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Geometric characterizations of p-Poincaré inequalities in the metric setting

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Abstract. Here we obtain some characterizations of Poincaré inequalities in locally complete doubling metric measure spaces. In the limiting case $p = \infty$ we prove that a locally complete metric space endowed with a doubling measure satisfies an ∞-Poincaré inequality if, and only if, every two points can be joined by a quasiconvex curve which "almost avoids" any fixed null set. As an application, we characterize doubling measures on the real line satisfying an ∞-Poincaré inequality. For Ahlfors Q-regular spaces, we obtain a characterization of p-Poincaré inequality for $p > Q$ in terms of the p-modulus of quasi-convex curves connecting two given points of the space. Finally, a related result is given in the case that $Q - 1 < p \leq Q$.

Key words p-Poincaré inequality in metric measure spaces, modulus of curves, quasiconvexity, singular doubling measures in $\mathbb{R}$, Lip-lip condition

AMS Subject Classification *Primary* 31E05, 46E35; *Secondary* 30L10

1 Introduction

During the last decade, different theories have been proposed for developing a first order analysis on metric measure spaces, see for example [13], [16], [15], [6], and [27] for a sample. The common idea underpinning some of these non-linear theories is that, for doing differential calculus in this abstract setting, one needs plenty of curves well distributed along the space. One way of making precise this idea consists on assuming that the space supports a p-Poincaré inequality for some $1 \leq p < \infty$. This a priori analytical property involves the metric, the measure, and the (upper) gradients, and encodes many geometric information. The exponent p from the p-Poincaré inequality actually also plays a geometrical role. The bigger the exponent p, the weaker the p-Poincaré inequality. The limiting case $p = \infty$ has been studied in [10] and has surprisingly different properties than the finite p-Poincaré inequality case.

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It is well known that complete metric spaces endowed with a doubling measure and supporting a p-Poincaré inequality are quasi convex (a property that does not depend on p), that is, that given two points one can find a rectifiable curve joining them whose length is bounded by the distance between the points up to a uniform constant; see for example [15], [6], and [19]. One in fact gains more information; there is a large family of quasi-convex curves.

One of the key tools used to define the notion of a large family of curves is the p-modulus of a family of curves, an outer measure defined on the set of all rectifiable curves. The presence of a p-Poincaré inequality on a metric space implies that the corresponding p-modulus of the collection of quasi-convex curves connecting two balls of positive measure has to be positive, that is, the space is p-thick quasi convex. For $p = \infty$ this property turns out to be special in that ∞-Poincaré inequality is characterized in terms of the positivity of the modulus; see [10].

In the first part of this present work, we give a new characterization of ∞-Poincaré inequality which avoids functions on the space and its (upper) gradients. In Theorem 3.1 we prove that a locally complete doubling metric space has an ∞-Poincaré inequality if and only if one can find quasi convex curves transversal to a given zero measure set, that is, if given a zero measure set N and two points, one can find a quasi convex curve γ connecting the two points such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) = 0$. This purely geometric property is a very simple, but powerful tool, useful in different applications. Two such applications are studied in Section 4 of this paper, and in Section 3, this theorem is used to answer two questions posed in [10] and [11].

An immediate consequence of Theorem 3.1 (Corollary 3.7) is that ∞-capacity of points is always positive when the space supports a Poincaré inequality. In particular every function in $N^{1,\infty}(X)$ is Lipschitz continuous. This solves an open problem posed in [10] and gives a complete understanding of the Newtonian function class for $p = \infty$.

It is worth mentioning that one can also apply Theorem 3.1 for p-finite type problems. As pointed out before, complete metric spaces endowed with a doubling measure and supporting a p-Poincaré inequality for $1 \le p \le \infty$ are quasi-convex. As far as we know, completeness has been a crucial hypothesis for all the different proofs of this fact in the literature. As a byproduct of the main result, we can weaken the hypothesis of completeness to local completeness, see Remark 3.3.

It was proved by Buckley, Björn, and Keith in [5] that $(\mathbb{R}, |\cdot|, \mu)$, with μ doubling, will support a p-Poincaré inequality for some $1 \le p < \infty$ if and only if $\mu \ll \mathscr{L}^1$ and the Radon-Nikodym derivative of μ with respect to $\mathscr{L}^1$ is a Muckenhoupt $\mathcal{A}_p$-weight. In contrast, we prove in Theorem 4.1 that to obtain an ∞-Poincaré inequality, it is both necessary and sufficient to have $\mathscr{L}^1 \ll \mu$. This would complete the picture for $n = 1$. In higher dimensions it is not known so far whether doubling measures on $\mathbb{R}^n$ supporting a p-Poincaré inequality for some $1 \le p < \infty$ must necessarily be absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^n$. We will show that in higher dimensional Euclidean setting, if the measure μ satisfies $\mathscr{L}^n \ll \mu$, then $(\mathbb{R}^n, |\cdot|, \mu)$ supports an ∞-Poincaré inequality. We do not know whether the converse is true. However, we use Theorem 3.1 to prove that certain singular measures μ on $\mathbb{R}^n$ cannot support an ∞-Poincaré inequality, see Example 4.5.

Cheeger [6] proved that doubling p-Poincaré spaces, $1 \le p < \infty$, admit a measurable differentiable structure for which Rademacher's Theorem holds. Subsequently, Keith [17] obtained the same conclusion under a weaker hypothesis called the $\mathrm{Lip} - \mathrm{lip}$ condition, a condition that does not depend on p. Recently, Bate [3] and Gong [12] have proved independently that the $\mathrm{Lip} - \mathrm{lip}$ condition is not only a sufficient, but also a necessary condition for a Cheeger differentiable structure. In Example 4.7, we construct a complete doubling metric measure space supporting an ∞-Poincaré inequality but with no measurable differentiable structure. This in turn implies, by Theorem 4.6, that the space does not have the $\mathrm{Lip} - \mathrm{lip}$ condition. Therefore, without any extra-hypothesis, there is no relation between the $\mathrm{Lip} - \mathrm{lip}$ condition and the ∞-Poincaré inequality. This solves an open question posed in [11].

In the case $p < \infty$, the property of being p-thick quasi convex is, in contrast to the $p = \infty$ case, too weak in order to characterize p-Poincaré inequalities, see [11]. The main reason is that one would need a more quantitative estimate for the p-moduli of curve families. Estimates of this nature that characterize p-Poincaré inequalities have been previously given in Heinonen–Koskela [16] (Loewner property), Keith [18] (Riesz measures), Bonk–Kleiner [4, Theorem 1.3], Semmes [26] (pencil of curves), and Maz'ya [21] (capacitary estimate). In the particular case of graphs with polynomial volume growth, Coulhon–Koskela [8] obtain a characterization in terms of modulus of families of curves for all the range of exponents $1 \leq p < \infty$. In the spirit of [8], in Theorem 5.1 and Theorem 5.5 we give characterizations of p-Poincaré inequalities for the range of exponents $Q - 1 < p < \infty$. For the range $Q - 1 < p \leq Q$ this characterization is in terms of the p-modulus of curves connecting two continua and their diameter and relative distance. We believe this characterization is not true when $p < Q - 1$. The discrete setting of [8] is special in that the local dimension associated with the graph related to the discrete setting of [8] is one-dimensional, and hence locally $Q = 1$; hence in the discrete setting [8] demonstrates a lower bound for p-moduli of curve families joining two continua in a graph, supporting a p-Poincaré inequality, in terms of the relative separation of the two continua even when $1 \leq p < Q - 1$, but such lower bound fails in more general metric measure spaces, for then it is possible to have a 1-dimensional continuum of positive diameter but with p-capacity zero.

2 Thick quasiconvex spaces: preliminaries

Along this paper we will assume that $X = (X, d, \mu)$ is a metric measure space, that is, (X, d) is a metric space equipped with a Borel measure μ, which is positive and finite on each ball. Recall that μ is said to be *doubling* if there is a constant C_μ such that, for each ball $B(x, r)$ in X,

$$\mu(B(x, 2r)) \leq C_\mu \cdot \mu(B(x, r)).$$

A *curve* in X is a continuous function $\gamma : I \to X$ for some compact interval $I \subset \mathbb{R}$. Such a curve is *rectifiable* if its length

$$\ell(\gamma) := \sup_{t_0 < t_1 < \cdots < t_n} \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j))$$

is finite. In the above definition of length $\ell(\gamma)$, the supremum is taken over all finite subdivisions $t_0 < t_1 < \cdots < t_n$ of the interval I. A rectifiable curve γ can be re-parametrized so that it is arc-length parametrized, that is, $I = [0, \ell(\gamma)]$ and for each $s \in I$, with $I_s := \{t \in I : t \leq s\}$, we have

$$\ell(\gamma|_{I_s}) = s.$$

Henceforth in the paper we will assume all rectifiable curves to be arc-length parametrized as above. The integral of a Borel function $\rho : X \to [0, \infty]$ over an arc-length parametrized curve γ is defined as:

$$\int_\gamma \rho\, ds = \int_0^{\ell(\gamma)} \rho(\gamma(t))\, dt.$$

The space X is said to be *quasiconvex* if there exists a constant $C \geq 1$ such that given two points $x, y \in X$, one can find a $C-$quasiconvex curve joining them, that is, a rectifiable curve γ such that $\ell(\gamma) \leq Cd(x, y)$.

Given $E \subset X$, let Γ_E^+ denote the family of curves γ in X such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$, where $\mathscr{L}^1$ is the usual 1-dimensional Lebesgue measure on the line. On the other hand, Γ_E denotes the family of curves γ such that $\gamma \cap E \neq \emptyset$.

Definition 2.1 If Γ is a family of curves in X, then set $F(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geq 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty(X)},$$

and for $1 \leq p < \infty$ the p-modulus of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p\, d\mu.$$

Note that

$$\lim_{p \to \infty} [\mathrm{Mod}_p(\Gamma)]^{1/p} = \mathrm{Mod}_\infty(\Gamma).$$

We next recall a characterization of path families whose ∞-modulus is zero.

Lemma 2.2 [9, Lemma 5.7] *Let Γ be a family of curves in X. The following conditions are equivalent:*

(a) $\mathrm{Mod}_\infty \Gamma = 0$.

(b) *There is a Borel function $\rho \geq 0$ with $\|\rho\|_{L^\infty(X)} = 0$ such that $\int_\gamma \rho\, ds = +\infty$ for each $\gamma \in \Gamma$.*

Definition 2.3 For $1 \leq p \leq \infty$ we say that (X, d, μ) is a *p-thick quasiconvex* space if there exists $C \geq 1$ such that for all $x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x,y)$, and all measurable sets $E \subset B(x,\varepsilon)$, $F \subset B(y,\varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have that

$$\mathrm{Mod}_p(\Gamma(E,F,C)) > 0,$$

where $\Gamma(E,F,C)$ denotes the collection of all curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq Cd(p,q)$. Here we do not require quantitative control on the modulus of the curve family, but we do require a quantitative control over the length of the curves, the control being exercised by the constant C.

Remark 2.4 We already know that every complete thick quasiconvex space X supporting a doubling measure is quasiconvex; see [10]. It was shown in [10] and [11] that if X supports a p-Poincaré inequality for some $1 \leq p \leq \infty$, then X is a p-thick quasiconvex space. It was also proved in [10] that ∞-thick quasiconvexity is also sufficient for the validity of an ∞-Poincaré inequality. However, the examples in [11] show that p-thick quasiconvexity is not sufficient for the validity of a p-Poincaré inequality when $1 \leq p < \infty$.

The proof of one of the main results of this paper, Theorem 3.1, will also show that we can replace completeness of X with local completeness of X in the results mentioned above.

A non-negative Borel measurable function g on X is said to be a *p-weak upper gradient* of a function $u : X \to [-\infty, \infty]$ if there is a family Γ of non-constant curves with $\mathrm{Mod}_p(\Gamma) = 0$ such that whenever γ is a rectifiable curve in X with $\gamma \notin \Gamma$, we have

$$|u(y) - u(x)| \leq \int_\gamma g\, ds,$$

where x and y denote the end points of γ. The above inequality should also be interpreted to mean that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$, $u(y)$ is not finite.

Definition 2.5 We say that X supports a *p-Poincaré inequality*, $1 \leq p \leq \infty$, if there are constants $C, \lambda \geq 1$ such that for each measurable function u on X, each p-weak upper gradient g of u, and each ball $B \subset X$ we have

$$\fint_B |u - u_B|\, d\mu \leq C\, \mathrm{rad}(B) \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}.$$

Here λB denotes the ball concentric with B (with respect to the pre-determined center) but with radius λ-times the radius of B. When $p = \infty$, the term inside the parantesis on the right-hand side of the above inequality should be interpreted to mean $\|\rho\|_{L^\infty(\lambda B)}$. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$u_A = \fint_A u := \frac{1}{\mu(A)} \int_A u\, d\mu.$$

The class $N^{1,p}(X)$ consists of all functions $u \in L^p(X)$ that have an upper gradient in $L^p(X)$. For $u \in N^{1,p}(X)$ there is an associated norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is over all upper gradients g of u. It was shown in [27] that $N^{1,p}(X)$ is a Banach space when $1 \le p < \infty$, and it was shown in [9] that $N^{1,\infty}(X)$ is a Banach space. From the results in [27] and [14] we know that when $1 \le p < \infty$, given $u \in N^{1,p}(X)$ there is a unique p-weak upper gradient $g_u \in L^p(X)$ of u such that whenever $g \in L^p(X)$ is a p-weak upper gradient of u we have $g_u \le g$ almost everywhere in X. Such g_u is called the *minimal p-weak upper gradient* of u. Given the non-locality of the norm of $L^\infty(X)$, such minimal weak upper gradients of functions in $N^{1,\infty}(X)$ are not readily verified to exist; however, using the approach of quasi-Banach function lattices, the paper [20] proved the existence of minimal p-weak upper gradients even for the case $p = \infty$.

The papers [1] and [2] together show that if the measure μ is doubling, then for $1 < p < \infty$ Lipschitz functions are dense in $N^{1,p}(X)$ and $N^{1,p}(X)$ is reflexive. The case $p = \infty$ is slightly different; see [9], [11], and [10]. The results in [10] show that when X is complete and μ is doubling, X supports an ∞-Poincaré inequality if and only if for each $u \in N^{1,\infty}(X)$ there is a function $u_0 \in \mathrm{LIP}^\infty(X)$ such that $u = u_0$ μ-a.e. in X and the respective energy seminorms are comparable. Here $\mathrm{LIP}^\infty(X)$ denotes the space of all bounded Lipschitz functions on X endowed with the norm given by

$$\|u\|_{\mathrm{LIP}^\infty(X)} = \sup_{x \in X} |u(x)| + \sup_{x,y \in X; y \ne x} \frac{|u(y) - u(x)|}{d(x,y)}.$$

Associated with (locally) Lipschitz functions u on X there are two local "Lipschitz constant" functions that act like the (modulus of the) derivative of u:

$$\mathrm{Lip}\, u(x) := \limsup_{r \to 0^+} \sup_{0 < d(y,x) \le r} \frac{|u(y) - u(x)|}{d(x,y)},$$

and

$$\mathrm{lip}\, u(x) := \liminf_{r \to 0^+} \sup_{0 < d(y,x) \le r} \frac{|u(y) - u(x)|}{d(x,y)}.$$

It was shown in [6] that for complete metric spaces, $\mathrm{Lip}\, u$ and $\mathrm{lip}\, u$ are almost everywhere comparable to each other if μ is doubling and supports a p-Poincaré inequality for some $1 \le p < \infty$. In Section 4 we will show that the above two "constant" functions are not necessarily related under ∞-Poincaré inequality, even if μ is doubling.

3 Thick quasiconvex spaces: the main theorem

In this section we will state and prove the first of the three main theorems of this paper. The two corollaries of the following theorem, one of the three main results of this paper, answers two open questions posed in [10] and [11].

Theorem 3.1 *Suppose that X is a locally complete metric space supporting a doubling Borel measure μ which is nontrivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports an ∞-Poincaré inequality.*

(b) *X is ∞-thick quasiconvex.*

(c) *X is connected and $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms.*

(d) *X supports an ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

(e) *(X, d, μ) is a* very thick quasiconvex *space, that is, there exists $C \geq 1$ such that for all $x, y \in X$, with $d(x, y) > 0$ we have that*

$$\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) > 0,$$

where $\Gamma(\{x\}, \{y\}, C)$ denotes the set of C-quasiconvex curves in X connecting x and y.

(f) *There is a constant C such that, for every null set N of X, and for every pair of points $x, y \in X$ there is a C-quasiconvex path γ in X connecting x to y such that $\gamma \notin \Gamma_N^+$.*

Furthermore, under any of the above equivalent conditions, there is a constant $C > 0$ such that whenever $x, y \in X$ are distinct,

$$\frac{1}{d(x,y)} \geq \mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) \geq \frac{1}{C\, d(x,y)}.$$

Remark 3.2 The implication $(d) \implies (b)$ does not require the local completeness hypothesis. The equivalence of $(a), (b), (c)$, and (d) has already been established in [10]. Therefore, to prove the above theorem, it suffices to establish the equivalence of $(b), (e)$, and (f).

Proof. $(f) \implies (e)$ Assume (X, d, μ) is not a very thick quasi-convex space with respect to the constant C, where C is the constant from Condition (f). Then there exist $x,y \in X$ such that $\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) = 0$. By Lemma 2.2 (b), there exists a non-negative Borel measurable function $g \in L^\infty(X)$ such that $\int_\gamma g\, ds = \infty$ for each $\gamma \in \Gamma(\{x\}, \{y\}, C)$ and $\|g\|_{L^\infty(X)} = 0$. Observe that $N = \{x \in X : g(x) > 0\}$ has zero measure. Then for each quasiconvex curve connecting x to y, $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) > 0$. Hence $\Gamma(\{x\}, \{y\}, C) \subset \Gamma_N^+$, which then violates the hypothesis of (f). Therefore (e) holds true whenever (f) is true, with the constant associated with Condition (e) no larger than the constant associated with Condition (f).

$(e) \implies (f)$ Assume that (X, d, μ) is a very thick quasi-convex space. Let N be a zero measure set. Because $\mu(N) = 0$, we know that $\mathrm{Mod}_\infty(\Gamma_N^+) = 0$ (because $\infty \cdot \chi_{N_0} \in F(\Gamma_N^+)$, where N_0 is a Borel set containing N such that $\mu(N_0) = 0$). Therefore $\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C) \setminus \Gamma_N^+) > 0$ and hence we have condition (f), with the associated constant no more than the constant from Condition (e).

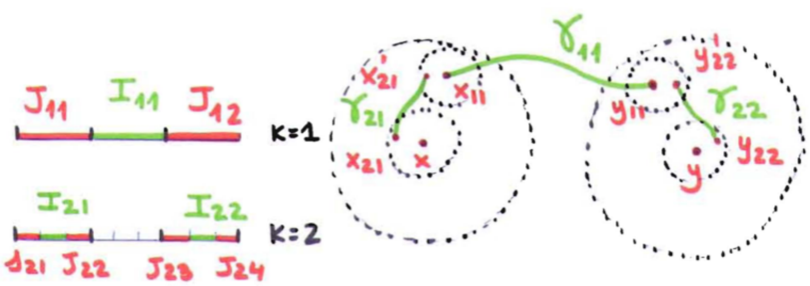
$(b) \implies (f)$ Fix $x, y \in X$, with $d(x,y) > 0$. Since X is locally complete, we can choose $0 < \varepsilon_1 \le \frac{1}{4} d(x,y)$ such that $\overline{B}(x_1, 6C\varepsilon_1)$ and $\overline{B}(x_2, 6C\varepsilon_1)$ are complete. Let $N \subset X$ such that $\mu(N) = 0$. Note that $\mathrm{Mod}_\infty(\Gamma_N^+) = 0$. Since the space is C-thick quasiconvex, we have

$$\mathrm{Mod}_\infty(\Gamma(B(x,\varepsilon_1), B(y,\varepsilon_1), C) \setminus \Gamma_N^+) > 0.$$

Thus there exist points $x_1 \in B(x,\varepsilon_1)$, $y_1 \in B(y,\varepsilon_1)$, and a curve $\gamma_1 : I_1 \to X$ with $\gamma_1 \notin \Gamma_N^+$ connecting x_1 and y_1, such that

$$\ell(\gamma_1) \le C d(x_1, y_1) \le 2C d(x,y).$$

It now suffices to be able to connect x_1 to x by a curve β_1 of length $\ell(\beta_1) \le C\, d(x_1, x)$, and connect y_1 to y by a curve β_2 of length $\ell(\beta_2) \le C\, d(y_1, y)$, such that $\mathscr{L}^1(\beta_1^{-1}(N) \cup \beta_2^{-1}(N)) = 0$. The concatenation of the three curves γ_1, β_1, and β_2 would then give the desired curve γ connecting x to y such that $\mathscr{L}^1(\gamma^{-1}(N)) = 0$. The curves β_1 and β_2 are constructed in a manner similar to the construction of a Cantor set, as follows.



Let $I_0 = [0,1]$, and for each $k \in \mathbb{N}$ let $J_{k,1}, \cdots, J_{k,2^k}$ be the subintervals remaining at the k-th step of constructing the standard 1/3-rd Cantor set in I_0. The complement in $I_0 \setminus \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{2^{i-1}} I_{i,j}$ of these 2^k subintervals are labelled (from the smaller left-hand endpoint to larger) by $I_{k,1}, \cdots, I_{k,2^{k-1}}$.

Note that $I_{1,1} = [1/3, 2/3]$, $I_{2,1} = [1/9, 2/9]$, $I_{2,2} = [7/9, 8/9]$ etc., and for each $k \in \mathbb{N}$ the intervals $I_{k,j}$, $j = 1, \cdots, 2^{k-1}$, are of length 3^{-k}. With this notation we may think of $\gamma_1 = \gamma_{1,1}$ as a Lipschitz map from $I_{1,1}$ with Lipschitz constant at most $3C\, d(x_1, x_2) \le 6C\, d(x,y)$ connecting $x_1 = x_{1,1}$ to $y_1 = y_{1,1}$.

Set $\varepsilon_2 = \frac{1}{4} \min\{\varepsilon_1, d(x, x_1), d(y, y_1)\}$. Then there exist points $x_{2,1} \in B(x, \varepsilon_2)$, $x'_{2,1} \in B(x_{1,1}, \varepsilon_2)$, $y_{2,2} \in B(y, \varepsilon_2)$ $y'_{2,2} \in B(y_{1,1}, \varepsilon_2)$, and paths $\gamma_{2,1} : I_{2,1} \to X$ connecting $x_{2,1}$ to $x'_{2,1}$ and $\gamma_{2,2} : I_{2,2} \to X$ connecting $y_{2,2}$ to $y'_{2,2}$ with $\gamma_{2,1}, \gamma_{2,2} \notin \Gamma_N^+$ such that

$$\ell(\gamma_{2,1}) \le C d(x_{2,1}, x_{2,2}) \le 2C d(x_{1,1}, x) \quad \text{and} \quad \ell(\gamma_{2,2}) \le C d(y_{2,3}, y_{2,4}) \le 2C d(y_{1,2}, y).$$

Iterating this construction, we have maps $\gamma_{k,j} : I_{k,j} \to X$ connecting $x_{k,j} \in B(x'_{k-1,m_j}, \varepsilon_k)$ to $x'_{k,l} \in B(x_{k-1,m_j+1}, \varepsilon_k)$ with $\gamma_{k,j} \notin \Gamma_N^+$ such that

$$\ell(\gamma_{k,j}) \le C\, d(x_{k,j}, x'_{k,j}) \le 2C\, d(x'_{k-1,m_j}, x_{k-1,m_j+1})$$

when $j = 1, \cdots, 2^{k-2}$, and $\gamma_{k,j}$ connecting $y_{k,j} \in B(y'_{k-1,m_j}, \varepsilon_k)$ to $y'_{k,l} \in B(y_{k-1,m_j+1}, \varepsilon_k)$ with $\gamma_{k,j} \notin \Gamma_N^+$ such that

$$\ell(\gamma_{k,j}) \leq C\, d(y_{k,j}, y'_{k,j}) \leq 2C\, d(y'_{k-1,m_j}, y_{k-1,m_j+1})$$

when $j = 2^{k-2}+1, \cdots, 2^{k-1}$. Thus we can create a collection of intervals $\{I_i\}_{i\in\mathbb{N}} := \{I_{k,j}\}_{\substack{k\in\mathbb{N}\\ j=1\cdots 2^{k-1}}}$ with each $I_i \subset I_0$, and a $6C\, d(x,y)$-Lipschitz continuous function

$$\tilde{\gamma} : \bigcup_{i\in\mathbb{N}} I_i \to Z = \gamma_1 \cup \overline{B}(x, \varepsilon_1) \cup \overline{B}(y, \varepsilon_1).$$

Since $(Z, d_{|Z})$ is complete there exists a $6C\, d(x,y)$-Lipschitz continuous extension $\gamma : I_0 \to Z$, and by construction we have $\gamma \notin \Gamma_N^+$. Furthermore, we have that $\mathscr{L}^1(I_0 \setminus \bigcup_{i\in\mathbb{N}} I_i) = 0$ and

$$\ell(\gamma) = \ell(\gamma_1) + \sum_{i\in\mathbb{N}} \ell(\gamma|_{I_i}) \leq 2Cd(x,y).$$

$(e) \implies (b)$ is straightforward. This completes the proof of the first part of the theorem.

We next prove the second part of the theorem. To this end, we assume that Conditions (a)–(f) hold. Fixing $x_0, y_0 \in X$ such that $x_0 \neq y_0$, we denote the collection of all rectifiable curves in $B(x_0, 4Cd(x_0,y_0))$ connecting x_0 to y_0 by Γ_{x_0,y_0}. Let $g \in L^\infty(X)$ be a nonnegative Borel measurable function on X such that for all $\gamma \in \Gamma_{x_0,y_0}$, the integral $\int_\gamma g\, ds \geq 1$ and set $g_0 = g$ in $B(x_0, 2Cd(x_0,y_0))$ and $g_0 = g + 1/[2Cd(x_0,y_0)]$ on $X \setminus B(x_0, 2Cd(x_0,y_0))$. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } B(x_0,r)} \int_\gamma g_0\, ds,$$

where $r < d(x_0,y_0)/4$ and consider $u = \min\{\tilde{u}, 2\}$. By the definition of u and by Condition (f), we can see that u is Lipschitz continuous on X. Indeed, if $z, w \in X$, then setting N to be the collection of all points $y \in X$ for which $g(y) > \|g\|_{L^\infty(X)}$ and noting that $\mu(N) = 0$, there must be a C-quasiconvex curve γ in X connecting z to w with $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N) = 0$. Hence by the fact that g_0 is an upper gradient of u, we have $|u(z) - u(w)| \leq \int_\gamma g_0\, ds \leq \|g_0\|_{L^\infty(X)}\, d(z,w)$. Then it follows that $u = 0$ on $B(x_0, r)$ and by the choice of g and g_0, $u(y_0) \geq 1$. Note that u is measurable and that g is an upper gradient of u; hence $u \in N^{1,\infty}(X)$.

Now $\forall i \in \mathbb{Z}$ define $B_i = B(x, 2^{1-i}d(x_0,y_0))$ if $i \geq 0$, and $B_i = B(y, 2^{1+i}d(x_0,y_0))$ if $i \leq -1$. Since we know that x_0 and y_0 are Lebesgue points for u, we have that

$$\begin{aligned} 1 \leq |u(x_0) - u(y_0)| \leq \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} u d\mu - \fint_{B_{i+1}} u d\mu \Big| &\leq C_\mu \sum_{i\in\mathbb{Z}} \fint_{B_i} \Big| u - \fint_{B_i} u d\mu \Big| d\mu \\ &\leq C_\mu C d(x_0,y_0) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\ &\leq C d(x_0,y_0) \|g\|_{L^\infty(X)}. \end{aligned}$$

Hence

$$\|g\|_{L^\infty(X)} \geq \frac{1}{C\, d(x_0,y_0)}.$$

Taking the infimum over all such g we obtain the inequality

$$\mathrm{Mod}_\infty(\Gamma_{x_0,y_0}) \geq \frac{1}{C\,d(x_0,y_0)}.$$

For $m \in \mathbb{N}$ we set $\Lambda(x_0,y_0,m) = \Gamma_{x_0,y_0} \setminus \Gamma(\{x_0\},\{y_0\},m)$. Each curve in $\Lambda(x_0,y_0,m)$ has length at least $m\,d(x_0,y_0)$, and so the function $\rho_m = [m\,d(x_0,y_0)]^{-1}\chi_{B(x_0,2(C+\lambda)d(x_0,y_0))} \in F(\Lambda(x_0,y_0,m))$. It follows that

$$\mathrm{Mod}_\infty(\Lambda(x_0,y_0,m)) \leq \frac{1}{m\,d(x_0,y_0)}.$$

So if $m = 2C$, then we have that

$$\mathrm{Mod}_\infty(\Gamma(\{x_0\},\{y_0\},2C) \geq \mathrm{Mod}_\infty(\Gamma_{x_0,y_0}) - \mathrm{Mod}_\infty(\Lambda(x_0,y_0,m)) \geq \frac{1}{2C\,d(x_0,y_0)}.$$

For the upper bound consider $u_0(z) = \frac{d(x_0,z)}{d(x_0,y_0)}$ which has $g_{u_0} = \frac{1}{d(x_0,y_0)}$ as an upper gradient. Moreover,

$$1 = |u_0(x_0) - u_0(y_0)| \leq \int_\gamma g_{u_0},$$

so $g_{u_0} = \frac{1}{d(x_0,y_0)}$ is an upper gradient for u_0 and so

$$\mathrm{Mod}_\infty(\Gamma(\{x_0\},\{y_0\},C) \leq \mathrm{Mod}_\infty(\Gamma_{x_0,y_0}) \leq \|g_{u_0}\|_{L^\infty} = \frac{1}{d(x_0,y_0)}.$$

This completes the proof of Theorem 3.1. □

Remark 3.3 A careful look at the proof of $(b) \implies (f)$ of Theorem 3.1 reveals that a locally complete metric space (X,d) supporting a doubling Borel measure μ and a weak p-Poincaré inequality for some $1 \leq p \leq \infty$ is quasiconvex. Previous results required completeness of the space X, see [15], [6], [18], and [19].

Remark 3.4 It was proven in [10, Corollary 4.15] that the Sierpinski carpet endowed with the Euclidean distance and the s-dimensional Hausdorff measure with $s = \frac{\log 8}{\log 3}$ does not support an ∞-Poincaré inequality. Theorem 3.1 proves that for each $m \in \mathbb{N}$, there exists a null set N of X and a pair of points $x,y \in X$ such that every m-quasiconvex path γ in X connecting x to y belongs to $\gamma \in \Gamma_N^+$. This fact could help to understand the set of rectifiable curves in fractal type sets with no Poincaré inequalities.

Remark 3.5 Notice that Theorem 3.1 does not hold for $1 \leq p < \infty$. In particular, the implication $(b) \implies (e)$ is false. For example $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ has a 1-Poincaré inequality but the p-modulus of curves passing through a point is zero when $1 \leq p \leq n$.

Definition 3.6 The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions u in $N^{1,p}(X)$ such that $u_{|E} \geq 1$.

Corollary 3.7 *Under the hypothesis of Theorem 3.1, if X supports a weak ∞-Poincaré inequality, then $\mathrm{Mod}_\infty(\Gamma_{x_0}) > 0$ for each $x_0 \in X$, where Γ_{x_0} denotes the collection of all non-constant curves passing through the point x_0. In particular, $\mathrm{Cap}_\infty(\{x_0\}) > 0$ so each equivalence class $[f] \in N^{1,\infty}(X)$ has exactly one element in it. Thus every Newtonian function in $N^{1,\infty}(X)$ is Lipschitz continuous.*

Proof. Observe that for a set $F \subset X$ with $\mu(F) = 0$ we have $\mathrm{Cap}_\infty(F) = 0$ if and only if $\mathrm{Mod}_\infty(\Gamma_F) = 0$. Since the measure of a singleton set in a doubling measure space is zero, the result follows. □

Remark 3.8 As the slit disc in the Euclidean plane shows, the converse of the above corollary does not hold.

Corollary 3.9 *Under the hypothesis of Theorem 3.1, if X supports a weak ∞-Poincaré inequality then there exists a constant $C > 0$ such that for each $u \in \mathrm{LIP}^\infty(X)$*

$$\sup_{x \in X} \mathrm{Lip}\, u(x) \leq C \|\mathrm{Lip}\, u\|_{L^\infty(X)}.$$

Proof. Let $u \in \mathrm{LIP}^\infty(X)$ and $K = \|\mathrm{Lip}\, u\|_{L^\infty(X)} < \infty$. Then there exists a null set N such that $\mathrm{Lip}\, u(z) \leq K$ for each $z \in X \setminus N$. Now, given $x, y \in X$, take γ in X connecting x and y, parametrized by the arc-length such that $\ell(\gamma) \leq Cd(x,y)$ and $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) = 0$. Then since $\mathrm{Lip}\, u(\gamma(t)) \leq K$ for $\mathscr{L}^1-$a.e. $t \in [0, \ell(\gamma)]$, we have

$$|u(x) - u(y)| \leq \int_\gamma \mathrm{Lip}\, u\, ds = \int_0^{\ell(\gamma)} \mathrm{Lip}\, u(\gamma(t)) dt \leq K\ell(\gamma) \leq KCd(x,y).$$

Therefore, $\sup_{x \in X} \mathrm{Lip}\, u(x) \leq C \|\mathrm{Lip}\, u\|_{L^\infty(X)}$. □

4 Singular measures and Lip-lip condition

In this section we give some applications of Theorem 3.1. Note that the thick quasiconvexity is stable under perturbation of the measure by an absolutely continuous measure.

Theorem 4.1 *Let μ be a doubling measure on $\mathbb{R}$. Then $(\mathbb{R}, |\cdot|, \mu)$ supports an ∞-Poincaré inequality if and only if $\mathscr{L}^1 \ll \mu$.*

Proof. Let us prove first that $(\mathbb{R}, |\cdot|, \mu)$ supporting ∞-Poincaré inequality implies $\mathscr{L}^1 \ll \mu$. Assume that there exists a measurable set N in $\mathbb{R}$ such that $\mathscr{L}^1(N) > 0$ and $\mu(N) = 0$. By the Borel regularity of $\mathscr{L}^1$ we may assume that N is a closed subset of $\mathbb{R}$. Take any two points $x, y \in \mathbb{R}$ such that $\mathscr{L}^1([x, y] \cap N) > 0$. In this case, any curve γ connecting the two points satisfies $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(z) = \int_x^z \chi_{N\cap[x,y]}(\omega)\, d\omega.$$

Then f is absolutely continuous on $\mathbb{R}$, and in fact is Lipschitz continuous, and by the choice of x and y we have that $f(x) \neq f(y)$. Thus f must be in $N^{1,\infty}_{\text{loc}}(\mathbb{R}, \mu)$. On the other hand, f is locally constant on the open set $\mathbb{R} \setminus N$, and so the fact that $\mu(N) = 0$ implies that the constant function 0 is a p-weak upper gradient of f in $(\mathbb{R}, |\cdot|, \mu)$. This would not be possible if $(\mathbb{R}, |\cdot|, \mu)$ supports an ∞-Poincaré inequality.

Now we prove the converse. Since $(\mathbb{R}, |\cdot|, \mathscr{L}^1)$ supports an ∞-Poincaré inequality, there is a constant $C > 0$ such that given $x, y \in \mathbb{R}$ we have $\text{Mod}_\infty(\Gamma(\{x\}, \{y\}, C), \mathscr{L}^1) > 0$; indeed, we in fact have $\text{Mod}_\infty(\{[x, y]\}, \mathscr{L}^1) > 0$. On the other hand, because $\mathscr{L}^1 \ll \mu$, for each $\rho \in F(\Gamma(\{x\}, \{y\}, C))$ we have that $\|\rho\|_{L^\infty(\mathbb{R},\mathscr{L}^1)} \leq \|\rho\|_{L^\infty(\mathbb{R},\mu)}$, and so we also have that $\text{Mod}_\infty(\Gamma(\{x\}, \{y\}, C), \mu) > 0$, that is, $(\mathbb{R}, |\cdot|, \mu)$ is also thick quasiconvex. It follows from Theorem 3.1 that this space also has ∞-Poincaré inequality. □

Remark 4.2 A result due to Buckley, Björn, and Keith in [5] tells us that when μ is doubling, $(\mathbb{R}, |\cdot|, \mu)$ supports a p-Poincaré inequality for some $1 \leq p < \infty$ if and only if $\mu \ll \mathscr{L}^1$ and the Radon-Nikodym derivative of μ with respect to $\mathscr{L}^1$ is a Muckenhoupt $\mathcal{A}_p$-weight. In contrast, Theorem 4.1 tells us that to obtain ∞-Poincaré inequality, it is both necessary and sufficient to have $\mathscr{L}^1 \ll \mu$. In particular, if ν is a singular doubling measure on $\mathbb{R}$, then $\mu = \mathscr{L}^1 + \nu$ would satisfy this condition, and so we have $(\mathbb{R}, |\cdot|, \mu)$ supporting an ∞-Poincaré inequality even though μ is not absolutely continuous with respect to $\mathscr{L}^1$. The Riesz measure constructed in [30] is a singular doubling measure on $\mathbb{R}$. See also [29].

We now consider higher dimensional Euclidean spaces. Let μ be a doubling measure on $\mathbb{R}^n$. By the Radon Nykodym theorem, given a measure μ in $\mathbb{R}^n$, there exist μ_s and μ_a such that

$$\mu = \mu_a + \mu_s, \qquad \mu_a \ll \mathscr{L}^n, \qquad \mu_s \perp \mathscr{L}^n.$$

Moreover, since the measure μ is doubling, the support of μ is $\mathbb{R}^n$.

Lemma 4.3 *Suppose that $\mathscr{L}^n \ll \mu$. Then $(\mathbb{R}^n, |\cdot|, \mu)$ supports ∞-Poincaré inequality.*

Proof. With the decomposition $\mu = \mu_a + \mu_s$, there is a Borel set $E \subset \mathbb{R}^n$ with $\mathscr{L}^n(E) = 0$ such that $\mu_s(\mathbb{R}^n \setminus E) = 0$. Hence the condition $\mathscr{L}^n \ll \mu$ implies that $\mathscr{L}^n \ll \mu_a$.

Since $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ supports an ∞-Poincaré inequality, from the above discussion it follows that so does $(\mathbb{R}^n, |\cdot|, \mu_a)$.

Fix $x, y \in \mathbb{R}^n$ and let Γ be the collection of all 2-quasiconvex curves in $\mathbb{R}^n$ connecting x to y. Then, for each admissible function $g \in F(\Gamma)$, where $F(\Gamma)$ is as in Definition 2.1,

$$\|g\|_{L^\infty(\mu)} \geq \|g\|_{L^\infty(\mu_a)} > 0,$$

where we used Theorem 3.1 to obtain the last inequality. Hence $\mathrm{Mod}_\infty(\Gamma, \mu) > 0$ and by applying Theorem 3.1 again, we see that $(\mathbb{R}^n, |\cdot|, \mu)$ also supports an ∞-Poincaré inequality.
□

Remark 4.4 Unlike in the $\mathbb{R}$-case of Theorem 4.1, we do not know whether a doubling measure μ on $\mathbb{R}^n$, $n \geq 2$, supporting an ∞-Poincaré inequality, must necessarily satisfy $\mathscr{L}^n \ll \mu$. That is, the converse of the above lemma is not known when $n \geq 2$.

The next example illustrates another application of Theorem 3.1. Given the above remark, we cannot immediately claim that the singular measure μ on $\mathbb{R}^n$ cannot support an ∞-Poincaré inequality; we instead use Theorem 3.1. Recall that there are doubling measures on $\mathbb{R}$ that are mutually singular to $\mathscr{L}^1$, such as the Riesz measure, see [30], [29], and [11].

Example 4.5 Let μ be the product measure given by $\mu = \mu_1 \times \mu_2 \cdots \times \mu_n$, where $\mu_i \perp \mathscr{L}^1$ is a doubling measure on $\mathbb{R}$ for each $1 \leq i \leq n$. Then $(\mathbb{R}^n, |\cdot|, \mu)$ does not support ∞-Poincaré inequality. Indeed, since μ_i is singular, there exists a set E_i such that $\mu_i(E_i) = 0$ whereas $\mathscr{L}^1(E_i) > 0$. Consider the n-dimensional set

$$E = (E_1 \times \mathbb{R} \times \mathbb{R}.. \times \mathbb{R}) \cup (\mathbb{R} \times E_2 \times \mathbb{R} \cdots \times \mathbb{R}) \cup \cdots \cup (\mathbb{R} \times \cdots \mathbb{R} \times E_n),$$

which has $\mu(E) = 0$ and $\mathscr{L}^n(E) > 0$. Choose two points $x, y \in \mathbb{R}^n$ with $x \neq y$ and a curve γ connecting x to y. Then for any $1 \leq i \leq n$ such that $x_i \neq y_i$ (where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$),

$$\mathcal{H}^1(E \cap \gamma) \geq \mathcal{H}^1(P_i(E \cap \gamma)) = \mathscr{L}^1(P_i(E \cap \gamma)) > 0$$

where P_i denotes the projections onto the i-th axis. We can deduce that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$ and so $\gamma \in \Gamma_E^+$. Therefore $(\mathbb{R}^n, |\cdot|, \mu)$ does not support ∞-Poincaré inequality because μ violates Condition (f) of Theorem 3.1.

We conclude this section by considering the so-called Lip–lip property of Keith [17]. In [6] Cheeger proved that doubling p-Poincaré spaces admit a (non-degenerate) differentiable structure for which Lipschitz functions are differentiable μ-a.e. in the sense that there exists a countable collection of pairs $\{(X_\alpha, \mathbf{x}_\alpha)\}$ of measurable sets $X_\alpha \subset X$ (*charts*) and Lipschitz maps

$$\mathbf{x}_\alpha = (x_\alpha^1, \ldots, x_\alpha^{N_\alpha}) : X \longrightarrow \mathbb{R}^{N_\alpha}$$

(*coordinates*), that satisfy the following conditions:

(*i*) $\mu\Big(X \setminus \bigcup_\alpha X_\alpha\Big) = 0;$

(*ii*) There exists $N \geq 1$ such that $N_\alpha \leq N$ for each $(X_\alpha, \mathbf{x}_\alpha)$;

(*iii*) If $u : X \to \mathbb{R}$ is Lipschitz, then for each $(X_\alpha, \mathbf{x}_\alpha)$ there exists a unique (up to a set of zero measure) measurable function $d^\alpha u : X_\alpha \longrightarrow \mathbb{R}^{N_\alpha}$ such that

$$\limsup_{\substack{y \to x \\ y \neq x}} \frac{|u(y) - u(x) - d^\alpha u(x) \cdot (\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{d(y, x)} = 0 \tag{1}$$

for μ-a.e. $x \in X_\alpha$.

If the above holds, we say that (X, d, μ) supports a *measurable differentiable structure.*

Observe that the exponent p is present in the hypothesis of this result, but it has no role in the conclusions. Keith, in [17] weakened the hypotheses so as not to depend on p. He defined the $\mathrm{Lip} - \mathrm{lip}$ condition as follows: a metric measure space X is said to satisfy a $\mathrm{Lip} - \mathrm{lip}$ *condition* if there exists a constant $K \geq 1$ such that whenever $u : X \longrightarrow \mathbb{R}$ is a Lipschitz function, we have

$$\mathrm{Lip}\, u(x) \leq K \,\mathrm{lip}\, u(x)$$

for μ-a.e. $x \in X$. The thesis [17, Section 1.4] conjectures that this condition can be understood as a version of Cheeger's theorem for $p = \infty$.

It is known that complete doubling metric measure spaces which admit a weak p-Poincaré inequality for any $1 \leq p < \infty$ satisfy the $\mathrm{Lip} - \mathrm{lip}$ condition as well. On the other hand, it is clear that the $\mathrm{Lip} - \mathrm{lip}$ condition does not imply the validity of a p-Poincaré inequality for any $1 \leq p \leq \infty$. A non-empty non-quasiconvex open set of $\mathbb{R}^n$ has the $\mathrm{Lip} - \mathrm{lip}$ condition but does not support any p-Poincaré inequality, $1 \leq p \leq \infty$.

Very recently it has been proved that the Lip-lip condition is not also a sufficient but also a necessary condition for the validity of a Rademacher theorem in the metric measure setting. The complete characterization is the following.

Theorem 4.6 ([3, Corollary 10.4], [12, Theorem 1.3]) *Let (X, d) be a complete metric space endowed with a Radon measure μ. Then (X, d, μ) supports a Cheeger differentiable structure if and only if the measure μ is pointwise doubling and if there exists a countable collection of measurable sets $\{Z_n\}$ with associated constants M_n such that $\mu\Big(X \setminus \bigcup_n Z_n\Big) = 0$ and for each $n \in \mathbb{N}$, the space (Z_n, d, μ) satisfies a* $\mathrm{Lip} - \mathrm{lip}$ *condition with constant M_n.*

See also [25, Page 7].

In the next example we will construct a complete doubling metric measure space supporting an ∞-Poincaré inequality but with no measurable differentiable structure which in turn implies by Theorem 4.6 that the space does not have the Lip-lip condition. Therefore, without any extra-hypothesis, there is no relation between the Lip-lip condition and the ∞-Poincaré inequality.

Example 4.7 Take any singular doubling measure with constant C in $\mathbb{R}$ denoted μ_s and define $\mu = \mu_s + \mathscr{L}^1$. Observe that μ is a doubling measure. Indeed,

$$\begin{aligned}\mu(B(x,2r)) \le& \mu_s(B(x,2r)) + \mathscr{L}^1(B(x,2r)) \le C\mu_s(B(x,r)) + 2\mathscr{L}^1(B(x,2r)) \\ \le& \max\{2,C\}(\mu_s(B(x,r)) + \mathscr{L}^1(B(x,r))).\end{aligned}$$

By Theorem 4.1, $(\mathbb{R}, d = |\cdot|, \mu)$ supports an $\infty-$Poincaré inequality. On the other hand, since $\mu_s \perp \mathscr{L}^1$, there exists a set N such that $\mu(N) > 0$ whereas $\mathscr{L}^1(N) = 0$. A classical result by Choquet [7] states that given a set $E \subset \mathbb{R}$, there exists a Lipschitz function $u_0 : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at any point of $x \in E$ if and only if $\mathscr{L}^1(E) = 0$. Using this result we can construct a Lipschitz function u_0 that is Euclidean differentiable nowhere in N. Assume that $(\mathbb{R}, |\cdot|, \mu)$ has a measurable differentiable structure in the sense of Cheeger. For simplicity assume that $\mathbb{R}$ is decomposed in one single chart denoted by X_α. Then, there exists a unique measurable function $du_0 : X_\alpha \longrightarrow \mathbb{R}^{N_\alpha}$ such that

$$\lim_{\substack{y \to x \\ y \neq x}} \frac{|u_0(y) - u_0(x) - du_0(x) \cdot (\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{|y - x|} = 0 \tag{2}$$

for μ-a.e. $x \in X_\alpha$. In particular, by [24, Corollary 6.30] combined with [25, Lemma 4.1], or else by [12, Corollary 6.5], we know that we can choose the coordinate functions to be a certain collection of distance functions. More precisely, there exist points $x_1, x_2, ..., x_{N_\alpha} \in \mathbb{R}$ such that $x_\alpha(x) = (|x_1 - x|, |x_2 - x|...., |x_{N_\alpha} - x|)$. Denote by Z the set of points where u_0 is non-differentiable with respect to μ. Observe that the function $x_\alpha : \mathbb{R} \to \mathbb{R}$ is Euclidean differentiable on $\mathbb{R}^n \setminus \{x_1, x_2, ..., x_{N_\alpha}\}$. Since μ being doubling cannot charge finite sets, we know that there is a point x_0 in $(X_\alpha \cap N) \setminus Z$ (that is not any of $x_1, x_2, ..., x_{N_\alpha}$), such that (2) holds for $x = x_0$, that is, u_0 is differentiable at x_0 with respect to the chart X_α. In particular u_0 is differentiable at x_0 with respect to the standard Euclidean coordinate functions, with Euclidean derivative given by

$$\sum_{i=1}^{N_\alpha} \alpha_i \frac{1}{|x_0 - x_i|}(x_0 - x_i),$$

where $(\alpha_1, \alpha_2, ..., \alpha_{N_\alpha})$ is the metric derivative of u_0 (with respect to the chart (X_α, x_α)) at x_0, yielding a contradiction.

5 Characterization of p-Poincaré inequality for large p

Poincaré and Sobolev inequalities for functions in the Sobolev classes have proven to be useful tools in the study of solutions to PDEs, and hence it is of interest to know what Euclidean domains, and more generally, metric measure spaces, support such inequalities. The first to study such inequalities and the associated embedding theorems was Sobolev, see [28]. Characterizations of such inequalities in terms of isoperimetric inequality and condenser inequalities, in the setting of Euclidean spaces and manifolds were given by Mazýa [22], [23]; a nice exposition can be found in [21]. However, in this section we are concerned more with obtaining a characterization of p-Poincaré inequalities in terms of p-moduli of curve families. In the case that the metric measure space is complete and Ahlfors Q-regular, a geometric (Loewner property) characterization of Q-Poincaré inequality was first given in [16]. In this section, we focus on Ahlfors Q-regular metric measure spaces with $Q > 1$, and wish to characterize p-Poincaré inequality in terms of p-moduli of curve families that connect two sets, for the two cases $p > Q$ and $Q - 1 < p \leq Q$. Such a characterization for graphs was obtained by Coulhon and Koskela [8].

Given the characterization of ∞-Poincaré inequality from Theorem 3.1, it is natural to ask whether there is a similar characterization of p-Poincaré inequality for large enough p. Given the Morrey embedding theorem, we consider $p > Q$ with Q the Ahlfors regularity exponent of μ. Recall that a measure μ is *Ahlfors Q-regular* if there is a constant $C > 0$ such that whenever $x \in X$ and $0 < r < \operatorname{diam} X$, $r^Q/C \leq \mu(B(x,r)) \leq Cr^Q$.

A version of the following theorem holds if μ is known to be doubling, where Q is the logarithm of the doubling constant of μ. However, for the sake of simplicity we focus only on Ahlfors regular measures. Interested readers can easily modify the argument, but in this case the constant C_u depends not only on $\|g_u\|_{L^p(B(x,\tau d(x,y)))}$ but also on the choice of a compact subset $K \subset X$ that contains $B(x, \tau d(x,y))$.

Theorem 5.1 *Let X be a complete Ahlfors Q-regular space and $p > Q$. Then the following conditions are equivalent:*

(1) *X supports a p-Poincaré inequality.*

(2) *X is connected and there are constants $C > 0, \tau > 1$ such that every $u \in N^{1,p}(X)$ is $(1 - \frac{Q}{p})$-Hölder continuous. In fact, for all $x, y \in X$ we have*

$$|u(x) - u(y)| \leq C_u \, d(x,y)^{1-\frac{Q}{p}},$$

with $C_u \leq C\|g_u\|_{L^p(B(x,\tau d(x,y)))}$, where g_u is the minimal p-weak upper gradient of u.

(3) *There is a constant $C > 0$ such that, for every pair of distinct points $x, y \in X$,*

$$\operatorname{Mod}_p(\Gamma(\{x\},\{y\},C)) \geq \frac{C}{d(x,y)^{p-Q}},$$

where $\Gamma(\{x\},\{y\},C)$ denotes the family of C-quasiconvex curves connecting x to y.

Remark 5.2 Let Γ_{x_0} denote the collection of all non-constant rectifiable curves intersecting x_0. Condition (3) directly implies that $\operatorname{Mod}_p(\Gamma_{x_0}) > 0$ and therefore $\operatorname{Cap}_p(\{x_0\}) > 0$.

Proof. That (1) $\Longrightarrow$ (2) follows from the Morrey embedding theorem, see for example [27, Theorem 5.1.] or [15, Theorem 5.1]. To show that (2) $\Longrightarrow$ (1), we suppose (2) holds. Let $u \in N^{1,p}(X)$ with minimal p-weak upper gradient g_u, and let B be a ball in X. Then by Condition (2),

$$|u(x) - u(y)| \leq C\, d(x,y)^{1-Q/p} \left(\int_{B(x,\tau d(x,y))} g_u^p \, d\mu \right)^{1/p}$$

whenever $x, y \in B$. Note that $B(x, \tau d(x,y)) \subset 2\tau B$ whenever $x, y \in B$. Therefore

$$|u(x) - u(y)| \le C\, d(x,y)^{1-Q/p} \left(\int_{2\tau B} g_u^p \, d\mu \right)^{1/p}.$$

Let R be the radius of B. Then by the fact that $p > Q$ and the Ahlfors Q-regularity of μ, for $x, y \in B$ we have

$$|u(x) - u(y)| \le C\, R \left(\frac{1}{\mu(B)} \int_{2\tau B} g_u^p \, d\mu \right)^{1/p}.$$

Integrating over x and y in B, we obtain

$$\fint_B |u - u_B| \, d\mu \le \fint_B \fint_B |u(x) - u(y)| \, d\mu(x) \, d\mu(y) \le C\, R \left(\fint_{2\tau B} g_u^p \, d\mu \right)^{1/p}.$$

That is, u, g_u satisfy the p-Poincaré inequality on B, with $\lambda = 2\tau$. This proves (1).

We now show that if X satisfies the condition that whenever $u \in N^{1,p}(X)$ there is a constant $C_u > 0$ such that for all $x, y \in X$ we have $|u(x) - u(y)| \le C_u d(x,y)^{1-Q/p}$, then there are constants $C > 0$ and $\tau > 1$ such that we can take $C_u^p \le C \int_{B(x,\tau d(x,y))} g_u^p \, d\mu$. We prove this by contradiction as follows. Fix a ball $B \subset X$, $x_0, y_0 \in 1/4B$ with $x_0 \ne y_0$, and let $\mathcal{A}$ be the collection of all $u \in N^{1,p}(X)$ such that $u(x_0) = 0, u(y_0) = 1$, and $C_u \le 1$ for points $x, y \in B$. It suffices to show that

$$\inf_{u \in \mathcal{A}} \int_{4B} g_u^p \, d\mu =: C_{\mathcal{A},B} > 0.$$

Suppose $C_{\mathcal{A},B} = 0$. Then for each positive integer n we have $u_n \in \mathcal{A}$ such that $\int_{4B} g_{u_n}^p \, d\mu < 1/n$. Because of the hypothesis, $\mathcal{A}$ is an equicontinuous equibounded family of functions on the compact set $\overline{2B}$, and so by the Arzela-Ascoli theorem there is a subsequence of $(u_n)_n$, denoted $(u_{n_k})_k$, converging uniformly in $\overline{2B}$ to a function u_∞; this limit function satisfies $u_\infty(x_0) = 0$, $u_\infty(y_0) = 1$, and u is $(1 - Q/p)$-Hölder continuous with Hölder continuity constant at most 1 on $\overline{2B}$. It follows that u is non-constant on $\overline{2B}$. On the other hand, by the reflexivity of $N^{1,p}(\overline{2B})$ (since $1 < p < \infty$; see [1]), and by the fact that $(u_{n_k})_k$ is bounded in $N^{1,p}(\overline{2B})$, we can apply Mazur's lemma to this subsequence. Thus we obtain a convex combination sequence of this subsequence that converges in $N^{1,p}(\overline{2B})$ to some $v_\infty \in N^{1,p}(\overline{2B})$. By passing to a further subsequence if needed, we can also assume that this convergence of convex combination subsequence is also pointwise almost everywhere in $\overline{2B}$. Thus $v_\infty = u_\infty$ almost everywhere in $\overline{2B}$, and so $v_\infty \ge 3/4$ almost everywhere in a small neighborhood of y_0 and $v_\infty \le 1/4$ almost everywhere in a small neighborhood of x_0. That is, v_∞ is non-constant in $\overline{2B}$. On the other hand, $g_{v_\infty} = 0$ by the choice of the sequence, and so the collection of rectifiable curves connecting $A := \{x \in X : v_\infty(x) > 3/4\}$ to its complement has p-modulus zero. This means that χ_A is in the class $N^{1,p}(X)$, which is not possible as X is connected and hence χ_A cannot be Hölder continuous.

Let us prove now that (2) $\Longrightarrow$ (3). Fix $x_0, y_0 \in X$. We denote the collection of all rectifiable curves in $B(x_0, 2\tau d(x_0, y_0))$ with end points x_0 and y_0 by $\Gamma(x_0, y_0)$. Let g be a nonnegative

Borel measurable function on X such that for all $\gamma \in \Gamma(x_0, y_0)$, the integral $\int_\gamma g\, ds \geq 1$. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path in } B(x_0, 2\tau d(x_0,y_0)) \text{ connecting } z \text{ to } B(x_0,r)} \int_\gamma g\, ds,$$

where $r < d(x, y)/4$. With η a Lipschitz function with $\eta = 1$ on $B(x_0, \tau d(x_0, y_0))$, $\eta = 0$ on $X \setminus B(x_0, 2\tau d(x_0, y_0))$, and $0 \leq \eta \leq 1$ on X, we set $u = \eta \min\{\tilde{u}, 2\}$. Then it follows that $u = 0$ on $B(x_0, r)$ and by the choice of g, $u(y_0) \geq 1$. Notice that u is measurable, g is an upper gradient of u and $u \in N^{1,p}(X)$.

By hypothesis, every $u \in N^{1,p}(X)$ is $(1 - \frac{Q}{p})$-Hölder continuous, and so

$$1 \leq |u(x_0) - u(y_0)| \leq C \|g\|_{L^p(B(x_0, \tau d(x_0,y_0)))}\, d(x_0, y_0)^{1-\frac{Q}{p}}.$$

Taking the infimum over all such g we obtain the estimate

$$\mathrm{Mod}_p(\Gamma_{x_0,y_0}) \geq \frac{1}{C\, d(x_0, y_0)^{p-Q}}.$$

Given a positive integer m, recall that $\Gamma(\{x_0\}, \{y_0\}, m)$ denotes the collection of all rectifiable curves in $B(x_0, 2\tau d(x_0, y_0))$ of length at most $m\, d(x_0, y_0)$. Set

$$\Lambda(x_0, y_0, m) = \Gamma_{x_0,y_0} \setminus \Gamma(\{x_0\}, \{y_0\}, m).$$

Then by the subadditivity property of the modulus, we have

$$\mathrm{Mod}_p(\Gamma(\{x_0\}, \{y_0\}, m)) + \mathrm{Mod}_p(\Lambda(x_0, y_0, m)) \geq \frac{1}{C\, d(x_0, y_0)^{p-Q}}.$$

On the other hand, since $\rho := m^{-1}\, d(x_0, y_0)^{-1} \chi_{B(x_0, 2\tau d(x_0,y_0))}$ is admissible for computing the p-modulus of $\Lambda(x_0, y_0, m)$, we have

$$\mathrm{Mod}_p(\Lambda(x_0, y_0, m)) \leq \frac{\mu(B(x_0, 2\tau d(x_0, y_0)))}{m^p\, d(x_0, y_0)^p} \leq \frac{1}{C_1\, m^p\, d(x_0, y_0)^{p-Q}},$$

where C_1 is a constant depending only on the Q-Ahlfors regular constant. Hence, when

$$m > \left(\frac{2C}{C_1}\right)^{1/p},$$

we must then have

$$\mathrm{Mod}_p(\Gamma(\{x_0\}, \{y_0\}, m)) \geq \frac{1}{2C\, d(x_0, y_0)^{p-Q}}.$$

Thus, choosing the quasiconvexity constant to be $1 + \left(\frac{2C}{C_1}\right)^{1/p}$, we obtain the desired conclusion.

To complete the proof of the theorem, let us prove that (3) $\Longrightarrow$ (2). Let $u \in N^{1,p}(X)$ and denote by g_u its the minimal p-weak upper gradient. Fix $x_0, y_0 \in X$ Lebesgue points for u. Recall that $\mu-$a.e. point is a Lebesgue point of every locally integrable function in X. Consider now the function $v = \dfrac{|u - u(x_0)|}{|u(y_0) - u(x_0)|}$ and observe that $v(x_0) = 0$, $v(y_0) = 1$. The function $g_v = \dfrac{g_u}{|u(y_0) - u(x_0)|}$ is the minimal p-weak upper gradient of v, and so for any rectifiable curve connecting x_0 and y_0 we have

$$1 = |v(x_0) - v(y_0)| \leq \int_\gamma g_v,$$

which means in particular that g_v is an admissible function for computing the p-modulus Γ_{x_0,y_0}. Note that the curves in Γ_{x_0,y_0} stay inside $B(x_0, 2C\, d(x_0, y_0))$. Hence, by hypothesis we have

$$\frac{1}{|u(y_0) - u(x_0)|^p} \int_{B(x_0,2C\,d(x_0,y_0))} g_u^p d\mu = \int_{B(x_0,2C\,d(x_0,y_0))} g_v^p d\mu \geq \mathrm{Mod}_p(\Gamma_{x_0,y_0}) \geq \frac{C}{d(x_0, y_0)^{p-Q}},$$

and so

$$|u(y_0) - u(x_0)|^p \leq C d(x_0, y_0)^{p-Q} \|g_u\|^p_{L^p(B(x_0,2C\,d(x_0,y_0)))}.$$

Rising to the power $1/p$ both sides of the previous inequality we get that u is $(1 - \frac{Q}{p})$-Hölder continuous $\mu-$a.e. We can obtain a Hölder continuous extension of u from its Lebesgue set to the whole X which defines the same element in $N^{1,p}(X)$ (notice that by Remark 5.2, points have positive capacity). □

Recall the following characterization of the p-Poincaré inequality in terms of the restricted maximal function defined by

$$M_R(f)(x) = \sup_{0<r<R} \fint_{B(x,r)} |f| d\mu.$$

Lemma 5.3 [16, Lemma 5.15] *Let X be a complete doubling metric measure space, and $1 < p < \infty$. Then X admits a $p-$Poincaré inequality if and only if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every continuous function u in a ball B_R and every (p-weak) upper gradient g of u we have*

$$|u(x) - u(y)| \leq C d(x, y)(M_{CR}(g^p)(x) + M_{CR}(g^p)(y))^{\frac{1}{p}},$$

for $\mu-$a.e $x, y \in B_R$.

We next recall the notions of Hausdorff content and Hausdorff measure. Given $s > 0$, $0 < R \leq \infty$, and $E \subset X$, the s-dimensional Hausdorff R-content (simply called the s-dimensional Hausdorff content when $R = \infty$) is the number

$$\mathcal{H}^s_R(E) = \inf \Big\{ \sum_j \mathrm{diam}(B_j)^s \;:\; E \subset \bigcup_j B_j, \mathrm{diam}(B_j) < R \Big\}.$$

The s-dimensional Hausdorff measure of E is the number

$$\mathcal{H}^s(E) = \lim_{R \to 0^+} \mathcal{H}^s_R(E).$$

Theorem 5.4 [16, Theorem 5.9] *Suppose that (X, d, μ) is a doubling metric measure space with $\mu(B(x, r)) \geq Cr^Q$ for some $Q \geq 1$ and admitting a p-Poincaré inequality for some $1 \leq p \leq Q$. Let E and F be two compact subsets of a ball B_R in X such that $0 < \mathcal{H}^s(E), \mathcal{H}^s(F) < \infty$ for some $Q \geq s > Q - p$. Then, there exists a constant $C \geq 1$, depending only on s and on the data associated with X so that*

$$\int_{B_{CR}} g^p d\mu \geq C^{-1} \frac{\min\{\mathcal{H}^s_\infty(E), \mathcal{H}^s_\infty(F)\}}{R^{p-Q+s}}$$

whenever u is a continuous function in the ball B_{CR} with $u_{|E} \leq 0$ and $u_{|F} \geq 1$ and g is a p-weak upper gradient for u.

We next focus on the case $Q - 1 < p \leq Q$.

Theorem 5.5 *Let X be a connected, complete Ahlfors $Q-$regular metric measure space. Then X supports a p-Poincaré inequality for some $Q - 1 < p \leq Q$ if and only if there exists a constant $C > 0$ such that for every two disjoint non degenerate continua E, F in a ball B_R we have*

$$\mathrm{Mod}_p(\Gamma(E, F, B_{CR})) \geq \frac{1}{CR^{1-Q+p}} \min\{\mathrm{diam}(E), \mathrm{diam}(F)\}, \tag{3}$$

where $\mathrm{Mod}_p(\Gamma(E, F, B_{CR}))$ denotes the modulus of rectifiable curves connecting E to F inside B_{CR}.

Proof. From Theorem 5.4 we know that if X supports a p-Poincaré inequality, then Condition (3) holds (use $s = 1$ and the fact that $\mathcal{H}^s_\infty(E) \geq \mathrm{diam}(E)$).

For the converse, we model our proof along that of Lemma 5.17 of [16]. We suppose that Condition (3) holds. We fix a ball B in X, and let $x, y \in C_q^{-1}B$, where C_q is the quasiconvexity constant of X. Let u be a continuous function on X with upper gradient ρ, and let γ be a C-quasiconvex curve in X connecting x to y. Then $\gamma \subset B$. By rescaling u if needed, we can assume that $|u(x) - u(y)| = 1$. Let $M > \max\{2, C_q\}$ be a large constant (to be determined later), and for each $j \in \mathbb{N} \cup \{0\}$ we set

$$A_j = B(x, M^{-3j}\tau) \setminus B(x, M^{-3j-2}\tau),$$

where $\tau > 0$ is chosen so that the sphere centered at x with radius τ intersects γ at its midpoint. It is easy to see that $|x - y|/2 \leq \tau \leq |x - y|$. For each j let γ_j denote a subcurve of γ lying in A_j and connecting the two spheres, centered at x, of radii $M^{-3j}\tau$ and $M^{-3j-2}\tau$. Let Γ_j denote the collection of curves in B connecting γ_j to γ_{j+1} in $B(x, M^{-3j+2}\tau)$. Then by (3) we have

$$\mathrm{Mod}_p(\Gamma_j) \geq \frac{1}{C\, M^3}.$$

Let

$$a_j = \inf_{\beta \in \Gamma_j} \int_\beta \rho\, ds.$$

Then ρ/a_j is admissible for computing $\mathrm{Mod}_p(\Gamma_j)$, and so

$$M_{2\tau}\rho^p(x) \geq \fint_{B(x,M^{-3j+2}\tau)} \rho^p \, d\mu \geq \frac{a_j^p \, M^{(3j-2)Q}}{C\, M^3\, \tau^Q}.$$

Case 1: There is a choice of j for which

$$a_j\, M^{(3j-2)Q/p} \geq c\, \tau^{(Q-p)/p}\, M^{3/p}$$

for some c that depends only on the data of X and the choice of M. In this case, we have that

$$C|x-y|\, (M_{2\tau}\rho^p(x))^{1/p} \geq c = c\, |u(x)-u(y)|,$$

from which we obtain the Hajłasz type inequality

$$|u(x)-u(y)| \leq \frac{C}{c}\, |x-y|\, \left[(M_{2\tau}\rho^p(x))^{1/p} + (M_{2\tau}\rho^p(y))^{1/p} \right]. \tag{4}$$

Note that if we have the above inequality for μ-almost all $x \in C_q^{-1}B$, then we obtain p-Poincaré inequality, see [15] or [16].

Case 2: There is a set of positive measure in $C_q^{-1}B$ for which no such choice of j exists. Let $0 < c < M^{-3/p}/3$. Fix x, y from that set. Then for each j we have that

$$a_j < c\, \tau^{(Q-p)/p}\, M^{3/p}\, M^{(-3j+2)Q}.$$

Then there is a curve $\beta_j \in \Gamma_j$, connecting γ_j to γ_{j+1}, so that

$$\int_{\beta_j} \rho\, ds < c\, \tau^{(Q-p)/p}\, M^{3/p}\, M^{(-3j+2)Q}.$$

If β_j and β_{j+1} intersect for each j, then we can concatenate them to obtain (using a similar argument with x replaced with y) a rectifiable curve β connecting x to y such that

$$|u(x)-u(y)| = 1 \leq \int_\beta \rho\, ds < c\, \tau^{(Q-p)/p}\, M^{3/p} \sum_{j=0}^{\infty} M^{(-3j+2)Q} \leq c\, \tau^{(Q-p)/p}\, M^{3/p} < \tau^{(Q-p)/p},$$

which is not possible since $\tau < 1$. So there is some j for which β_j and β_{j+1} do not intersect. For such j we let Λ_j be the collection of all rectifiable curves in $B(x, M^{-3j+2}\tau)$ connecting β_j to β_{j+1}. We now set

$$b_j = \inf_{\alpha \in \Lambda_j} \int_\alpha \rho\, ds.$$

As in Case 1, if we have that for some choice of j,

$$b_j\, M^{(3j-2)Q/p} \geq c\, \tau^{(Q-p)/p}\, M^{3/p},$$

then we have the Hajłasz-type inequality (4). So we assume that for each j,

$$b_j < c\, \tau^{(Q-p)/p}\, M^{3/p}\, M^{(-3j+2)Q},$$

and hence we can choose a curve α_j connecting β_j and β_{j+1} so that

$$\int_{\alpha_j} \rho\, ds < c\,\tau^{(Q-p)/p}\, M^{3/p}\, M^{(-3j+2)Q}.$$

Now we can concatenate β_j, α_j, and β_{j+1} for all such j, and concatenate β_j and β_{j+1} when they do intersect, to obtain a curve connecting x to y on which the path integral of ρ is smaller than 1, violating the upper gradient property of ρ again. Therefore, in Case 2 there is some choice of j for which we have

$$b_j\, M^{(3j-2)Q/p} \geq c\,\tau^{(Q-p)/p}\, M^{3/p},$$

and so we have (4).

Combining Cases 1 and 2, we see that for each $x, y \in C_q^{-1}B$,

$$|u(x) - u(y)| \leq \frac{C}{c}\,|x-y|\, \left[(M_{2\tau}\rho^p(x))^{1/p} + (M_{2\tau}\rho^p(y))^{1/p}\right].$$

It follows that X supports a p-Poincaré inequality. $\square$

Remark 5.6 The case $p = Q$ corresponds to Loewner spaces of Heinonen–Koskela [16]. Recall that an Ahlfors Q-regular space is said to be a *Q-Loewner space* if there is a function $\psi : (0,\infty) \to (0,\infty)$ such that whenever $E, F \subset B \subset X$ (with B a ball in X) are disjoint continua (compact connected sets with at least two points), then

$$\mathrm{Mod}_Q(\Gamma(E,F,CB)) \geq \psi(\mathrm{dist}(E,F)/\min\{\mathrm{diam}(E),\, \mathrm{diam}(F)\}).$$

We point out here that the Q-Loewner property characterization of Q-Poincaré inequality for Ahlfors Q-regular spaces is stronger than ours since we require a specific type of function ψ, namely $\psi(t) = 1/t$. For a related characterization of Q-Poincaré inequality see [4].

References

[1] L. Ambrosio, M. Colombo, S. di Marino: Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope. Preprint 2012.

[2] L. Ambrosio, N. Gigli, G. Savaré: Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. *Rev. Mat. Iberoam.* **29** (2013), no. 3, 969–996.

[3] D. Bate: Structure of measures in Lipschitz differentiability spaces. arXiv **1208.1954v3** (2012).

[4] M. Bonk, B. Kleiner: Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.* **9** (2005), 219–246.

[5] Björn, J., Buckley, S., and Keith, S. Admissible measures in one dimension. *Proc. Amer. Math. Soc. 134*, 3 (2006), 703–705 (electronic).

[6] J. Cheeger: Differentiability of Lipschitz Functions on metric measure spaces. *Geom. Funct. Anal.* **9** (1999), 428–517.

[7] G. Choquet: Applications des propriétés descriptives de la fonction contingent à la théorie des fonctions de variable réelle et à la géometrie différentielle des variétés cartésiennes, J. Math. Pures Appl., 26 (1947), 115–226.

[8] T. Coulhon, P. Koskela: Geometric interpretations of L^p Poincaré inequalities on graphs with polynomial volume growth. *Milan J. Math.* **72** (2004), 209–248.

[9] E. Durand-Cartagena, J. A. Jaramillo: Pointwise Lipschitz functions on metric spaces. *J. Math. Anal. Appl.* **363** (2010), 525–548.

[10] E. Durand-Cartagena, J. A. Jaramillo, N. Shanmugalingam: The ∞-Poincaré inequality in metric measure spaces. *Michigan Math. Journal.* **60** (2011).

[11] E. Durand-Cartagena, N. Shanmugalingam, A. Williams: p-Poincaré inequality versus ∞-Poincaré inequality: some counterexamples. *Math. Z.* **271** (2012), no. 1-2, 447–467.

[12] J. Gong: The Lip-lip condition on metric measure spaces, arXiv **1208.2869v1** (2012).

[13] P. Hajłasz: Sobolev spaces on an arbitrary metric space. *Potential Anal.* **5** (1996), no. 4, 403–415.

[14] P. Hajłasz: Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173–218, *Contemp. Math.* **338** Amer. Math. Soc., Providence, RI, 2003.

[15] P. Hajłasz, P. Koskela: Sobolev met Poincaré. *Mem. Amer. Math. Soc.* **145** (2000), no. 688, x+101 pp.

[16] J. Heinonen, P. Koskela: Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* **181** (1998), 1–61.

[17] S. Keith: A differentiable structure for metric measure spaces. *Adv. Math.* **183** (2004) 271–315.

[18] S. Keith: Modulus and the Poincaré inequality on metric measure spaces. *Math. Z.* **245** (2003), 255–292.

[19] R. Korte: Geometric implications of the Poincaré inequality. *Results Math.* **50** (2007), no. 1-2, 93–107.

[20] L. Malý: Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices, *Ann. Acad. Sci. Fenn. Math.* **38** (2013), 727–745.

[21] V. Maz'ya: Sobolev spaces. Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin (1985), xix+486 pp.

[22] V. Maz'ya: Classes of domains and imbedding theorems for function spaces. *Dokl. Akad. Nauk SSSR* **133** 527–530 (Russian); translated as *Soviet Math. Dokl.* **1** (1960), 882–885.

[23] V. Maz'ya: Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 307–340, *Contemp. Math.*, **338** Amer. Math. Soc., Providence, RI, 2003.

[24] A. Schioppa: On the relationship between derivations and measurable differentiable structures on metric measure spaces. ArXiv **1205.3235v1** (2012), *Ann. Acad. Sci. Fenn.*, to appear.

[25] A. Schioppa: Derivations and Alberti representations. ArXiv **1311.2439** (2013).

[26] S. Semmes: Finding Curves on General Spaces through Quantitative Topology, with Applications to Sobolev and Poincaré Inequalities. *Selecta Math., New Series* Vol. **2**, no. 2 (1996), 155–295.

[27] N. Shanmugalingam: Newtonian Spaces: An extension of Sobolev spaces to Metric Measure Spaces. *Rev. Mat. Iberoamericana*, **16** (2000), 243–279.

[28] S. L. Sobolev: Applications of functional analysis in mathematical physics. Translated from the Russian by F. E. Browder. Translations of Mathematical Monographs, Vol. 7 American Mathematical Society, Providence, R.I. (1963), vii+239 pp.

[29] E. M. Stein: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[30] A. Zygmund: Trigonometric series. 2nd ed. Vols. I, II. Cambridge University Press, New York 1959 Vol. I. xii+383 pp.; Vol. II. vii+354 pp.

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