THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Existence of evolutionary variational solutions via the calculus of variations

V. Bögelein, F. Duzaar and P. Marcellini

REPORT No. 12, 2013/2014, fall
ISSN 1103-467X
ISRN IML-R- -12-13/14- -SE+fall

EXISTENCE OF EVOLUTIONARY VARIATIONAL SOLUTIONS VIA THE CALCULUS OF VARIATIONS

VERENA BÖGELEIN, FRANK DUZAAR, AND PAOLO MARCELLINI

ABSTRACT. In this paper we introduce a purely variational approach to time dependent problems, yielding the existence of global parabolic minimizers, that is

$$\int_0^T \int_\Omega \big[u \cdot \partial_t \varphi + f(x, Du)\big]\, dxdt \le \int_0^T \int_\Omega f(x, Du + D\varphi)\, dxdt,$$

whenever $T > 0$ and $\varphi \in C_0^\infty(\Omega \times (0,T), \mathbb{R}^N)$. For the integrand $f\colon \Omega \times \mathbb{R}^{Nn} \to [0,\infty]$ we merely assume convexity with respect to the gradient variable and coercivity. These evolutionary variational solutions are obtained as limits of maps depending on space and time minimizing certain convex variational functionals. In the simplest situation, with some growth conditions on f, the method provides the existence of global weak solutions to Cauchy-Dirichlet problems of parabolic systems of the type

$$\partial_t u - \operatorname{div} D_\xi f(x, Du) = 0 \qquad \text{in } \Omega \times (0,\infty).$$

1. INTRODUCTION

In this paper we are concerned with the existence for evolutionary problems possessing a variational structure, in the sense that we are aiming to construct solutions which inherit a certain minimizing property. The advantage of these parabolic minimizers or variational solutions stems from the fact that they might exist even in situations where the associated parabolic system makes no sense. Here we should recall the stationary case, where it is possible to establish the existence of minimizers by the Direct Methods of the Calculus of Variations in fairly general situations, whereas additional stronger assumptions are needed to guarantee that the minimizers fulfill the Euler-Lagrange system. It is exactly this point we address in this paper: we construct variational solutions (parabolic minimizers) to evolutionary problems under general assumptions on the integrand, where a priori it is not clear at all that these minimizers also solve the associated evolutionary system.

1.1. **The main result.** To explain our ideas and results in more detail, we start for simplicity with a variational integrand $f\colon \Omega \times \mathbb{R}^{Nn} \to [0,\infty]$, a given inhomogeneity $h\colon \Omega \to \mathbb{R}^N$ and an initial datum $u_o\colon \Omega \to \mathbb{R}^N$. Here, Ω denotes a bounded domain in $\mathbb{R}^n$ with $n \ge 2$ and $\Omega_\infty := \Omega \times (0,\infty)$ stands for the infinite space-time cylinder over Ω. We note that $N \ge 1$, so that the problem could be vector-valued. For points in $\mathbb{R}^{n+1}$ we usually write $z = (x,t)$. Differentiation with respect to the spatial variable x will be denoted by Du, while $\partial_t u$ stands for the differentiation with respect to time. Associated to data (f, h, u_o), the Cauchy-Dirichlet problem takes the form

$$\begin{cases} \partial_t u - \operatorname{div} D_\xi f(x, Du) = h(x) & \text{in } \Omega_\infty, \\ u = u_o & \text{on } \partial_{\mathcal{P}} \Omega_\infty, \end{cases} \tag{1.1}$$

where $u\colon \Omega_\infty \subset \mathbb{R}^{n+1} \to \mathbb{R}^N$ and $\partial_{\mathcal{P}}\Omega_\infty := [\partial\Omega \times (0,\infty)] \cup [\overline{\Omega} \times \{0\}]$ denotes the parabolic boundary of Ω_∞. In the case $N > 1$ we are dealing with parabolic systems. More

---

Date: October 26, 2013.

2000 *Mathematics Subject Classification.* 35A01, 35D05, 35K45, 35K87, 49J40.

Key words and phrases. Existence, parabolic systems, parabolic minimizers, evolutionary variational solutions, evolutionary solutions via calculus of variations.

generally, we consider a Carathéodory-function $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$ fulfilling the following convexity and coercivity assumptions:

$$\left\{\begin{array}{c}\mathbb{R}^N\times\mathbb{R}^{Nn}\ni(u,\xi)\mapsto f(x,u,\xi)\text{ is convex for a.e. }x\in\Omega;\\ f(x,u,\xi)\ge\nu|\xi|^p-g(x)(1+|u|),\quad\forall(x,u,\xi)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn},\end{array}\right.\tag{1.2}$$

for some $\nu>0$ and $p>1$. Moreover, we assume $g\in L^{p'}(\Omega)$, where $p'=\frac{p}{p-1}$ denotes the Hölder conjugate of p. Note that the convexity assumption on the integrand f with respect to u already implies a linear growth from below. For the initial and boundary datum $u_o\in W^{1,p}(\Omega,\mathbb{R}^N)$ we assume that

$$u_o\in L^2(\Omega,\mathbb{R}^N)\quad\text{and}\quad\int_\Omega f(x,u_o,Du_o)\,dx<\infty.\tag{1.3}$$

Corresponding to the integrand f and the initial datum u_o we can state on a purely formal level the following Cauchy-Dirichlet problem on Ω_∞:

$$\left\{\begin{array}{ll}\partial_t u-\operatorname{div}D_\xi f(x,u,Du)=-D_uf(x,u,Du) & \text{in }\Omega_\infty,\\ u=u_o & \text{on }\partial_{\mathcal{P}}\Omega_\infty.\end{array}\right.\tag{1.4}$$

In the following definition we describe the concept of *variational solutions* to Cauchy-Dirichlet problems, as for instance those considered in (1.4). Here we follow an idea by Lichnewsky and Temam [17], which was first used in the context of the evolutionary parametric minimal surface equation. Variational solutions are sometimes also called *parabolic minimizers*. We will come back to the slight differences in possible definitions later, see § 3. At this point we only give the following definition.

Definition 1.1 (Variational Solutions)**.** Suppose that $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\overline{\mathbb{R}}$ is a variational integrand satisfying the convexity and coercivity assumption (1.2). Moreover, assume that the Cauchy-Dirichlet datum u_o fulfills (1.3). We identify a measurable map $u\colon\Omega_\infty\to\mathbb{R}^N$ in the class

$$u\in L^p\big(0,T;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N)\big)\cap C^0\big([0,T];L^2(\Omega,\mathbb{R}^N)\big),\quad\text{for any }T>0$$

as a *variational solution* associated to the Cauchy-Dirichlet problem (1.1) if and only if the variational inequality

$$\begin{aligned}\int_0^T\int_\Omega f(x,u,Du)\,dxdt\le&\int_0^T\int_\Omega\big[\partial_t v\cdot(v-u)+f(x,v,Dv)\big]\,dxdt\\ &+\tfrac12\|v(\cdot,0)-u_o\|^2_{L^2(\Omega)}-\tfrac12\|(v-u)(\cdot,T)\|^2_{L^2(\Omega)}\end{aligned}\tag{1.5}$$

holds true, whenever $T>0$ and $v\in L^p(0,T;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ with $\partial_t v\in L^2(\Omega_T,\mathbb{R}^N)$. □

Here and in the following we use the shorthand notation

$$W^{1,p}_{u_o}(\Omega,\mathbb{R}^N):=u_o+W^{1,p}_0(\Omega,\mathbb{R}^N).$$

At this stage we should mention that we could have started in the definition of variational solutions by a map in $L^1(0,T;W^{1,1}(\Omega,\mathbb{R}^N))\cap C^0([0,T],L^2(\Omega,\mathbb{R}^N))$ for any $T>0$. This can be inferred by testing the minimality condition (1.5) with the admissible comparison map $v(\cdot,t)\equiv u_o$. This yields that the left-hand side of (1.5) is finite, and the growth condition from below implies an L^p–$W^{1,p}$-bound. Moreover, from this bound one also obtains that the initial condition $u(\cdot,0)=u_o$ holds true in the usual L^2-sense; cf. Lemma 2.1 for the details and the proof. The main result of the paper is the following existence result.

Theorem 1.2. *Suppose that $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\overline{\mathbb{R}}$ is a variational integrand satisfying the convexity and coercivity assumptions* (1.2) *and that the Cauchy-Dirichlet datum u_o*

fulfills the requirements of (1.3)*. Then, there exists a variational solution* u *in the sense of Definition* 1.1*. Moreover,* u *satisfies*

$$\partial_t u \in L^2\big(\Omega_\infty, \mathbb{R}^N\big) \quad \textit{and} \quad u \in C^{0,\frac{1}{2}}\big([0,T]; L^2(\Omega, \mathbb{R}^N)\big) \quad \forall T > 0$$

and the time derivative $\partial_t u$ *fulfills the quantitative bound*

$$\int_0^\infty \int_\Omega |\partial_t u|^2 \,dxdt \le \int_\Omega f(x, u_o, Du_o)\,dx. \tag{1.6}$$

Further, for any $0 \le t_1 < t_2 < \infty$ *we have the energy estimate*

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_\Omega f(x, u, Du)\,dxdt \le 2e \int_\Omega f(x, u_o, Du_o)\,dx. \tag{1.7}$$

Finally, if the integrand f *is strictly convex, then the variational solution is unique.*

1.2. **Examples.** The assumptions of our theorem cover a large variety of interesting variational functionals already considered in the literature. Amongst them there are variational integrands fulfilling a standard growth condition from below and above, functionals of non-standard p, q growth, functionals with exponential growth and Orlicz-type functionals:

$$\begin{aligned} f_1(x, Du) &:= \alpha(x)|Du|^p + \beta(x)|Du|^q \\ f_2(x, Du) &:= \alpha_1(x)|u_{x_1}|^{p_1} + \cdots + \alpha_n(x)|u_{x_n}|^{p_n} \\ f_3(Du) &:= |Du|^p \log(1 + |Du|) \\ f_4(Du) &:= \exp(|Du|^r). \end{aligned}$$

Here, $1 < p < q$, $1 < p_1 < \cdots < p_n$ and $r \ge 1$ are arbitrary integrability exponents, and the functions $\alpha, \beta \colon \Omega \to [0, \infty)$ are non-negative and bounded with $\alpha(x) + \beta(x) \ge \nu > 0$ a.e. on Ω, whereas $\alpha_i(x) \ge \nu > 0$ for $i = 1, \dots, n$ and a.e. $x \in \Omega$. Also functionals of splitting type, such as

$$f(x, u, Du) := f(x, Du) + g(x, u)$$

are covered, where $g \colon \mathbb{R}^n \to \mathbb{R}$ is merely convex with respect to u. For example, a lower order term of the type $g(x, u) := |u|^p$ with an arbitrary $p \ge 1$ or $g(x, u) := -h(x) \cdot u$ is included. In the peculiar case of the variational functional $f(x, u, Du) := \frac{1}{p}|Du|^p + \frac{\lambda}{q}|u|^q$ with $\lambda \in \mathbb{R}_+$, our result guarantees the existence of a variational solution. In a second step it can be shown that the variational solution is a global weak solution to the associated Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \Delta_p u = -\lambda |u|^{q-2} u & \text{in } \Omega_\infty, \\ u = u_o & \text{on } \partial_{\mathcal{P}} \Omega_\infty. \end{cases}$$

We note that this result holds true without imposing any constraint on the exponent q of the non-linearity, and therefore covers classes of parabolic equations with super-critical nonlinearity.

1.3. **Passing to the parabolic system.** The passage from the minimality condition (1.5) to the associated parabolic system is possible under certain additional assumptions on the integrand f. For simplicity we shall restrict ourselves to a classical case, where the integrand $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is a Carathéodory-function, satisfying (1.2) and in addition the following growth condition from above

$$f(x, u, \xi) \le L\big(|\xi|^p + |u|^p + G(x)\big), \tag{1.8}$$

where $L \ge \nu$ and $G \in L^{p'}(\Omega)$. By its convexity f is an almost everywhere differentiable, locally Lipschitz function with respect to (u, ξ). Thus, the compositions $D_\xi f(x, u, Du)$

and $D_u f(x,u,Du)$ are well defined. Together with assumption (1.8) it is easy to show the following growth conditions

$$|D_\xi f(x,u,\xi)| + |D_u f(x,u,\xi)| \le c(p,L)\big[|\xi|^{p-1} + |u|^{p-1} + |g(x)| + |G(x)| + 1\big],$$

whenever $(x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$, cf. [20, Lemma 2.1]. Moreover, if f satisfies the growth condition (1.8), then one can show that the time derivative $\partial_t u$ of the variational solution belongs to the distributional space $L^{p'}(0,T;W^{-1,p'}(\Omega,\mathbb{R}^N))$ for any $T > 0$. Here, $p' = \frac{p}{p-1}$ denotes again the Hölder conjugate of p. For the particular solution constructed in Theorem 1.2 such an argumentation is unnecessary, since we even have $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$. In the minimality condition (1.5) we use the testing function $v \equiv u + s\varphi$, with $0 < s < 1$ and $\varphi \in C_0^\infty(\Omega_\infty,\mathbb{R}^N)$. The resulting inequality is divided by s. Afterwards, we let $s \downarrow 0$, which amounts in taking the one-sided derivative of the mapping

$$[0,1) \ni s \mapsto \int_0^\infty \int_\Omega f(x, u + s\varphi, Du + sD\varphi)\, dxdt.$$

By the fact that $\partial_t u \in L^{p'}(0,T;W^{1,p'}(\Omega,\mathbb{R}^N))$, respectively $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ the result is that

$$\int_0^\infty \int_\Omega \Big[\partial_t u \cdot \varphi + D_\xi f(x,u,Du) \cdot D\varphi + D_u f(x,u,Du) \cdot \varphi\Big]\, dxdt \le 0$$

holds true for any $\varphi \in C_0^\infty(\Omega_\infty,\mathbb{R}^N)$. Here, we can replace φ by $-\varphi$ to obtain the reversed inequality, so that the variational solution solves the associated parabolic system and therefore is a global solution to the Cauchy-Dirichlet problem (1.4). Moreover, since also $u = u_o$ on $\partial_{\mathcal{P}}\Omega_\infty$, the variational solution is a global weak solution to the associated Cauchy-Dirichlet problem.

As an application for variational integrands $f(x,\xi) - h(x) \cdot u$ with a principal part f satisfying a standard growth condition of the type

$$\nu|\xi|^p \le f(x,\xi) \le L(1 + |\xi|^p)$$

for some $p > 1$ and structural constants $0 < \nu \le L$, we have that variational solutions are global weak solutions to the Cauchy-Dirichlet problem (1.1).

More generally, the results of [4] allow the derivation of the parabolic system, if the integrand f is of class C^2 and furthermore satisfies a non-standard growth condition of the type

$$\nu|\xi|^p \le f(\xi) \le L(1 + |\xi|^q)$$

whenever $2 \le p \le q < p + \min\{1, \frac{4}{n}\}$. At this stage we should mention that it is an interesting problem to establish – maybe under further structure conditions on the integrand f – that a variational solution also solves the associated parabolic system. For example in the elliptic case it is known that minimizers to the integrand $f(\xi) = \exp(|\xi|^p)$ have a locally bounded gradient in the interior of Ω, cf. [21, 22]. Therefore, it could be of interest to establish the same result for parabolic systems, maybe more general for integrands of the type $f(\xi) = \phi(|\xi|)$, where ϕ is not necessarily a Δ_2-function. We will take this issue up in a forthcoming paper. At this stage we should mention that in the scalar case for parabolic equations with non-standard p,q growth without variational structure the existence problem has been treated in [3] by use of a completely different approach; cf. [7] for a related existence result.

1.4. **The method of the proof and some comments on the history of the problem.** As mentioned already before, our method will be of purely variational nature, and goes back to a conjecture of De Giorgi [8, 9] concerning the existence of global weak solutions to the Cauchy problem for non-linear hyperbolic wave equations on $\mathbb{R}^n$. More precisely, De Giorgi suggested to construct such solutions as limits of minimizers of convex variational integrals on $\mathbb{R}^n \times (0,\infty)$. The proposed approach can be viewed as a link between the powerful methods of the Classical Calculus of Variations and the theory of Non-linear

Hyperbolic Wave Equations. In [26] Serra & Tilli solved the De Giorgi conjecture (up to subsequences) for the Laplacian as the principal part in an affirmative way; see also [28] for the construction of weak solutions on finite time intervals.

In the present paper we use a similar approach, in order to treat general non-linear parabolic evolutionary problems related to variational integrands $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\overline{\mathbb{R}}$. Under the weak assumptions on f from (1.2) the Classical Calculus of Variations ensures the existence of minimizers to the Dirichlet problem associated to the variational functional, so that it is natural to develop a related theory for evolutionary Cauchy-Dirichlet problems via a modification of De Giorgi's ingenious idea. A few words concerning the method are in order. For a given time independent datum $u_o:\Omega\to\mathbb{R}^N$ we consider mappings $u\colon\Omega_\infty\to\mathbb{R}^N$ satisfying the Cauchy-Dirichlet boundary condition $u=u_o$ on $\partial_{\mathcal{P}}\Omega_\infty$. For such mappings (of course we have to impose certain integrability conditions) we consider for given $\varepsilon\in(0,1]$ the following convex variational integrals:

$$\mathcal{F}_\varepsilon(v):=\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\Big[\tfrac12|\partial_t v|^2+\tfrac1\varepsilon f(x,u,Du)\Big]\,dxdt.$$

The growth assumption from below and the convexity of f allow the application of standard methods from the Classical Calculus of Variations ensuring the existence of minimizers u_ε in certain classes of mappings. Essentially, these classes are defined by requiring that $v\in W^{1,1}(\Omega\times(0,T);\mathbb{R}^N)$ for any time $T>0$, $v=u_o$ on $\partial_{\mathcal{P}}\Omega_\infty$ in the sense of traces, and finally that $\mathcal{F}_\varepsilon$ is coercive.

To explain why the sequence u_ε is expected to converge to a solution of the Cauchy-Dirichlet problem

$$\begin{cases}\partial_t u-\operatorname{div}D_\xi f(x,u,Du)=-D_u f(x,u,Du) & \text{in }\Omega_\infty,\\ u=u_o & \text{on }\partial_{\mathcal{P}}\Omega_\infty,\end{cases}$$

one computes the Euler-Lagrange system of the functional $\mathcal{F}_\varepsilon$ in its classical form. From the classical form one easily deduces that the minimizers u_ε formally solve

$$-\varepsilon\,\partial_{tt}u_\varepsilon+\partial_t u_\varepsilon-\operatorname{div}D_\xi f(x,u_\varepsilon,Du_\varepsilon)=-D_u f(x,u_\varepsilon,Du_\varepsilon),$$

and moreover fulfill the Cauchy-Dirichlet boundary condition $u_\varepsilon=u_o$ on $\partial_{\mathcal{P}}\Omega_\infty$. Therefore, it seems to be natural to consider the limit $\varepsilon\downarrow0$. Roughly speaking, in the wave-type systems above, the term $\varepsilon\partial_t^2u_\varepsilon$ should disappear in the limit $\varepsilon\downarrow0$. Formally, this would lead to a solution u of the Cauchy-Dirichlet problem, provided we could establish the convergence $u_\varepsilon\to u$ in an appropriate sense. At this stage we should mention that the argument is purely heuristical. The assumptions on the integrand f are so weak, that in general we cannot expect that minimizers satisfy the Euler-Lagrange system. Furthermore, even for the functional $f(\xi)=\frac1p|\xi|^p$, with $p\neq2$, we would not be allowed to pass to the limit $\varepsilon\downarrow0$ in the Euler-Lagrange system, since this would require the a.e. convergence $Du_\varepsilon\to Du$.

The main idea to overcome this difficulty, is to stay on the level of minimizers, i.e. not to pass to the Euler-Lagrange system. This has to be understood in the following sense: The mappings u_ε minimize a variational functional $\mathcal{F}_\varepsilon$. Therefore, a limit u as $\varepsilon\downarrow0$ should also inherit a certain minimizing property. We note that this is a delicate point even if the mappings u_ε would minimize the same functional. In our case, we are using a sequence of convex functionals $\mathcal{F}_\varepsilon$ to obtain solutions of an evolutionary problem. Though, the interplay between the Calculus of Variations and the Parabolic Theory must stay on the level of minimization. The link between the convex functionals $\mathcal{F}_\varepsilon$ and the evolutionary problem, is the notion of *evolutionary variational solutions* as in Definition 1.1, going back to the paper of Lichnewsky & Temam [17]. It is exactly the notion of solution which allows us to argue on the level of functionals and which avoids the use of the Euler-Lagrange system associated to $\mathcal{F}_\varepsilon$.

In the present paper we follow this path, by first deriving a certain a priori bound for the energy and the time derivative of the sequence. This comes out by comparing the mappings with u_o, and by using the inner variation with respect to time. The latter was one of the main ingredients of the approach by Sera & Tilli [26]. In our case, the inner variation with respect to time leads to a conservation law, which finally yields suitable energy bounds, and moreover the bound for the time derivative, both uniformly with respect to ε. This allows the passage to a weakly converging subsequence. The considered notion of weak convergence is strong enough, to have the lower semi-continuity of the variational functional associated to the integrand f. At this stage, a direct comparison argument in the functionals $\mathcal{F}_\varepsilon$ together with the convexity allows the passage to the limit, establishing that the weak limit is a variational solution in the sense of Definition 1.1.

The main advantage of our strategy is, that only rather weak assumptions on the integrand f must be imposed, to ensure the existence of a variational solution. It is worth to note that the indicated method works for general convex variational functionals of higher order. However, in this paper we only consider first order integrands as in (1.2) to keep the presentation as simple as possible. We will take this up in subsequent work.

Finally, we should mention that the above described approach has been used before in [1, 2, 23, 24, 27] on finite space-time cylinders $\Omega_T = \Omega \times (0,T)$, to construct solutions of certain parabolic problems. The functionals $\mathcal{F}_\varepsilon$ are termed *Weighted Energy Dissipation Functional*. In the above mentioned papers the corresponding Euler-Lagrange equation, leading to an elliptic regularization of the original evolutionary problem, is utilized in order to pass to the limit $\varepsilon \downarrow 0$. Therefore, the applications are mostly limited to special variational functionals of standard growth and the overall set up has a more abstract point of view (Hilbert spaces, methods from Convex Analysis such as sub-differentials, abstract theory to nonlinear evolutionary PDE's).

Acknowledgement. This work is partially supported by the Institute Mittag-Leffler (Djursholm, Sweden), that the three authors were visiting in Autumn 2013. We would like to thank the staff at the Institute for their kind hospitality.

2. Preliminaries and notations

2.1. **Notations.** The spaces $L^p(\Omega,\mathbb{R}^N)$, $W^{1,p}(\Omega,\mathbb{R}^N)$ and $W^{1,p}_0(\Omega,\mathbb{R}^N)$ denote the usual Lebesgue and Sobolev spaces, and we write $W^{1,p}_{u_o}(\Omega,\mathbb{R}^N) := u_o + W^{1,p}_0(\Omega,\mathbb{R}^N)$. Moreover, by Ω_T, with $T\in(0,\infty)$ we denote the space-time cylinder $\Omega\times(0,T)$; when $T=\infty$ we write Ω_∞ for $\Omega\times(0,\infty)$.

2.2. **The initial condition.** As mentioned in the introduction, variational solutions in the sense of Definition 1.5 fulfill the initial condition $u(\cdot,0)=u_o$ in the usual L^2-sense. This follows from the fact that the difference $\|u(\cdot,T)-u_o\|^2_{L^2}$ grows at most linearly with respect to $T>0$, cf. estimate (2.1) below.

Lemma 2.1. *Let f be a variational integrand satisfying the convexity and coercivity assumption* (1.2)*. Then any variational solution u in the sense of Definition* (1.1) *fulfills the initial condition in the L^2-sense, i.e. we have*

$$\lim_{t\downarrow 0}\|u(\cdot,t)-u_o\|^2_{L^2(\Omega)}=0.$$

Proof. First of all, testing the minimality condition (1.5) with the admissible comparison map $v(\cdot,t)\equiv u_o$, $t>0$, we we find that

$$\int_0^T\int_\Omega f(x,u,Du)\,dxdt+\tfrac12\|u(\cdot,T)-u_o\|^2_{L^2(\Omega)}\le T\int_\Omega f(x,u_o,Du_o)\,dx<\infty$$

holds true for any $T > 0$. Next, we use the growth assumption (1.2) to estimate the variational integral from below. This implies that

$$\nu \int_0^T \int_\Omega |Du|^p \, dxdt - \int_0^T \int_\Omega g(1+|u|) \, dxdt \le T \int_\Omega f(x, u_o, Du_o) \, dx$$

holds true. Now, we use in turn Young's and Poincaré's inequality in order to bound the negative term in the preceding inequality. This procedure leads to the inequality

$$\begin{aligned} &\int_0^T \int_\Omega g(1+|u|) \, dxdt \\ &\qquad \le \frac{\nu}{2} \int_0^T \int_\Omega |Du|^p \, dxdt + c\,T \int_\Omega \big(|u_o|^p + |Du_o|^p + |g|^{p'} + 1\big) \, dx \end{aligned}$$

for a constant c depending only on ν, p and $\operatorname{diam}(\Omega)$. Inserting this above we obtain that

$$\begin{aligned} &\int_0^T \int_\Omega |Du|^p \, dxdt + \|u(\cdot, T) - u_o\|^2_{L^2(\Omega)} \\ &\qquad \le c\,T \int_\Omega \Big[f(x, u_o, Du_o) + |u_o|^p + |Du_o|^p + |g|^{p'} + 1 \Big] \, dx \,, \end{aligned} \tag{2.1}$$

for any $T > 0$. Here, we discard the energy term in the left-hand side and then let $T \downarrow 0$ in the right-hand side. This proves that u satisfies the initial boundary condition $u(\cdot, 0) = u_o$ in the L^2-sense as claimed. □

2.3. **Mollification in time.** Due to their lack of regularity with respect to time, the variational solutions in the sense of Definition 1.1 are in general not admissible as comparison maps in (1.5). However, if for example the integrand f has p-growth from above, i.e.

$$f(x, u, \xi) \le L(|\xi|^p + |u|^p + 1)$$

holds true, then one can show that the time derivative $\partial_t u$ of the variational solution belongs to the distributional space $L^{p'}(0, T; W^{-1,p'}(\Omega, \mathbb{R}^N))$ for any $T > 0$. Here, $p' = \frac{p}{p-1}$ denotes again the Hölder conjugate of p. However, in the general case, where we only assume p-growth from below, this is not clear and therefore we shall use a certain mollification in time. The precise construction of the regularization is as follows: For $T > 0$ and $v \in L^1(\Omega_T, \mathbb{R}^N)$, $v_o \in L^1(\Omega, \mathbb{R}^N)$ and $h \in (0, T]$ we define

$$[v]_h(\cdot, t) := e^{-\frac{t}{h}} v_o + \tfrac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(\cdot, s) \, ds, \tag{2.2}$$

for $t \in [0, T]$. One of the basic features of this mollification is, that $[v]_h$ (formally) solves the ordinary differential equation

$$\partial_t [v]_h = -\tfrac{1}{h} \big([v]_h - v\big)$$

with initial condition $[v]_h(\cdot, 0) = v_o$. The basic properties of the mollification $[\,\cdot\,]_h$ are provided in the following lemma, cf. [14, Lemma 2.2], or [4, Appendix B] for the proofs of the particular statements.

Lemma 2.2. *Suppose that $v \in L^1(\Omega_T, \mathbb{R}^N)$ and moreover $v_o \in L^1(\Omega, \mathbb{R}^N)$. Then, the mollification $[v]_h$ defined in (2.2) admits the following properties:*

(i) *Assume that $v \in L^p(\Omega_T, \mathbb{R}^N)$ and $v_o \in L^p(\Omega, \mathbb{R}^N)$ for some $p \ge 1$. Then, we also have $[v]_h \in L^p(\Omega_T, \mathbb{R}^N)$ and the following quantitative estimate holds true:*

$$\|[v]_h\|_{L^p(\Omega_T)} \le \|v\|_{L^p(\Omega_T)} + h^{\frac{1}{p}} \|v_o\|_{L^p(\Omega)}.$$

Moreover, $[v]_h \to v$ in $L^p(\Omega_T, \mathbb{R}^N)$ as $h \downarrow 0$.

(ii) *Suppose that* $v \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ *and* $v_o \in W^{1,p}(\Omega,\mathbb{R}^N)$ *with* $p \geq 1$. *Then also* $[v]_h \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ *and the following quantitative estimate holds true:*

$$\|[v]_h\|_{L^p(0,T;W^{1,p}(\Omega))} \leq \|v\|_{L^p(0,T;W^{1,p}(\Omega))} + h^{\frac{1}{p}}\|v_o\|_{W^{1,p}(\Omega)}.$$

Moreover, $[v]_h \to v$ *in* $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ *as* $h \downarrow 0$.

(iii) *Assume that* $v \in L^p(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N))$ *and* $v_o \in W^{1,p}_0(\Omega,\mathbb{R}^N)$. *Then, also* $[v]_h \in L^p(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N))$.

(iv) *In the case that* $v \in C^0([0,T];L^2(\Omega,\mathbb{R}^N))$ *and moreover* $v_o \in L^2(\Omega,\mathbb{R}^N)$, *we have* $[v]_h \in C^0([0,T];L^2(\Omega,\mathbb{R}^N))$, $[v]_h(\cdot,0) = v_o$ *and moreover* $[v]_h \to v$ *in* $C^0([0,T];L^2(\Omega,\mathbb{R}^N))$ *as* $h \downarrow 0$.

(v) *Suppose that* $v \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ *and* $v_o \in L^2(\Omega,\mathbb{R}^N)$. *Then also* $\partial_t[v]_h \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$. *Moreover we have*

$$\partial_t[v]_h = -\tfrac{1}{h}\big([v]_h - v\big).$$

(vi) *If* $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ *and* $v_o \in L^2(\Omega,\mathbb{R}^N)$, *then we have* $\partial_t[v]_h \to \partial_t v$ *in* $L^2(\Omega_T,\mathbb{R}^N)$ *and the inequality*

$$\|\partial_t[v]_h\|_{L^2(\Omega_T)} \leq \|\partial_t v\|_{L^2(\Omega_T)}$$

holds true for any $h \in (0,T]$. □

The next Lemma ensures the convergence $f(x,[v]_h,D[v]_h) \to f(x,v,Dv)$ in the limit $h \downarrow 0$, provided that $f(x,v,Dv) \in L^1$.

Lemma 2.3. *Let* $T > 0$ *and assume that*

$$v \in L^1\big(0,T;W^{1,1}(\Omega,\mathbb{R}^N)\big), \quad \textit{with } f(\cdot,v,Dv) \in L^1(\Omega_T)$$

and

$$v_o \in W^{1,1}(\Omega,\mathbb{R}^N), \quad \textit{with } f(\cdot,v_o,Dv_o) \in L^1(\Omega).$$

Then, we have $f(\cdot,[v]_h,D[v]_h) \in L^1(\Omega_T)$ *and moreover*

$$\lim_{h\downarrow 0}\int_0^T\int_\Omega f\big(x,[v]_h,D[v]_h\big)\,dxdt = \int_0^T\int_\Omega f(x,v,Dv)\,dxdt.$$

Proof. For simplicity we omit in our notation the v-dependence of the integrand f. This is justified by the convexity assumption on the integrand with respect to (v,Dv). The differences in the computations are just at a formal level. We first observe that

$$\frac{1}{h(1-e^{-\frac{t}{h}})}\int_0^t e^{\frac{s-t}{h}}\,ds = 1.$$

This allows us to interpret the mollification $[v]_h$ – modulo a multiplicative factor – as a mean with respect to the measure $e^{\frac{s-t}{h}}ds$. Accordingly to this interpretation we first rewrite $f(x,D[v]_h)$ and afterwards use the convexity of f and Jensen's inequality. This procedure yields the following point wise bound:

$$\begin{aligned}
&f\big(\cdot,D[v]_h\big)(x,t)\\
&= f\bigg(x, e^{-\frac{t}{h}}Dv_o(x) + \frac{1-e^{-\frac{t}{h}}}{h(1-e^{-\frac{t}{h}})}\int_0^t Dv(x,s)e^{\frac{s-t}{h}}\,ds\bigg)\\
&\leq e^{-\frac{t}{h}}f\big(x,Dv_o(x)\big) + \big(1-e^{-\frac{t}{h}}\big)\,f\bigg(x,\frac{1}{h(1-e^{-\frac{t}{h}})}\int_0^t Dv(x,s)e^{\frac{s-t}{h}}\,ds\bigg)\\
&\leq e^{-\frac{t}{h}}f\big(x,Dv_o(x)\big) + \frac{1}{h}\int_0^t f\big(x,Dv(x,s)\big)e^{\frac{s-t}{h}}\,ds\\
&= [f(\cdot,Dv)]_h(x,t).
\end{aligned}$$

Here $[f(\cdot, Dv)]_h$ is defined according to definition (2.2) with v_o replaced by $f(\cdot, Dv_o)$. Since $f(\cdot, Dv) \in L^1(\Omega_T)$ and $f(\cdot, Dv_o) \in L^1(\Omega)$ by assumption, we obtain via Lemma 2.2 (i) the uniform bound

$$\|[f(x, Dv)]_h\|_{L^1(\Omega_T)} \leq \|f(x, Dv)\|_{L^1(\Omega_T)} + h\|f(x, Dv_o)\|_{L^1(\Omega)} < \infty.$$

Since $h\|f(x, Dv_o)\|_{L^1(\Omega)} \to 0$ in the limit $h \downarrow 0$, a variant of the dominated convergence theorem implies that

$$\lim_{h\downarrow 0} \int_0^T \int_\Omega f\big(x, D[v]_h\big)\, dxdt = \int_0^T \int_\Omega f(x, Dv)\, dxdt$$

holds true. This proves the claim of the Lemma. □

3. Variational Solutions Versus Parabolic Minimizers

In Definition 1.1 we introduced – following an idea of Lichnewsky & Temam [17] – the notion of *variational solutions*. Nowadays the notion of *parabolic minimizers* introduced independently by Wieser [29] has been used by several authors. They studied different regularity properties such as the self-improving property of higher integrability in the vectorial case, local boundedness and Hölder continuity in the scalar case or partial regularity in the vectorial case, cf. [11, 12, 25]. Moreover, there was some interest in extending the notion to the metric space setting, cf. [13, 15, 16, 18, 19]. In this Section we aim to establish that variational solutions actually are parabolic minimizers. But before going into the details we first give the Definition of a parabolic minimizer, introduced by Wieser [29]. In the sequel we assume that $f\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \overline{\mathbb{R}}$ is a variational integrand satisfying (1.2) and that the Cauchy-Dirichlet datum u_o satisfies $(1.3)_2$.

Definition 3.1. A measurable map $u\colon \Omega_\infty \to \mathbb{R}^N$ is termed *parabolic minimizer* associated to the variational integrand f and the Cauchy-Dirichlet datum u_o if and only if

$$u \in L^p\big(0, T; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)\big) \qquad \forall T > 0$$

and moreover the following minimality condition

$$\int_0^T \int_\Omega u \cdot \partial_t \varphi + f(x, u, Du)\, dxdt \leq \int_0^T \int_\Omega f(x, u + \varphi, Du + D\varphi)\, dxdt \tag{3.1}$$

holds true, whenever $T > 0$ and $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. □

Note that Definition 3.1 is local with respect to the initial boundary, i.e. u is not necessarily equal to u_o at the initial time $t = 0$. We should also mention here, that the Definition of parabolic minimizers was given up to now only in the context of variational integrands satisfying a standard p-growth condition form above and below. Actually, the notion was that of a parabolic Q-minimizer, i.e. a mapping u as in Definition 3.1, but with a right-hand side

$$Q \int_0^T \int_\Omega f(x, u + \varphi, Du + D\varphi)\, dxdt$$

for some fixed $Q \geq 1$. The case $Q = 1$ is of course the one of parabolic minimizers.

If $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$, or $\partial_t u \in L^p(0, T; W^{-1,p'}(\Omega, \mathbb{R}^N))$ for any $T > 0$, then one can easily show that the minimality conditions (1.5) and (3.1) are equivalent (provided that the function u in (3.1) satisfies $u(\cdot, 0) = u_o$). This can be seen by substituting $\varphi = v - u$, respectively $v = u + s\varphi$, with $s > 0$, together with a Minty type argument. Note that the variational solution constructed in Theorem 1.2 satisfies $\partial_t u \in L^2(\Omega_\infty, \mathbb{R}^N)$ and therefore it is a parabolic minimizer in the sense of Definition 3.1. Note also that $\partial_t u \in L^p(0, T; W^{-1,p'}(\Omega, \mathbb{R}^N))$ if f has p-growth from above, see also § 2.3. Since, in general we do not want to impose a condition on the time derivative $\partial_t u$, we need to regularize u with respect to time, in order to prove that variational solutions are also parabolic minimizers. For the sake of completeness, we provide this argument in the following.

Proposition 3.2. *If u is a variational solution in the sense of Definition 1.1, then it is also a parabolic minimizer in the sense of Definition 3.1.*

Proof. We consider a fixed $T > 0$ and a testing function $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. Our aim now, is to prove that inequality (3.1) holds true. To establish this, we would like to choose $v = u + s\varphi$ with $s > 0$ as comparison function in (1.5). Then, the result would follow by a Minty-type argument. However, since $v = u + s\varphi$ is not an admissible choice in (1.5), we have to use a certain mollification with respect to time. More precisely, in (1.5) we choose the comparison map $v = v_h$, where $v_h := [u]_h + s[\varphi]_h$ with $s > 0$. Here $[u]_h$ is defined according to (2.2) and with u_o as the choice for v_o, while $[\varphi]_h$ is defined in the same way but with $v_o = 0$. The fact that v is admissible in (1.5) is an immediate consequence of Lemma 2.2, (iii) and (v). Then, for the first term on the right-hand side of (1.5) we deduce that

$$
\begin{aligned}
\int_0^T \int_\Omega & \partial_t v_h \cdot (v_h - u)\, dxdt \\
&= \int_0^T \int_\Omega \Big[\partial_t [u]_h \cdot ([u]_h - u) + s\partial_t [u]_h \cdot [\varphi]_h + s\partial_t[\varphi]_h \cdot (v_h - u)\Big]\, dxdt \\
&= \int_0^T \int_\Omega \Big[-\tfrac{1}{h} |[u]_h - u|^2 - s[u]_h \cdot \partial_t [\varphi]_h + s(v_h - u)\cdot \partial_t[\varphi]_h\Big]\, dxdt \\
&\quad + s \int_\Omega \big([u]_h \cdot [\varphi]_h\big)(\cdot, T)\, dx \\
&\le \int_0^T \int_\Omega s(s[\varphi]_h - u)\cdot \partial_t [\varphi]_h \, dxdt + s\int_\Omega \big([u]_h \cdot [\varphi]_h\big)(\cdot, T)\, dx\,.
\end{aligned}
$$

In the second last line we used Lemma 2.2 (v) and performed an integration by parts with respect to time. Note that no boundary term occurs for $t = 0$, since $[\varphi]_h(\cdot, 0) = 0$ by Lemma 2.2 (iv). Inserting this into the minimality condition (1.5), we get

$$
\begin{aligned}
\int_0^T \int_\Omega f(x, u, Du)\, dxdt \le \int_0^T \int_\Omega & \Big[s(s[\varphi]_h - u)\cdot \partial_t[\varphi]_h + f(x, v_h, Dv_h)\Big]\, dxdt \\
&+ s\int_\Omega \big([u]_h \cdot [\varphi]_h\big)(\cdot, T)\, dx - \tfrac12 \|(v_h - u)(\cdot, T)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Now, in view of Lemma 2.2 (i), (iv), (vi) and Lemma 2.3 we can pass to the limit $h \downarrow 0$ in the preceding inequality. Note that the second boundary term is non-positive and can therefore be discarded in advance. Moreover, observe that the first boundary term after the passage to the limit vanishes, since $\varphi(\cdot, T) \equiv 0$ on Ω. Altogether, this leads us to

$$
\begin{aligned}
\int_0^T \int_\Omega & f(x, u, Du)\, dxdt \\
&\le \int_0^T \int_\Omega \Big[s(s\varphi - u)\cdot \partial_t \varphi + f(x, u + s\varphi, Du + sD\varphi)\Big]\, dxdt \\
&\le \int_0^T \int_\Omega \Big[s(s\varphi - u)\cdot \partial_t \varphi + (1-s) f(x, u, Du) + s f(x, u + \varphi, Du + D\varphi)\Big]\, dxdt.
\end{aligned}
$$

In the last line we used the convexity of f. Subtracting $(1 - s)\int_0^T \int_\Omega f(x, u, Du)\, dxdt$ on both sides (note that the finiteness of the integral follows by choosing $v(\cdot, t) \equiv u_o$ as comparison map in (1.5)) and dividing by $s > 0$ we get

$$
\int_0^T \int_\Omega f(x, u, Du)\, dxdt \le \int_0^T \int_\Omega \big[(s\varphi - u)\cdot \partial_t \varphi + f(x, u + \varphi, Du + D\varphi)\big]\, dxdt.
$$

Now, we pass to the limit $s \downarrow 0$ and finally come up with the inequality

$$\int_0^T \int_\Omega \big[u \cdot \partial_t \varphi + f(x, u, Du)\big]\, dxdt \le \int_0^T \int_\Omega f(x, u + \varphi, Du + D\varphi)\, dxdt.$$

The previous inequality holds true for any $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. But this means that u is a parabolic minimizer in the sense of Definition 3.1 and completes the proof. □

In our existence theorem we construct variational solutions with a time derivative $\partial_t u \in L^2(\Omega_\infty, \mathbb{R}^N)$. In this case we are able to show that the minimality condition (3.1) is satisfied on a.e. time slice $\Omega \times \{t\}$. The precise statement is as follows:

Proposition 3.3. *Let u be as in Definition 3.1, that is $u \in L^p(0, T; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N))$ for any $T > 0$. If in addition $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ for any $T > 0$, then the minimality condition (3.1) is equivalent to the following minimality condition on time slices:*

$$\int_\Omega f(\cdot, u, Du)(\cdot, t)\, dx \le \int_\Omega \big[\partial_t u(\cdot, t) \cdot \eta + f(\cdot, u + \eta, Du + D\eta)(\cdot, t)\big]\, dx \tag{3.2}$$

holds true for any $\eta \in C_0^\infty(\Omega, \mathbb{R}^N)$ and a.e. $t \in (0, \infty)$.

Proof. First, we suppose that u satisfies (3.1). We test (3.1) with a testing function of splitting type, i.e. with $\varphi(x, t) = \eta(x)\zeta(t)$, where $\zeta \in C_0^\infty((0, \infty))$ with $\zeta \ge 0$ and $\eta \in C_0^\infty(\Omega, \mathbb{R}^N)$. This implies that

$$\begin{aligned}
&\int_{\operatorname{spt}\zeta} \int_\Omega f(x, u, Du)\, dxdt \\
&\quad \le \int_{\operatorname{spt}\zeta} \int_\Omega \big[-\zeta' u \cdot \eta + f(x, u + \zeta\eta, Du + \zeta D\eta)\eta\big]\, dxdt \\
&\quad \le \int_{\operatorname{spt}\zeta} \int_\Omega \Big[\zeta \partial_t u \cdot \eta + (1 - \zeta) f(x, u, Du) + \zeta f(x, u + \eta, Du + D\eta)\Big]\, dxdt.
\end{aligned}$$

In the last line we used the convexity of f and performed an integration by parts. The latter is possible due to the assumption on the time derivative of u. Subtracting the integral of the left-hand side on both sides of the inequality we infer that

$$0 \le \int_{\operatorname{spt}\zeta} \int_\Omega \zeta \Big[\partial_t u \cdot \eta + f(x, u + \eta, Du + D\eta) - f(x, u, Du)\Big]\, dxdt$$

holds true for any $0 \le \zeta \in C_0^\infty((0, \infty))$. But this implies for a.e. $t > 0$ that

$$\int_\Omega f(\cdot, u, Du)(\cdot, t)\, dx \le \int_\Omega \big[\partial_t u(\cdot, t) \cdot \eta + f(\cdot, u + \eta, Du + D\eta)(\cdot, t)\big]\, dx$$

holds true for any $\eta \in C_0^\infty(\Omega, \mathbb{R}^N)$. This proves the claim.

On the other hand, if u satisfies the minimality condition (3.2) for a.e. time slice, we simply choose $\varphi(\cdot, t)$ as testing function η on the time slice $\Omega \times \{t\}$, where φ is a smooth comparison function with compact support in Ω_∞. Integrating the result with respect to time, proves that u satisfies (3.1). □

4. Existence via Elliptic Convex Minimization

In this Chapter we give the proof of Theorem 1.2. Henceforth, we assume that the variational integrand $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \overline{\mathbb{R}}$ satisfies (1.2) and that the Cauchy-Dirichlet datum is as in (1.3).

4.1. **A sequence of minimizers to a variational functional on Ω_∞.** In this chapter we consider for $\varepsilon\in(0,1]$ variational integrals of the form

$$\mathcal{F}_\varepsilon(v):=\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big[\tfrac12|\partial_t v|^2+\tfrac1\varepsilon f(x,v,Dv)\big]\,dxdt.$$

In order to deal with the existence problem associated to these functionals we first introduce a suitable function space, in which the minimization of $\mathcal{F}_\varepsilon$ will be achieved. A measurable function $v\colon\Omega_\infty\to\mathbb{R}^N$ is said to belong to $\mathcal{K}_\varepsilon$ if and only if $v\in W^{1,1}(\Omega_T,\mathbb{R}^N)$ for any $T>0$ and moreover

$$\|v\|_{\mathcal{K}_\varepsilon}:=\bigg[\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}|\partial_t v|^2\,dxdt\bigg]^{\frac12}+\bigg[\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big[|v|^p+|Dv|^p\big]\,dxdt\bigg]^{\frac1p}<\infty.$$

We note that the time independent extension $\overline{u}_o(\cdot,t):=u_o$ for any $t>0$ of $u_o\in W^{1,p}(\Omega,\mathbb{R}^N)\cap L^2(\Omega,\mathbb{R}^N)$ to Ω_∞ belongs to $\mathcal{K}_\varepsilon$. The subspace of functions with zero trace on the lateral boundary, i.e. those functions $v\in\mathcal{K}_\varepsilon$ satisfying $v=0$ on $\partial_{\mathcal{P}}\Omega_\infty$ (as usual this has to be understood in the sense of traces), shall be abbreviated by $\mathcal{N}_\varepsilon$. We keep in mind that $(\mathcal{K}_\varepsilon,\|\cdot\|_{K_\varepsilon})$, and therefore also $(\mathcal{N}_\varepsilon,\|\cdot\|_{K_\varepsilon})$, are Banach spaces. Furthermore, for a map $v\in\mathcal{K}_\varepsilon$ there holds

$$\int_0^T\int_\Omega\big(|\partial_t v|^2+|v|^p+|Dv|^p\big)\,dxdt<\infty\qquad\forall\,T>0.$$

Again, considering the function $\overline{u}_o$ as above, we observe that the class of mappings $v\in u_o+\mathcal{N}_\varepsilon$ with finite energy $\mathcal{F}_\varepsilon(v)<\infty$ is non-empty, that is

$$\big(u_o+\mathcal{N}_\varepsilon\big)\cap\{\mathcal{F}_\varepsilon(u)<\infty\}\neq\emptyset.$$

Furthermore, by assumption (1.2) we infer the following bound from below for the functional $\mathcal{F}_\varepsilon$:

$$\mathcal{F}_\varepsilon(v)\ge\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big[\tfrac12|\partial_t v|^2+\nu|Dv|^p-g(1+|v|)\big]\,dxdt\qquad\forall\,v\in u_o+\mathcal{N}_\varepsilon.$$

For functions $v\in u_o+\mathcal{N}_\varepsilon$ a slice wise application of Poincaré's inequality to $(v-u_o)(\cdot,t)$ for a.e. $t\in(0,\infty)$ implies that

$$\begin{aligned}\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}|v|^p\,dxdt&\le c\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big(|Dv|^p+|u_o|^p+|Du_o|^p\big)\,dxdt\\&=c\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}|Dv|^p\,dxdt+c\int_\Omega\big(|u_o|^p+|Du_o|^p\big)\,dx,\end{aligned}$$

holds true for a constant $c=c(p,\operatorname{diam}(\Omega))\ge1$. Therefore, for $\delta\in(0,1]$ we obtain by Young's inequality that

$$\begin{aligned}&\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}g(1+|v|)\,dxdt\\&\qquad\le\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big[\delta(1+|v|^p)+c_\delta|g|^{p'}\big]\,dxdt\\&\qquad\le c\,\delta\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}|Dv|^p\,dxdt+c_\delta\int_\Omega\big[1+|g|^{p'}+|u_o|^p+|Du_o|^p\big]\,dx,\end{aligned}$$

for a constant $c=c(p,\operatorname{diam}(\Omega))\ge1$. Inserting this above and choosing $\delta>0$ small enough, we obtain that

$$\begin{aligned}\mathcal{F}_\varepsilon(v)&\ge\frac{\nu}{2c}\int_0^\infty\int_\Omega e^{-\frac{t}{\varepsilon}}\big[|\partial_t v|^2+|v|^p+|Dv|^p\big]\,dxdt-c\Big[1+\|g\|^{p'}_{L^{p'}}+\|u_o\|^p_{W^{1,p}}\Big]\\&\ge\tfrac{\nu}{2c}\min\big\{\|v\|^2_{\mathcal{K}_\varepsilon},\|v\|^p_{\mathcal{K}_\varepsilon}\big\}-c\Big[1+\|g\|^{p'}_{L^{p'}(\Omega)}+\|u_o\|^p_{W^{1,p}(\Omega)}\Big]\end{aligned}$$

holds true for any $v \in u_o + \mathcal{N}_\varepsilon$ with a constant $c = c(\nu, p, \operatorname{diam}(\Omega)) \geq 1$. This ensures that the functional $\mathcal{F}_\varepsilon$ is coercive with respect to $\|\cdot\|_{\mathcal{K}_\varepsilon}$ on $u_o + \mathcal{N}_\varepsilon$. Furthermore, the convexity of the integrand f implies the strict convexity of the functional $\mathcal{F}_\varepsilon$. Therefore, we can apply the lower semicontinuity result [10, Theorem 4.3] to conclude the following existence result for $\mathcal{F}_\varepsilon$-minimizing maps:

Lemma 4.1. *For any given $\varepsilon \in (0, 1]$, the variational functional $\mathcal{F}_\varepsilon$ admits a unique minimizer $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$.*

4.2. **Energy bounds.** In this section we establish certain energy bounds for minimizers $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ of $\mathcal{F}_\varepsilon$, which later on allow us to extract a converging subsequence in the limit $\varepsilon \downarrow 0$. We start with the following simple uniform energy bound.

Lemma 4.2. *For any minimizer $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ of $\mathcal{F}_\varepsilon$ we have*

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \int_\Omega f(x, u_o, Du_o)\, dx.$$

Proof. From the minimality of u_ε, note that the time independent extension $\overline{u}_o$ of u_o to Ω_∞ is an admissible comparison function, we conclude

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \mathcal{F}_\varepsilon(\overline{u}_o) = \frac{1}{\varepsilon}\int_0^\infty \int_\Omega e^{-\frac{t}{\varepsilon}} f(x, \overline{u}_o, D\overline{u}_o)\, dxdt = \int_\Omega f(x, u_o, Du_o)\, dx.$$

This proves the claim. □

By the definition of $\mathcal{F}_\varepsilon$ the preceding lemma immediately implies

Corollary 4.3. *Any minimizer $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ of $\mathcal{F}_\varepsilon$ satisfies*

$$\int_0^\infty \int_\Omega e^{-\frac{t}{\varepsilon}} |\partial_t u_\varepsilon|^2\, dxdt \leq 2\int_\Omega f(x, u_o, Du_o)\, dx.$$

and

$$\int_0^\infty \int_\Omega e^{-\frac{t}{\varepsilon}} f(x, u_\varepsilon, Du_\varepsilon)\, dxdt \leq \varepsilon \int_\Omega f(x, u_o, Du_o)\, dx.$$

In the sequel we shall improve the preceding bounds for the time and the spatial derivative, in the sense that the weight $e^{-\frac{t}{\varepsilon}} \leq 1$ can be removed in the integrals on the left-hand side. Here, we use the fact that minimizers of variational functionals often satisfy certain conservation laws (Noether theorem). These conservation laws usually follow from *inner variations*. Here, we use inner variations with respect to the time variable. The argument is inspired by the paper of Sera & Tilli [26], where the authors use inner variations to get uniform estimates for a sequence of approximating solutions to the non-homogeneous wave equation. Before stating the result, we need to define for $t > 0$ the following auxiliary functions:

$$\begin{aligned}
\mathcal{L}_\varepsilon(t) &:= \int_\Omega \tfrac{1}{2}|\partial_t u_\varepsilon(\cdot, t)|^2\, dx,\\
\mathcal{H}_\varepsilon(t) &:= \int_\Omega \big[\tfrac{1}{2}|\partial_t u_\varepsilon(\cdot, t)|^2 + \tfrac{1}{\varepsilon} f(\cdot, u_\varepsilon, Du_\varepsilon)(\cdot, t)\big]\, dx,\\
\mathcal{I}_\varepsilon(t) &:= \int_t^\infty e^{-\frac{s}{\varepsilon}} \mathcal{H}_\varepsilon(s)\, ds.
\end{aligned}$$

Note that $\mathcal{L}_\varepsilon(t)$ and $\mathcal{H}_\varepsilon(t)$ are non-negative and locally integrable on $(0, \infty)$, and that $\mathcal{I}_\varepsilon(t)$ is non-negative, continuous and decreasing for $t \geq 0$. Furthermore, we have $\mathcal{I}_\varepsilon(0) = \mathcal{F}_\varepsilon(u_\varepsilon)$ and $\mathcal{I}_\varepsilon(t) \to 0$ as $t \uparrow \infty$.

Lemma 4.4. *Let $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ be a minimizer of $\mathcal{F}_\varepsilon$. Then, for a.e. $t \in (0, \infty)$ there holds*

$$\frac{d}{dt}\big(e^{\frac{t}{\varepsilon}} \mathcal{I}_\varepsilon(t)\big) = -2\mathcal{L}_\varepsilon(t) \leq 0.$$

In particular, the function $t \mapsto e^{\frac{t}{\varepsilon}} \mathcal{I}_\varepsilon(t)$ is decreasing.

Proof. As already mentioned above, we use inner variations with respect to time. To avoid an overburdened notation we delete the subscript ε from our notation and simply write $u \equiv u_\varepsilon$ as well as $\mathcal{F}, \mathcal{L}, \mathcal{H}, \mathcal{I}$ instead of $\mathcal{F}_\varepsilon, \mathcal{L}_\varepsilon, \mathcal{H}_\varepsilon, \mathcal{I}_\varepsilon$. Let $g \in C_0^\infty(\mathbb{R}_+)$. For $\delta \in \mathbb{R}$, we define the function $\varphi_\delta \in C_0^\infty(\mathbb{R}_+)$ by

$$\varphi_\delta(s) := s + \delta g(s).$$

For sufficiently small values of $|\delta| \ll 1$, we have that φ_δ is a diffeomorphism of $\mathbb{R}_+$ onto itself. The inverse function of φ_δ we denote by ψ_δ, i.e. we write $\psi_\delta := \varphi_\delta^{-1}$. Then, we have $t = \varphi_\delta(\psi_\delta(t)) = \psi_\delta(t) + \delta g(\psi_\delta(t))$, so that

$$\psi_\delta(t) = t - \delta g(\psi_\delta(t)). \tag{4.1}$$

Now, we define the inner variation $u_\delta(x, s) := u(x, \varphi_\delta(s))$, and compute its energy:

$$\begin{aligned}
\mathcal{F}(u_\delta) &= \int_0^\infty \int_\Omega e^{-\frac{s}{\varepsilon}} \Big[\tfrac12 |\partial_s u_\delta|^2 + \tfrac1\varepsilon f(\cdot, u_\delta, Du_\delta)\Big]\, dxds \\
&= \int_0^\infty \int_\Omega e^{-\frac{s}{\varepsilon}} \Big[\tfrac12 |\partial_t u(x, \varphi_\delta(s))|^2 \varphi_\delta'(s)^2 \\
&\qquad\qquad + \tfrac1\varepsilon f(x, u(x, \varphi_\delta(s)), Du(x, \varphi_\delta(s)))\Big]\, dxds \\
&= \int_0^\infty \int_\Omega e^{-\frac{\psi_\delta(t)}{\varepsilon}} \Big[\tfrac12 |\partial_t u(x,t)|^2 \varphi_\delta'(\psi_\delta(t))^2 \\
&\qquad\qquad + \tfrac1\varepsilon f(x, u(x,t), Du(x,t))\Big] \psi_\delta'(t)\, dxdt \\
&= \int_0^\infty \int_\Omega e^{-\frac{\psi_\delta(t)}{\varepsilon}} \bigg[\frac{1}{2\psi_\delta'(t)} |\partial_t u|^2 + \frac{\psi_\delta'(t)}{\varepsilon} f(\cdot, u, Du)\bigg]\, dxdt.
\end{aligned}$$

Since $\varphi_\delta', \psi_\delta' \in [\frac12, \frac32]$ for $|\delta| \ll 1$ small enough and $e^{-\frac{\psi_\delta(t)}{\varepsilon}} \le e^{\frac{\delta \|g\|_\infty}{\varepsilon}} e^{-\frac{t}{\varepsilon}}$, we find that $\mathcal{F}(u_\delta) < \infty$ for $|\delta| \ll 1$. To proceed further, we observe from (4.1) that the function ψ_δ fulfills the identities $\psi_\delta(t)|_{\delta=0} = t$, $\psi_\delta'(t)|_{\delta=0} = 1$, $\frac{d}{d\delta}\psi_\delta(t)|_{\delta=0} = -g(t)$ and $\frac{d}{d\delta}\psi_\delta'(t)|_{\delta=0} = -g'(t)$ for any $t \in \mathbb{R}_+$. Taking these identities and the fact that $u \equiv u_\varepsilon$ is a minimizer of $\mathcal{F} \equiv \mathcal{F}_\varepsilon$ into account, we can use u_δ as a comparison map for u. Hence, $\delta \mapsto \mathcal{F}(u_\delta)$ has a minimum at $\delta = 0$, and therefore we must have (note that the function $\delta \mapsto \mathcal{F}(u_\delta)$ is differentiable with respect to δ)

$$\begin{aligned}
0 = \frac{d}{d\delta}\Big|_{\delta=0} \mathcal{F}(u_\delta) &= \int_0^\infty \int_\Omega \frac{e^{-\frac{t}{\varepsilon}} g(t)}{\varepsilon} \big[\tfrac12 |\partial_t u|^2 + \tfrac1\varepsilon f(\cdot, u, Du)\big]\, dxdt \\
&\quad + \int_0^\infty \int_\Omega e^{-\frac{t}{\varepsilon}} \big[\tfrac12 g'(t) |\partial_t u|^2 - \tfrac1\varepsilon g'(t) f(\cdot, u, Du)\big]\, dxdt.
\end{aligned}$$

The preceding identity is often called first variation with respect to inner variations or sometimes also second Euler-Lagrange equation. Taking the definitions of $\mathcal{L}, \mathcal{H}$ and $\mathcal{I}$ into account, and observing that $\mathcal{I}'(t) = -e^{-\frac{t}{\varepsilon}} \mathcal{H}(t)$ for a.e. $t \in \mathbb{R}_+$, we can re-write the second Euler-Lagrange equation from above in the form

$$\begin{aligned}
0 &= \int_0^\infty \Big[-g(t) \tfrac1\varepsilon \mathcal{I}'(t) + g'(t) \mathcal{I}'(t) + g'(t) 2 e^{-\frac{t}{\varepsilon}} \mathcal{L}(t)\Big]\, dt \\
&= \int_0^\infty g'(t) \Big[\tfrac1\varepsilon \mathcal{I}(t) + \mathcal{I}'(t) + 2e^{-\frac{t}{\varepsilon}} \mathcal{L}(t)\Big]\, dt,
\end{aligned}$$

where we performed in the last line an integration by parts and also used the fact that g has compact support in $\mathbb{R}_+$. Since $g \in C_0^\infty(\mathbb{R}_+)$ was arbitrary, we conclude by the classical Du Bois-Raymond Lemma, that the expression appearing in the parenthesis must be constant, i.e. that

$$\tfrac1\varepsilon \mathcal{I}(t) + \mathcal{I}'(t) + 2e^{-\frac{t}{\varepsilon}} \mathcal{L}(t) \equiv C \qquad \text{for a.e. } t \in (0, \infty)$$

holds true for some constant $C \geq 0$. Since $\mathcal{I}'(t) + e^{-\frac{t}{\varepsilon}}\mathcal{L}(t) \in L^1(\mathbb{R}_+)$ and $\mathcal{I}(t) \to 0$ as $t \to \infty$, we can conclude that $C = 0$. Therefore, multiplying the preceding identity by $e^{\frac{t}{\varepsilon}}$ we find that

$$\frac{d}{dt}\big(e^{\frac{t}{\varepsilon}}\mathcal{I}(t)\big) = -2\mathcal{L}(t) \qquad \text{for a.e. } t \in (0,\infty).$$

Since $\mathcal{L}$ is non-negative, this proves the assertion of the lemma. □

Lemma 4.4 yields several important bounds for $\mathcal{F}_\varepsilon$-minimizers $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$. We start with the following easy consequence:

Corollary 4.5. *Let $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ be a minimizer of $\mathcal{F}_\varepsilon$. Then, for any $t \in (0,\infty)$ there holds*

$$e^{\frac{t}{\varepsilon}}\mathcal{I}_\varepsilon(t) \leq \int_\Omega f(x,u_o,Du_o)\,dx.$$

Proof. From Lemma 4.4 we know that the function $t \mapsto e^{\frac{t}{\varepsilon}}\mathcal{I}(t)$ is decreasing. This, together with Lemma 4.2 yields that

$$e^{\frac{t}{\varepsilon}}\mathcal{I}_\varepsilon(t) \leq e^0\mathcal{I}_\varepsilon(0) = \mathcal{F}_\varepsilon(u_\varepsilon) \leq \int_\Omega f(x,u_o,Du_o)\,dx,$$

proving the assertion of the lemma. □

Lemma 4.6. *Any minimizer $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ of $\mathcal{F}_\varepsilon$ satisfies*

$$\int_0^\infty \int_\Omega |\partial_t u_\varepsilon|^2\,dxdt \leq \int_\Omega f(x,u_o,Du_o)\,dx.$$

Proof. First, from Lemma 4.4 we recall the identity $2\mathcal{L}_\varepsilon(t) = -\frac{d}{dt}(e^{\frac{t}{\varepsilon}}\mathcal{I}_\varepsilon(t))$ for a.e. $t \in (0,\infty)$. Integrating over $(t_1,t_2) \Subset (0,\infty)$ and using Corollary 4.5, we find that

$$2\int_{t_1}^{t_2} \mathcal{L}(t)\,dt = -e^{\frac{t}{\varepsilon}}\mathcal{I}(t)\big|_{t_1}^{t_2} \leq e^{\frac{t_1}{\varepsilon}}\mathcal{I}(t_1) \leq \int_\Omega f(x,u_o,Du_o)\,dx.$$

Since the right-hand side is independent of t_1 and t_2 we are allowed to pass to the limits $t_1 \downarrow 0$ and $t_2 \uparrow \infty$. This proves the result. □

Lemma 4.7. *Let $u_\varepsilon \in u_o + \mathcal{N}_\varepsilon$ be a minimizer of $\mathcal{F}_\varepsilon$. Then, for any $0 \leq t_1 < t_2$ with $t_2 - t_1 \geq \varepsilon$ there holds*

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \int_\Omega f(x,u_\varepsilon,Du_\varepsilon)\,dxdt \leq 2e\int_\Omega f(x,u_o,Du_o)\,dx.$$

Proof. From Corollary 4.5 we obtain for any $t \in (0,\infty)$ that

$$\begin{aligned}\int_t^{t+\varepsilon}\int_\Omega f(x,u_\varepsilon,Du_\varepsilon)\,dxds &\leq e^{\frac{t+\varepsilon}{\varepsilon}}\int_t^{t+\varepsilon}\int_\Omega e^{-\frac{s}{\varepsilon}} f(x,u_\varepsilon,Du_\varepsilon)\,dxds\\ &\leq \varepsilon e^{\frac{t+\varepsilon}{\varepsilon}}\int_t^{t+\varepsilon} e^{-\frac{s}{\varepsilon}}\mathcal{H}_\varepsilon(s)\,ds\\ &\leq \varepsilon e\, e^{\frac{t}{\varepsilon}}\mathcal{I}_\varepsilon(t) \leq \varepsilon e\int_\Omega f(x,u_o,Du_o)\,dx.\end{aligned}$$

For $0 \leq t_1 < t_2$ with $t_2 - t_1 \geq \varepsilon$ we now choose $K \in \mathbb{N}$ in such a way that $(K-1)\varepsilon < t_2 - t_1 \leq K\varepsilon$ holds true. Then, we have $K\varepsilon \leq t_2 - t_1 + \varepsilon \leq 2(t_2 - t_1)$. Therefore, we conclude from the last inequality that

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega f(x,u_\varepsilon,Du_\varepsilon)\,dxdt &\leq \sum_{i=0}^{K-1}\int_{t_1+i\varepsilon}^{t_1+(i+1)\varepsilon}\int_\Omega f(x,u_\varepsilon,Du_\varepsilon)\,dxdt\\ &\leq eK\varepsilon\int_\Omega f(x,u_o,Du_o)\,dx\end{aligned}$$

$$\le 2e(t_2-t_1)\int_\Omega f(x,u_o,Du_o)\,dx.$$

This proves the assertion of the lemma. □

Corollary 4.8. *Under the assumptions (1.2) and (1.3), any minimizer $u_\varepsilon \in u_o+\mathcal{N}_\varepsilon$ of $\mathcal{F}_\varepsilon$ satisfies*

$$\int_0^T\int_\Omega \big[|u_\varepsilon|^p+|Du_\varepsilon|^p\big]\,dxdt \le c\,T\int_\Omega \big[1+|g|^{p'}+|u_o|^p+f(x,u_o,Du_o)\big]\,dx$$

for any $T\ge\varepsilon$ and with a constant $c=c(\nu,p,\operatorname{diam}(\Omega))$.

Proof. First we use the growth assumption (1.2) and Lemma 4.7 to infer that

$$\int_0^T\int_\Omega \big[\nu|Du_\varepsilon|^p-g(1+|u_\varepsilon|)\big]\,dxdt \le 2eT\int_\Omega f(x,u_o,Du_o)\,dx.$$

Arguing similarly to the proof in § 4.1 we find that

$$\int_0^T\int_\Omega |u_\varepsilon|^p\,dxdt \le c\int_0^T\int_\Omega |Du_\varepsilon|^p\,dxdt + c\,T\int_\Omega \big(|u_o|^p+|Du_o|^p\big)\,dx$$

holds true with a constant c depending only on p and $\operatorname{diam}(\Omega)$. Therefore, for $\delta\in(0,1]$ we obtain by Young's inequality that

$$\begin{aligned}\int_0^T\int_\Omega g(1+|u_\varepsilon|)\,dxdt &\le \int_0^T\int_\Omega \big[\delta(1+|u_\varepsilon|^p)+c_\delta|g|^{p'}\big]\,dxdt\\ &\le c\,\delta\int_0^T\int_\Omega |Du_\varepsilon|^p\,dxdt + c_\delta\int_\Omega\big[1+|g|^{p'}+|u_o|^p+|Du_o|^p\big]\,dx\end{aligned}$$

for a constant $c_\delta=c_\delta(p,\operatorname{diam}(\Omega)),1/\delta)$. Joining the preceding estimates and choosing $\delta>0$ small enough, we obtain the claim of the lemma. □

4.3. **Passage to the limit.** Here, we will pass to the limit $\varepsilon\downarrow 0$ and thereby prove the part of Theorem 1.2 concerning the existence of variational solutions.

By Lemma 4.6 and Corollary 4.8 we know that the family $(u_\varepsilon)_{\varepsilon>0}$ of $\mathcal{F}_\varepsilon$-minimizing functions is bounded in $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ for any fixed $T>0$, and that the corresponding time derivatives $\partial_t u_\varepsilon$ are bounded in $L^2(\Omega_\infty,\mathbb{R}^N)$ (both assertions holding uniformly with respect to $\varepsilon\in(0,1]$). Therefore, there exists a (not re-labelled) subsequence $\varepsilon_j\downarrow 0$, which we denote still by ε, and a measurable function $u\colon\Omega_\infty\to\mathbb{R}^N$ with the following properties: For any $T>0$ we have $u\in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$. Moreover, the time derivative of u exists and satisfies $\partial_t u\in L^2(\Omega_\infty,\mathbb{R}^N)$. Further, we have $u=u_o$ on the parabolic boundary $\partial_{\mathcal{P}}\Omega_\infty$ in the sense of traces. Finally, in the limit $\varepsilon\downarrow 0$ we have

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } L^p(\Omega_T,\mathbb{R}^N)\\ Du_\varepsilon\rightharpoonup Du & \text{weakly in } L^p(\Omega_T,\mathbb{R}^{Nn})\\ \partial_t u_\varepsilon\rightharpoonup\partial_t u & \text{weakly in } L^2(\Omega_\infty,\mathbb{R}^N).\end{cases}$$

As mentioned above, the limit $\varepsilon\downarrow 0$ has to be understood in the sense that there exists a sequence $\varepsilon_j\downarrow 0$ such that the above convergences hold true as $j\to\infty$. Moreover, by lower semicontinuity u satisfies the energy estimates (1.6) and (1.7). In particular, estimate (1.6) implies that for any $0\le t_1<t_2$ we have

$$\begin{aligned}\|u(\cdot,t_2)-u(\cdot,t_1)\|^2_{L^2(\Omega)} &= \int_\Omega\bigg|\int_{t_1}^{t_2}\partial_t u(\cdot,t)\,dt\bigg|^2\,dx\\ &\le |t_2-t_1|\int_{t_1}^{t_2}\int_\Omega|\partial_t u|^2\,dxdt\\ &\le |t_2-t_1|\int_\Omega f(x,u_o,Du_o)\,dx.\end{aligned}$$

Choosing $t_1 = 0$, the preceding inequality implies for any $t > 0$ that

$$\begin{aligned}\int_\Omega |u(\cdot,t)|^2\,dx &\le 2\int_\Omega |u_o|^2\,dx + 2\int_\Omega |u(\cdot,t)-u_o|^2\,dx\\ &\le 2\int_\Omega |u_o|^2\,dx + 2t\int_\Omega f(x,u_o,Du_o)\,dx.\end{aligned}$$

Therefore, we conclude that

$$u \in C^{0,\frac12}\big([0,T];L^2(\Omega)\big) \qquad \text{for any } T>0.$$

For further reference we note that the same computations can be performed for u replaced by u_ε. In particular, choosing $t_1 = 0$ in the second last inequality, we find that

$$\int_\Omega |u_\varepsilon(\cdot,t)-u_o|^2\,dx \le t\int_\Omega f(x,u_o,Du_o)\,dx \qquad \text{for any } t>0. \tag{4.2}$$

At this point it remains to show that the limit function u is a variational solution in the sense of Definition 1.1. To this aim we fix $T > 0$ and let $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$. For $\theta \in (0,\frac{T}{2})$ we define the following cut-off function with respect to time:

$$\zeta_\theta(t) := \begin{cases} \frac{1}{\theta}t & \text{if } t\in[0,\theta)\\ 1 & \text{if } t\in[\theta,T-\theta]\\ \frac{1}{\theta}(T-t) & \text{if } t\in(T-\theta,T].\end{cases}$$

Then, for any choice of $\varepsilon,\delta \in (0,1)$ the function

$$\widetilde\varphi_{\varepsilon,\delta}(\cdot,t) := \begin{cases} \delta e^{\frac{t}{\varepsilon}}\zeta_\theta(t)\varphi(\cdot,t) & \text{if } t\in[0,T]\\ 0 & \text{if } t>T\end{cases}$$

belongs to $\mathcal{N}_\varepsilon$, and therefore $u_\varepsilon + \widetilde\varphi_{\varepsilon,\delta}$ is an admissible comparison function for the $\mathcal{F}_\varepsilon$-minimizing mapping u_ε. By the minimizing property of u_ε we therefore obtain that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \mathcal{F}_\varepsilon(u_\varepsilon + \widetilde\varphi_{\varepsilon,\delta})$$

holds true, and this can be re-written in the form

$$\begin{aligned}0 \le \int_0^T\int_\Omega e^{-\frac{t}{\varepsilon}}\Big[&\tfrac12\big[|\partial_t u_\varepsilon + \delta\partial_t(e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi)|^2 - |\partial_t u_\varepsilon|^2\big]\\ &+ \tfrac{1}{\varepsilon}\big[f(x,u_\varepsilon+\delta e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi, Du_\varepsilon+\delta e^{\frac{t}{\varepsilon}}\zeta_\theta D\varphi) - f(x,u_\varepsilon,Du_\varepsilon)\big]\Big]\,dxdt.\end{aligned}$$

Evaluating the terms containing the time derivative, and using the convexity of f with respect to the variables (u,ξ), i.e. the fact that

$$\begin{aligned}f(\cdot,u_\varepsilon+\delta e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi, Du_\varepsilon+\delta e^{\frac{t}{\varepsilon}}\zeta_\theta D\varphi) - f(\cdot,u_\varepsilon,Du_\varepsilon)&\\ \le \delta e^{\frac{t}{\varepsilon}}\zeta_\theta\big(f(\cdot,u_\varepsilon+\varphi,Du_\varepsilon+D\varphi) - f(\cdot,u_\varepsilon,Du_\varepsilon)\big)&\end{aligned}$$

holds true (here we need to have that $\delta e^{\frac{T}{\varepsilon}} \le 1$, which is of course satisfied for δ small enough), we conclude that

$$\begin{aligned}0 \le \int_0^T\int_\Omega e^{-\frac{t}{\varepsilon}}\Big[&\tfrac12\delta^2|\partial_t(e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi)|^2 + \delta\partial_t u_\varepsilon\cdot\partial_t(e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi)\\ &+ \tfrac{1}{\varepsilon}\delta e^{\frac{t}{\varepsilon}}\zeta_\theta\big[f(\cdot,u_\varepsilon+\varphi,Du_\varepsilon+D\varphi) - f(\cdot,u_\varepsilon,Du_\varepsilon)\big]\Big]\,dxdt.\end{aligned}$$

We multiply the preceding inequality by ε/δ and subsequently let $\delta \downarrow 0$. This yields the estimate

$$\begin{aligned}0 \le \int_0^T\int_\Omega e^{-\frac{t}{\varepsilon}}\Big[&\varepsilon\partial_t u_\varepsilon\cdot\partial_t(e^{\frac{t}{\varepsilon}}\zeta_\theta\varphi)\\ &+ e^{\frac{t}{\varepsilon}}\zeta_\theta\big[f(\cdot,u_\varepsilon+\varphi,Du_\varepsilon+D\varphi) - f(\cdot,u_\varepsilon,Du_\varepsilon)\big]\Big]\,dxdt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \zeta_\theta \Big[\partial_t u_\varepsilon \cdot \varphi + f(\cdot, u_\varepsilon + \varphi, Du_\varepsilon + D\varphi) - f(\cdot, u_\varepsilon, Du_\varepsilon)\Big]\, dxdt \\
&\quad + \varepsilon \int_0^T \int_\Omega \Big[\zeta_\theta' \partial_t u_\varepsilon \cdot \varphi + \zeta_\theta \partial_t u_\varepsilon \cdot \partial_t \varphi\Big]\, dxdt.
\end{aligned}
$$

Now, we consider $v \in L^p(0,T; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N))$ with $\partial_t v \in L^2(\Omega_T, \mathbb{R}^N)$. Then, $\varphi = v - u_\varepsilon$ is an admissible choice in the preceding calculation. Therefore, the last inequality can be re-written in terms of v as follows:

$$
\begin{aligned}
&\int_0^T \int_\Omega f(\cdot, u_\varepsilon, Du_\varepsilon)\, dxdt \\
&\qquad \le \int_0^T \int_\Omega (1-\zeta_\theta) f(\cdot, u_\varepsilon, Du_\varepsilon)\, dxdt + \int_0^T \int_\Omega \zeta_\theta \partial_t u_\varepsilon \cdot (v - u_\varepsilon)\, dxdt \\
&\qquad\quad + \int_0^T \int_\Omega \zeta_\theta f(\cdot, v, Dv)\, dxdt \\
&\qquad\quad + \varepsilon \int_0^T \int_\Omega \big[\zeta_\theta' \partial_t u_\varepsilon \cdot (v - u_\varepsilon) + \zeta_\theta \partial_t u_\varepsilon \cdot \partial_t (v - u_\varepsilon)\big]\, dxdt \\
&\qquad =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon.
\end{aligned}
$$

The meaning of I_ε – IV_ε is obvious in this context. If $\theta \ge \varepsilon$, the term I_ε can be bounded with the help of Lemma 4.7 as follows:

$$
\begin{aligned}
\mathrm{I}_\varepsilon &\le \int_0^\theta \int_\Omega f(\cdot, u_\varepsilon, Du_\varepsilon)\, dxdt + \int_{T-\theta}^T \int_\Omega f(\cdot, u_\varepsilon, Du_\varepsilon)\, dxdt \\
&\le 4\theta e \int_\Omega f(\cdot, u_o, Du_o)\, dx.
\end{aligned}
$$

The term II_ε can be re-written in the form

$$
\mathrm{II}_\varepsilon = \int_0^T \int_\Omega \zeta_\theta \partial_t v \cdot (v - u_\varepsilon)\, dxdt - \frac{1}{2} \int_0^T \int_\Omega \zeta_\theta \partial_t |v - u_\varepsilon|^2\, dxdt\, .
$$

Since $\zeta_\theta(T) = 0 = \zeta_\theta(0)$, we obtain for the second term on the right-hand side by an integration by parts that

$$
\begin{aligned}
-\frac{1}{2} \int_0^T \int_\Omega \zeta_\theta \partial_t |v - u_\varepsilon|^2\, dxdt &= \frac{1}{2} \int_0^T \int_\Omega \zeta_\theta' |v - u_\varepsilon|^2\, dxdt \\
&= \frac{1}{2\theta} \int_0^\theta \int_\Omega |v - u_\varepsilon|^2\, dxdt - \frac{1}{2\theta} \int_{T-\theta}^T \int_\Omega |v - u_\varepsilon|^2\, dxdt.
\end{aligned}
$$

For the first term on the right-hand side we use estimate (4.2) and get

$$
\begin{aligned}
&\frac{1}{2\theta} \int_0^\theta \int_\Omega |v - u_\varepsilon|^2\, dxdt \\
&\qquad \le \bigg[\bigg(\frac{1}{2\theta} \int_0^\theta \int_\Omega |v - u_0|^2\, dxdt\bigg)^{\frac12} + \bigg(\frac{1}{2\theta} \int_0^\theta \int_\Omega |u_\varepsilon - u_0|^2\, dxdt\bigg)^{\frac12}\bigg]^2 \\
&\qquad \le \bigg[\bigg(\frac{1}{2\theta} \int_0^\theta \int_\Omega |v - u_0|^2\, dxdt\bigg)^{\frac12} + \bigg(\frac{\theta}{4} \int_\Omega f(x, u_o, Du_o)\, dx\bigg)^{\frac12}\bigg]^2.
\end{aligned}
$$

Collecting terms and using the weak convergence $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega_T, \mathbb{R}^N)$ we can pass to the limit $\varepsilon \downarrow 0$ in II_ε and obtain that

$$
\liminf_{\varepsilon \downarrow 0} \mathrm{II}_\varepsilon \le \int_0^T \int_\Omega \zeta_\theta \partial_t v \cdot (v - u)\, dxdt - \frac{1}{2\theta} \int_{T-\theta}^T \int_\Omega |v - u|^2\, dxdt
$$

$$+\bigg[\bigg(\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_0|^2\,dxdt\bigg)^{\frac12}+\bigg(\frac{\theta}{4}\int_\Omega f(x,u_o,Du_o)\,dx\bigg)^{\frac12}\bigg]^2.$$

Finally, since $\partial_t u_\varepsilon$ and u_ε are uniformly bounded in $L^2(\Omega_T,\mathbb{R}^N)$, we have that $\mathrm{IV}_\varepsilon\to 0$ as $\varepsilon\downarrow 0$. Inserting the previous observations above and using the lower semicontinuity of the convex functional $w\mapsto\int_0^T\int_\Omega f(x,w,Dw)\,dxdt$ with respect to weak convergence in $L^1(0,T;W^{1,1}(\Omega_T,\mathbb{R}^N))$, we arrive at

$$\begin{aligned}\int_0^T\int_\Omega f(\cdot,u,Du)\,dxdt &\le \liminf_{\varepsilon\downarrow 0}\int_0^T\int_\Omega f(\cdot,u_\varepsilon,Du_\varepsilon)\,dxdt\\ &\le \int_0^T\int_\Omega \zeta_\theta\big[\partial_t v\cdot(v-u)+f(\cdot,v,Dv)\big]\,dxdt+4\theta e\int_\Omega f(\cdot,u_o,Du_o)\,dx\\ &\quad+\bigg[\bigg(\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_0|^2\,dxdt\bigg)^{\frac12}+\bigg(\frac{\theta}{4}\int_\Omega f(x,u_o,Du_o)\,dx\bigg)^{\frac12}\bigg]^2\\ &\quad-\frac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u|^2\,dxdt.\end{aligned}$$

Note, that this last inequality holds true for any $\theta\in(0,\frac{T}{2})$, and therefore we can pass to the limit $\theta\downarrow 0$ in the right-hand side. We arrive at

$$\begin{aligned}\int_0^T\int_\Omega f(\cdot,u,Du)\,dxdt \le \int_0^T\int_\Omega &\big[\partial_t v\cdot(v-u)+f(\cdot,v,Dv)\big]\,dxdt\\ &+\tfrac12\|v(\cdot,0)-u_o\|^2_{L^2(\Omega)}-\tfrac12\|(v-u)(\cdot,T)\|^2_{L^2(\Omega)}.\end{aligned}$$

This proves the claim that is u is a variational solution to (1.5).

4.4. **Uniqueness for strictly convex integrands.** Here, we prove that the parabolic minimizer is unique, if f is strictly convex. To this aim, we suppose that

$$u_1,u_2\in L^p\big(0,T;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N)\big)\cap C^0\big([0,T];L^2(\Omega,\mathbb{R}^N)\big),\quad \text{for any } T>0$$

are two different variational solutions to (1.5). Adding the variational inequalities (1.5) for u_1 and u_2 for some fixed $T>0$ and taking into account the fact that $\|(v-u_i)(\cdot,T)\|^2_{L^2(\Omega)}\ge 0$ for $i=1,2$ yields for any $v\in L^p(0,T;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ with $\partial_t v\in L^2(\Omega_T,\mathbb{R}^N)$ that

$$\begin{aligned}&\int_0^T\int_\Omega\big[f(x,u_1,Du_1)+f(x,u_2,Du_2)\big]\,dxdt\\ &\qquad\le 2\int_0^T\int_\Omega\big[\partial_t v\cdot(v-w)+f(x,v,Dv)\big]\,dxdt+\|v(\cdot,0)-u_o\|^2_{L^2(\Omega)}.\end{aligned}$$

Here, we have abbreviated $w:=\frac{u_1+u_2}{2}$. At this point we would like to choose the comparison map $v=w$ in the previous inequality. However, this is not allowed, since in general we do not know that $\partial_t w$ belongs to $L^2(\Omega_T,\mathbb{R}^N)$. For this reason, we shall use the time-regularized function $[w]_h$ from (2.2) with $v_o=u_o$ and $h\in(0,T]$. By Lemma 2.2 we have $[w]_h\in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t[w]_h\in L^2(\Omega_T,\mathbb{R}^N)$ and $[w]_h=u_o$ on $\partial_{\mathcal{P}}\Omega_T$. Therefore, we are allowed to choose $v=[w]_h$ as comparison function in the last inequality and this leads to

$$\begin{aligned}&\int_0^T\int_\Omega\big[f(x,u_1,Du_1)+f(x,u_2,Du_2)\big]\,dz\\ &\le 2\int_0^T\int_\Omega\big[\partial_t[w]_h\cdot\big([w]_h-w\big)+f\big(x,[w]_h,D[w]_h\big)\big]\,dz=:2(\mathrm{I}_h+\mathrm{II}_h),\end{aligned}\tag{4.3}$$

with the obvious meaning of I_h and II_h. Due to Lemma 2.2 (v) we know that I_h is non-negative, since

$$\mathrm{I}_h = -\frac{1}{h}\int_{\Omega_T} \big|[w]_h - w\big|^2\,dz \le 0.$$

In order to treat II_h we first observe by the convexity of f that

$$\int_0^T\int_\Omega f(x,w,Dw)\,dxdt \le \frac{1}{2}\int_0^T\int_\Omega \big[f(x,u_1,Du_1)+f(x,u_2,Du_2)\big]\,dxdt < \infty,$$

i.e. $f(x,w,Dw)$ in $L^1(\Omega_T,\mathbb{R}^{Nn})$. This allows us to apply Lemma 2.3 to infer that

$$\lim_{h\downarrow 0}\mathrm{II}_h = \int_0^T\int_\Omega f(x,w,Dw)\,dxdt = \int_0^T\int_\Omega f\big(x,\tfrac12(u_1+u_2),\tfrac12(Du_1+Du_2)\big)\,dxdt$$

holds true. Using this observation and the fact that $\mathrm{I}_h\le 0$ from above we can pass in (4.3) to the limit $h\downarrow 0$ to obtain that

$$\begin{aligned}\int_0^T\int_\Omega &[f(x,u_1,Du_1)+f(x,u_2,Du_2)]\,dxdt\\ &\le 2\int_0^T\int_\Omega f\big(x,\tfrac12(u_1+u_2),\tfrac12(Du_1+Du_2)\big)\,dxdt\\ &< \int_0^T\int_\Omega [f(x,u_1,Du_1)+f(x,u_2,Du_2)]\,dxdt.\end{aligned}$$

In the last step we used the strict convexity of f and the assumption that $u_1 \not\equiv u_2$. Thus, we arrived at the desired contradiction. This proves the uniqueness of variational solutions and thus finishes the proof of Theorem 1.2.

REFERENCES

[1] G. Akagi and U. Stefanelli, A variational principle for doubly nonlinear evolution. *Appl. Math. Lett.* 23(9):1120–1124, 2010.

[2] G. Akagi and U. Stefanelli, Weighted energy-dissipation functionals for doubly nonlinear evolution. *J. Funct. Anal.* 260(9):2541–2578, 2011.

[3] V. Bögelein, F. Duzaar and P. Marcellini, Parabolic equations with p,q-growth. *J. Math. Pures Appl.*, 100(4):535–563, 2013.

[4] V. Bögelein, F. Duzaar and P. Marcellini, Parabolic systems with p,q-growth: a variational approach. *Arch. Ration. Mech. Anal.*, 210(1):219–267, 2013.

[5] H. Brezis and I. Ekeland, Un principe variationnel associé à certaines équations paraboliques. Le cas indépendant du temps. *C. R. Acad. Sci. Paris Sér. A-B* 282(17):971–974, 1976.

[6] H. Brezis and I. Ekeland, Un principe variationnel associé à certaines équations paraboliques. Le cas dépendant du temps. *C. R. Acad. Sci. Paris Sér. A-B* 282(20):1197–1198, 1976.

[7] Y. Cai and S. Zhou, Existence and uniqueness of weak solutions for a non-uniformly parabolic equation. *J. Funct. Anal.* 257(10):3021–3042, 2009.

[8] E. De Giorgi, Conjectures concerning some evolution problems. (Italian) A celebration of John F. Nash, Jr. *Duke Math. J.* 81(2):255–268, 1996.

[9] E. De Giorgi, *Selected papers.* Springer-Verlag, Berlin, 2006 (edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo).

[10] E. Giusti, *Direct Methods in the Calculus of Variations*. World Scientific Publishing Company, Tuck Link, Singapore, 2003.

[11] J. Habermann, Global gradient estimates for non-quadratic vector-valued parabolic quasi-minimizers. preprint.

[12] J. Habermann, Vector-valued parabolic ω-minimizers. preprint.

[13] J. Habermann, Higher Integrability for upper gradients of vector-valued parabolic quasi-minimizers on metric spaces. preprint.

[14] J. Kinnunen and P. Lindqvist, Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)* 185(3):411–435, 2006.

[15] J. Kinnunen, N. Marola, M. Miranda and F. Paronetto, Harnack's inequality for parabolic De Giorgi classes in metric spaces. *Adv. Differential Equations* 17:801–832, 2012.

[16] J. Kinnunen and M. Masson, Parabolic omparison principle and quasiminimizers in metric measure spaces. *Proc. Amer. Math. Soc.*, to appear.

[17] A. Lichnewsky and R. Temam, Pseudosolutions of the time-dependent minimal surface problem. *J. Differential Equations*, 30(3):340–364, 1978.

[18] M. Masson, M. Miranda Jr, F. Paronetto and M. Parviainen, Local higher integrability for parabolic quasiminimizers in metric spaces. *Ricerche Mat*, to appear.

[19] M. Masson and M. Parviainen, Global higher integrability for parabolic quasiminimizers in metric spaces. *J. Anal. Math.*, to appear.

[20] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Rational Mech. Analysis* 105(3):267–284, 1989.

[21] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 23(1):1–25, 1996.

[22] P. Marcellini and G. Papi, Nonlinear elliptic systems with general growth. *J. Differential Equations* 221:412–443, 2006.

[23] A. Mielke and M. Ortiz, A class of minimum principles for characterizing the trajectories of dissipative systems. *ESAIM Control Optim. Calc. Var.* 14(3) (2008) 494516.

[24] A. Mielke and U. Stefanelli, Weighted energy-dissipation functionals for gradient flows. *ESAIM Control Optim. Calc. Var.* 17(1):52–85, 2011.

[25] M. Parviainen, Global higher integrability for parabolic quasiminimizers in nonsmooth domains. *Calc. Var. Partial Differential Equations* 31(1):75–98, 2008.

[26] E. Serra and P. Tilli, Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. *Ann. of Math. (2)* 175(3):1551–1574, 2012.

[27] E. Spadaro and U. Stefanelli, A variational view at the time-dependent minimal surface equation. *J. Evol. Equ.* 11(4):793–809, 2011.

[28] U. Stefanelli, The De Giorgi conjecture on elliptic regularization *Math. Models Methods Appl. Sci.* 21: 1377-1394, 2011.

[29] W. Wieser, Parabolic Q-minima and minimal solutions to variational flow. *Manuscripta Math.* 59(1):63–107, 1987.

VERENA BÖGELEIN, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, CAUERSTRASSE 11, 91058 ERLANGEN, GERMANY
E-mail address: boegelein@math.fau.de

FRANK DUZAAR, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, CAUERSTRASSE 11, 91058 ERLANGEN, GERMANY
E-mail address: duzaar@math.fau.de

PAOLO MARCELLINI, DIPARTIMENTO DI MATEMATICA E INFORMATICA "U.DINI", UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A , 50134 FIRENZE, ITALY
E-mail address: marcellini@math.unifi.it