrho, \tilde\theta \varrho^2})}{\varrho^N} \frac{d\varrho}{\varrho} \end{aligned}$$

$$
\begin{aligned}
&= \mu^+ - \xi\omega + \boldsymbol{\gamma}\, \mathbb{I}_2^{\mu}(4r, 16\tilde{\theta} r^2) \\
&\le \mu^+ - \xi\omega + \boldsymbol{\gamma}\, B^{-1}\xi\omega.
\end{aligned}
$$

Here, we also used (4.19) in the last line, and the constant $\boldsymbol{\gamma}$ depends on N, m, C_o, C_1, λ and A. In the latter case, by the construction of a_1 we have that $\boldsymbol{K}_o(a_1) = \kappa$; therefore, taking (4.14), (4.15), and the definition of a_o into account, we obtain

$$
\begin{aligned}
\kappa &= \frac{1}{r_o^{N+2}} \iint_{\frac{1}{2}Q_o \cap \{u > a_o\}} u^{m-1} \left(\frac{u - a_o}{a_1 - a_o} \right)^{1+\lambda} dxdt \\
&\le \boldsymbol{\alpha}(N) 2^{N+2} \Big(\frac{13}{12}\Big)^{m-1} \omega^{m-1} \left(\frac{\mu^+ - (\mu^+ - \xi\omega)}{a_1 - a_o} \right)^{1+\lambda} \nu_+ \Big(\frac{\omega}{A}\Big)^{1-m},
\end{aligned}
$$

implying the inequality

$$
a_1 - a_o \le \boldsymbol{\gamma}\xi\omega \left(\frac{\nu_+}{\kappa A^{1-m}} \right)^{\frac{1}{1+\lambda}} \le \boldsymbol{\gamma}\xi\omega(\nu_+)^{\frac{1}{1+\lambda}},
$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda, A)$. Here, we note that κ has already been fixed in dependence on $N, m, C_o, C_1, A, \lambda$. We substitute this inequality back into (4.34) and estimate $\sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N} \le \boldsymbol{\gamma} B^{-1}\xi\omega$, as before. In this way, we obtain

$$
a_J \le \mu^+ - \xi\omega + \boldsymbol{\gamma}\xi\omega(\nu_+)^{\frac{1}{1+\lambda}} + B^{-1}\xi\omega,
$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda, A)$. Combining the two alternatives we obtain that the preceding inequality holds true in any case for any $J \in \mathbb{N}$. Since $\{a_i\}$ is a monotone increasing sequence, the previous bound implies that the limit $\lim_{j\to\infty} a_j = a_\infty$ exists, is finite, that also $\lim_{j\to\infty} d_j = \lim_{j\to\infty}(a_{j+1} - a_j) = 0$, and

$$
a_\infty \le \mu^+ - \xi\omega\Big[1 - \boldsymbol{\gamma}(\nu_+)^{\frac{1}{1+\lambda}} - \boldsymbol{\gamma} B^{-1}\Big].
$$

Since $a_\infty \ge a_o > 0$, and $Q_j \downarrow \{z_1\}$ we can conclude that

$$
\begin{aligned}
&\left(\frac{u(z_1)}{a_\infty} \right)^{m-1} (u(z_1) - a_\infty)_+^{1+\lambda} \\
&\qquad = \lim_{j\to\infty} ⨌_{Q_j} \left(\frac{u}{a_j} \right)^{m-1} (u - a_j)_+^{1+\lambda} dxdt \\
&\qquad = \lim_{j\to\infty} \frac{d_j^{1+\lambda}}{\boldsymbol{\alpha}(N)\tilde{\theta} r_j^{N+2} a_j^{m-1}} \iint_{Q_j} u^{m-1} \left(\frac{(u - a_j)_+}{d_j} \right)^{1+\lambda} dxdt \\
&\qquad \le \frac{1}{\boldsymbol{\alpha}(N)\tilde{\theta} a_o^{m-1}} \lim_{j\to\infty} \frac{d_j^{1+\lambda}}{r_j^{N+2}} \iint_{Q_j} u^{m-1} \left(\frac{(u - a_j)_+}{d_j} \right)^{1+\lambda} dxdt \\
&\qquad \le \frac{1}{\boldsymbol{\alpha}(N)\tilde{\theta} a_o^{m-1}} \lim_{j\to\infty} d_j^{1+\lambda}\kappa = 0.
\end{aligned}
$$

Therefore, we conclude that $u(z_1) \le a_\infty$, and by the previous bound on a_∞ we obtain

$$
u(z_1) \le \mu^+ - \xi\omega\Big[1 - \boldsymbol{\gamma}(\nu_+)^{\frac{1}{1+\lambda}} - \boldsymbol{\gamma} B^{-1}\Big].
$$

Now, we choose B large enough such that

$$
B \ge (\nu_+)^{-\frac{1}{1+\lambda}}, \tag{4.35}
$$

With this choice of B we have

$$u(z_1) \le \mu^+ - \xi\omega\Big[1 - \boldsymbol{\gamma}(\nu_+)^{\frac{1}{1+\lambda}}\Big],$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda, A)$. Finally, we choose ν_+ such that

$$1 - \boldsymbol{\gamma}(\nu_+)^{\frac{1}{1+\lambda}} \ge a \quad \Longleftrightarrow \quad \nu_+ \le \Big(\frac{1-a}{\boldsymbol{\gamma}}\Big)^{1+\lambda}. \tag{4.36}$$

This fixes ν_+ in dependence on $N, m, C_o, C_1, \lambda, A$, and a. Inserting this choice of ν_+ above, we finally arrive at

$$u(z_1) \le \mu^+ - a\xi\omega,$$

which proves the claim (4.20).

Finally, a few remarks concerning the dependencies of the constants are in order. First, for the constant ν_+ we imposed the smallness conditions (4.24) and (4.36), i.e.

$$\nu_+ \le \frac{\kappa}{2\boldsymbol{\gamma}} \quad \text{and} \quad \nu_+ \le \Big(\frac{1-a}{\boldsymbol{\gamma}}\Big)^{1+\lambda}.$$

Since κ and $\boldsymbol{\gamma}$ both depend on $N, m, C_o, C_1, \lambda, A$, we can choose ν_+ of the form $(\frac{1-a}{\boldsymbol{\gamma}})^{1+\lambda}$, with a constant $\boldsymbol{\gamma} \ge 1$ depending on $N, m, C_o, C_1, \lambda, A$, as stated in Remark 4.2. Second, for the constant B we required in (4.32) and (4.35) that

$$B \ge \frac{1}{\kappa} \quad \text{and} \quad B \ge \Big(\frac{1}{\nu_+}\Big)^{\frac{1}{1+\lambda}}.$$

This means, we get the functional dependence of B in the form $\frac{\boldsymbol{\gamma}}{1-a}$, with a constant $\boldsymbol{\gamma} \ge 1$ depending on $N, m, C_o, C_1, \lambda, A$, as stated in Remark 4.2. □

With respect to the notation of Lemma 4.1, assume

$$r = \tfrac{1}{2}\varrho, \quad a = \tfrac{1}{2}, \quad \xi = \tfrac{1}{2}, \quad A = 1, \quad \tilde{\theta} = \theta = \omega^{1-m},$$

and let ν_* be the corresponding value of ν_- given by (4.13), i.e.

$$\nu_* := \nu_-\big(\xi = \tfrac{1}{2}, a = \tfrac{1}{2}, A = 1\big).$$

Due to these choices, ν_* is now a quantity that depends only on N, m, C_o, and C_1. Furthermore, taking into account the definition of μ^+ and ω, from here on we assume that

$$\tfrac{1}{2}\omega \le \mu^+ - \tfrac{1}{4}\omega \le \tfrac{5}{6}\omega. \tag{4.37}$$

Note that this coincides with (4.14), which then holds. The left-hand inequality can be taken as holding in all cases; the case when the right-hand inequality fails to hold will be examined later. We have two alternatives, which we now discuss separately.

4.2. **The first alternative.** If

$$\big|\{u \le \tfrac{1}{2}\omega\} \cap Q_{\varrho,\theta\varrho^2}\big| \le \nu_*\big|Q_{\varrho,\theta\varrho^2}\big| \tag{4.38}$$

then, by Lemma 4.1 we have

$$u \ge \tfrac{1}{4}\omega \qquad \text{a.e. in } Q_{\frac{1}{2}\varrho,\frac{1}{4}\theta\varrho^2},$$

which implies

$$-\inf_{Q_{\frac{1}{2}\varrho,\frac{1}{4}\theta\varrho^2}} u \le -\tfrac{1}{4}\omega.$$

Adding the essential supremum of u over $Q_{\frac{1}{2}\varrho,\frac{1}{4}\theta\varrho^2}$ on the left-hand side and μ^+ on the right-hand side, by (4.37) we conclude

$$\operatorname*{osc}_{Q_{\frac{1}{2}\varrho,\frac{1}{4}\theta\varrho^2}} u \le \mu^+ - \tfrac{1}{4}\omega \le \tfrac{5}{6}\omega.$$

At this stage we recall that we used the short-hand notation introduced at the beginning of § 4.1, in partucular that $\varrho = \varrho_n$, $\theta = \theta_n$ and $\omega = \omega_n$. We have thus proved that

$$\operatorname*{osc}_{Q_{\frac12\varrho_n,\theta_n(\frac12\varrho_n)^2}} u \le \tfrac56\omega_n. \tag{4.39}$$

This finishes the induction step in the case of the first alternative.

4.3. **The second alternative.** If (4.38) does not hold true, then

$$\big|\{u \le \tfrac12\omega\} \cap Q_{\varrho,\theta\varrho^2}\big| > \nu_*\big|Q_{\varrho,\theta\varrho^2}\big|,$$

which we rewrite as

$$\big|\{u > \tfrac12\omega\} \cap Q_{\varrho,\theta\varrho^2}\big| \le (1-\nu_*)\big|Q_{\varrho,\theta\varrho^2}\big|. \tag{4.40}$$

In the following, we will examine the consequences of (4.40). Due to (4.37), (4.40) yields

$$\big|\{u > \mu^+ - \tfrac14\omega\} \cap Q_{\varrho,\theta\varrho^2}\big| \le (1-\nu_*)\big|Q_{\varrho,\theta\varrho^2}\big|. \tag{4.41}$$

We have the following

Lemma 4.3. *There exists a time level $\bar s \in [-\theta\varrho^2, -\frac12\nu_*\theta\varrho^2]$, such that*

$$\big|\{u(\cdot,\bar s) > \mu^+ - \tfrac14\omega\} \cap B_\varrho\big| \le \frac{1-\nu_*}{1-\frac12\nu_*}|B_\varrho|. \tag{4.42}$$

Proof. If (4.42) were not to hold for some s in the indicated range, then

$$\big|\{u(\cdot,s) > \mu^+ - \tfrac14\omega\} \cap B_\varrho\big| > \frac{1-\nu_*}{1-\frac12\nu_*}|B_\varrho| \qquad \forall s \in [-\theta\varrho^2, -\tfrac12\nu_*\theta\varrho^2],$$

and therefore,

$$\begin{aligned}\big|\{u > \mu^+ - \tfrac14\omega\} \cap Q_{\varrho,\theta\varrho^2}\big| &\ge \int_{-\theta\varrho^2}^{-\frac12\nu_*\theta\varrho^2} \big|\{u(\cdot,s) > \mu^+ - \tfrac14\omega\} \cap B_\varrho\big|\,ds \\ &> \theta\varrho^2(1-\nu_*)\big|B_\varrho\big| = (1-\nu_*)\big|Q_{\varrho,\theta\varrho^2}\big|,\end{aligned}$$

which contradicts (4.41). □

In the following, whenever there is no risk of confusion, for simplicity, we will omit the reference point of the potential and we will write $\mathbf{I}_2^\mu(r,\theta)$, instead of $\mathbf{I}_2^\mu(z_o,r,\theta)$. It is a straightforward consequence of the definition, that

$$\mathbf{I}_2^\mu(2r, 4\theta r^2) \ge \frac{\mu(Q_{r,\theta r^2})}{r^N}. \tag{4.43}$$

Lemma 4.4. *There exists a positive integer $s_1 \ge 4$ depending only on N, m, C_o, C_1 such that either*

$$\omega < 2^{s_1}\mathbf{I}_2^\mu(2\varrho, 4\theta\varrho^2), \tag{4.44}$$

or

$$\left|\left\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{s_1}}\right\} \cap B_\varrho\right| \le \left(1 - \tfrac14\nu_*^2\right)|B_\varrho| \tag{4.45}$$

holds true for all $t \in [\bar s, 0]$.

Proof. We let $k = \mu^+ - \frac{1}{4}\omega$ and define

$$H_k^+ := \sup_{B_\varrho \times [\bar{s},0]} (u-k)_+ \le \tfrac{1}{4}\omega.$$

Furthermore, we let $c = \frac{\omega}{2^{\ell+2}}$ for some integer $\ell \in \mathbb{N}$, with $\ell \ge 2$. In order to apply Proposition 3.3, we must have $0 < c < H_k$. Therefore, the integer ℓ must later on be chosen large enough in a universal way; then s_1 will be $\ell + 2$. We will apply Proposition 3.3, on the cylinder $B_\varrho \times [\bar{s}, 0]$, and with the logarithmic function $\psi \circ u = \psi_{(a,b,c)} \circ u$ for the choice $(a, b, c) = \big(H_k, k, \frac{\omega}{2^{\ell+2}}\big)$; i.e. we consider

$$\psi(u)(x,t) := \big(\psi_{(H_k,k,\frac{\omega}{2^{\ell+2}})} \circ u\big)(x,t) = \ln_+ \left[\frac{H_k^+}{H_k^+ - \big(u(x,t)-k\big)_+ + \frac{\omega}{2^{\ell+2}}}\right].$$

The cutoff function $x \mapsto \zeta(x) \in [0,1]$ is taken to be 1 in the ball $B_{(1-\sigma)\varrho}$, where $\sigma \in (0,1)$ has to be chosen, vanishing on the boundary of B_ϱ, and such that $|D\zeta| \le \frac{1}{\sigma\varrho}$. With these choices, we have

$$\begin{aligned}\int_{B_{(1-\sigma)\varrho}\times\{t\}} \psi^2(u)\,dx &\le \int_{B_\varrho\times\{\bar{s}\}} \psi^2(u)\,dx + \frac{\boldsymbol{\gamma}}{\sigma^2\varrho^2}\int_{B_\varrho\times[\bar{s},0]} u^{m-1}\psi(u)\,dxdt \\ &\quad + \frac{2^{\ell+3}}{\omega}\ln\Big(\frac{2^{\ell+2}H_k^+}{\omega}\Big)\int_{B_\varrho\times[\bar{s},0]} \chi_{\{u>k\}}\,d\mu, \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3\end{aligned} \tag{4.46}$$

for all $t \in [\bar{s}, 0]$. We now proceed to estimate all the terms separately. By their definition, and the fact that $H_k^+ \le \frac{1}{4}\omega$, it is immediate to see that

$$\psi(u) \le \ln\Big(\frac{H_k^+ 2^{\ell+2}}{\omega}\Big) \le \ln 2^\ell = \ell \ln 2. \tag{4.47}$$

For the first integrals on the right-hand side we use the preceding inequality and the fact that the logarithmic function $\psi(u)$ vanishes whenever $u \le \mu^+ - \frac{1}{4}\omega + 2^{-(\ell+2)}\omega$. This leads to the following estimate of I_1:

$$\begin{aligned}\mathrm{I}_1 &\le \ell^2 \ln^2 2 \Big|\Big\{u(\cdot,\bar{s}) > \mu^+ - \tfrac{1}{4}\omega + \frac{\omega}{2^{\ell+2}}\Big\} \cap B_\varrho\Big| \\ &\le \ell^2 \ln^2 2 \big|\big\{u(\cdot,\bar{s}) > \mu^+ - \tfrac{1}{4}\omega\big\} \cap B_\varrho\big| \\ &\le \ell^2 \ln^2 2\, \frac{1-\nu_*}{1-\frac{1}{2}\nu_*}|B_\varrho|,\end{aligned}$$

where in the last line we used Lemma 4.3. Next, we estimate the integral I_2. Here, we use again the bound for $\psi(u)$ from above, the second inequality in (4.37) (more precisely that $u \le \mu^+ \le \frac{13}{12}\omega$), and finally $\bar{s} \le \theta\varrho^2 = \omega^{1-m}\varrho^2$. This procedure yields the estimate

$$\mathrm{I}_2 \le \frac{\boldsymbol{\gamma}\,\ell\ln 2}{\sigma^2\varrho^2}\big(\tfrac{13}{12}\omega\big)^{m-1}\omega^{1-m}\varrho^2|B_\varrho| \le \frac{\boldsymbol{\gamma}\,\ell\ln 2}{\sigma^2}|B_\varrho|,$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1)$. Finally, we come to the estimate of the integral I_3. Using the second inequality of (4.47) we obtain

$$\mathrm{I}_3 \le \frac{2^{\ell+3}\,\ell\ln 2}{\omega}\,\mu(Q_{\varrho,\theta\varrho^2}) \le \frac{2^{\ell+3}\ell\ln 2}{\boldsymbol{\alpha}(N)\omega}\,\mathbf{I}_2^\mu(2\varrho, 4\theta\varrho^2)|B_\varrho|,$$

where we have used (4.43) for the last estimate. Assume for the moment that ℓ has been chosen, and that

$$\frac{2^\ell}{\omega}\,\mathbf{I}_2^\mu(2\varrho,4\theta\varrho^2)<1\,.$$

Otherwise, (4.44) holds trivially for $s_1=\ell+2$. Then, we have $\mathrm{I}_3\le\boldsymbol{\gamma}(N)\ell|B_\varrho|$. Joining the estimates for I_1 - I_3 with (4.46) we arrive at

$$\int_{B_{(1-\sigma)\varrho}\times\{t\}}\psi^2(u)\,dx\le\ell^2\ln^2 2\,\frac{1-\nu_*}{1-\frac12\nu_*}|B_\varrho|+\frac{\boldsymbol{\gamma}\,\ell}{\sigma^2}|B_\varrho|, \tag{4.48}$$

for a constant $\boldsymbol{\gamma}=\boldsymbol{\gamma}(N,m,C_o,C_1)$. Now we estimate the left-hand side in (4.48) from below, integrating over the smaller set $B_{(1-\sigma)\varrho}\cap\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\}$. Since $\psi(u)(x,t)$ is a decreasing function of H_k^+ for any fixed x in such a set, we can estimate

$$\psi(u)^2\ge\ln^2\frac{\frac{\omega}{4}}{\frac{\omega}{2^{\ell+1}}}=(\ell-1)^2\ln^2 2. \tag{4.49}$$

Inserting (4.49) into (4.48), and dividing through by $(\ell-1)^2\ln^2 2$, we obtain

$$\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\right\}\cap B_{(1-\sigma)\varrho}\right|\le\left(\frac{\ell}{\ell-1}\right)^2\frac{1-\nu_*}{1-\frac12\nu_*}|B_\varrho|+\frac{\boldsymbol{\gamma}\ell}{\sigma^2(\ell-1)^2}|B_\varrho|.$$

On the other hand, we have

$$\begin{aligned}&\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\right\}\cap B_\varrho\right|\\&\qquad\le\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\right\}\cap B_{(1-\sigma)\varrho}\right|+\left|B_\varrho\backslash B_{(1-\sigma)\varrho}\right|\\&\qquad\le\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\right\}\cap B_{(1-\sigma)\varrho}\right|+N\boldsymbol{\alpha}(N)\sigma|B_\varrho|.\end{aligned}$$

Therefore, we conclude that

$$\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{\ell+2}}\right\}\cap B_\varrho\right|\le\left(\frac{\ell}{\ell-1}\right)^2\frac{1-\nu_*}{1-\frac12\nu_*}|B_\varrho|+\frac{\boldsymbol{\gamma}\ell}{\sigma^2(\ell-1)^2}|B_\varrho|+\boldsymbol{\gamma}\sigma|B_\varrho|,$$

holds true for all $t\in[\bar{s},0]$. Now choose σ small enough, such that $\boldsymbol{\gamma}(N)\sigma\le\frac38\nu_*^2$, and then ℓ so large that

$$\left(\frac{\ell}{\ell-1}\right)^2<\left(1-\tfrac12\nu_*\right)(1+\nu_*)\quad\text{and}\quad\frac{\boldsymbol{\gamma}\ell}{\sigma^2(\ell-1)^2}\le\tfrac38\nu_*^2.$$

Now, the claim follows with $s_1:=\ell+2$. Since ν_* depends on N,m,C_o,C_1, then also s_1 depends on these quantities. In particular, s_1 is independent of ω, ϱ, $\bar{s}$. □

Corollary 4.1. *Under the assumptions of Lemma 4.4, either*

$$\omega<2^{s_1}\mathbf{I}_2^\mu(2\varrho,4\theta\varrho^2),$$

or

$$\left|\left\{u(\cdot,t)>\mu^+-\frac{\omega}{2^j}\right\}\cap B_\varrho\right|\le\left(1-\tfrac14\nu_*^2\right)|B_\varrho|$$

holds true for all $j\ge s_1$, and for all $t\in[-\frac12\nu_*\theta\varrho^2,0]$.

We now let $\theta_*:=\frac12\nu_*\theta$ and work in the cylinder $Q_{\varrho,\theta_*}=B_\varrho\times\left(-\frac12\nu_*\theta\varrho^2,0\right]$. We have the following.

Lemma 4.5. *For every $\bar\nu\in(0,1)$, there exists a positive integer q_*, depending only on m, N, C_o, C_1, and $\bar\nu$, such that either*

$$\omega < 2^{s_1+q_*}\,\mathbf{I}^{\mu}_2(4\varrho,16\theta\varrho^2),$$

or

$$\Big|\Big\{u>\mu^+-\frac{\omega}{2^{s_1+q_*}}\Big\}\cap Q_{\varrho,\theta_*}\Big| < \bar\nu\big|Q_{\varrho,\theta_*}\big|.$$

Proof. Without loss of generality we can assume that the first alternative of Corollary 4.1 is violated, i.e. that $\omega\ge 2^{s_1}\mathbf{I}^{\mu}_2(2\varrho,4\theta\varrho^2)$. Otherwise, the first alternative of the claim trivially holds for any choice of $q_*\in\mathbb{N}$. Now, for q_* to be chosen we consider the energy estimate (3.2) for the truncated function $(u-k_j)_+$ with levels

$$k_j=\mu^+-\frac{\omega}{2^j}\qquad\text{for } j=s_1,s_1+1,\dots,s_1+q_*,$$

over the pair of cylinders

$$Q=Q_{\varrho,\theta_*}=B_\varrho\times\big(-\tfrac12\nu_*\theta\varrho^2,0\big],\qquad Q'=B_{2\varrho}\times(-\nu_*\theta\varrho^2,0].$$

The cutoff function $(x,t)\mapsto\zeta(x,t)\in[0,1]$ is taken to be 1 in Q, vanishing on the parabolic boundary of Q', and such that $|D\zeta|\le\frac{2}{\varrho}$, and $0\le\zeta_t\le\frac{2}{\nu_*\theta\varrho^2}$. With these stipulations, the energy estimate (3.2) takes the form

$$\begin{aligned}C_o\Big(\frac{\omega}{2}\Big)^{m-1}&\iint_Q|D(u-k_j)_+|^2dxdt\\&\le\gamma\bigg[\frac{\omega^{m-1}}{\nu_*\varrho^2}\iint_{Q'}(u-k_j)_+^2\,dxdt+\int_{Q'}(u-k_j)_+d\mu\bigg].\end{aligned}$$

Here, we used in turn the fact that $u\le\mu^+\le\frac{13}{12}\omega<2\omega$ (see (4.37)) in order to estimate the function u^{m-1} from above, the definition $\theta=\omega^{1-m}$, and the fact that in the left-hand side integral we only have to integrate over the set of those points in Q where $u>k_j$. Since $k_j\ge\mu^+-\frac14\omega\ge\frac12\omega$ by the first inequality of (4.37), the estimate from below follows. The right-hand side is now easily estimated since $(u-k_j)_+\le\frac{\omega}{2^j}$. Therefore, we obtain

$$\begin{aligned}C_o\Big(\frac{\omega}{2}\Big)^{m-1}&\iint_Q|D(u-k_j)_+|^2dxdt\\&\le\gamma\bigg[\frac{\omega^{m-1}}{\nu_*\varrho^2}\Big(\frac{\omega}{2^j}\Big)^2|Q'|+\frac{\omega}{2^j}\,\mu(Q')\bigg]\\&\le\frac{\gamma\omega^{m-1}}{\nu_*\varrho^2}\Big(\frac{\omega}{2^j}\Big)^2|Q'|+\frac{\gamma\omega^{m-1}}{\nu_*\varrho^2}\Big(\frac{\omega}{2^j}\Big)^2|Q|\frac{2^{s_1+q_*}}{\omega}\frac{\mu(Q')}{\varrho^N}\\&\le\frac{\gamma\omega^{m-1}}{\nu_*\varrho^2}\Big(\frac{\omega}{2^j}\Big)^2|Q|\bigg[1+\frac{2^{s_1+q_*}}{\omega}\frac{\mu(Q')}{\varrho^N}\bigg].\end{aligned}$$

Assume for the moment that $q_*\in\mathbb{N}$ has been already determined, and that

$$\frac{2^{s_1+q_*}}{\omega}\frac{\mu(Q_{2\varrho,4\theta\varrho^2})}{\varrho^N}\le 1.$$

Otherwise, by (4.43) the first alternative of the claim would follow. Hence, we obtain that

$$\iint_Q|D(u-k_j)_+|^2dxdt\le\frac{\gamma}{\nu_*\varrho^2}\Big(\frac{\omega}{2^j}\Big)^2|Q|. \tag{4.50}$$

Next, we apply DeGiorgi's isoperimetric inequality (see [7], Chapter 2, Lemma 2.2) for $t\in\big(-\frac12\nu_*\theta\varrho^2,0\big]$ over the ball B_ϱ for the levels $k_j=:k<\ell:=k_{j+1}$. Taking into account

Corollary 4.1, we obtain for a.e. $t \in \big(-\frac{1}{2}\nu_*\theta\varrho^2, 0\big]$ (note that the second alternative must hold, due to the assumption from the beginning of the proof))

$$\begin{aligned}
&\frac{\omega}{2^{j+1}}\big|\big\{u(\cdot,t) > k_{j+1}\big\} \cap B_\varrho\big| \\
&\qquad\le \frac{\gamma\varrho^{N+1}}{\big|\big\{u(\cdot,t) < k_j\big\} \cap B_\varrho\big|}\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap B_\varrho} |Du(\cdot,t)|\,dx \\
&\qquad\le \frac{4\gamma\varrho^{N+1}}{\nu_*^2|B_\varrho|}\int_{[k_j<u(\cdot,t)<k_{j+1}]\cap B_\varrho} |Du(\cdot,t)|dx \\
&\qquad= \frac{\gamma\varrho}{\nu_*^2}\int_{[k_j<u(\cdot,t)<k_{j+1}]\cap B_\varrho} |Du(\cdot,t)|dx.
\end{aligned}$$

We define

$$A_j := \{u > k_j\} \cap Q,$$

and integrate the preceding inequality with respect to time over $\big(-\frac{1}{2}\nu_*\theta\varrho^2, 0\big]$, and apply Cauchy-Schwartz's inequality. This gives

$$\begin{aligned}
\frac{\omega}{2^{j+1}}|A_{j+1}| &\le \frac{\gamma\varrho}{\nu_*^2}\bigg(\iint_{[k_j<u(\cdot,t)<k_{j+1}]\cap Q}|Du|^2dxdt\bigg)^{\frac{1}{2}}\big|A_j \setminus A_{j+1}\big|^{\frac{1}{2}} \\
&\le \frac{\gamma\varrho}{\nu_*^2}\bigg(\iint_Q |D(u-k_j)_+|^2dxdt\bigg)^{\frac{1}{2}}\big|A_j \setminus A_{j+1}\big|^{\frac{1}{2}} \\
&\le \frac{\gamma}{\sqrt{\nu_*{}^5}}\frac{\omega}{2^j}|Q|^{\frac{1}{2}}\big|A_j \setminus A_{j+1}\big|^{\frac{1}{2}}.
\end{aligned}$$

Here, we used (4.50) in the last line. From the preceding inequality we easily get

$$|A_{j+1}|^2 \le \frac{\gamma}{\nu_*^5}\,|Q|\,\big|A_j \setminus A_{j+1}\big|.$$

We add up these recursive inequalities for $j = s_1, s_1+1, \ldots, s_1+q_*-1$ and use the fact that the right-hand side forms a telescopic series. This procedure yields the measure bound

$$(q_*-1)|A_{s_1+q_*}|^2 \le \frac{\gamma}{\nu_*^5}\,|Q|^2,$$

from which easily follows

$$|A_{s_1+q_*}| \le \sqrt{\frac{\gamma}{\nu_*^5(q_*-1)}}\,|Q|.$$

Now, for fixed $\bar\nu \in (0,1)$ as in the statement of the Lemma, we choose $q_* \in \mathbb{N}$ according to the requirement that

$$\sqrt{\frac{\gamma}{\nu_*^5(q_*-1)}} \le \bar\nu. \tag{4.51}$$

With this choice of q_*, the claim follows. □

Now, referring to the notation of Lemma 4.2, we let

$$r = \tfrac{1}{2}\varrho, \quad a = \tfrac{1}{2}, \quad \xi = 2^{-(s_1+q_*)}, \quad A = (\tfrac{1}{2}\nu_*)^{\frac{1}{m-1}}, \quad \tilde\theta = \tfrac{1}{2}\nu_*\theta.$$

Applying Lemma 4.2, and taking into account the first alternative of Lemma 4.5, we conclude that either

$$\omega < C\,\mathbb{I}^{\mu}_{2,Q_{\varrho,\frac{1}{2}\nu_*\theta\varrho^2}}(4\varrho, 16\theta\varrho^2), \qquad C := \max\{B, 2^{s_1+q_*}\} \tag{4.52}$$

where B is the constant in (4.16), or

$$u \le \mu^+ - \frac{\omega}{2^{s_1+q_*+1}} \qquad \text{a.e. in } B_{\frac12\varrho} \times \big(-\tfrac12\nu_*\theta(\tfrac12\varrho)^2, 0\big], \tag{4.53}$$

provided $\bar\nu$ is chosen equal to ν_+ as defined in (4.18), of course with the choices $a = \frac12$ and $A = (\frac12\nu_*)^{\frac{1}{m-1}}$, and then in turn the integer q_* is chosen according to (4.51). Then, C depends only on m, N, C_o, C_1. The estimate (4.53) can be rewritten as

$$\sup_{Q_{\frac12\varrho,\frac12\nu_*\theta(\frac12\varrho)^2}} u \le \mu^+ - \frac{\omega}{2^{\ell_*+q_*+1}}.$$

But this implies that either (4.52) holds true, or

$$\begin{aligned} \operatorname*{osc}_{Q_{\frac12\varrho,\frac12\nu_*\theta(\frac12\varrho)^2}} u &= \sup_{Q_{\frac12\varrho,\frac12\nu_*\theta(\frac12\varrho)^2}} u - \inf_{Q_{\frac12\varrho,\frac12\nu_*\theta(\frac12\varrho)^2}} u \\ &\le \mu^+ - \frac{\omega}{2^{\ell_*+q_*+1}} - \mu^- \\ &\le \Big(1 - \frac{1}{2^{\ell_*+q_*+1}}\Big)\omega. \end{aligned}$$

Now, recalling the abbreviations we introduced at the beginning of § 4.1, in particular that $\varrho = \varrho_n$, $\omega = \omega_n$ and $\theta = \theta_n = \omega_n^{1-m}$, we have thus proved that either

$$\omega_n < C\,\mathbb{I}^{\mu}_{2,Q_{\varrho_n,\frac12\nu_*\theta_n\varrho_n^2}}\big(4\varrho_n, 16\theta_n\varrho_n^2\big),$$

or

$$\operatorname*{osc}_{Q_{\frac12\varrho_n,\frac12\nu_*\theta_n(\frac12\varrho_n)^2}} u \le \Big(1 - \frac{1}{2^{\ell_*+q_*+1}}\Big)\omega_n$$

holds true. Since $\operatorname{osc}_{Q_{\frac12\varrho_n,\frac12\nu_*\theta_n(\frac12\varrho_n)^2}} u \le \operatorname{osc}_{Q_n} u \le \omega_n$ by our induction assumption, we conclude that in any case there holds

$$\operatorname*{osc}_{Q_{\frac12\varrho_n,\frac12\nu_*\theta_n(\frac12\varrho_n)^2}} u \le \max\Big\{\Big(1 - \frac{1}{2^{\ell_*+q_*+1}}\Big)\omega_n,\ C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\varrho_n, 16\theta_n\varrho_n^2\big)\Big\}. \tag{4.54}$$

4.4. **Pasting the two alternatives together.** Recalling (4.1), we define

$$\delta := 1 - \frac{1}{2^{\ell_*+q_*+1}}, \quad \omega_{n+1} := \max\Big\{\delta\omega_n, C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\varrho_n, 16\theta_n\varrho_n^2\big)\Big\}.$$

Moreover, in view of the estimates for the oscillation of u from the first alternative in (4.39) and from the second alternative in (4.54), we need to define $\eta \in (0,1)$, in such a way that

$$Q_{n+1} = B_{\eta\varrho_n} \times \big(-\omega_{n+1}^{1-m}(\eta\varrho_n)^2, 0\big] \subset B_{\frac12\varrho_n} \times \big(-\tfrac12\nu_*\omega_n^{1-m}(\tfrac12\varrho_n)^2, 0\big].$$

Since $m \ge 1$, by its definition, we always have $\omega_{n+1}^{1-m} \le (\delta\omega_n)^{1-m}$. Therefore, the requirement is satisfied, if we set

$$\eta := \sqrt{\tfrac18\nu_*\delta^{m-1}} < \tfrac12.$$

With these choices for δ and η, we can paste together the first alternative (4.39) (note that $\delta \ge \frac56$) and the second alternative (4.54) to conclude that

$$\operatorname*{osc}_{Q_{n+1}} u \le \omega_{n+1}$$

holds true. The induction argument is now completed, provided the assumption (4.37) is satisfied.

4.5. **The Proof of Proposition 4.1 concluded.** If (4.37) does not hold for some index n, then we have

$$\mu_n^+ > \tfrac{13}{12}\omega_n,$$

which implies that $\mu_n^- \geq \frac{1}{12}\omega_n$. However, this inequality implies that u is uniformly bounded away from zero in Q_n, and therefore equation (1.2) under the structure condition (1.3) is non-degenerate in Q_n, and behaves like a quasilinear parabolic equation with growth of order 2, with a measure data right-hand side, as considered, for example, in [8, 9]. By these results, u is continuous in Q_n.

We briefly outline, how to make this quantitative, following the same approach used in [7], Appendix B, § 13. Assume first that (4.37) fails to hold for $n = 0$. Then, with $\mu_o^\pm$ and ω_o defined by (4.10), modify the construction of Q_o in (4.11) as follows: Instead of Q_o we consider the smaller cylinder

$$Q_{\varrho_o,\theta^*\varrho_o^2} = B_{\varrho_o} \times (-\theta^*\varrho_o^2, 0] \subset Q_o \quad \text{where } \theta^* := \big(\tfrac{12}{13}\mu_o^+\big)^{1-m}.$$

Next, we introduce the change of variables by letting $\Phi(x,s) := (x, \theta^* s)$. Then Φ maps $Q_{\varrho_o,\varrho_o^2}$ into $Q_{\varrho_o,\theta^*\varrho_o^2}$. Then, we scale u down to the new cylinder by letting

$$v(x,s) := \frac{u(x,\theta^* s)}{\mu_o^+} \quad \text{for } (x,s) \in Q_{\varrho_o,\varrho_o^2}. \tag{4.55}$$

The vector-field $\mathbf{A}$ and the measure μ are also transformed by defining

$$\tilde{\mathbf{A}}(x,s,\xi) := \frac{\theta^*}{\mu_o^+}\,\mathbf{A}\big(x, \theta^* s, u(x,\theta^* s), \mu_o^+\xi\big), \quad \text{and} \quad \tilde{\mu} := \frac{\theta^*}{\mu_o^+}\,\Phi^\sharp\mu,$$

where $\Phi^\sharp\mu$ denotes the pull-back of the measure μ. Now, it is straightforward to check that v satisfies

$$v_s - \operatorname{div}\tilde{\mathbf{A}}(x,s,Dv) = \tilde{\mu} \quad \text{weakly in } Q_{\varrho_o,\varrho_o^2}.$$

Moreover, the transformed vector-field $\tilde{\mathbf{A}}\colon Q_{\varrho_o,\varrho_o^2} \times \mathbb{R}^N \to \mathbb{R}^N$ fulfills for a.e $(x,t) \in Q_{\varrho_o,\varrho_o^2}$ and every $\xi \in \mathbb{R}^N$ the monotonicity and boundedness condition

$$\begin{cases} \tilde{\mathbf{A}}(x,s,\xi)\cdot\xi \geq \big(\tfrac{1}{12}\big)^{m-1} mC_o|\xi|^2, \\ \big|\tilde{\mathbf{A}}(x,s,\xi)\big| \leq \big(\tfrac{13}{12}\big)^{m-1} mC_1|\xi|, \end{cases}$$

for the same structural constants $m \geq 1$, C_o, C_1 as in (1.3). Therefore, v is a weak solution of a non-singular parabolic equation in the parabolic cylinder $Q_{\varrho_o,\varrho_o^2}$. By the classical theory, there exist $\delta_o \in (0,1)$ and $\boldsymbol{\gamma} > 1$, that can be determined a priori only in terms of the data N, m, C_o, C_1, and which are independent of ϱ_o and μ_o^+, and a sequence of radii $\varrho_\ell = 4^{-\ell}\varrho_o$ such that

$$\operatorname*{osc}_{Q_{\varrho_{\ell+1}}} v \leq \delta_o \operatorname*{osc}_{Q_{\varrho_\ell}} v + \frac{\boldsymbol{\gamma}}{(\mu_o^+)^m}\,\mathbb{I}^{\mu}_{2,Q_{\varrho_\ell,\varrho_\ell^2}}(4\varrho_\ell, 16\varrho_\ell^2).$$

Returning to the function u and the cylinder $Q_{\varrho_o,\theta^*\varrho_o^2}$, this establishes the induction argument for this sequence of cylinders.

Now, suppose that (4.37) continues to hold at each step until $n-1$, and that it fails to hold at step n. In this case, we modify the construction by considering the smaller cylinder

$$Q_{\varrho_n,\theta_n^*\varrho_n^2} = B_{\varrho_n} \times \big(-\theta_n^*\varrho_n^2, 0\big] \subset Q_n \quad \text{where } \theta_n^* := (\tfrac{12}{13}\mu_n^+)^{1-m}.$$

As in the case $n = 0$ we use a change of variables similar to (4.55) to transform the equation into a non-singular one, to which the classical theory, which we have just discussed, can be applied. This completes the proof of Proposition 4.1. □

Remark 4.3. Roughly speaking, if at step n (4.37) does not hold true, then on Q_n the solution u is bounded away from zero and therefore the equation behaves like a non-singular parabolic equation with measure data on the right-hand side, to which the classical regularity theory can be applied. Hence, at scale n, i.e. on Q_n, the behavior changes from degenerate to non-degenerate, and Q_n becomes a *non-intrinsic* cylinder. At such a scale, which could be called *switching* scale, the standard parabolic theory is applicable and yields the decay of the oscillation on a sequence of non-intrinsic cylinders (i.e. standard parabolic cylinders). More precisely, the same arguments given in the final part of the proof of Theorem 1.1 hold true, and the continuity of u remains a direct consequence of the uniform vanishing on compact sets $E_o \subset E_T$ of the potential $\mathbf{I}_2^\mu(z, r, r^2)$. □

5. Applications

In this final chapter we list some simple consequences of Theorem 1.1. A first, actually immediate, corollary concerns measures which have integrable densities.

Corollary 5.1. *Let u be a non-negative, locally bounded, weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field* $\mathbf{A}$ *fulfills the growth and ellipticity conditions* (1.3)*. Assume that $\mu \in L^{\frac{N+2}{2}}(E_T)$. If the functions*

$$z \;\to\; \int_0^r \|\mu\|_{L^{\frac{N+2}{2}}(Q_{\varrho,\varrho^2}(z))} \frac{d\varrho}{\varrho}$$

converge locally uniformly to zero in E_T as $r \to 0$, then u is locally continuous in E_T.

A further, also rather straightforward, corollary of Theorem 1.1, concerns measures with special density properties. Let us consider a function $h : [0, \infty) \to [0, \infty)$, such that

$$\int_0^r h(\varrho)\frac{d\varrho}{\varrho} < \infty \quad \text{for some} \quad r > 0. \tag{5.1}$$

Then we have

Corollary 5.2. *Let u be a non-negative, locally bounded, weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field* $\mathbf{A}$ *fulfills the growth and ellipticity conditions* (1.3)*. Assume that μ satisfies*

$$\mu\big(Q_{\varrho,\varrho^2}(z)\big) \le C\varrho^N h(\varrho)$$

for every cylinder $Q_{\varrho,\varrho^2}(z) \Subset E_T$, where the function $h(\cdot)$ satisfies (5.1)*; then u is locally continuous in E_T.*

The preceding Corollary covers for example the case of measures μ satisfying

$$\mu\big(Q_{\varrho,\varrho^2}(z)\big) \le \gamma\varrho^{N+\varepsilon} \quad \text{for some } \varepsilon > 0,$$

whenever $Q_{\varrho,\varrho^2}(z) \Subset E_T$. As fas as we know, this result is new.

A third consequence concerns measures, which have densities in Lorentz spaces. In order to explain the result we have in mind, we need to recall a few basic definitions relevant to Lorentz spaces. Let $\mu \colon E_T \to \mathbb{R}$ be a measurable map such that

$$\big|\big\{z \in E_T : |\mu(z)| > \sigma\big\}\big| < \infty \quad \text{for } \sigma > 0.$$

We assume that μ is extended to the whole $\mathbb{R}^{N+1}$ letting $\mu \equiv 0$ outside E_T. The decreasing rearrangement $\mu^* \colon [0, \infty] \to [0, \infty]$ is pointwise defined by

$$\mu^*(s) := \sup\big\{\sigma \ge 0 : \big|\big\{z \in E_T : |\mu(z)| > \sigma\big\}\big| > s\big\}\,.$$

Now, the usual definition of the Lorentz space $L(p,q)$, for $p \in (0,\infty)$ and $q \in (0,\infty)$, is as follows:

$$\mu \in L(p,q) \quad \Longleftrightarrow \quad [\mu]_{p,q} \overset{\text{def}}{=} \left(\frac{q}{p}\int_0^\infty \big(\mu^*(\varrho)\varrho^{\frac{1}{p}}\big)^q \,\frac{d\varrho}{\varrho}\right)^{\frac{1}{q}} < \infty. \tag{5.2}$$

Later on we will need a characterization of Lorentz spaces, using an averaged version of μ^*. This characterization goes back to Hunt [10]. For $s > 0$, we consider the following maximal operator

$$\mu(s)^{**} \overset{\text{def}}{=} \frac{1}{s}\int_0^s \mu^*(\sigma)\,d\sigma.$$

With μ^{**} at hand, for $q < \infty$ one defines

$$\|\mu\|_{\gamma,q} \overset{\text{def}}{=} \left(\frac{q}{p}\int_0^\infty \big(\mu^{**}(\varrho)\varrho^{\frac{1}{p}}\big)^q \,\frac{d\varrho}{\varrho}\right)^{\frac{1}{q}}. \tag{5.3}$$

Then, [17, Theorem 3.21] shows that

$$[\mu]_{p,q} \le \|\mu\|_{p,q} \le \boldsymbol{\gamma}(p,q)[\mu]_{p,q}$$

holds true for $p > 1$. Therefore it is natural to work with the quantity $\|\cdot\|_{p,q}$ when dealing with Lorentz spaces, at least when $p > 1$. Since in our application the index p is always greater then one (actually we have $p = \frac{N+2}{2}$), we can use this second more convenient characterization. We note that the quantity $\|\cdot\|_{p,q}$ makes $L(p,q)$ to a Banach space when $p > 1$. From [9, §3.1, Lemma 2] we recall the following.

Lemma 5.1. *Assume that* $\mu \in L\big(\frac{N+2}{2},1\big)$*. Then for every* $r > 0$ *there holds*

$$\sup_{z\in E_T} \mathbf{I}_2^\mu(z,r,r^2) \le \boldsymbol{\gamma}(N)\int_0^{2\boldsymbol{\alpha}(N)r^{N+2}} \mu^{**}(s)\varrho^{\frac{2}{N+2}}\,\frac{d\varrho}{\varrho}.$$

Theorem 1.1 has now the following immediate corollary:

Corollary 5.3. *Let* u *be a non-negative, locally bounded, weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field* $\mathbf{A}$ *fulfills the growth and ellipticity conditions* (1.3)*. Assume that* $\mu \in L(\frac{N+2}{2},1)$ *holds locally in* E_T*. Then* u *is locally continuous in* E_T*.*

Proof. Without loss of generality, we may assume that μ is extended to $\mathbb{R}^{N+1}$ by zero outside of E_T and that $\mu \in L\big(\frac{N+2}{2},1\big)(\mathbb{R}^{N+1})$. In order to conclude the claim, we only have to switch from E_T to any subset $E_o \Subset E_T$, and then apply the following argument. This localization of μ to E_o is always possible, by setting μ to zero outside E_o. Now the proof goes as follows: Since $\mu \in L\big(\frac{N+2}{2},1\big)$ we infer from (5.3) that

$$\int_0^\infty \mu^{**}(s)\varrho^{\frac{2}{N+2}}\,\frac{d\varrho}{\varrho} < \infty.$$

Therefore, by Lemma 5.3 we can conclude that

$$\lim_{r\downarrow 0}\sup_{z\in E_T} \mathbf{I}_2^\mu(z,r,r^2) \le \lim_{r\downarrow 0}\boldsymbol{\gamma}(N)\int_0^{2\boldsymbol{\alpha}(N)r^{N+2}} \mu^{**}(s)\varrho^{\frac{2}{N+2}}\,\frac{d\varrho}{\varrho} = 0.$$

Now, the claim follows from Theorem 1.1. □

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