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Preservation of bounded geometry under sphericalization and flattening

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Abstract

The sphericalization of a metric space produces a bounded metric space from an unbounded metric space, while the flattening procedure produces an unbounded metric space from a bounded metric space. This corresponds to obtaining the Riemann sphere from the complex plane, and obtaining the complex plane from the Riemann sphere. In this paper we show that sphericalization and flattening procedures on a complete metric measure space preserve properties such as Ahlfors regularity and doubling property. We also show that if the metric space has a doubling measure and is in addition quasiconvex and annular quasiconvex, then the sphericalization and flattening procedures preserve the property of supporting a p-Poincaré inequality.

1 Introduction

The landmark papers [13] and [12] together pointed to a need to develop a theory of analysis on metric measure spaces in order to study quasiconformal mappings between metric measure spaces. It was shown in [12] that in considering two metric measure spaces equipped with Ahlfors regular measures supporting a

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Poincaré inequality, then much of the theory of quasiconformal mappings between Euclidean spaces also extends to quasiconformal mappings between them; see also [14]. Using the notion of upper gradients developed in [13] and [12], the paper [22] developed a notion of Sobolev spaces in such a metric measure space setting; and in [23], [5], and [16] (among many other papers) it was shown that much of the Euclidean non-linear potential theory has analog in metric spaces equipped with a doubling measure supporting a Poincaré inequality. See also [8] for a discussion on some implications of the Poincaré inequality in the study of parabolic equations in the setting of manifolds. Hence it makes sense to identify and/or construct metric spaces that are equipped with a doubling (or, Ahlfors regular) measure supporting a Poincaré inequality.

Given a metric measure space whose measure is doubling and supports a Poincaré inequality, it is natural to ask whether geometric modifications of the space also yield a doubling space supporting a Poincaré inequality. Recall that the Euclidean spaces, and their sphericalizations (that is, Euclidean spheres) satisfy such doubling condition and support Poincaré inequalities. Following the notion of "uniformization" found in [1] and [2], the papers [4] and [15] considered sphericalizations of metric spaces. In this paper we will consider such a sphericalization of a metric measure space, and propose a measure on the sphericalized space by using the measure on the original space. The main goal of this paper is to prove the following theorem. See Section 2 for the definitions and descriptions of the notions used in the following theorem.

Theorem 1.1. *Suppose that (X, d, μ) is a complete metric measure space such that μ is a Borel regular measure for which non-empty open sets have positive measure and bounded sets have finite measure. Suppose also that (X, d) is quasiconvex and annular quasiconvex. Then for $1 \le p < \infty$, (X, d, μ) is doubling and supports a p-Poincaré inequality if and only if for each (or equivalently, some) $a \in X$ the sphericalized space $(\dot{X}, \hat{d}_a, \mu_a)$ supports a p-Poincaré inequality.*

The proof of Theorem 1.1 is obtained by combining Proposition 3.6, Theorem 3.15, Proposition 4.4 and Theorem 4.12, together with the fact that the flattening of a sphericalized space (flattened using the point $c = \infty$) is biLipschitz equivalent to the original metric space (see [4]).

The assumptions of quasiconvexity and annular quasiconvexity are necessarily found in metric measure spaces whose measure is doubling and supports a p-Poincaré inequality for sufficiently small $p \ge 1$ (namely, if $1 \le p < s$ where s is the "upper dimension" associated with the doubling property of the measure, see (2.3)). The fact that doubling metric measure spaces supporting a Poincaré inequality are quasiconvex was first proved by Semmes, see [9] or [6], and the corresponding result about annular quasiconvexity for spaces supporting a p-Poincaré inequality for sufficiently small p was proved by Korte in [17]. Thus the assumptions of quasiconvexity and annular quasiconvexity of (X, d, μ) in Theorem 1.1 are not overly restrictive. The facts that quasiconvexity and annular quasiconvexity are preserved under sphericalization and under flattening was first proved in [4], see Theorem 2.14 below. The assumption that X is annular quasiconvex cannot be removed in the above theorem, see Example 3.23 below. Note that quasiconvexity follows from annular quasiconvexity for connected spaces; however, for clarity of exposition, we have explicitly stated quasiconvexity as a separate assumption on the space.

The above theorem also gives us a way of constructing new metric measure spaces with doubling measure supporting a Poincaré inequality, from other doubling metric measure spaces supporting a Poincaré inequality. Hence, together with the results in [6] (persistence of doubling and validity of Poincaré inequality under measured Gromov-Hausdorff limits), [3] (boundaries of certain hyperbolic buildings), [19] (fractal constructions from Cantor sets cross an interval in the plane), and [20] (fat Sierpinski carpets), the results of this paper gives us a way of identifying more metric measure spaces that have a doubling measure supporting a Poincaré inequality. For example, flattening of the fat Sierpinski carpets of [20] give an unbounded fractal space that supports a Poincaré inequality.

The natural identification between the metric space X and its sphericalization (or flattening) is known only to be quasimöbius, see for example [4]. It is known (at least amongst the class of Ahlfors Q-regular metric measure spaces with the same index Q) that quasiconformal mapping between metric spaces preserve the property of validity of a Poincaré inequality. More specifically, it is known that if $F : X \to Y$ is a quaisconformal mapping between two Ahlfors Q-regular metric measure spaces, then if one of them support a p-Poincaré inequality for $p \leq Q$ (resp. $p > Q$), then the other also satisfies a p'-Poincaré inequality for some $p' \leq Q$ (resp. $p' > Q$), see [12]. Similar invariance results are not known for quasimöbius mappings. Hence the proof of Theorem 1.1 is not trivial.

While most results in analysis on metric spaces require only the doubling property of the measure and the validity of a Poincaré inequality, some results (such as those related to quasiconformal mappings) are known only to hold when the measure is in addition Ahlfors regular; see for example the results in [12], [11], [1], and [18]. Hence in this paper we also consider the question of whether Ahlfors regularity is preserved under sphericalization and flattening procedures, see Proposition 4.1 and Proposition 3.2.

In Section 2 we describe the basic concepts and notions used in this paper. In Section 3 we consider the preservation of the geometric properties of Ahlfors regularity, doubling, and Poincaré inequality under sphericalization, while in Section 4 we prove the preservation of these properties under flattening. A combination of these results yield the validity of Theorem 1.1 above.

2 Basic concepts

In this section we gather some basic concepts of analysis on metric spaces such as, quasiconvexity, doubling property, Ahlfors regularity, Poincaré inequality, that are the main objects of study in this paper. For a detailed description of these concepts the reader is referred to [10].

Let (X, d) be a metric space. An open ball centered at $x \in X$ and of radius $r > 0$ is $B(x, r) = \{y \in X : d(x, y) < r\}$ and the closed ball is $B(x, r) = \{y \in X : d(x, y) \leq r\}$. Note that the closed ball contains the topological closure of the open ball, but could potentially be a larger set. If B is a ball centered at x and of radius r and $\lambda > 0$, we denote by λB the ball centered at x and of radius λr. Given $x \in X$ and $A \subset X$, the distance from x to A is denoted by $\operatorname{dist}(x, A)$. The distance between the sets $A, B \subset X$ is denoted by $\operatorname{dist}(A, B)$. The diameter of a set $A \subset X$ is denoted by $\operatorname{diam}(A)$. The space X is said to be *proper* if the closed bounded subsets of X are compact.

The space X is said to be *locally compact* if given x in X, there is a neighborhood V of x such that $\overline{V}$ is compact. If X is unbounded, we let $\dot{X} = X \cup \{\infty\}$ be its one-point extension. The topology on $\dot{X}$ is the union of the topology on X and the collection of all sets that are complements in $\dot{X}$ of compact subsets of X. Recall that if X is locally compact, then $\dot{X}$ is the one-point compactification of X (see, for instance, [21, Theorem 29.1]).

2.1 Conditions on measures

Let (X, d, μ) be a metric measure space, with μ a Borel measure on X. A measure is a Borel measure if open subsets of X are measurable and every set is contained in a Borel set with the same measure. The space (X, μ) is said to be *Ahlfors Q-regular* if $Q > 0$ and

$$\frac{1}{C} R^Q \leq \mu(B_R) \leq C R^Q \tag{2.1}$$

for some $C \geq 1$ and for all closed balls B_R of radius $0 < R < \operatorname{diam}(X)$. If (X, μ) is Ahlfors Q-regular, then the Hausdorff dimension of X is equal to Q. In particular, if the measure μ in (2.1) is not specified, then it is understood that μ is the Hausdorff measure. The space (X, μ) is *doubling* if balls have finite positive measure and there is a constant $C_\mu \geq 1$ such that

$$\mu(2B) \leq C_\mu \mu(B) \tag{2.2}$$

for all balls B. Ahlfors regular measures are necessarily doubling. Condition (2.2) implies that there are constants $C > 0$ and $s > 0$, depending only on C_μ, such that

$$\frac{\mu(B(x,r))}{\mu(B(y,R)} \geq C\left(\frac{r}{R}\right)^s \tag{2.3}$$

whenever $0 < r \leq R$ and $x \in B(y,R)$. See [10] for a proof of this.

2.2 Quasiconvexity

By a *curve* in X we mean a continuous map $\gamma\colon I \to X$, where $I \subset \mathbb{R}$ is an interval. When $I = [a,b]$ for some $a, b \in \mathbb{R}$ with $a < b$, the *length* $l_d(\gamma)$ of γ with respect to the metric d is defined by

$$\ell_d(\gamma) = \sup \sum_{k=0}^{n-1} d(\gamma(t_k), \gamma(t_{k+1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$. We say that γ is *rectifiable* if $\ell_d(\gamma) < \infty$. We simply write $\ell(\gamma)$ if the metric is clear from the context.

The space metric (X,d) is said to be *quasiconvex* if there is a constant $C \geq 1$ such that each pair of points x and y in the space can be joined by a curve γ with

$$\ell_d(\gamma) \leq C d(x,y). \tag{2.4}$$

We also say that (X,d) is C*-quasiconvex*. We say that X is *annular quasiconvex* if there is a constant $A \geq 1$ such that whenever $x \in X$ and $r > 0$, for each pair of points $x, y \in B(x,r) \setminus B(x,r/2)$ there is a curve $\gamma_{x,y}$ connecting x to y such that $\gamma_{x,y}$ lies in the annulus $B(x,Ar) \setminus B(x,r/A)$ and $\ell_d(\gamma_{x,y}) \leq A\, d(x,y)$.

This concept was introduced in [17] and has been used in [15] and [4]. It was shown in [4] that quasiconvexity and annular quasiconvexity are preserved under sphericalization, see Theorem 2.14 below.

2.3 Upper gradients and Poincaré inequality

For a rectifiable curve $\gamma\colon [a,b] \to X$ let $s_\gamma\colon [a,b] \to [0,\ell(\gamma)]$ be the associated *length function*. That is, $s_\gamma(t) = \ell(\gamma|[a,t])$. There exists a unique (1-Lipschitz continuous) map $\gamma_s\colon [0,\ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve γ_s is called the *arc length parametrization* of γ. If γ is a rectifiable curve in X, the line integral over γ of a Borel function $\rho\colon X \to [0,\infty]$ is defined by

$$\lambda_\rho(\gamma) = \int_\gamma \rho ds = \int_0^{\ell(\gamma)} (\rho \circ \gamma_s)(t) dt. \tag{2.5}$$

We also refer to $\lambda_\rho(\gamma)$ as the *ρ-length* of γ. If $\rho \equiv 1$, then $\lambda_\rho(\gamma) = \ell_d(\gamma)$ is the length of γ with respect to the metric d. Given a real-valued function u in a metric space X, a Borel function $g\colon X \to [0,\infty]$ is an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_\gamma g ds \tag{2.6}$$

for each rectifiable curve γ joining x and y in X.

Let $1 \leq p < \infty$. Suppose also that each ball in the metric measure space (X, d, μ) has finite and positive measure. We say that X admits a *p-Poincaré inequality* if there is a constant $C \geq 1$ such that

$$\fint_B |u - u_B| d\mu \leq C \mathrm{diam}(B) \Big(\fint_B g^p d\mu \Big)^{1/p} \tag{2.7}$$

for all balls B in X, for all measurable functions u on B, and for all upper gradients g of u. Here, and in what follows, we use the notation

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu$$

for the mean value of u in B.

2.4 Sphericalization and flattening

The goal of this paper is to study whether certain properties of a given metric measure space are preserved under sphericalization. Classical sphericalization of the Euclidean space yields the sphere equipped with the chordal metric. The sphericalization of a given metric space is modeled upon that of the Euclidean space. One of the two goals of this paper is to show that properties such as doubling of the measure and Ahlfors regularity of the measure are preserved under sphericalization and that if the measure is doubling and (X, d) is quasiconvex and annular quasiconvex, then the property of support of p-Poincaré inequality ($1\,ep < \infty$) is also preserved under sphericalization. The second goal of this paper is to show similar invariance of these properties under flattening. Thus it would follow that a quasiconvex annular quasiconvex metric measure space (X, d, μ) is doubling/Ahlfors regular and supports a p-Poincaré inequality if and only if $(\dot{X}, \hat{d}_a, \mu_a)$ has these properties.

Recall that the sphericalization of the Euclidean space $\mathbb{R}^n$ is the one-point compactification of $\mathbb{R}^n$ equipped with the chordal metric χ given by

$$\chi(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}},$$

where for each $x \in \mathbb{R}^n$ the value $\chi(x, \infty) = 2(1 + |x|^2)^{-1/2}$. Interpreting $|x|$ as the distance (in the original Euclidean metric) of x from the special point

0, we consider the following spherical metric d_a on $X \cup \{\infty\}$, for some fixed special point $a \in X$. Recall that $\dot{X} = X \cup \{\infty\}$ is equipped with the one-point compactification topology.

Definition 2.8. *Given (X, d) and a base point $a \in X$, a density function $d_a \colon \dot{X} \times \dot{X} \to [0, \infty)$ is defined by*

$$d_a(x, y) = \begin{cases} \frac{d(x,y)}{[1+d(x,a)][1+d(y,a)]} & \text{if } x, y \in X, \\ \frac{1}{1+d(x,a)} & \text{if } x \in X,\ y = \infty, \\ 0 & \text{if } x = \infty = y. \end{cases} \tag{2.9}$$

There exists a metric $\hat{d}_a$ on $\dot{X}$ whose induced topology agrees with the topology of $\dot{X}$ satisfying

$$\frac{1}{4} d_a(x, y) \le \hat{d}_a(x, y) \le d_a(x, y) \tag{2.10}$$

for all $x, y \in \dot{X}$ (see [1, Lemma 2.2]). This metric is obtained by

$$\hat{d}_a(x, y) = \inf_{(x = x_0, x_1, \cdots, x_k, x_{k+1} = y)} \sum_{j=0}^{k} d_a(x_j, x_{j+1}),$$

where the infimum is taken over all finite sequences $(x = x_0, x_1, \cdots, x_k, x_{k+1} = y)$ from $\dot{X}$.

Definition 2.11. *The metric space $(\dot{X}, \hat{d}_a)$ is said to be the sphericalization of (X, d). Since $\hat{d}_a$ is 4-biLipschitz equivalent to d_a, and since there is no closed form formula for $\hat{d}_a$, for convenience we will use d_a in defining balls in $\dot{X}$. Balls in $\dot{X}$, with respect to d_a, will be denoted $B_a = B_a(x, r)$, while the balls in X, with respect to the original metric d, will be denoted $B = B(x, r)$.*

Given that (X, d) is a rectifiably connected unbounded metric space, set $\rho_0 \colon \dot{X} \to [0, \infty]$ to be the Borel function defined by

$$\rho_0(x) = \frac{1}{[1 + d(x, a)]^2}. \tag{2.12}$$

Given a rectifiable curve γ in X, let $\gamma_s \colon [0, \ell(\gamma)] \to X$ be its arc length parametrization with respect to the original metric d. Under sphericalization γ corresponds to $\dot{\gamma} \colon [0, \ell(\gamma)] \to \dot{X}$ defined by $\dot{\gamma}(t) = \gamma_s(t)$. If γ is a curve in X, by an abuse of notation we will denote the corresponding curve in $\dot{X}$ also by γ. The length $\ell_{\hat{d}_a}(\gamma)$ of $\dot{\gamma}$ with respect to the metric $\hat{d}_a$ is equal to $\lambda_{\rho_0}(\gamma)$ given by (2.10), that is,

$$\ell_{\hat{d}_a}(\gamma) = \lambda_{\rho_0}(\gamma) = \int_\gamma \rho_0 ds = \int_0^{\ell(\gamma)} \frac{1}{[1 + d(\gamma_s(t), a)]^2} ds(t) \le \ell(\gamma). \tag{2.13}$$

In particular, γ is rectifiable with respect to the metric $\hat{d}_a$ if it is rectifiable with respect to the original metric d.

Theorem 2.14 (Buckley–Herron–Xie). *Suppose (X,d) is A-quasiconvex and annular A-quasiconvex. Then $(\dot{X},\hat{d}_a)$ is A'-quasiconvex and is A'-annular quasiconvex, where A' depends only on A and a. Conversely, if $(\dot{X},\hat{d}_a)$ is A-quasiconvexy and A-annular quasiconvex for some $a\in X$, then there is a constant A'', depending only on A and a, such that (X,d) is A''-quasiconvex and A''-annular quasiconvex.*

Note that if X is proper (that is, closed and bounded subsets of X are compact), then $\dot{X}$ is compact. Furthermore, $\operatorname{diam}(\dot{X})=1$.

Suppose now in addition that X is also equipped with a Borel-regular measure μ such that measures of non-empty open sets are positive and measures of bounded sets are finite. Then there is an analogous measure on $\dot{X}$, given as follows: given a set $A\subset\dot{X}$,

$$\mu_a(A):=\int_{A\setminus\{\infty\}}\frac{1}{\mu(B(a,1+d(z,a)))^2}\,d\mu(z). \tag{2.15}$$

We will show in Proposition 3.6 that if μ is doubling, then so is μ_a, from which it will follow that because $\dot{X}$ is bounded, $\mu_a(\dot{X})$ is finite. We will show here that under a more restrictive hypothesis that X is path-connected and μ is doubling, then $\mu_a(\dot{X})<\infty$. Indeed, setting $B_i=B(a,2^i)$ for $i\in\mathbb{N}\cup\{0\}$ and $B_{-1}=\emptyset$, we have

$$\begin{aligned}
\mu_a(\dot{X}) &= \int_X \frac{1}{\mu(B(a,1+d(z,a)))^2}\,d\mu(z)\\
&= \sum_{i=0}^{\infty}\int_{B_i-B_{i-1}}\frac{1}{\mu(B(a,1+d(z,a)))^2}\,d\mu(z)\\
&\le \frac{\mu(B(a,2)}{\mu(B(a,1))^2}+\sum_{i=2}^{\infty}\int_{B_i-B_{i-1}}\frac{1}{\mu(B(a,1+d(z,a)))^2}\,d\mu(z)\\
&\le \frac{C_\mu}{\mu(B(a,1))}+\sum_{i=1}^{\infty}\int_{B_i-B_{i-1}}\frac{1}{\mu(B(a,2^{i-1}))^2}\,d\mu(z)\\
&\le \frac{C_\mu}{\mu(B(a,1))}+\sum_{i=1}^{\infty}\frac{\mu(B(a,2^i))}{\mu(B(a,2^{i-1}))^2}\\
&\le \frac{C_\mu}{\mu(B(a,1))}+\frac{C_\mu^2}{\mu(B(a,1))}\sum_{i=1}^{\infty}\frac{\mu(B(a,1))}{\mu(B(a,2^i))}
\end{aligned}$$

Since X is path-connected, there are constants $C_1>0$ and $\alpha>0$ such that whenever $x\in X$ and $0<r\le R$, we have

$$\frac{\mu(B(x,r))}{\mu(B(x,R))}\le C_1\left(\frac{r}{R}\right)^\alpha.$$

Using this in the above, we have

$$\mu_a(\dot{X})\le\frac{C_\mu}{\mu(B(a,1))}+\frac{C_1\,C_\mu^2}{\mu(B(a,1))}\sum_{i=1}^{\infty}2^{-i\alpha}<\infty.$$

So far we have considered sphericalization of a metric space. Now we turn to the reciprocal procedure of flattening.

Definition 2.16. *Given (X, d) and a base point $c \in X$, a density function $d^c\colon X^c \times X^c \to [0, \infty]$ is defined by*

$$d^c(x, y) = \frac{d(x, y)}{d(x, c)d(y, c)} \text{ if } x, y \in X^c := X \setminus \{c\}. \tag{2.17}$$

According to [4], we have a metric space $(X_c, \bar{d})$ associated to d_c such that

$$\frac{1}{4} d^c(x, y) \leq \bar{d}(x, y) \leq d^c(x, y)$$

for all $x, y \in X_c$(see [4],Lemma 3.2). The metric is defined by

$$\bar{d}(x, y) = \inf_{(x=x_0, x_1, \cdots, x_k, x_{k+1}=y)} \sum_{j=0}^{k} d^c(x_j, x_{j+1})$$

Definition 2.18. *The metric space $(X^c, \bar{d})$ is said to be flattening of (X, d). Balls in X^c, with respect to the "metric" d_c, will be denoted $B_c(x, r)$, while the balls in X, with respect to the metric d, will be denoted $B(x, r)$.*

Given a measure μ on X, there is an associated measure μ^c on X^c, given by

$$\mu^c(A) := \int_A \frac{1}{\mu(B(c, d(c, z)))^2} \, d\mu(z) \tag{2.19}$$

whenever $A \subset X^c$ is a Borel set.

Under the flattening procedure the length of a curve is changed as follows. Given a curve $\gamma : [a, b] \to X^c$ that is rectifiable in the metric d (note that $X^c \subset X$), the length of γ in the new metric $\bar{d}$ is given by

$$\ell_{d_c}(\dot{\gamma}) = \int_0^{\ell(\gamma)} \frac{1}{d(\gamma_s(t), c)^2} \, ds(t),$$

where γ_s is the arc-length reparametrization, in the metric d, of γ. The following theorem is also from [4].

Theorem 2.20 (Buckley-Herron-Xie)**.** *If (X, d) is A-quasiconvex and annular A-quasiconvex and $c \in X$, then $(X^c, \bar{d})$ is A'-quasiconvex and A'-annular quasiconvex, with A' depending solely on A. Conversely, if $(X^c, \bar{d})$ is A-quasiconvex and annula A-quaisconvex, then (X, d) is A''-quasiconvex and annular A''-quasiconvex, with A'' depending soely on A.*

2.5 Standing assumptions

Throughout this paper we will assume that (X, d, μ) is a metric measure space with (X, d) complete and proper, A-quasiconvex, and annular A-quaisconvex, and that μ is a Borel regular measure such that non-empty open sets have positive measure and bounded sets have finite measure. We also fix a special point $a \in X$.

We will denote constants whose particular values are of no particular interest to us by C. One should keep in mind that each occurrence of C, even within the same line, might have a different value from the other occurrences, though they might be related.

3 Preservation of bounded geometry under sphericalization

The goal in this section is to show that properties of X such as Ahlfors regularity and doubling property are preserved under sphericalization, and that the validity of a p-Poincaré inequality also is preserved under additional connectivity assumptions on X and the doubling property of μ. We first consider the Ahlfors regularity property.

3.1 Preservation of Ahlfors Q-regularity

If the measure μ on X is Ahlfors Q-regular, then we can define a measure μ_a of balls in $\dot{X}$ by

$$\mu_a(B_a(x,r)) = \int_{B_a(x,r)} \rho d\mu = \int_{B_a(x,r)} \frac{1}{[1+d(y,a)]^{2Q}} d\mu(y). \tag{3.1}$$

This measure is comparable to the measure defined in (2.15), and so there is no inconsistency in using the above definition of μ_a in this subsection. Here, and in what follows, we use the notation $B_a(x,r) = \{y \in \dot{X} \colon d_a(x,y) < r\}$.

Proposition 3.2. *Suppose that μ is Ahlfors Q-regular and X is path-connected. Then μ_a also is Ahlfors Q-regular on the metric space $(\dot{X}, \hat{d}_a)$.*

Proof. Since $\dot{X}$ is compact, to verify Ahlfors regularity, we may without loss of generality we can assume that $r \le 1/2$. Set $R = 1/r - 1$. Then $1/2r \le R < 1/r$.

First, we consider a ball centered at ∞. We have

$$\begin{aligned} B_a(\infty, r) &= \left\{ x \in X \colon \frac{1}{1+d(x,a)} < r \right\} \cup \{\infty\} \\ &= \left\{ x \in X \colon d(x,a) > \frac{1}{r} - 1 \right\} \cup \{\infty\} = \dot{X} \setminus \overline{B}(a,R). \end{aligned}$$

Put $B_i = B(a, 2^i R)$, $i = 0, 1, 2, \ldots$. Then

$$\mu_a(B_a(\infty,r)) = \sum_{i=1}^{\infty} \mu_a(B_i \backslash B_{i-1}) = \int_{B_i \backslash B_{i-1}} \frac{1}{[1+d(y,a)]^{2Q}} d\mu(y) \approx \frac{\mu(B_i \setminus B_{i-1})}{[1+2^i R]^{2Q}}.$$

Here, we used the fact that because X is path-connected, $\mu(B_i \setminus B_{i-1}) \approx \mu(B_i)$.

Hence, because $R \geq 1$, we have

$$\mu_a(B_a(\infty, r)) \approx \sum_{i=1}^{\infty} \frac{\mu(B_i \setminus B_{i-1})}{[1 + 2^i R]^{2Q}} \approx \sum_{i=1}^{\infty} \frac{(2^i R)^Q}{(1 + 2^i R)^{2Q}}$$
$$\approx \sum_{i=1}^{\infty} \frac{1}{(2^i R)^Q} \approx \frac{1}{R^Q} \sum_{i=1}^{\infty} 2^{-iQ} \approx \frac{1}{R^Q} \approx r^Q,$$

as required.

Next we need to consider $x \in \dot{X}$ such that $x \neq \infty$. We break the proof into four cases.

Case 1: $d_a(x, \infty) \leq \frac{r}{2}$. Then

$$\frac{1}{1 + d(x, a)} \leq \frac{r}{2}, \text{ that is, } d(x, a) > \frac{1}{r}.$$

If $y \in B_a(x, r)$, then by (2.10) and the triangle inequality of $\hat{d}$, we see that $d_a(y, \infty) \leq 32r$. Therefore $B_a(x, r) \subset B_a(\infty, 32r)$. Similarly, we also have $B_a(\infty, r/32) \subset B_a(x, r)$, since $d_a(x, \infty) < r/2$. It follows from the above argument that

$$\frac{1}{C} r^Q \leq \mu_a(B_a(\infty, r/32)) \leq \mu_a(B_a(x, r)) \mu_a(B_a(\infty, 32r)) \leq C\, r^Q.$$

Case 2: $r/2 < d_a(x, \infty) < 4r$. As in the discussion above, $B_a(x, r) \subset B_a(\infty, 512r)$, and so $\mu(B_a(x, r)) \leq C r^Q$, for some $C > 0$. To obtain a lower bound, we argue as follows. We have by the assumption of this case, $(4r)^{-1} \leq 1 + d(x, a) \leq 2\, r^{-1}$. Let $C_0 > 4$. Then for $y \in B(x, \frac{1}{C_0 r})$,

$$1 + d(y, a) \leq 1 + d(x, a) + d(x, y) \leq \frac{2}{r} + \frac{1}{C_0 r} < \frac{3}{r},$$

and

$$1 + d(y, a) \geq 1 + d(x, a) - d(x, y) \geq \frac{1}{4r} - \frac{1}{C_0 r} = \frac{C_0 - 4}{4 C_0 r}.$$

So

$$d_a(x, y) = \frac{d(x, y)}{(1 + d(x, a))(1 + d(y, a))} < \frac{1}{C_0 r}\, 4r\, \frac{4 C_0 r}{C_0 - 4}.$$

Hence, if $C_0 = 20$, then $d_a(x, y) < r$, and it follows that $B(x, \frac{1}{20r}) \subset B_a(x, r)$. Therefore, by the Ahlfors Q-regularity of μ and the above estimates,

$$\mu_a(B_a(x, r)) \geq \int_{B(x, 1/20r)} \frac{1}{[1 + d(y, a)]^{2Q}}\, d\mu(y) \geq \frac{r^{2Q}}{3^{2Q}}\, \mu(B(x, \tfrac{1}{20r})) \geq \frac{1}{C}\, r^Q.$$

Case 3: $d_a(x, \infty) \geq 4r$ and $d(x, a) \leq 1$. Then for $y \in B_a(x, r)$,

$$\frac{4r\, d(x, y)}{1 + d(y, a)} \leq \frac{d(x, y)}{(1 + d(x, a))(1 + d(y, a))} < r,$$

that is, $4\,d(x,y) \le 1 + d(y,a)$. It follows that $4[1 + d(y,a) - 1 - d(x,a)] \le 1 + d(y,a)$, and it follows from $d(x,a) \le 1$ that $1 \le 1 + d(y,a) < 3$. Hence if $y \in B_a(x,r)$, we have that

$$r > \frac{d(x,y)}{(1+d(x,a))(1+d(y,a))} \ge \frac{1}{6}\, d(x,y),$$

that is, $B_a(x,r) \subset B(x,6r)$. Therefore, by the fact that $1 + d(y,a) \ge 1$ and by the Ahlfors Q-regularity of μ,

$$\mu_a(B_a(x,r)) \le \int_{B(x,6r)} \frac{1}{[1+d(y,a)]^{2Q}}\, d\mu(y) \le \mu(B(x,6r)) \le C\, r^Q.$$

Furthermore, if $y \in B(x,r)$, then $d_a(x,y) \le d(x,y) < r$, that is, $B(x,r) \subset B_a(x,r)$. Hence, by the Ahlfors Q-regularity of μ and by the above estimates,

$$\mu_a(B_a(x,r)) \ge \int_{B(x,r)} \frac{1}{[1+d(y,a)]^{2Q}}\, d\mu(y) \ge \frac{1}{3^{2Q}}\, \mu(B(x,r)) \ge \frac{1}{C}\, r^Q.$$

Case 4: $d_a(x,\infty) \ge 4r$ and $d(x,a) > 1$. In this case, as in Case 3 we know that $4\,d(x,y) < 1 + d(y,a)$ when $y \in B_a(x,r)$. Therefore, $4[1+d(y,a)-1-d(x,a)] < 1 + d(y,a)$, that is,

$$1 + d(y,a) < \frac{4}{3}\,[1 + d(x,a)].$$

Similarly, because $4[1 + d(x,a) - 1 - d(y,a)] < 1 + d(y,a)$, we obtain

$$\frac{4}{5}\,[1 + d(x,a)] < 1 + d(y,a).$$

Set

$$\Lambda(x) := \{y \in X \,:\, 4\,[1 + d(x,a)]/5 < 1 + d(y,a) < 4\,[1 + d(x,a)]/3\}.$$

We have from the above argument that $B_a(x,r) \subset \Lambda(x)$ and that when $y \in B_a(x,r)$,

$$d(x,y) < r[1 + d(x,a)][1 + d(y,a)] < \frac{4}{3}\, r\,[1 + d(x,a)]^2.$$

Conversely, if $y \in \Lambda(x) \cap B(x, 4r[1 + d(x,a)]^2/5)$, then

$$d_a(x,y) = \frac{d(x,y)}{[1+d(x,a)][1+d(y,a)]} < \frac{4r}{5}[1+d(x,a)]^2 \frac{5}{4[1+d(x,a)]^2} = r.$$

It follows that

$$\Lambda(x)\cap B(x,4r[1+d(x,a)]^2/5) \subset B_a(x,r) \subset \Lambda(x)\cap B(x,4r[1+d(x,a)]^2/3). \quad (3.3)$$

Hence

$$\begin{aligned}\mu_a(B_a(x,r)) &\le \int_{\Lambda(x)\cap B(x,4r[1+d(x,a)]^2/3)} \frac{1}{[1+d(y,a)]^{2Q}}\, d\mu(y)\\ &\le \left(\frac{5}{4}\right)^{2Q} \frac{1}{[1+d(x,a)]^{2Q}}\, \mu(B(x,4r[1+d(x,a)]^2/3))\\ &\le C\, r^Q.\end{aligned}$$

On the other hand, if $y \in B(x, 4r[1+d(x,a)]^2/5)$, then by the fact that $d_a(x,\infty) \geq 4r$ (and hence $4r[1+d(x,a)] \leq 1$), we see that

$$\begin{aligned}\frac{4}{5}[1+d(x,a)] < 1+d(x,a)-d(x,y) \leq 1+d(y,a) &\leq 1+d(x,a)+d(x,y)\\ &< \frac{6}{5}[1+d(x,a)]\\ &< \frac{4}{3}[1+d(x,a)],\end{aligned}$$

that is, $B(x, 4r[1+d(x,a)]^2/5) \subset \Lambda(x)$. It follows from (3.3) that

$$\mu_a(B_a(x,r)) \geq \int_{B(x,4r[1+d(x,a)]^2/5)} \frac{1}{[1+d(y,a)]^{2Q}}\, d\mu(y) \geq \frac{1}{C}\, r^Q.$$

So we have proved the Ahlfors regularity of the sphericalization of the metric measure space. □

3.2 Preservation of doubling property

We now turn our attention to measures μ that are doubling on X but might not be Ahlfors regular. It is natural to ask whether doubling property carries over to the sphericalized space. The goal of this subsection is to answer this in the affirmative.

Suppose that μ is a doubling measure. We define the measure μ_a in the sphericalization $\dot{X}$ by

$$\mu_a(A) := \int_A \rho_a d\mu = \int_A \frac{1}{\mu(B(a, 1+d(y,a)))^2} d\mu(y). \tag{3.4}$$

Note that when μ is Ahlfors Q-regular, the above definition agrees with that of (3.1). To prove that doubling property is preserved under sphericalization, we need the following lemma. The conclusion of this lemma would be direct if we assume in addition that X is a path-connected space, or even uniformly perfect. The goal of this lemma is to avoid this additional assumption (note that in general path-connectivity is a consequence of the validity of a p-Poincaré inequality).

Lemma 3.5. *Let (X, d, μ) be a metric space with μ doubling. Then for any ball $B = B(x, r_0)$ with $\mu(B(x,r_0)) \leq \frac{1}{C_\mu}\mu(X)$, we can find some $r_1 \geq r_0$ such that*

$$C_\mu\, \mu(B(x,r_0)) < \mu(B(x,r_1)) \leq C_\mu^3\, \mu(B(x,r_0)).$$

which C_μ is the doubling constant.

Proof. Let $r_1 = 2\sup\{r \geq r_0 : \mu(B(x,r)) \leq C_\mu\, \mu(B(x,r_0))\}$. Then

$$C_\mu\, \mu(B(x,r_0)) < \mu(B(x,r_1)) \leq C_\mu^2\, \mu(B(x,r_1/4)) \leq C_\mu^3\, \mu(B(x,r_0)).$$

□

Proposition 3.6. *Suppose that μ is a doubling measure on (X,d). That is, $\mu(2B) \le C_\mu \mu(B)$ for all balls B in X. Then μ_a is a doubling measure on $\dot{X}$.*

Proof. Consider a ball $B_a(x,R)$. Since $\dot{X}$ is compact, it suffices to prove the doubling condition for balls of radius R that satisfies $0 < R < 1/32$.

Case 1: $d_a(x,\infty) \ge 4R$. Then

$$4R \le d_a(x,\infty) \text{ if and only if } 1 + d(x,a) \le \frac{1}{4R}. \tag{3.7}$$

For $y \in B_a(x,2R)$, we have

$$d_a(x,y) = \frac{d(x,y)}{(1+d(x,a))(1+d(x,a))} < 2R,$$

and so by (3.7),

$$1 + d(y,a) - 1 - d(x,a) \le d(x,y) < 2R[1+d(x,a)][1+d(y,a)] \le \frac{1+d(y,a)}{2},$$

that is, $1 + d(y,a) < 2[1+d(x,a)]$. Similarly, we obtain

$$1 + d(x,a) - 1 - d(y,a) < \frac{1+d(y,a)}{2},$$

that is, $2[1+d(x,a)] < 3[1+d(y,a)]$. Combining these two conclusions, we see that whenever $y \in B_a(x,2R)$,

$$\frac{2}{3}[1+d(x,a)] < 1 + d(y,a) < 2[1+d(x,a)] \le \frac{1}{2R}.$$

Therefore, when $y \in B_a(x,2R)$,

$$\frac{1}{2}\frac{d(x,y)}{[1+d(x,a)]^2} \le d_a(x,y) = \frac{d(x,y)}{[1+d(x,a)][1+d(y,a)]} \le \frac{3}{2}\frac{d(x,y)}{[1+d(x,a)]^2},$$

and it follows that

$$\begin{aligned}\mu_a(B_a(x,2R)) = \int_{B_a(x,2R)} \frac{1}{\mu(B(a,d(y,a)+1))^2} d\mu(y) &\le \frac{\mu(B_a(x,2R))}{\mu(B(a,2[1+d(x,a)]/3))^2} \\ &\le \frac{\mu(B(x,4R[1+d(x,a)]^2))}{\mu(B(a,2[1+d(x,a)]/3))^2}.\end{aligned}$$

Similarly, for $y \in B_a(x,R)$, we have

$$d(x,y) < R[1+d(x,a)][1+d(y,a)] \le \frac{1+d(y,a)}{4},$$

and hence

$$\frac{4}{5}[1+d(x,a)] < 1 + d(y,a) < \frac{4}{3}[1+d(x,a)].$$

It follows that when $y \in B(x,R)$,

$$\frac{3}{4}\frac{d(x,y)}{[1+d(x,a)]^2} \le d_a(x,y) \le \frac{5}{4}\frac{d(x,y)}{[1+d(x,a)]^2}.$$

Hence,

$$\mu_a(B_a(x,R)) = \int_{B_a(x,R)} \frac{1}{\mu(B(a,d(y,a)+1))^2} d\mu(y) \geq \frac{\mu(B_a(x,R))}{\mu(B(a,4[1+d(x,a)]/3))}$$
$$\geq \frac{\mu(B(x,4R[1+d(x,a)]^2/5))}{\mu(B(a,4[1+d(x,a)]/3))}$$
$$\geq \frac{1}{C_\mu^5} \frac{\mu(B(x,4R[1+d(x,a)]^2))}{\mu(B(a,2[1+d(x,a)]/3))^2}$$
$$\geq \mu_a(B_a(x,2R)).$$

This completes the proof of the doubling property in this case.

Case 2: $d_a(x,\infty) \leq \frac{R}{2}$. Then $1+d(x,a) \geq 2/R$. In this case, we want to find some spherical balls $B_a(\infty, KR)$ and $B_a(\infty, R/L)$ such that

$$B_a(\infty, R/L) \subset B_a(x,R) \subset B_a(x,2R) \subset B_a(\infty, KR)$$

and we want to choose such positive numbers K, L independently of x, R. Since $1+d(x,a) \geq 2/R$, if $d_a(y,\infty) \leq R/L$, then

$$d_a(x,y) = \frac{d(x,y)}{(1+d(x,a))(1+d(y,a))} \leq \frac{1+d(x,a)+1+d(y,a)}{(1+d(x,a))(1+d(y,a))} \leq R/2 + R/L.$$

Therefore the choice of $L = 2$ gives $B_a(\infty, R/2) \subset B_a(x,R)$ as desired.

Next, suppose $y \in B_a(x,2R)$. Then $d(x,y) < 2R(1+d(x,a))(1+d(y,a))$, and so we have

$$\frac{1}{1+d(y,a)} - \frac{1}{1+d(x,a)} = \frac{(1+d(x,a))-(1+d(y,a))}{(1+d(x,a))(1+d(y,a)} \leq \frac{d(x,y)}{(1+d(x,a))(1+d(y,a)}$$
$$< 2R,$$

so we have

$$\frac{1}{1+d(y,a)} < 2R + \frac{1}{1+d(x,a)} \leq 3R.$$

It follows that $B_a(x,2R) \subset B_a(\infty, 3R)$, that is,

$$B_a(\infty, R/2) \subset B_a(x,R) \subset B_a(x,2R) \subset B_a(\infty, 3R). \tag{3.8}$$

Note that for $y \in X$, $\frac{1}{1+d(y,a)} < \rho$ if and only if $d(y,a) > \frac{1}{\rho} - 1$. That is, when $0 < \rho < 1/2$,

$$X \setminus B(a, \tfrac{1}{\rho} - 1) = B_a(\infty, \rho) \setminus \{\infty\}. \tag{3.9}$$

We now apply Lemma 3.5 with $r_0 = \frac{1}{3R} - 1$. By an inductive application of Lemma 3.5, we can choose $r_1, r_2, \ldots$ such that

$$C_\mu \mu(B(a,r_n)) \leq \mu(B(a,r_{n+1})) \leq C_\mu^3 \mu(B(a,r_n)) \tag{3.10}$$

for $n = 0, 1, 2, \ldots$. Note that by the proof of Lemma 3.5 we have $r_{n+1} \geq 2r_n$, it follows that $\lim_n r_n = \infty$. Hence we have

$$B_a(\infty, 3R) \setminus \{\infty\} = \bigcup_{n=0}^{\infty} B(a, r_{n+1}) \setminus B(a, r_n),$$

and therefore by (3.9) and by (3.10), and by the fact that $C_\mu > 1$,

$$\begin{aligned}
\mu_a(B_a(\infty, 3R)) &= \int_{B_a(\infty,3R)} \frac{1}{\mu(B(a, d(y,a)+1))^2} d\mu(y) \\
&= \sum_{n=0}^{\infty} \int_{B(a,r_{n+1})\setminus B(a,r_n)} \frac{1}{\mu(B(a, d(y,a)+1))^2} d\mu(y) \\
&\leq \sum_{n=0}^{\infty} \frac{\mu(B(a, r_{n+1})) - \mu(B(a, r_n))}{\mu(B(a, r_n))^2} \\
&\leq \sum_{n=0}^{\infty} \frac{C_\mu^3 - 1}{C_\mu^n \, \mu(B(a, r_0))} \\
&\leq \frac{C}{\mu(B(a, \frac{1}{3R} - 1))} \leq \frac{C}{\mu(B(a, \frac{2}{R} - 6))} \leq \frac{C}{\mu(B(a, \frac{1}{R}))}.
\end{aligned}$$

In the last step above, we used the doubling property of μ. Similarly, in Lemma 3.5 we choose $s_0 = \frac{2}{R} - 1$ and $s_1, s_2, \ldots$ such that $C\mu(B(a, s_k)) \leq \mu(B(a, s_{k+1})) \leq C^3 \mu(B(a, s_k))$, and because

$$B_a(\infty, R) \setminus \{\infty\} = \bigcup_{i=0}^{\infty} B(a, s_{i+1}) \setminus B(a, s_i),$$

we obtain the following estimate:

$$\begin{aligned}
\mu_a(B_a(\infty, R/2)) &= \int_{B_a(\infty,R/2)} \frac{1}{\mu(B(a, d(y,a)+1))^2} d\mu(y) \\
&= \sum_{n=0}^{\infty} \int_{B(a,s_{n+1})\setminus B(a,s_n)} \frac{1}{\mu(B(a, d(y,a)+1))^2} d\mu(y) \\
&\geq \sum_{n=0}^{\infty} \frac{\mu(B(a, s_{n+1})) - \mu(B(a, s_n))}{\mu(B(a, 2s_{n+1}))^2} \\
&\geq \sum_{n=0}^{\infty} \frac{1 - C^{-3}}{C^{3n} \mu(B(a, s_0))} \geq \frac{1}{C\mu(B(a, \frac{2}{R} - 1))} \geq \frac{1}{C\, \mu(B(a, \frac{1}{R}))}.
\end{aligned}$$

Because $R < 1/32$, we have that $s_0 = 2R^{-1} - 1 > 63$, and so for each n we have that $s_n \geq s_0 > 2$. We used this fact in obtaining the third inequality in the above series of inequalities. Combining the above with the estimate for $\mu_a(B_a(\infty, R/2))$ and the nested balls identity (3.8) tells us that $\mu_a(B(x, 2R)) \leq C\, \mu_a(B(x, R))$.

Case 3: $R/2 \le d_a(x,\infty) \le 4R$. Then $\frac{1}{4R} \le 1+d(x,a) \le \frac{2}{R}$. For $y \in B_a(x, 8R)$, we have $d(x,y) \le 8R(1+d(x,a))(1+d(y,a))$. By triangle inequality,

$$(d(x,a)+1) - (d(y,a)+1) \le d(x,y) \le 8R(1+d(x,a))(1+d(y,a)).$$

It follows from the hypothesis of this case that $(4R)^{-1} \le 17[1+d(y,a)]$, that is,

$$B_a(x,8R) \subset B_a(\infty, 68R) = X \setminus B(a, (68R)^{-1} - 1) \cup \{\infty\}.$$

We apply Lemma 3.5 inductively, by first choosing $r_0 = \frac{1}{68R} - 1$ to obtain $r_1 \ge 2r_0$, and then use r_1 to obtain $r_2 \ge 2r_1$ and thus get a sequence $r_1, r_2, \ldots$ such that

$$C_\mu \mu(B(a,r_n)) \le \mu(B(a,r_{n+1})) \le C_\mu^3 \mu(B(a,r_n))$$

for $n = 0, 1, 2, \ldots$. Now we obtain the following estimate:

$$\begin{aligned}
\mu_a(B_a(\infty, 68R)) &= \sum_{n=0}^{\infty} \int_{B(a,r_{n+1})\setminus B(a,r_n)} \frac{1}{\mu(B(a,d(y,a)+1))^2} d\mu(y) \\
&\le \sum_{n=0}^{\infty} \frac{\mu(B(a,r_{n+1})) - \mu(B(a,r_n))}{\mu(B(a,r_n))^2} \\
&\le \sum_{n=0}^{\infty} \frac{(C_\mu^3 - 1)\mu(B(a,r_n))}{\mu(B(a,r_n))^2} \\
&\le \sum_{n=0}^{\infty} \frac{C_\mu^3 - 1}{\mu(B(a,r_n))} \\
&\le \sum_{n=0}^{\infty} \frac{C_\mu^3 - 1}{C_\mu^n \mu(B(a,r_0))} \le \frac{C}{\mu(B(a,r_0))} = \frac{C}{\mu(B(a,\frac{1}{68R}-1)}.
\end{aligned}$$

Next we want to find some $L > 0$ such that $B(x, 1/(LR)) \subset B_a(x, R/4)$. Since $d(x,a)+1 \ge 1/(4R)$, when $y \in B(x, 1/(LR))$ we see that

$$d(y,a)+1 \ge d(x,a) - d(x,y) + 1 \ge \frac{1}{4R} - \frac{1}{LR} \ge \frac{L-4}{4LR},$$

and so we have

$$\begin{aligned}
d(x,y) < \frac{1}{LR} &\le \frac{1}{LR}\frac{4LR\,[d(y,a)+1]}{L-4}\, 4R\,[d(x,a)+1] \\
&\le \frac{16R}{L-4}\,[1+d(y,a)]\,[1+d(x,a)].
\end{aligned}$$

It follows that

$$d_a(x,y) < \frac{16}{L-4} R.$$

Setting $L = 68$ we therefore conclude that $B(x, 1/(68R)) \subset B_a(x, R/4)$. We also have from the hypothesis of this case that $d(y,a)+1 \le d(x,a)+1+d(x,y) \le 3/R$, and hence

$$\begin{aligned}\mu_a(B_a(x,R)) \geq \mu_a(B_a(x,R/4)) &= \int_{B_a(x,R/4)} \frac{1}{\mu(B(a,d(y,a)+1))^2} d\mu(y) \\ &\geq \int_{B(x,\frac{1}{68R})} \frac{1}{\mu(B(a,d(y,a)+1))^2} d\mu(y) \\ &\geq \frac{\mu(B(x,\frac{1}{68R}))}{\mu(B(a,\frac{3}{R}))^2}.\end{aligned}$$

By the hypothesis of this case again, we have $d(x,a) \leq 2/R$, and so by the doubling property of μ we have $C\,\mu(B(x,(1/68R))) \geq \mu(B(a,(1/68R)))$. Therefore,

$$\begin{aligned}\mu_a(B_a(x,2R)) \leq \mu_a(B_a(x,8R)) &\leq C\mu_a(B_a(\infty,68R)) \\ &\leq \frac{C}{\mu(B(a,\frac{1}{68R}))} \\ &\leq C\,\frac{\mu(B(x,\frac{1}{68R}))}{\mu(B(a,\frac{3}{R}))^2} \leq C\mu_a(B_a(x,R)).\end{aligned}$$

□

3.3 Poincaré inequality

Now that we have demonstrated the preservation of doubling measure space under sphericalization, we with to investigate the preservation of the Poincaré Inequality. In addition to the standard assumptions outlined at the end of Section 2, we assume here that the measure μ is doubling. Then by the argument in the previous subsection, the sphericalized metric measure space $(\dot{X}, d_a, \mu_a)$ is also doubling, where

$$d\mu_a(y) = \frac{d\mu(y)}{\mu(B(a,d(y,a)+1))^2}$$

as defined in (3.4).

Assume that (X,d,μ) supports a p-Poincaré Inequality, that is, as in (2.7), there are constants $C,\lambda > 0$ such that for all $u \in \mathrm{Lip}_{loc}(X)$,

$$⨍_B |u(y) - u_B| d\mu(y) \leq C\ \mathrm{rad}(B)\Big(⨍_{\lambda B} g^p(y) d\mu(y)\Big)^{1/p}.$$

We want to show that the Lipschitz functions in $(\dot{X}, d_a, \mu_a)$ also supports a p-Poincaré Inequality.

First, given an upper gradient g of a function $u \in \mathrm{Lip}_{loc}(X)$, we need to find a transformation of g that yields an upper gradient of u in $\dot{X}$, an vice versa. This is the goal of the next lemma. Recall that because d is locally biLipschitz equivalent to $\hat{d}_a$ and to d_a, we know that if $u \in \mathrm{Lip}(\dot{X})$, then $u \in \mathrm{Lip}_{loc}(X)$, and so has an upper gradient in X .

Lemma 3.11. *Suppose that u is a Lipschitz function on $\dot{X}$. If g is an upper gradient of u in X, then the function $\hat{g}$ given by*

$$\hat{g}(x) = g(x)(1 + d(x,a))^2 \tag{3.12}$$

is an upper gradient of u in $\dot{X}$. Furthermore, if h is an upper gradient of u in $\dot{X}$, then the function $\check{h}$ given by

$$\check{h}(x) = \frac{h(x)}{(1 + d(x,a))^2} \tag{3.13}$$

is an upper gradient of u in X.

Proof. We will prove the first part of the lemma, the proof of the second part being similar – we leave that to the interested reader.

Given a compact rectifiable curve $\gamma \subset \dot{X} \setminus \{\infty\}$, we know that γ is also rectifiable in X. We want to choose $\hat{g}$ so that, with ds_a denoting the arc-length element in $(\dot{X}, \hat{d}_a)$ an ds the arc-length element in (X, d),

$$\int_\gamma \hat{g}\, ds_a = \int_\gamma g\, ds,$$

from which we would have the desired upper gradient inequality for γ in the metric of $\dot{X}$:

$$|u(x) - u(y)| \le \int_\gamma g\, ds = \int_\gamma \hat{g}\, ds_a.$$

Note that by (2.13),

$$ds_a(y) = \frac{ds(y)}{(1 + d(y,a))^2},$$

so the choice of $\hat{g}$ given in (3.12) satisfies the upper gradient inequality on γ with respect to the metric $\hat{d}_a$.

Now we consider a non-constant compact rectifiable curve in $\dot{X}$ that intersects ∞. Then the set $\gamma^{-1}(\{\infty\})$ is a closed subset of the domain interval of γ. Without loss of generality, by cutting the curve up into two segments if necessary, that $\gamma : [a,b] \to \dot{X}$ satisfies $\gamma(a) = \infty$ and $\gamma(b) \in \dot{X} \setminus \{\infty\} = X$. Let $t_0 = \sup\{t \in [a,b] : \gamma(t) = \infty\}$. Then $\gamma(t_0) = \infty$ and $\gamma|_{(t_0,b]} \subset \dot{X} \setminus \{\infty\}$. We can find a monotone decreasing sequence of numbers t_k with $t_0 < t_k < b$ such that $\lim_k t_k = t_0$. Then $\gamma|_{[t_k,b]}$ is a compact rectifiable curve in X, and hence by the above paragraph,

$$|u(\gamma(t_k)) - u(\gamma(b))| \le \int_{\gamma|_{[t_k,b]}} \hat{g}\, ds_a \le \int_\gamma \hat{g}\, ds_a.$$

Given that u is Lipschitz continuous on $\dot{X}$, it follows that

$$|u(\gamma(a)) - u(\gamma(b))| = \lim_{k\to\infty} |u(\gamma(t_k)) - u(\gamma(b))| \le \int_\gamma \hat{g}\, ds_a,$$

that is, $\hat{g}$ satisfies the upper gradient inequality on γ with respect to the metric $\hat{d}_a$ even when γ intersects ∞. This completes the proof of the first part of the lemma. $\square$

The next lemma gives a useful integral inequality when we have a weak type inequality. We will use this in the proof of the Poincaré inequality on $\dot{X}$, by first proving an estimate on super level sets of quantities of the form $|u - u_B|$.

Lemma 3.14. ([10, Lemma 4.22]) *Let* (Y, μ) *be a measure space and* f *be a measurable function on* Y *such that for some* $q_0 > 1, C_0 > 0$ *and all* $r > 0$,

$$\mu(\{y \in Y : |f(y)| \le t\}) \le C_0 t^{-q_0}.$$

Then for each q *with* $1 \le q < q_0$*, we have*

$$\left(\int_Y |f|^q d\mu\right)^{1/q} \le \left(\frac{q_0}{q_0 - q}\right)^{1/q} C_0^{1/q_0} \mu(Y)^{1/q - 1/q_0}.$$

Now we are ready to state and prove the following main theorem of this section.

Theorem 3.15. *Suppose that the metric measure space* (X, d, μ) *supports a p-Poincaré Inequality, the measure* μ *is doubling, and that* $(\dot{X}, d_a, \mu_a)$ *satisfies annular quasi-convexity with the annular constant* A*. Then* $(\dot{X}, d_a, \mu_a)$ *also supports a p-Poincaré Inequality, with constants that depend quantitatively on* A *and on the doubling and Poincaré constants of* (X, d, μ).

Proof. As with the other invariance results in this paper, we break the proof up into cases: first we consider balls that are far away from ∞, then balls centered at ∞, and then more general balls. Because $\dot{X}$ is bounded, it suffices to prove the Poincaré inequality for balls $B_a(x, r)$, $x \in \dot{X}$, for $0 < r < 1/(100\lambda A^2)$ where λ is the scaling constant in the p-Poincaré inequality associated with (X, d, μ), and A is the quasiconvexity and annular quasiconvexity constant of $\dot{X}$. Balls of radius $r \ge 1/(100\lambda A^2)$ can then be compared to balls centered at ∞ as in Case 2, and thus we can obtain the inequality for large balls as a consequence. Hence in this proof we will only consider balls $B_a(x, r)$ with $0 < r < 1/(100\lambda A^2)$.

Let $u \in \mathrm{Lip}(\dot{X})$ and g be an upper gradient of u in X with respect to the original metric d. Furthermore, let $x \in X$ and $0 < r < 1/(100\lambda A^2)$.

Case 1: $d_a(x, \infty) \ge 8\lambda r$. Recall that we assume $0 < r < 1/(100\lambda A^2)$. We choose a positive integer $i \ge 3$ so that

$$2^i \lambda r \le d_a(x, \infty) = \frac{1}{1 + d(x, a)} \le 2^{i+1} \lambda r. \tag{3.16}$$

Then $1/(2^{i+1}\lambda r) \le 1 + d(x, a) \le 1/(2^i \lambda r)$. If $y \in X$ such that $d_a(x, y) < r$, then

$$d(x, y) < r(1 + d(x, a))(1 + d(y, a)) \le \frac{1 + d(y, a)}{2^i \lambda} \le \frac{1 + d(x, a) + d(x, y)}{2^i \lambda},$$

and so

$$d(x,y) \le \frac{1+d(x,a)}{2^i\lambda - 1} \le \frac{1}{2^{2i-1}\lambda^2 r},$$

that is, $B_a(x,r) \subset B(x, 2^{1-2i}\lambda^{-2}/r)$. Furthermore, if $z \in B(x, 2^{-2i-3}\lambda^{-2}/r)$, then by the fact that $d_a(x,\infty) \ge 8\lambda r$,

$$d(x,z) < \frac{1}{2^{2i+3}\lambda^2 r} \le \frac{1}{2^{i+2}\lambda}(1+d(x,a))\frac{r}{r},$$

and

$$1 + d(z,a) \ge 1 + d(x,a) - d(x,z) \ge \frac{1}{2^{i+1}\lambda r} - \frac{1}{2^{2i+3}\lambda^2 r} > \frac{1}{2^{i+2}\lambda r}.$$

Combining the above two estimates, we obtain

$$d(x,z) < r\,(1+d(x,a))(1+d(z,a)),$$

that is, $B(x, 2^{-2i-3}\lambda^{-2}/r) \subset B_a(x,r)$. Thus we have

$$B(x, \tfrac{1}{2^{2i+3}\lambda^2 r}) \subset B_a(x,r) \subset B(x, \tfrac{1}{2^{2i-1}\lambda^2 r}).$$

We simplify notation by setting

$$B_s = B(x, \tfrac{1}{2^{2i+3}\lambda^2 r}),\ B_l = B(x, \tfrac{1}{2^{2i-1}\lambda^2 r}).$$

Then we have $B_s \subset B_a(x,r) \subset B_l$. Note that when $z \in \lambda B_l$,

$$1 + d(z,a) \le 1 + d(x,a) + d(z,x) < \frac{1}{2^i\lambda r} + \frac{1}{2^{2i-1}\lambda r} \le \frac{2}{2^i\lambda r} \le 4(1+d(x,a)),$$

and

$$1 + d(z,a) \ge 1 + d(x,a) - d(z,x) > \frac{1}{2^i\lambda r} - \frac{1}{2^{2i-1}\lambda r} \ge \frac{1/2}{2^i\lambda r} \ge \frac{1+d(x,a)}{4}.$$

Hence for $z \in \lambda B_l$ we have

$$\frac{1}{2^{i+3}\lambda r} \le \frac{1+d(x,a)}{4} \le 1 + d(z,a) \le 4(1+d(x,a)) \le \frac{1}{2^{i-2}\lambda r}. \tag{3.17}$$

It follows from the above and the doubling property of μ that for $z \in \lambda B_l$,

$$C^{-1}\,\frac{d\mu(z)}{\mu(B(a,1/(2^i r)))^2} \le d\mu_a(z) = \frac{d\mu(z)}{\mu(B(a,1+d(a,z))^2} \le C\,\frac{d\mu(z)}{\mu(B(a,1/(2^i r)))^2}. \tag{3.18}$$

It follows that

$$\mu_a(B_a(x,r)) \le C\,\frac{\mu(B_l)}{\mu(B(a,1/(2^i r)))^2} \le C\,C_\mu^4 \frac{\mu(B_s)}{\mu(B(a,1/(2^i r)))^2},$$

and

$$\mu_a(B_a(x,r)) \ge C^{-1}\,\frac{\mu(B_s)}{\mu(B(a,1/(2^i r)))^2},$$

from which we obtain

$$\frac{1}{C}\,\frac{\mu(B_s)}{\mu(B(a,1/(2^i r)))^2} \le \mu_a(B_a(x,r)) \le C\,\frac{\mu(B_s)}{\mu(B(a,1/(2^i r)))^2}. \tag{3.19}$$

From (3.17) again, for $z \in \lambda B_l$ we also get

$$\frac{1}{C}\,\frac{1}{2^{2i}r^2}\,g(z) \le \hat{g}(z) = g(z)(1+d(a,z))^2 \le C\,\frac{1}{2^{2i}r^2}\,g(z). \tag{3.20}$$

Now, by applying (3.19), (3.18), and the p-Poincaré inequality of (X,d,μ) in order, we obtain

$$\begin{aligned}
\fint_{B_a(x,r)} |u - u_{B_a(x,r)}|\,d\mu_a &\le 2\fint_{B_a(x,r)} |u - u_{B_l}|\,d\mu_a \\
&\le \frac{C\,\mu(B(a,1/(2^i r)))^2}{\mu(B_l)} \int_{B_l} |u - u_{B_l}|\,d\mu_a \\
&\le C\fint_{B_l} |u - u_{B_l}|\,d\mu \\
&\le C\,\frac{1}{2^{2i-1}\lambda^2 r}\left(\fint_{\lambda B_l} g^p\,d\mu\right)^{1/p}.
\end{aligned}$$

In the above, $u_{B_l} = \mu(B_l)^{-1}\int_{B_l} u\,d\mu$. Now by applying (3.18) again as well as (3.20), we obtain the inequality

$$\fint_{B_a(x,r)} |u - u_{B_a(x,r)}|\,d\mu_a \le \frac{C\,r}{2^{-1}\lambda^2}\left(\frac{1}{\mu_a(B_a(x,r))}\int_{\lambda B_l} \hat{g}^p\,d\mu_a\right)^{1/p}.$$

From (3.17) and the definition of B_l, if $z \in \lambda B_l$ we have

$$d_a(x,z) = \frac{d(x,z)}{(1+d(x,a))(1+d(z,a))} \le C\,\frac{1}{2^{2i-1}\lambda^2 r}\,2^{2i}\lambda^2 r^2 \le C\,r.$$

That is, $\lambda B_l \subset C B_a(x,r)$. Hence by the doubling property of μ_a (proved in Subsection 3.2),

$$\fint_{B_a(x,r)} |u - u_{B_a(x,r)}|\,d\mu_a \le \frac{C\,r}{2^{-1}\lambda^2}\left(\fint_{CB_a(x,r)} \hat{g}^p\,d\mu_a\right)^{1/p},$$

which is the p-Poincaré inequality on $B_a(x,r)$ as desired.

Case 2: $x = \infty$. Recall again that we assume $0 < r < 1/(10\lambda A^2)$. We choose $z \in B_a(\infty,r)$ such that $d_a(z,\infty) = 3r/4$, and set $\rho = \frac{r}{20A^2\lambda}$. Then from Case 1 above, we know that

$$\fint_{B_a(z,\rho)} |u - u_{B_a(z,\rho)}|\,d\mu_a \le C\,r\left(\fint_{B_a(z,C\rho)} \hat{g}^p\,d\mu_a\right)^{1/p}. \tag{3.21}$$

For each $y \in B_a(\infty,r)\setminus\{\infty\}$, we choose the unique positive integer l such that

$$2^{-l-1}r \le d_a(y,\infty) < 2^{-l}r.$$

By the quasiconvexity of $\dot{X}$ (see Theorem 2.14), there is a quasiconvex curve in $\dot{X}$ connecting y to z such that $\ell_{\hat{d}_a}(\gamma) \le A\, d_a(y,z)$. For each $i = 1, \cdots, l$ let z_j be the last location where γ enters $B_a(\infty, 2^{1-i}r)$. Next by the annular quasiconvexity we can choose a curve β_i connecting z_i to z_{i+1}, with $\ell_{\hat{d}_a}(\beta_i) \le A\, d_a(z_i, z_{i+1}) \approx 2^{-i}r$ and $\beta_i \subset B_a(\infty, 2^{-i}Ar) \setminus B_a(\infty, 2^{-i}r/A)$. Let β be the concatenation of the curves β_i, $i = 1, \cdots, l$. Then $\beta \subset B_a(\infty, Ar) \setminus B_a(\infty, 2^{-l-1}r/A)$ and $\ell_{\hat{d}_a}(\beta) \le A^2\, d_a(z,y)$. For each curve β_i, we have

$$\ell_{\hat{d}_a}(\beta_i) \le A d_a(z_{i+1}, z_i) \le \frac{3}{2}\, 2^{-i} r.$$

we set $\rho_i = 2^{-i}\rho$. For every i between $0 \le i \le l$, we can subdivide the subcurve β_i at points $x_{i,0}, x_{i,1}, x_{i,2} \cdots, x_{i,j}, \cdots, x_{i,m_i} \in \beta_i$ with $x_{i,0} = z_i, x_{i,m_i} = z_{i+1}$ such that $\rho_i/4 \le \ell_{\hat{d}_a}(\beta_{(x_{i,j}, x_{i,j+1})}) \le \rho_i/2$ for $0 \le j \le m_i$. Here, $\beta_{(x_{i,j}, x_{i,j+1})}$ denotes the segment of β_i with end points $x_{i,j}$ and $x_{i,j+1}$. We have

$$m_i \le \frac{2^{-i}Ar}{\rho_i} + 1 \le \frac{Ar}{\rho} + 1 \le 20A^3\lambda + 1.$$

We also set $B_{0,1} = B(z,\rho)$ and $m_0 = 1$. For $i = 1, \cdots, l$ and $j = 1, \cdots, m_i$, we set $B_{i,j} = B_a(x_{i,j}, \rho_i)$. For each positive integer j we also set $B_{l+1,j} = B_a(y, 2^{-j-l-1}r)$. For $y \in B_a(\infty, r) \setminus \{\infty\}$ let $\mathcal{C}_y$ denote the family of balls $B_{i,j}$, $i = 0, \cdots, l+1$, $1 \le j < m_i + 1$ (with $m_i = \infty$ when $i = l+1$). Given a ball $B \in \mathcal{C}_y$, we denote its successor in the lexicographic ordering of $\mathcal{C}_y$ by B^*. Note that each $B \in \mathcal{C}_y$, $2B$ satisfies the condition considered in Case 1. Hence for each $B \in \mathcal{C}_y$,

$$\fint_{2B} |u - u_{2B}|\, d\mu_a \le C\, \mathrm{rad}_a(B) \left(\fint_{2\lambda B} \hat{g}^p\, d\mu_a \right)^{1/p}.$$

Here $\mathrm{rad}_a(B)$ is the radius of B in the metric d_a. So $\mathrm{rad}_a(B_{i,j}) \approx 2^{-i}\rho$.

Because u is Lipschitz continuous on $\dot{X}$, each point $y \in B_a(\infty, r) \setminus \{\infty\}$ is a Lebesgue point of u. Hence

$$|u(y) - u_{B_{0,1}}| \le \sum_{B \in \mathcal{C}_y} |u_B - u_{B^*}|.$$

Note that for $B \in \mathcal{C}_y$, we have $B \subset 2B^*$ and $B^* \subset 2B$. Hence

$$\begin{aligned} |u(y) - u_{B_{0,1}}| &\le 2 \sum_{B \in \mathcal{C}_y} \fint_{2B} |u - u_{2B}| \\ &\le C \sum_{B \in \mathcal{C}_y} \mathrm{rad}_a(B) \left(\fint_{2\lambda B} \hat{g}^p\, d\mu_a \right)^{1/p}. \end{aligned}$$

Fix $t > 0$, and suppose that $y \in B_a(\infty, r) \setminus \{\infty\}$ such that $|u(y) - u_{B_z}| \geq t$. Let $0 < \varepsilon < 1$, to be chosen later. Then we can choose a positive number C_ε (independently of t) such that $C_\varepsilon \sum_{i=0}^{\infty} 2^{-i\varepsilon} = 1$. Therefore

$$C_\varepsilon t \sum_{i=0}^{\infty} 2^{-i\varepsilon} = t \leq C \sum_{B \in \mathcal{C}_y} \operatorname{rad}_a(B) \left(\fint_{2\lambda B} \hat{g}^p d\mu_a \right)^{1/p}.$$

Hence there exists $B_y \in \mathcal{C}_y$ such that, with $B_y = B_{i_y, j}$ for some $j < m_{i_y} + 1$,

$$C_\varepsilon t 2^{-i_y \varepsilon} \leq C \operatorname{rad}_a(B_y) \left(\fint_{2\lambda B_y} \hat{g}^p \, d\mu_a \right)^{1/p} \leq C\, 2^{-i_y} \rho \left(\fint_{2\lambda B_y} \hat{g}^p \, d\mu_a \right)^{1/p}.$$

$$C_\epsilon 2^{-i\epsilon} t \leq C \rho_i \left(\fint_{3\lambda B_x} \hat{g}^p d\mu_a \right)^{1/p}$$

Recall that $2^{-i} = \rho_i/\rho_0 = 20\lambda A^2 \rho_i / r$, and so

$$t \leq \frac{C}{C_\varepsilon} 2^{-i_y(1-\varepsilon)} \frac{r}{20\lambda A^2} \left(\fint_{2\lambda B_y} \hat{g}^p \, d\mu_a \right)^{1/p} \leq \frac{C}{C_\varepsilon} \left(\frac{\rho_{i_y}}{r} \right)^{1-\varepsilon} r \left(\fint_{2\lambda B_y} \hat{g}^p \, d\mu_a \right)^{1/p}.$$

Since $2\lambda B_y \subset A\, B_a(\infty, r) \subset 3AB_{0,1}$, by (2.3) we have that

$$\frac{\mu_a(2\lambda B_y)}{\mu_a(3AB_{0,1})} \geq C \left(\frac{2\lambda \rho_{i_y}}{3Ar} \right)^s,$$

that is,

$$\frac{\rho_{i_y}}{r} \leq C \left(\frac{\mu_a(3\lambda B_y)}{\mu_a(3AB_{0,1})} \right)^{1/s}.$$

Hence, by the doubling property of μ_a (proved in Subsection 3.2), and by denoting C/C_ε also by C (recall that C denotes constants whose precise value might be different in each occurance, see the discussion in Section 2),

$$t \leq C\, r \left(\frac{\mu_a(2\lambda B_y)}{\mu_a(B_0)} \right)^{(1-\epsilon)/s} \left(\fint_{2\lambda B_y} \hat{g}^p d\mu_a \right)^{1/p}.$$

We choose $0 < \varepsilon < 1$ such that $\tau := 1 - (1-\epsilon)p/s > 0$. Then

$$\mu_a(2\lambda B_y)^\tau \leq \frac{C r^p}{t^p \mu_a(B_0)^{1-\tau}} \int_{2\lambda B_y} \hat{g}^p d\mu_a. \tag{3.22}$$

Let

$$E_t = \{ y \in B(\infty, r) \, : \, y \neq \infty \text{ and } |u(y) - u_{B_{0,1}}| \geq t \}.$$

By the argument above, for every $y \in E_t$, there exists $B_y \in \mathcal{C}_y$ satisfying (3.22) and $x \in B_y$. By the 5-covering lemma [10, Theorem 1.2], there is a countable a pairwise disjoint subcollection of the collection $40\lambda A^2 B_y$, $y \in E_t$, denoted

$\{40\lambda A^2 B_k\}_{k=1}^{\infty}$ such that $E_t \subset \bigcup_k 200\lambda A^2 B_k$. By the doubling property of μ_a and the pairwise disjointness property of the collection $\{2\lambda B_k\}_k$, we see that

$$\begin{aligned}\mu_a(E_t) &\le \sum_{k=1}^{\infty} \mu_a(200\lambda A^2 B_k) \le C \sum_{k=1}^{\infty} \mu_a(2\lambda B_k) \\ &\le \frac{Cr^{p/\tau}}{t^{p/\tau}\mu_a(B_0)^{1/\tau-1}} \sum_{k=1}^{\infty} \left(\int_{2\lambda B_k} \hat{g}^p \, d\mu_a\right)^{1/\tau} \\ &\le \frac{Cr^{p/\tau}}{t^{p/\tau}\mu_a(B_0)^{1/\tau-1}} \left(\int_{3A\lambda B_0} \hat{g}^p \, d\mu_a\right)^{1/\tau}.\end{aligned}$$

From Lemma 3.14, we can conclude that

$$\fint_{B_a(\infty,r)} |u - u_{B_{0,1}}| d\mu_a \le Cr \left(\fint_{3A\lambda B_{0,1}} \hat{g}^p d\mu_a\right)^{1/p}.$$

Therefore, by the fact that $3A\lambda B_{0,1} \subset B_a(\infty, 6A\lambda r)$, we have

$$\fint_{B_a(\infty,r)} |u - u_{B_a(\infty,r)}| \, d\mu_a \le 2\fint_{B_a(\infty,r)} |u - u_{B_{0,1}}| d\mu_a \le Cr \left(\fint_{6A\lambda B_a(\infty,r)} \hat{g}^p d\mu_a\right)^{1/p}$$

as desired.

Case 3: $d_a(x,\infty) < 8\lambda r$. In this case we use the conclusion of Case 3 above as an aid, since $B_a(x,r) \subset B_a(\infty, 16\lambda r)$, $B_a(\infty, 96A\lambda^2 r) \subset B_a(x, 105A\lambda^2 r)$ and the ball $B_a(\infty, 16\lambda r)$ satisfies the hypothesis of Case 3. Hence by the doubling property of μ_a,

$$\begin{aligned}\fint_{B_a(x,r)} |u - u_{B_a(x,r)}| \, d\mu_a &\le 2\fint_{B_a(x,r)} |u - u_{B_a(\infty,16\lambda r)}| \, d\mu_a \\ &\le C\fint_{B_a(\infty,16\lambda r)} |u - u_{B_a(\infty,16\lambda r)}| \, d\mu_a \\ &\le Cr \left(\fint_{96A\lambda^2 B_a(\infty,r)} \hat{g}^p d\mu_a\right)^{1/p} \\ &\le Cr \left(\fint_{B_a(x,105A\lambda^2 r)} \hat{g}^p d\mu_a\right)^{1/p}.\end{aligned}$$

In conclusion, we have shown the desired Poincaré inequality in the following three cases: $B_a(x,r)$ is far away from ∞, that is, $d_a(x,\infty) \ge 8\lambda r$ (Case 1); $B_a(x,r) = B_a(\infty,r)$ (Case 2), $B_a(x,r)$ is close to ∞, that is, $d_a(x,\infty) < 8\lambda r$ (Case 3). Each of these cases built upon the previous case, and together cover all the possibilities. This completes the proof of Theorem 3.15. □

The requirement that (either X or) $\dot{X}$ is annular quasiconvex cannot be removed in Theorem 3.15, as the following example shows.

Example 3.23. Let X be the 2-dimensional Euclidean infinite strip $\mathbb{R}\times[-1,1]$, equipped with the Euclidean metric and the 2-dimensional Lebesgue measure. Then the sphericalization of X with respect to the base point $a=\vec{0}$ gives a subset of the 2-dimensional Euclidean sphere obtained as the region trapped between two circles on the sphere that intersect only at one point, with the two circles tangential to each other at the point of intersection. Such a region cannot support a p-Poincaré inequality for any $1 \le p < 3$, even though, as a convex region in the plane, X supports a 1-Poincaré inequality and the Lebesgue measure on X is doubling. A direct computation using the sphericalization quantity d_a shows that the cusp made by the two circles in the sphericalization is a quadratic cusp; hence in this example $\dot{X}$ does support a p-Poincaré inequality for $p > 3$, see for example [7, Example 2.2]. We do not know whether the removal of the assumption of annular quasiconvexity would still force the sphericalized space to support a q-Poincaré inequality for sufficiently large q.

4 Preservation of bounded geometry under flattening

In Section 3 we proved that bounded geometry is preserved under the procedure of sphericalization. In doing so, we had to consider two special points, the base point $a \in X$ and the compactification point ∞. In proving a similar result about the flattening procedure, we have to distinguish just one point, the "blow-up point" $c \in X$. While this makes the arguments somewhat simpler, it also makes the arguments different from the analogs for sphericalization. Recall the measure μ^c on X^c obtained from μ in (2.19).

4.1 Preservation of Ahlfors Q-regularity

Proposition 4.1. *Suppose that μ is Ahlfors Q-regular and X is path-connected. Then μ^c is also Ahlfors Q-regular on the metric space $(X^c, \bar{d})$.*

Recall the definition of Ahlfors Q-regularity from (2.1). In the case of Ahlfors regularity, the measure μ^c as defined in (2.19) is equivalent to the following:

$$\mu^c(A) = \int_A \frac{1}{d(z,c)^{2Q}}\, d\mu(z)$$

whenever $A \subset X^c$ is a Borel set.

Proof. We split the proof into two parts to prove the proposition.

Case 1: $rd(x,c) \leq \frac{1}{2}$. In this case, for $y \in B^c(x,r)$ we have $d(x,y) < rd(x,c)d(y,c)$, and so $d(y,c) - d(x,c) \leq d(x,y) < rd(x,c)d(y,c)$, that is,

$$d(y,c) \leq \frac{d(x,c)}{1 - rd(x,c)} < 2d(x,c).$$

Furthermore, we also have $d(x,c) - d(y,c) \leq d(x,y) < rd(x,c)d(y,c)$, and so

$$d(y,c) \geq \frac{d(x,c)}{1 + rd(x,c)} > 2d(x,c)/3.$$

Thus when $y \in B^c(x,r)$ we have

$$\frac{2\,d(x,c)}{3} < d(y,c) < 2\,d(x,c). \tag{4.2}$$

We now want two balls $B(x,r_1), B(x,r_2)$ (in the original metric space (X,d)) with comparable radii r_1 and r_2 such that $B(x,r_1) \subset B^c(x,r) \subset B(x,r_2)$.

Let $y \in B^c(x,r)$. Then by (4.2) we have $d(x,y) < rd(x,c)d(y,c) \leq 2rd(x,c)^2$, so $B^c(x,r) \subset B(x, 2rd(x,c)^2)$ and $r_2 = 2rd(x,c)^2$. We now show that we can take $r_1 = 2rd(x,c)^2/3$. For $y \in B(x, 2rd(x,c)^2/3)$ we have $d(x,y) < 2rd(x,c)^2/3$, and so by (4.2) again,

$$d^c(x,y) = \frac{d(x,y)}{d(x,c)\,d(y,c)} < \frac{2r}{3}\,\frac{d(x,c)}{d(y,c)} < r,$$

that is, $B(x, 2rd(x,c)^2/3) \subset B^c(x,r)$ as desired. Thus we have

$$B(x, 2rd(x,c)^2/3) \subset B^c(x,r) \subset B(x, 2rd(x,c)^2).$$

Now we estimate

$$\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(z)}{d(z,c)^{2Q}}.$$

For the lower bound of $\mu^c(B^c(x,r))$, from (4.2) we have

$$\begin{aligned}\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(z)}{d(z,c)^{2Q}} &\geq \int_{B(x,2rd(x,c)^2/3)} \frac{d\mu(z)}{d(z,c)^{2Q}} \\ &\geq \int_{B(x,2rd(x,c)^2/3)} \frac{d\mu(z)}{2d(x,c)^{2Q}} \\ &\geq \frac{\mu(B(x, 2rd(x,c)^2/3))}{2d(x,c)^{2Q}} \\ &\geq r^Q/C.\end{aligned}$$

To obtain an upper bound for $\mu^c(B^c(x,r))$, we argue as follows:

$$\begin{aligned}\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(z)}{d(z,c)^{2Q}} &\leq \int_{B^c(x,r)} \frac{9}{4d(x,c)^{2Q}}\, d\mu(z) \\ &\leq \int_{B(x,2rd(x,c)^2)} \frac{9}{4d(x,c)^{2Q}}\, d\mu(z) \\ &\leq \frac{9\,\mu(B(x, 2rd(x,c)^2)}{4\,d(x,c)^{2Q}} \\ &\leq Cr^Q.\end{aligned}$$

Combining the above two sets of inequalities we obtain the Ahlfors Q-regularity of balls $B^c(x,r)$ in the case that $rd(x,c) \le 1/2$.
Case 2: $rd(x,c) \ge 2$. In this case, when $y \in X^c \setminus B^c(x,r)$, we have

$$rd(x,c)d(y,c) \le d(x,y) \le d(x,c) + d(y,c). \tag{4.3}$$

Hence, when $y \notin B^c(x,r)$,

$$d(y,c) < \frac{d(x,c)}{rd(x,c)-1} \le \frac{2}{r}.$$

It follows that

$$X \setminus \overline{B}(c, 2/r) \subset B^c(x,r).$$

Furthermore, when $z \in B^c(x,r)$, we have $d(x,c)-d(y,c) \le d(x,y) < rd(x,c)d(y,c)$, and so

$$d(y,c) > \frac{d(x,c)}{rd(x,c)+1} \ge \frac{2}{3r}.$$

Thus we now see that

$$X \setminus \overline{B}(c, 2/r) \subset B^c(x,r) \subset X \setminus \overline{B}(c, \tfrac{2}{3r}).$$

So for the upper bound, we have from the Ahlfors Q-regularity of μ that

$$\begin{aligned}
\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(z)}{d(c,z)^{2Q}} &\le \int_{X\setminus B(c,2/(3r))} \frac{d\mu(z)}{d(c,z)^{2Q}} \\
&\le \sum_{i=1}^{\infty} \int_{B(c,2^{i+1}/(3r))\setminus B(c,2^i/(3r))} \frac{d\mu(z)}{d(c,z)^{2Q}} \\
&\le \sum_{i=1}^{\infty} \int_{B(c,2^{i+1}/(3r))\setminus B(c,2^i/(3r))} \frac{d\mu(z)}{(2^i/(3r))^{2Q}} \\
&\le C \sum_{i=1}^{\infty} \frac{(2^{i+1}/(3r))^Q}{(2^i/(3r))^{2Q}} \le Cr^Q.
\end{aligned}$$

For a lower bound, we have

$$\begin{aligned}
\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(z)}{d(c,z)^{2Q}} &\ge \int_{X\setminus B(c,2/r)} \frac{d\mu(z)}{d(c,z)^{2Q}} \\
&\ge \sum_{i=1}^{\infty} \int_{B(c,2^{i+1}/r)\setminus B(c,2^i/r)} \frac{d\mu(z)}{d(c,z)^{2Q}} \\
&\ge \sum_{i=1}^{\infty} \int_{B(c,2^{i+1}/3r)\setminus B(c,2^i/r)} \frac{d\mu(z)}{(2^i/r)^{2Q}} \\
&\ge C \sum_{i=1}^{\infty} \frac{(2^i/r)^Q}{(2^i/r)^{2Q}} \le Cr^Q.
\end{aligned}$$

This completes the proof of the Ahlfors Q-regularity of μ^c when $rd(x,c) \ge 2$.

Case 3: $1/2 \leq rd(x,c) \leq 2$. In this case, we have $B^c(x,r/4) \subset B^c(x,r) \subset B^c(x,4r)$, and the left-most ball satisfies the hypothesis of Case 1 while the right-most ball satisfies the hypothesis of Case 2. Thus an application of these two cases yields the desired Ahlfors Q-regularity estimates for the ball $B^c(x,r)$ in the event that $1/2 < rd(x,c) < 2$.

The three cases above together exhaust all possibilities, thus completing the proof. □

4.2 Preservation of doubling measure

Now we wish to prove the preservation of doubling property under flattening. Recall the definition of (2.19):

$$\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2}.$$

Proposition 4.4. *Suppose that μ is doubling. Then μ^c is also doubling on (X^c, d^c, μ^c).*

Proof. We again consider three cases.
Case 1: Suppose we have $rd(x,c) \leq 1/4$, then $2rd(x,c) \leq 1/2$. For $y \in B^c(x,r)$, we have

$$d(x,c) - d(y,c) \leq d(x,y) \leq rd(x,c)d(y,c) \leq \frac{1}{4}d(y,c),$$

so that $d(y,c) \geq \frac{4}{5}d(x,c)$. Similarly, we have

$$d(y,c) - d(x,c) \leq d(x,y) \leq rd(x,c)d(y,c) \leq \frac{1}{4}d(y,c).$$

Therefore when $y \in B^c(x,r)$,

$$\frac{4}{5}d(x,c) \leq d(y,c) \leq \frac{4}{3}d(x,c).$$

On the other hand, when $y \in B(x, 4rd(x,c)^2/5)$, we have by the hypothesis assumed in Case 1,

$$d(x,c) - d(y,c) \leq d(x,y) < rd(x,c)\frac{4}{5}d(x,c) \leq \frac{1}{5}d(x,c), \tag{4.5}$$

and hence $\frac{4}{5}d(x,c) \leq d(y,c)$. Therefore we can conclude that $d(x,y) < rd(x,c)d(y,c)$, that is, $B(x, 4rd(x,c)^2/5) \subset B^c(x,r)$. Hence by (4.5),

$$\begin{aligned}\mu^c(B^c(x,r)) = \int_{B^c(x,r)} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} \geq& \int_{B^c(x,r)} \frac{d\mu(y)}{\mu\left(B(c,4d(x,c)/3)\right)^2} \\ \geq& C\frac{\mu(B(x,4rd(x,c)^2/5))}{\mu\left(B(c,d(x,c))\right)^2}.\end{aligned} \tag{4.6}$$

For $y \in B^c(x,2r)$, we have

$$d(y,c) - d(x,c) \le d(x,y) < 2rd(x,c)d(y,c) \le \frac{1}{2}d(y,c),$$

so that $d(y,c) < 2d(x,c)$. Similarly, we have

$$d(x,c) - d(y,c) \le d(x,y) < 2rd(x,c)d(y,c) \le \frac{1}{2}d(y,c).$$

Thus, when $y \in B^c(x,2r)$,

$$\frac{2}{3}d(x,c) < d(y,c) < 2d(x,c). \tag{4.7}$$

Applying the above, when $y \in B^c(x,2r)$ we have

$$d(x,y) < 2rd(x,c)d(y,c) \le 4rd(x,c)^2,$$

so we have $B^c(x,2r) \subset B(x,4rd(x,c)^2)$. Therefore, we have from (4.7) that

$$\begin{aligned}\mu^c(B^c(x,2r)) = \int_{B^c(x,2r)} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} &\le \int_{B^c(x,2r)} \frac{d\mu(y)}{\mu\left(B(c,2d(x,c)/3\right)^2} \\ &\le C\,\frac{\mu(B^c(x,2r))}{\mu\left(B(c,d(x,c))\right)^2} \\ &\le C\,\frac{\mu(B(x,4rd(x,c)^2))}{\mu\left(B(c,d(x,c))\right)^2}.\end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8) together with the doubling property of μ, we have $\mu^c(B^c(x,2r)) \le C\mu^c(B^c(x,r))$ when $rd(x,c) \le 1/4$.

Case 2: $rd(x,c) \ge 2$. Then arguing exactly as in Case 2 of the proof of Proposition 4.1, we obtain

$$\begin{aligned} X \setminus \overline{B}(c,2/r) \subset B^c(x,r) \subset X \setminus \overline{B}(c,2/(3r)), \\ X \setminus \overline{B}(c,1/r) \subset B^c(x,2r) \subset X \setminus \overline{B}(c,1/(3r)). \end{aligned}$$

From Lemma 3.5, we can choose $r_{i,j}$ with $i = 1,2$ and $j = 1,2,\cdots$ such that

$$\begin{aligned} r_{1,0} = 1/(3r) \le r_{1,1} \le r_{1,2} \le \cdots \\ r_{2,0} = 2/r \le r_{2,1} \le r_{2,2} \le \cdots . \end{aligned}$$

and so that with $B_{i,j} = B(c, r_{i,j})$, we have

$$C_\mu\, \mu(B_{i,j}) \le \mu(B_{i,j+1}) \le C_\mu^3\, \mu(B_{i,j}).$$

Consequently,

$$
\begin{aligned}
\mu^c(B^c(x,2r)) \le \mu^c(X\setminus B(c,1/(3r))) &= \int_{X\setminus B(c,1/(3r))} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} \\
&\le \sum_{j=0}^{\infty} \int_{B_{1,j+1}\setminus B_{1,j}} \frac{d\mu(y)}{\mu(B(c,d(y,c)))^2} \\
&\le \sum_{j=0}^{\infty} \frac{\mu(B_{1,i+1})-\mu(B_{1,i})}{\mu(B_{1,i})^2} \\
&\le \sum_{j=0}^{\infty} \frac{C_\mu^3-1}{\mu(B_{1,i})} \\
&\le \sum_{j=0}^{\infty} \frac{C}{C_\mu^i \mu(B(c,1/(3r))} \le \frac{C}{\mu(B(c,2/r))}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mu^c(B_c(x,r)) \ge \mu^c(X\setminus B(c,2/r)) &= \int_{X\setminus B(c,2/r)} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} \\
&\ge \sum_{j=0}^{\infty} \int_{B_{2,j+1}\setminus B_{2,j}} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} \\
&\ge \sum_{j=0}^{\infty} \frac{\mu(B_{2,j+1})-\mu(B_{2,j})}{\mu\left(B_{2,j+1}\right)^2} \\
&\ge \sum_{j=0}^{\infty} \frac{C_\mu-1}{C_\mu\,\mu(B_{2,j+1})} \\
&\ge \sum_{j=0}^{\infty} \frac{C}{C_\mu^{3j+3}\mu(B(c,2/r))} \ge \frac{C}{\mu(B(c,2/r))}.
\end{aligned}
$$

By combining the two equations above, we obtain the desired doubling inequality $\mu_c(B_(x,2r)) \le C\mu_(B_c(x,r))$ when $rd(x,c)\ge 2$.

Case 3: $1/2 < rd(x,c) < 2$. In this case, when $y \in B^c(x,2r)$ we have $d(x,y) < 2rd(x,c)d(y,c)$, and so by the triangle inequality $d(x,y) \ge d(x,c)-d(y,c)$ we see that

$$
d(y,c) > \frac{d(x,c)}{2rd(x,c)+1} \ge \frac{1}{10\,r}.
$$

It follows that $B^c(x,2r) \subset X\setminus\overline{B}(c,1/(10r))$. By the choice of r_i from Lemma 3.5 and $B_i = B(c,r_i)$ such that $r_0 = 1/(10r)$ and $C_\mu\mu(B_i) \le \mu(B_{i+1}) \le C_\mu^3\,\mu(B_i)$, we have as in the argument of Case 2 that

$$
\begin{aligned}
\mu^c(B^c(x,2r)) &\le \mu^c(X\setminus\overline{B}(c,1/(10r))) \\
&\le \sum_{i=1}^{\infty} \int_{B_{i+1}\setminus B_i} \frac{1}{\mu(B(c,d(z,c)))^2}\,d\mu(z) \\
&\le \sum_{i=1}^{\infty} \frac{\mu(B_{i+1})-\mu(B_i)}{\mu(B_i)^2} \le \frac{C}{\mu(B(c,1/(10r)))}. \qquad (4.9)
\end{aligned}
$$

On the other hand, when $y \in B^c(x, r/8)$ we have from (4.5) of Case 1 (applied to $B^c(x, r/8)$ and noting that $r/8 < 1/4$) that

$$\mu^c(B^c(x, r/8)) \geq C\,\frac{\mu(B(x, rd(x,c)^2/5))}{\mu(B(c, d(x,c)))^2} \geq C^{-1}\,\frac{\mu(B(x, d(x,c)/10))}{\mu(B(c, d(x,c))^2}.$$

By the doubling property of μ and the hypothesis that forms Case 3, we have

$$\mu^c(B^c(x, r/8)) \geq C^{-1}\,\frac{1}{\mu(B(c, d(x,c)))} \geq C^{-1}\,\frac{1}{\mu(B(c, 2/r))} \geq C^{-1}\,\frac{1}{\mu(B(c, 1/(10r)))}.$$

Combining the above with (4.9) we obtain that

$$\mu^c(B^c(x, 2r)) \leq C\,\mu^c(B^c(x, r/8)) \leq C\,\mu^c(x, r)$$

as desired in the case that $1/2 < rd(x,c) < 2$.

From the three cases above the proof of the doubling property of μ^c is now complete. □

4.3 Poincaré Inequality

Now we are ready to consider the preservation of the validity of p-Poincaré inequality of a doubling metric measure space that is quasiconvex and annular quasiconvex, under the procedure of sphericalization. Recall from (2.19) that the measure after flattening is given by

$$d\mu^c(y) = \frac{d\mu(y)}{\mu\left(B(a, d(y,a))\right)^2}.$$

Lemma 4.10. *Suppose that u is a Lipschitz function on X^c. If g is an upper gradient of u in $X\backslash$, then the function $\bar{g}$ is given by*

$$\bar{g}(x) = g(x)(d(x,c))^2 \tag{4.11}$$

The proof is essentially similar the proof of (3.11), so we will leave it for the interested reader.

Theorem 4.12. *Suppose that the metric measure space (X, d, μ) supports a p-Poincaré Inequality, the measure μ is doubling, and that (X^c, d^c, μ^c) satisfies annular quasi-convexity with an annular constant A. Then (X^c, d^c, μ^c) also supports a p-Poincaré Inequality, with constants that depends on A and on the doubling and Poincaré constants of (X, d, μ).*

Proof. As before, we consider three cases.
Case 1: $6\lambda rd(x,c) \leq 1/2$. Then, as we have shown in Case 1 of Proposition 4.1,

$$B(x, 2rd(x,c)^2/3) \subset B^c(x, r) \subset B(x, 2r\lambda d(x,c)^2) \subset B^c(x, 6\lambda r).$$

Furthermore, from that argument, we also obtain that whenever $y \in B^c(x, 6\lambda r)$,

$$\frac{2}{3}d(x,c) < d(y,c) < 2d(x,c).$$

Hence we obtain

$$\mu^c(B^c(x,kr)) = \int_{B^c(x,r)} \frac{d\mu(y)}{\mu\left(B(c,d(y,c))\right)^2} \approx \frac{\mu(B^c(x,kr))}{\mu\left(B(c,d(x,c))\right)^2},$$

whenever $0 < k \leq 6\lambda$. In addition, for $y \in 6\lambda B^c(x,r)$ the upper gradient

$$\bar{g}(y) = g(y)(d(y,c))^2 \approx g(y)(d(x,c)^2).$$

Therefore, by the doubling property of μ we obtain a Poincaré Inequality on $B^c(x,r)$ when $6\lambda r d(x,c) \leq 1/2$ as follows:

$$\begin{aligned}
\fint_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c \leq& 2\fint_{B^c(x,r)} |u - u_{B(x,2rd(x,c)^2)}| d\mu^c \\
\leq& \frac{C}{\mu^c(B^c(x,r))} \int_{B^c(x,r)} \frac{|u - u_{B(x,2rd(x,c)^2)}|}{\mu\left(B(c,d(x,c))\right)^2} \, d\mu \\
\leq& \frac{C}{\mu(B^c(x,r))} \int_{B^c(x,r)} |u - u_{B(x,2rd(x,c)^2)}| \, d\mu \\
\leq& \frac{C}{\mu(B(x,2rd(x,c)^2/3))} \int_{B(x,2rd(x,c)^2)} |u - u_{B(x,2rd(x,c)^2)}| \, d\mu \\
\leq& Cr\, d(x,c)^2 \left(\fint_{B(x,2r\lambda d(x,c)^2)} g^p \, d\mu \right)^{1/p} \\
\leq& Cr \left(\int_{B(x,2r\lambda d(x,c)^2)} \bar{g}^p(y) \frac{\mu(B(c,d(x,c)))^2}{\mu(B^c(x,6\lambda r))} \, d\mu^c(y) \right)^{1/p} \\
\leq& Cr \left(\fint_{B^c(x,6\lambda r)} \bar{g}^p \, d\mu^c \right)^{1/p}
\end{aligned}$$

as desired. This completes the proof of p-Poincaré inequality for balls $B^c(x,r)$ when $6\lambda r d(x,c) < 1/2$.

Case 2: $\lambda r d(x,c) \geq 4\lambda$. In this case, as in Case 2 of the proof of Proposition 4.1 we see that

$$X \setminus \overline{B}(c, 2/r) \subset B^c(x,r) \subset X \setminus \overline{B}(c, 2/(3r)).$$

Therefore, by the connectedness of X we can find $z \in X^c$ such that $rd(z,c) = 4$ and note that $B^c(z, r/(96\lambda A)) \subset B^c(x,r)$. We know that $d(z,c) = 4/r$, and so for $y \in B^c(x,r)$ we have $d(y,c) > d(z,c)/6$.

Let l be the unique integer such that $2^l d(z,c) < d(y,c) \leq 2^{l+1} d(z,c)$. Then $l \geq -3$. By the quasiconvexity of X we can find a rectifiable (in X) curve γ connecting z to y such that $\ell_d(\gamma) \leq A\, d(z,y)$. As in the proof of Case 2 of Theorem 3.15, we then use the annular quasiconvexity of X to correct γ as follows. For $-3 \le j \le l$, let $y_j \in \gamma$ be a point such that $d(y_j, c) = 2^j d(z, c)$. By the annular quasi-convexity of X, we can find $\gamma_j \subset B(c, 2^{j+1}Ad(z, c)) \setminus B(c, 2^j d(z, c)/A)$ connecting y_j and y_{j+1} with $\ell_d(\gamma_j) \le Ad(y_j, y_{j+1})$. We set $r_j = 2^{-j} r_0$, where $r_0 = r/(96\lambda A)$. For

$$z'_j \in B(c, 2^{j+1}Ad(z, c)) \setminus B(c, 2^j d(z, c)/A),$$

we have

$$r_j d(z'_j, c) \le 2^{j+1}Ad(z, c)2^{-j} r/(96\lambda A) = rd(z, c)/(48\lambda) = 1/(12\lambda),$$

and hence $B^c(z'_j, r_j)$ satisfies the hypothesis of Case 1 above. Therefore $B^c(z'_j, r_j)$ satisfies the desired Poincaré Inequality by Case 1 with

$$B(z'_j, 2^j d(z, c)/(36\lambda A^3)) \subset B(z'_j, 2r_j d(z', c)^2/3) \subset B^c(z'_j, r_j) \subset B(z'_j, 2r_j d(z', c)^2).$$

We can therefore choose points $z_{0,j}, z_{1,j}, z_{2,j}, \cdots, z_{m_j,j}$ from the path γ_j such that

$$\ell(\gamma_j(z_{i,j}, z_{i+1,j})) = 2^j d(z, c)/(36\lambda A^3)$$

for $i \le m_j - 2$, and

$$\ell(\gamma_j(z_{m_j-1,j}, z_{m_j,j})) \le 2^j d(z, c)/(36\lambda A^3)$$

with $z_{0,j} = y_j, z_{m_j,j} = y_{j+1}$. We want to show m_j is bounded.

Since

$$\ell(\gamma_j) \le Ad(y_{0,j}, y_{1,j}) \le 2^{j+1}Ad(z, c),$$

and

$$(m_j - 1)2^j d(z, c)/(36\lambda A^3) \le \ell(\gamma_j) \le m_j 2^j d(z, c)/(36\lambda A^3.)$$

Then we have $m_j \le 1 + 72\lambda A^4$. We now consider the balls $B_{i,j} = B^c(z_{i,j}, r_j)$ for $i = 1, \cdots, m_j$ and $j = -3, \cdots, l$ and $B_{0,0} = B^c(z, r/(16A))$ so that these balls satisfy the hypothesis of Case 1 above. We also set $B_{i,l+1} = B^c(y, 2^{-i-l-1}r)$ for each positive integer i. Let $\mathcal{C}_y$ be the collection made from the above balls, ordered lexicographically.

Given $B \in \mathcal{C}_y$, we denote the successor in the lexicographic ordering of $\mathcal{C}_y$ by B^*. Note that for $B \in \mathcal{C}_y$, $2B$ satisfies the condition in Case 1 above. Hence for $B \in \mathcal{C}_y$,

$$\fint_{2B} |u - u_{2B}| d\mu^c \le C\mathrm{rad}^c(B) \left(\fint_{2\lambda B} \bar{g}^p d\mu^c \right)^{1/p}.$$

Here $\mathrm{rad}^c(B)$ is the radius of B in the flattening metric d^c. Now, as in the proof of Case 2 of Theorem 3.15, we obtain

$$|u(y) - u_{B_{0,1}}| \le \sum_{B \in \mathcal{C}_y} |u_B - u_{B^*}| \le C \sum_{B \in \mathcal{C}_y} \mathrm{rad}^c(B) \left(\fint_{2\lambda B} \bar{g}^p d\mu^c \right)^{1/p}.$$

Now, as in the covering argument proof of Case 2 of Theorem 3.15 we see that when $t > 0$, setting $E_t = \{y \in B^c(x,r) : |u(y) - u_{B_{0,1}}| \geq t\}$,

$$\mu^c(E_t) \leq \frac{Cr^{p/\tau}}{t_{p/\tau}\mu^(2AB^c(x,r))^{1/\tau-1}} \left(\int_{2\lambda AB^c(x,r)} \bar{g}^p d\mu^c\right)^{1/\tau}.$$

From Lemma (3.14), we can conclude the

$$\fint_{B^c(x,r)} |u - u_{B_{0,1}}| d\mu^c \leq Cr \left(\fint_{2\lambda AB^c(x,r)} \bar{g}^p d\mu^c\right)^{1/p}$$

Therefore, by the fact that $3A\lambda B_{0,1} \subset B^c(x, 6Ar)$, we have

$$\fint_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c \leq 2\fint_{B^c(x,r)} |u - u_{B_{0,1}}| d\mu^c \leq Cr \left(\fint_{6A\lambda B^c(x,r)} \bar{g}_p d\mu^c\right)^{1/p}$$

as desired.

Case 3: for $1/4 \leq \lambda r d(x,c) \leq 4\lambda$, we combine the outcome of Case 2 above to obtain

$$\begin{aligned}\fint_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c &\leq 2\fint_{B^c(x,r)} |u - u_{B^c(x,8r)}| d\mu^c \\ &\leq C\fint_{B^c(x,8r)} |u - u_{B^c(x,8r)}| d\mu^c \\ &\leq Cr \left(\fint_{48A\lambda B^c(x,r)} \bar{g}_p d\mu^c\right)^{1/p}.\end{aligned}$$

Here we used the fact that $B^c(x, 8r)$ satisfies the hypothesis of Case 2.

By combining the above three cases we have proved the theorem. □

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