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SHORT-TIME REGULARITY FOR THE H-SURFACE-FLOW

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ABSTRACT. We consider the heat flow associated with the H-surface system given by

$$\begin{cases} \partial_t u - \Delta u = -2(H \circ u) D_1 u \times D_2 u & \text{on } B \times (0,\infty) \\ u(\cdot,0) = u_o \text{ on } B, \qquad u = g \text{ on } \partial B \times (0,\infty) \end{cases}$$

for a prescribed bounded and continuous function $H\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying an isoperimetric condition of type c for $0 < c < 1$ and time-independent Dirichlet data g of class $C^{1,\gamma}$. Here, $B \subset \mathbb{R}^2$ denotes the unit disk. For the global solutions $u\colon B \to \mathbb{R}^3$ constructed in a preceding work, we prove short-time regularity in the sense that u and Du are locally Hölder continuous on $\overline{B} \times (0,T)$ up to a singular time $T > 0$. From this we deduce the existence of a global solution $\tilde{u}\colon B \times (0,\infty) \to \mathbb{R}^3$ that is regular except from finitely many singular times.

1. INTRODUCTION

1.1. **A brief history of the problem.** A classical problem in Geometric Analysis is given by the H-surface system

$$\Delta u = 2(H \circ u) D_1 u \times D_2 u \qquad \text{on } B,$$

where $B \subset \mathbb{R}^2$ denotes the unit disk and $H\colon A \to \mathbb{R}$ is a given function on a subset $A \subset \mathbb{R}^3$. The geometric significance of this system is that conformal solutions $u\colon B \to A$ parametrize immersed 2-dimensional disk type surfaces of prescribed mean curvature H. The existence problem of the associated Dirichlet problem has been studied intensively by many authors, e.g. by Heinz [19], Hildebrandt [20, 21], Gulliver & Spruck [17, 18], Steffen [32, 33] and Wente [37]. For existence results under a Plateau type boundary condition, we also refer to Struwe's monograph [36]. In general, a solution exists only under a certain smallness condition on the given function H. A famous example is the condition found by Hildebrandt [20, 21], who proved existence of an H-surface $u\colon B \to B^3_R$ in a ball $B^3_R \subset \mathbb{R}^3$ of radius $R > 0$, provided $|H| \le \frac{1}{R}$ on $A = B^3_R$. An existence result under a different kind of assumption was given by Wente [37], who imposed a bound on $\|H\|_{L^\infty}$ that depends on the area of a minimal surface with the prescribed boundary data. Steffen in [32, 33] unified both approaches observing that an isoperimetric condition of the type (c,s) on H with $c < 1$ is sufficient for the solvability of the Dirichlet problem. This is the most general condition ensuring existence in the elliptic case. In the present paper, we seek to analyze the associated parabolic problem under the same condition. More precisely, in the above terminology we have to assume an isoperimetric condition of type (c,∞), which is valid in most of the applications known from the elliptic case, see Theorem 1.3.

In a preceding work [3], we constructed global weak solutions $u\colon B \times [0,\infty) \to A \subset \mathbb{R}^3$ for the heat flow for H-surfaces, given by

$$\begin{cases} \partial_t u - \Delta u = -2(H \circ u) D_1 u \times D_2 u & \text{in } B \times (0,\infty), \\ u = u_o & \text{on } \partial_{\text{par}}\big(B \times (0,\infty)\big), \end{cases} \tag{1.1}$$

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for bounded continuous functions $H\colon A \to \mathbb{R}$ satisfying an isoperimetric condition of type (c, s) for $c < 1$ and $s \le \infty$. Moreover, the corresponding result for a Plateau type boundary condition was established in [11]. For related results in higher dimensions, we refer to [4] and [24]. The solutions exist in the space $u \in L^\infty(0,\infty; W^{1,2}(B^2,\mathbb{R}^3))$ and satisfy $\partial_t u \in L^2(B^2 \times (0,\infty))$. In the present article, we deal with the question of regularity of the constructed global weak solutions. The existence of regular solutions to (1.1) so far was only known under stronger assumptions than necessary for the existence result [3]. Namely, Rey [28] established the existence of a regular solution under the Hildebrandt type assumption $|u_o| < R$ and $|H| < \frac{1}{R}$ for a Lipschitz continuous function $H\colon B^3_R \to \mathbb{R}$. This condition is even strong enough to exclude the possibility of bubbling during the flow, which is the reason why Rey is able to construct even globally regular solutions. The result of Rey was later generalized to higher dimensions $n > 2$ in [23]; see also [25] where a global $C^{1,\alpha}$-regularity result is shown under a stronger smallness assumption of the type $|u||H| < 1/(2R)$. A result on short-time existence is due to Chen & Levine [6], who constructed solutions that stay regular up to a singular time $T > 0$. Chen & Levine have to assume the rather strong regularity assumption of Lipschitz continuity of H, but they do not need to assume any smallness condition on H. However, in order to extend their solution to a global solution with only finitely many singular times, they need to assume monotonicity of the Dirichlet energy $\mathbf{D}(u(\cdot,t))$ of the solution in time. This is an extra assumption that seems to be hard to verify for a given solution, and in fact, our analysis shows that it is more natural to expect monotonicity of the energy functionals

$$t \mapsto \mathbf{D}(u(\cdot,t)) + 2\mathbf{V}_H(u(\cdot,t),u_o), \tag{1.2}$$

where $\mathbf{V}_H$ denotes the oriented H-volume enclosed by the maps $u(\cdot,t)$ and u_o; see Section 2.3 for the definition. In order to exploit this monotonocity, one needs to bound the above functional from below, which is the point where the isoperimetric condition on H seems to be unavoidable. Moreover, we weaken the Lipschitz assumption on H and treat more generally bounded and continuous functions.

A related result is due to Struwe [35], who proved short-time existence for a free boundary condition and constant H satisfying a Hildebrandt type condition. All results on short-time existence rely on techniques developed by Struwe for the harmonic map heat flow in the salient paper [34].

1.2. **Statement of the results.** Before stating our results in a precise way, we gather the assumptions that we will impose throughout this article. For the obstacle A we assume that

$$A \subset \mathbb{R}^3 \text{ is closed, convex, with } C^2\text{-boundary and bounded principal curvatures.} \tag{1.3}$$

We write $\mathcal{H}_{\partial A}(a)$ for the minimum of the principal curvatures of ∂A in a boundary point $a \in \partial A$, taken with respect to the inward pointing unit normal vector. For the function H we suppose that

$$H\colon A \to \mathbb{R} \text{ is a continuous and bounded function} \tag{1.4}$$

and satisfies

$$|H| \le \mathcal{H}_{\partial A} \qquad \text{on } \partial A. \tag{1.5}$$

Moreover, we impose a **spherical isoperimetric condition of type c on A**, for some parameter $0 < c < 1$. This means, that for every spherical 2-current T with $\operatorname{spt} T \subset A$, we assume that

$$2\big|\langle Q, H\Omega\rangle\big| = 2\left|\int_A i_Q H\Omega\right| \le c\,\mathbf{M}(T) \tag{1.6}$$

holds true, where Q is the unique integer multiplicity rectifiable 3-current with boundary $\partial Q = T$, $\mathbf{M}(Q) < \infty$ and $\operatorname{spt} Q \subset A$. In the preceding formula we used the notation $i_Q \in L^1(\mathbb{R}^3,\mathbb{Z})$ for the multiplicity function associated to Q and Ω for the volume form on

$\mathbb{R}^3$. In the terminology of [32, 33] this would be called spherical isoperimetric condition of type (c, ∞). For the initial and (time-independent) boundary values, we assume that

$$u_o \in W^{1,2}(B, A). \tag{1.7}$$

According to [3], the preceding set of assumptions implies existence of global weak solutions to the H-surface flow with Cauchy-Dirichlet data u_o. In order to achieve $C^{1,\beta}$-regularity up to the lateral boundary, we assume moreover that

$$u_o \in g + W_0^{1,2}(B, \mathbb{R}^3) \qquad \text{with } g \in W^{2,q}(B, \mathbb{R}^3) \text{ for some } q > 2. \tag{1.8}$$

In particular, this implies $u_o|_{\partial B} \in C^{1,\gamma}(\partial B, \mathbb{R}^3)$ and $g \in C^{1,\gamma}(\overline{B}, \mathbb{R}^3)$ for $\gamma = 1 - \frac{2}{q} \in (0, 1)$ by Sobolev embedding.

Our first result concerns the short-time regularity of the global weak solutions constructed in [3].

Theorem 1.1. *Let the assumptions (1.3) – (1.8) be in force with $g \in C^{1,\gamma}(\overline{B}, \mathbb{R}^3)$ for some $\gamma \in (0, 1)$ and consider the solution $u \in L^\infty(0, \infty; W^{1,2}(B, A))$ of (1.1) constructed in* [3]. *Then there is a time $t_o > 0$, depending only on $\|H\|_{L^\infty}$, c and u_o such that $u \in L^2(0, t_o; W^{2,2}(B, \mathbb{R}^3))$ and $Du \in C^{\gamma,\gamma/2}(\overline{B} \times (0, t_o), \mathbb{R}^{2\cdot 3})$.*

We can exploit the above result by using the values of the flow at the first singular time as the initial values for another flow and repeating this procedure iteratively. This readily yields the existence of a global solution that is regular except for a discrete set of singular times. Moreover, we prove a monotonicity property of the type (1.2) for the approximating solutions. Combined with the isoperimetric condition, this enables us to reduce the singular times to a finite set. More precisely, we have the following result.

Theorem 1.2. *Under the assumptions (1.3) – (1.8), with boundary data $g \in C^{1,\gamma}(\overline{B}, \mathbb{R}^3)$ for some $\gamma \in (0, 1)$, there is a global weak solution $u \in L^\infty(0, \infty; W^{1,2}(B, A))$ with $\partial_t u \in L^2(B \times (0, \infty))$ that is regular away from a finite set of times $\Sigma := \{T_1, \ldots, T_K\} \subset (0, \infty)$ in the sense*

$$Du \in C^{\gamma,\gamma/2}_{\mathrm{loc}}\big(\overline{B} \times ((0, \infty) \setminus \Sigma)\big).$$

Furthermore, the number of finite times can be bounded by

$$\operatorname{card} \Sigma \le C\mathbf{D}(u_o)$$

with a constant $C = C(\|H\|_{L^\infty}, c)$. In particular, the solution is regular on $B \times (0, \infty)$ in the case of small initial energy $\mathbf{D}(u_o) < \frac{1}{C}$.

We conclude this section by mentioning some known cases in which the isoperimetric condition (1.6) is satisfied. For the proofs we refer to [32, 33, 12, 13].

Theorem 1.3. *Let $H : A \to \mathbb{R}$ be a bounded and continuous function on $A \subset \mathbb{R}^3$. Then each of the conditions*

$$\int_A |H|^3 \, dx < \tfrac{9\pi}{2}, \tag{1.9}$$

$$A \subseteq B_R \quad \text{and} \quad \int_{\{\xi \in A : |H(\xi)| \ge \frac{1}{R}\}} |H|^3 \, dx < \tfrac{9\pi}{2} \tag{1.10}$$

$$\sup_A |H| < \tfrac{3}{2} \sqrt[3]{\frac{4\pi}{3\mathcal{L}^3(A)}} \tag{1.11}$$

$$\mathcal{L}^3\{a \in A \colon |H(a)| \ge \tau\} \le c\tfrac{4\pi}{3}\tau^{-3} \qquad \text{for some } c < 1 \text{ and any } \tau > 0 \tag{1.12}$$

ensures that H satisfies a spherical isoperimetric condition of type c with $c < 1$.

We note that conditions (1.9) and (1.11) are generalizations of the Hildebrandt type condition $A = B_R$ and $|H| < \frac{3}{2R}$.

1.3. **The strategy of the proof.** Finally a few words concerning the strategy of the proof are in order.

A local energy inequality. The starting point for the regularity proof for solutions of the flow $(1.1)_1$ is the observation that due to the critical non-linearity on the right-hand side, regularity can be expected only under a certain smallness condition on the energy. More precisely, a common smallness condition for regularity close to a point $(x_o, t_o) \in \overline{B} \times (0, \infty)$ is

$$\sup_{t\in(t_o-R^2,t_o)} \int_{B_R(x_o)\cap B} |Du(\cdot,t)|^2\,dx \le \varepsilon_2 \tag{1.13}$$

for a universal constant $\varepsilon_2 > 0$. The condition (1.13) turns out to be sufficient for the Hölder continuity of the gradient once we know additionally

$$u \in L^2\big(t_o - R^2, t_o; W^{2,2}(B_R(x_o)\cap B, \mathbb{R}^3)\big). \tag{1.14}$$

We deduce both the conditions (1.13) and (1.14) from the approximating solutions u_h that were used in [3] to construct the solution u by letting $h \downarrow 0$. We recall that the maps u_h were constructed as solutions of a time-discretized version of the flow by solving a minimization problem at each time step. For a brief outline of the construction, sometimes called *minimal movements*, we refer to Section 3. The idea of the proof of (1.13) and (1.14) is to apply a time-discrete version of the difference quotient method to the maps u_h in order to derive a local energy estimate of the form

$$\begin{aligned}\int_{B_{r/2}(x_o)} |Du_h(\cdot,s)|^2\,dx + \int_{B_{r/2}(x_o)\times(0,s)} |D^2u_h|^2\,dz \\ \le C\int_{B_r(x_o)} |Du_o|^2\,dx + \frac{C}{r^2}\int_{B_r(x_o)\times(0,s)} |Du_h|^2\,dz\end{aligned} \tag{1.15}$$

for all sufficiently small times $s > 0$ and an interior point $x_o \in B$. A similar estimate holds at the boundary with additional terms containing the boundary values. The preceding estimate would imply both (1.13) and (1.14) for u_h uniformly in $h > 0$ – and therefore also for the limit map u – if we would choose $r, s > 0$ sufficiently small. However, the application of the difference quotient technique is not possible in general, because of the quadratic non-linearity on the right-hand side of the system $(1.1)_1$. In fact, we can deal with this non-linearity – using a Gagliardo-Nirenberg interpolation argument – only if we already know a smallness condition of the type (1.13) that we wish to prove (cf. Theorems 4.4 and 4.7). However, if we choose $r > 0$ so small that

$$\sup_{x_o\in\overline{B}} \int_{B_r(x_o)\cap B} |Du_o|^2\,dx \le \varepsilon_1, \tag{1.16}$$

then we can show that this smallness condition is preserved at least for a small time span, cf. Theorem 4.11. The idea is to consider the maximal time for which (1.15) still holds for every $x_o \in \overline{B}$. From (1.15) we infer smallness of the energy up to the time $t = s$, and using the minimizing property of u_h together with the isoperimetric condition, we are able to show that the smallness of the energy is preserved in the next time step, i.e. at $t = s + h$ (cf. Lemma 4.10). This enables us to apply (1.15) at the time $s + h$ instead of s. Inductively, this implies that (1.15) holds up to a time $t_o > 0$ independent from $h > 0$. This argument is carried out in detail in Theorem 4.11.

Unfortunately, in the application of Lemma 4.10 it is necessary to diminish the radius of the balls under consideration. Since this lemma has to be applied at each time step, the argument can not be made independent from h unless we assume the smallness of the energy – as stated in (1.16) – in all points $x_o \in \overline{B}$ simultaneously. In other words, it is not possible to localize this argument: if the energy begins to concentrate at one point $x_1 \in \overline{B}$, we can not exclude the possibility that this might influence the solution at any other point $x_2 \in \overline{B}$, arbitrarily short after the concentration in x_1 occurs. This means that we have to implement the difference quotient argument up to the boundary even when considering the question of interior regularity.

Short-time regularity. Having established the energy inequality (1.15) uniformly in $h > 0$, we can pass to the limit $h \downarrow 0$ to infer the desired properties (1.13) and (1.14) for the solution u of the flow. These assumptions enable us to prove the $C^{1,\gamma}$-regularity for the solution. To this end, we apply the method of caloric approximation that was used before in the elliptic setting of the harmonic map system by Simon [31], see also [9] for the p-harmonic case. In Section 5, we extend this argument to the H-surface flow. This becomes particularly involved in the boundary case, where we first have to flatten the boundary in order to apply the caloric approximation lemma at the flat boundary and then transform back to the unit disk to derive the excess decay estimates that lead to the regularity of the solution. In this manner, we establish Hölder continuity of the spatial gradient up to a positive time $t_o > 0$.

Regularity except from finitely many times. In order to construct a solution with only finitely many singular times, we consider the functional $E_H(\tilde{u}) := \mathbf{D}(\tilde{u}) + 2\mathbf{V}_H(\tilde{u}, u_o)$. From the minimizing property of the approximating solutions u_h, it is straightforward to deduce that the functionals $E_H(u_h(\cdot, t))$ are weakly decreasing in time for every $h > 0$ (cf. 3.4). The isoperimetric condition on H furthermore implies a lower bound on E_H. The key step in the proof is to show that at every singular time, the functional E_H is reduced by a fixed amount. This is a consequence of the ε-regularity theorem from Section 5, which implies that the Dirichlet energies of the u_h have to concentrate at a singularity, when $h \downarrow 0$. The argument is carried out in Section 6. It relies on a lower semicontinuity property of E_H that can be established with methods from [8], combined with a careful analysis of the energy concentration at the singularity.

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2. Preliminaries

2.1. **Notation.** We write $B_\varrho(x_o) \subset \mathbb{R}^2$ for the open disk with radius $\varrho > 0$ and center x_o. The backwards parabolic cylinder with center in $z_o = (x_o, t_o) \in \mathbb{R}^2 \times \mathbb{R}_+$ will be abbreviated by

$$Q_\varrho(z_o) = B_\varrho(x_o) \times (t_o - \varrho^2, t_o).$$

For mean values over such cylinders, we use the abbreviation $(u)_{z_o;\varrho} := \fint_{Q_\varrho(z_o)} u(z)\, dz$. Moreover, we use the notations

$$B^+_\varrho(x_o) := B_\varrho(x_o) \cap B \qquad \text{and} \qquad Q^+_\varrho(z_o) := B^+_\varrho(x_o) \times (t_o - \varrho^2, t_o).$$

For the *Dirichlet energy* of $u \in W^{1,2}(B, \mathbb{R}^3)$, we write

$$\mathbf{D}(u) := \tfrac{1}{2} \int_B |Du|^2\, dx \qquad \text{and} \qquad \mathbf{D}_G(u) := \tfrac{1}{2} \int_G |Du|^2\, dx,$$

for any subset $G \subset B$.

2.2. **Some analytical tools.** Next, we recall a well-known iteration lemma whose proof can for example be found in [16, Lemma 6.1].

Lemma 2.1. *Let $0 < r < R < \infty$ and let $f\colon [r, R] \to [0, \infty)$ be a bounded function which satisfies for all $r \le s < t \le R$*

$$f(s) \le \vartheta f(t) + \frac{B}{(t-s)^\beta} + C$$

for constants $B, C \geq 0$, $\beta > 0$ *and* $\vartheta \in (0,1)$. *Then we have the bound*

$$f(r) \leq c(\beta,\vartheta)\bigg[\frac{B}{(R-r)^\beta} + C\bigg].$$

The crucial tool to deal with the critical non-linearity of the H-surface system is the following Gagliardo-Nirenberg-type interpolation inequality.

Lemma 2.2 ([27]). *Consider a ball* $B_\varrho(x_o)$ *of dimension* $n \geq 2$ *and radius* $0 < \varrho \leq 1$, *integrability exponents* $1 \leq \sigma, q, r < \infty$ *and a parameter* $\vartheta \in (0,1)$ *that are related by the inequality* $-\frac{n}{\sigma} \leq \vartheta(1-\frac{n}{q}) - (1-\vartheta)\frac{n}{r}$. *There is a constant* $c = c(n,q,r)$ *such that for any* $v \in W^{1,q}(B_\varrho(x_o))$ *we have*

$$\fint_{B_\varrho(x_o)\cap B} \Big|\frac{v}{\varrho}\Big|^\sigma dx \leq c\bigg[\fint_{B_\varrho(x_o)\cap B} \Big|\frac{v}{\varrho}\Big|^q + |Dv|^q\, dx\bigg]^{\frac{\vartheta\sigma}{q}} \bigg[\fint_{B_\varrho(x_o)\cap B} \Big|\frac{v}{\varrho}\Big|^r dx\bigg]^{\frac{(1-\vartheta)\sigma}{r}}.$$

For a map $u \in W^{2,2}(B_\varrho(x_o),\mathbb{R}^N)$ on a ball $B_\varrho(x_o) \subset \mathbb{R}^2$, we apply this inequality to $v = |Du| \in W^{1,2}(B_\varrho(x_o))$ with the parameters $\sigma = 4$, $n = 2$, $q = r = 2$ and $\vartheta = \frac{1}{2}$. This leads to the estimate

$$\int_{B_\varrho(x_o)\cap B} |Du|^4\, dx \leq C \int_{B_\varrho(x_o)\cap B} |Du|^2\, dx \int_{B_\varrho(x_o)\cap B} |D^2u|^2 + \Big|\frac{Du}{\varrho}\Big|^2 dx, \tag{2.1}$$

which holds true with a universal constant C. Now we consider a time-dependent map $u \in L^2(t_1,t_2;W^{2,2}(B_\varrho(x_o),\mathbb{R}^N))$ in two space dimensions. Applying the preceding estimate on each time slice and integrating with respect to time, we obtain

$$\int_{t_1}^{t_2}\int_{B_\varrho(x_o)\cap B} |Du|^4\, dx\, dt$$
$$\leq C \sup_{t\in(t_1,t_2)} \int_{B_\varrho(x_o)\cap B} |Du(\cdot,t)|^2\, dx \int_{t_1}^{t_2}\int_{B_\varrho(x_o)\cap B} |D^2u|^2 + \Big|\frac{Du}{\varrho}\Big|^2 dx\, dt. \tag{2.2}$$

2.3. **The H-volume functional.** In this section we recall the definition and some properties of the H-volume functional, which go back to [32], respectively [13]. We also refer to our earlier work [3] for a more detailed overview. The definition of the H-volume functional relies on the theory of currents, for which the standard references are [15] and [30]. Every map $u \in W^{1,2}(B,\mathbb{R}^3)$ determines an associated 2-current J_u on $\mathbb{R}^3$ by

$$J_u(\omega) := \int_B u^\#\omega = \int_B \langle \omega\circ u, D_1u \wedge D_2u\rangle\, dx \qquad \text{for all } \omega \in \mathcal{D}^2(\mathbb{R}^3).$$

Here, $\mathcal{D}^k(\mathbb{R}^3)$ denotes the space of smooth k-forms with compact support in $\mathbb{R}^3$ for $k \in \{0,1,2,3\}$. A Lusin-type approximation argument (cf. [14, Sect. 6.6.3]) yields that J_u is an integer multiplicity rectifiable 2-current of finite mass. For a closed convex obstacle $A \subseteq \mathbb{R}^3$ and a fixed reference surface $u_o \in W^{1,2}(B,A)$ we define the set of admissible surfaces by

$$\mathcal{S}(u_o,A) := \Big\{u \in W^{1,2}(B,A) : u - u_o \in W^{1,2}_0(B,\mathbb{R}^3)\Big\}.$$

Then for any $u, v \in \mathcal{S}(u_o,A)$, the current $(J_u - J_v)(\omega)$ is determined by integration of $u^\#\omega - v^\#\omega$ over the set $G := \{x \in B : u(x) \neq v(x)\}$. This implies

$$\mathbf{M}(J_u - J_v) \leq \mathbf{D}_G(u) + \mathbf{D}_G(v), \tag{2.3}$$

where $\mathbf{M}$ denotes the mass of a current. From [13, Lemma 3.3], respectively [8] we recall that $J_u - J_v$ is a spherical 2-current with support in A, i.e. it can be written in the form $(J_u - J_v)(\omega) = \int_{S^2} f^\#\omega$ for all $\omega \in \mathcal{D}^2(\mathbb{R}^3)$ and a map $f \in W^{1,2}(S^2,A)$. Therefore, the following result is crucial for our purposes:

Lemma 2.3 ([3, Lemma 3.4])**.** *Let $A \subseteq \mathbb{R}^3$ be a closed convex set. Then for every spherical 2-current T with support in A, there exists a* ***unique*** *integer multiplicity rectifiable 3-current Q with $\mathbf{M}(Q) < \infty$, $\partial Q = T$ and $\operatorname{spt} Q \subseteq A$.*

The 3-currents Q as in the lemma can be represented by a multiplicity function $i_Q \in L^1(\mathbb{R}^3, \mathbb{Z})$ with $\operatorname{spt}(i_Q) \subset A$ in the sense

$$Q(\gamma) = \int_{\mathbb{R}^3} i_Q \gamma \qquad \text{for all } \gamma \in \mathcal{D}^3(\mathbb{R}^3).$$

Now we are prepared to give the definition of the oriented H-volume enclosed by two maps $u, v \in \mathcal{S}(u_o, A)$.

Definition 2.4. *For given functions $u, v \in \mathcal{S}(u_o, A)$, we write $I_{u,v}$ for the unique integer multiplicity rectifiable 3-current with boundary $\partial I_{u,v} = J_u - J_v$, finite mass $\mathbf{M}(I_{u,v}) < \infty$ and $\operatorname{spt} I_{u,v} \subseteq A$. Moreover, by $i_{u,v} \in L^1(\mathbb{R}^3, \mathbb{Z})$ we denote the associated multiplicity function. Then the H-volume enclosed by u and v is defined by*

$$\mathbf{V}_H(u, v) := I_{u,v}(H\Omega) = \int_{\mathbb{R}^3} i_{u,v} H\Omega\,.$$

Here, Ω denotes the volume form on $\mathbb{R}^3$.

The next lemma contains the properties of the H-volume functional that will be relevant for the present article. For the proof we refer to [13, Lemma 3.6(i)] or [8, Lemma 3.5(i)].

Lemma 2.5. *Let $u, \tilde{u}, v \in \mathcal{S}(u_o, A)$ be given. Then we have*

$$\mathbf{V}_H(\tilde{u}, u) + \mathbf{V}_H(u, v) = \mathbf{V}_H(\tilde{u}, v)\,.$$

If $H : A \to \mathbb{R}^3$ is bounded, we moreover know

$$\big|\mathbf{V}_H(\tilde{u}, u)\big| \le \|H\|_{L^\infty} \|u - \tilde{u}\|_{L^\infty} \big[\mathbf{D}_G(u) + \mathbf{D}_G(\tilde{u})\big],$$

where $G = \{x \in B : u(x) \neq \tilde{u}(x)\}$.

2.4. ***H*-volume and spherical isoperimetric condition.** The crucial assumption on the presribed mean curvature function $H\colon A \to \mathbb{R}^3$ which ensures existence of solutions to the H-surface flow is a spherical isoperimetric condition of type $c \in (0, 1)$, see (1.6) for the definition. We will exploit this assumption in form of the following bound for the H-volume $\mathbf{V}_H(u, v)$ for any $u, v \in \mathcal{S}(u_o, A)$:

$$2\big|\mathbf{V}_H(u, v)\big| \le c\,\mathbf{M}(J_u - J_v) \le c\big(\mathbf{D}_G(u) + \mathbf{D}_G(v)\big), \tag{2.4}$$

where $G = \{x \in B : u(x) \neq v(x)\}$. The first inequality is a consequence of the isoperimetric condition, and the second one follows from (2.3).

3. The construction

In [3], respectively in [11], we constructed approximations to the H-surface flow via the following recursive scheme: For a fixed step size $h > 0$, we define maps $u_{j,h} \in W^{1,2}(B, A)$ for any $j \in \mathbb{N}_0$ by letting $u_{0,h} := u_o$ and then, assuming that $u_{j-1,h}$ is already constructed, we choose $u_{j,h} \in W^{1,2}(B, A)$ as a minimizer of the functional

$$\mathcal{F}_{j,h}(\tilde{u}) := \mathbf{D}(\tilde{u}) + 2\mathbf{V}_H(\tilde{u}, u_o) + \tfrac{1}{2h} \int_B \big|\tilde{u} - u_{j-1,h}\big|^2\, dx. \tag{3.1}$$

The existence of a minimizer to this functional was established in [3, 11]. Then we let

$$u_h(x, t) := u_{j,h}(x) \qquad \text{for } j \in \mathbb{N} \text{ with } (j-1)h < t \le jh \tag{3.2}$$

and $u_h(x, t) := u_o(x)$ for $t \le 0$. Using $u_{j-1,h}$ as a competitor for the minimizer $u_{j,h}$ of the functional $\mathcal{F}_{j,h}$, we deduce

$$\mathbf{D}(u_{j,h}) + 2\mathbf{V}_H(u_{j,h}, u_o) + \tfrac{1}{2h} \int_B \big|u_{j,h} - u_{j-1,h}\big|^2\, dx \tag{3.3}$$

$$\leq \mathbf{D}(u_{j-1,h}) + 2\mathbf{V}_H(u_{j-1,h}, u_o)$$

for all $j \in \mathbb{N}$. This implies that

$$[0,\infty) \ni t \mapsto \mathbf{D}\big(u_h(\cdot,t)\big) + 2\mathbf{V}_H\big(u_h(\cdot,t), u_o\big) \qquad \text{is non-increasing} \tag{3.4}$$

for every $h > 0$. Moreover, we can iterate (3.3) and exploit the isoperimetric condition (1.6) as in [3, (8.4), (8.5)] to get the energy inequality

$$\sup_{j\in\mathbb{N}} \left[\mathbf{D}(u_{j,h}) + \sum_{\ell=1}^{j} \tfrac{1}{2h} \int_B |u_{\ell,h} - u_{\ell-1,h}|^2 \, dx\right] \leq C_1 \mathbf{D}(u_o) \tag{3.5}$$

for all $h > 0$, with the constant $C_1 = \frac{1+c}{1-c}$ which is in particular independent from h. Note that $0 < c < 1$ denotes the constant from the isoperimetric condition (1.6). The functions $u_{j,h}$ weakly solve the Euler-Lagrange system

$$\frac{u_{j,h} - u_{j-1,h}}{h} - \Delta u_{j,h} = -2(H \circ u_{j,h}) D_1 u_{j,h} \times D_2 u_{j,h} \tag{3.6}$$

and can therefore be thought of as solutions to a time-discrete H-surface flow. As shown in [3, Sect. 8], for a sequence $h_i \downarrow 0$ we can achieve convergence

$$\begin{cases} u_{h_i} \overset{*}{\rightharpoondown} u & \text{weakly* in } L^\infty(0,T;W^{1,2}(B,\mathbb{R}^3)), \\ u_{h_i} \to u & \text{strongly in } L^\infty(0,T;L^2(B,\mathbb{R}^3)) \end{cases} \tag{3.7}$$

for a map $u \in L^\infty(0,T;W^{1,2}(B,A)) \cap C^0([0,T],L^2(B,A))$. Moreover, according to [3, Sect. 8] the limiting map satisfies $\partial_t u \in L^2(B\times(0,\infty),\mathbb{R}^3)$ and weakly solves the H-surface system

$$\partial_t u - \Delta u = -2(H\circ u) D_1 u \times D_2 u \qquad \text{in } B \times (0,\infty). \tag{3.8}$$

4. Local Energy bounds

The aim of this section is the proof of the following theorem.

Theorem 4.1. *There exist universal constants $\delta = \delta(\|H\|_{L^\infty}, c, \mathbf{D}(u_o), \|Dg\|_{W^{1,2}}) > 0$ and $\varepsilon_1 = \varepsilon_1(\|H\|_{L^\infty}, c) > 0$, and moreover a radius $r_o = r_o(\|Dg\|_{L^4}) \in (0, \frac14]$ with the following property: Assume that for a time $t_1 \geq 0$, some radius $r \in (0, r_o]$ and $h \in (0, \delta r^2)$, we have that*

$$\varepsilon := \sup_{x_o \in \overline{B}} \int_{B_r(x_o)\cap B} |Du_h(\cdot,t_1)|^2 \, dx \leq \varepsilon_1 \tag{4.1}$$

holds true. Then, for every $t_2 \in (t_1, t_1 + \delta r^2 - h)$ and every disk $B_r(x_o)$ with $x_o \in \overline{B}$ we have

$$\begin{aligned} \sup_{t_1 \leq t \leq t_2} \int_{B_{r/2}(x_o)\cap B} |Du_h(\cdot,t)|^2 \, dx + \int_{t_1}^{t_2} \int_{B_{r/2}(x_o)} |D^2 u_h|^2 \, dx \, dt \\ \leq C\varepsilon + C\frac{t_2 - t_1}{r^2}\Big(\mathbf{D}(u_o) + \|Dg\|^2_{W^{1,2}}\Big) + Cr\|Dg\|^2_{L^4}, \end{aligned}$$

with a universal constant C depending only on $\|H\|_{L^\infty}$. In the case of an interior disk $B_r(y) \subset B$, the terms involving the boundary data g can be omitted in the last estimate.

The proof will be given in Section 4.3. We can exploit this theorem at all times $t \geq 0$ for which we know the smallness condition (4.1) to hold independently from $h > 0$. This is indeed the case for $t = 0$, since we chose the initial values u_o independently from h. Letting $h_i \downarrow 0$ and using the convergence $u_{h_i} \overset{*}{\rightharpoondown} u$ in L^∞-$W^{1,2}$, we deduce

Corollary 4.2. *For the constants $\delta, \varepsilon_1 > 0$, the radius $r_o \in (0, \frac{1}{4}]$ and C from Theorem 4.1, the following holds true: If $r \in (0, r_o)$ is chosen so small that*

$$\varepsilon := \sup_{x_o \in \overline{B}} \int_{B_r(x_o) \cap B} |Du_o|^2 \, dx \le \varepsilon_1$$

then for every $t_2 \in (0, \delta r^2)$ and every disk $B_r(x_o)$ with $x_o \in \overline{B}$ we have

$$\sup_{0 \le t \le t_2} \int_{B_{r/2}(x_o) \cap B} |Du(\cdot, t)|^2 \, dx + \int_0^{t_2} \int_{B_{r/2}(x_o)} |D^2 u|^2 \, dx \, dt$$
$$\le C\varepsilon + C\frac{t_2}{r^2}\Big(\mathbf{D}(u_o) + \|Dg\|^2_{W^{1,2}}\Big) + Cr\|Dg\|^2_{L^4}.$$

In particular, this shows that $u \in L^2\big(0, \delta r^2; W^{2,2}(B, \mathbb{R}^3)\big)$.

4.1. **Difference quotient technique for the time-discrete parabolic system.** Using the minimizing property of $u_{j,h}$ and elliptic estimates, in a preceding work we already proved in the interior $u_{j,h} \in W^{2,2}_{\mathrm{loc}}$ for every $j \in \mathbb{N}$ and $h > 0$ and the corresponding quantitative $W^{2,2}$-estimates, see [3, Lemma 7.3], and also [11, Lemma 8.1]. We can actually extend this kind of regularity up to the boundary.

Lemma 4.3. *For every $j \in \mathbb{N}$ and $h > 0$ we have $u_{j,h} \in W^{1,4}(B, \mathbb{R}^3) \cap W^{2,2}(B, \mathbb{R}^3)$.*

Proof. Since we know already that $u_j \in W^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^3) \hookrightarrow C^{0,\alpha}_{\mathrm{loc}}(B, \mathbb{R}^3)$ from [3, Lemma 7.3], and since the boundary datum $u_o|_{\partial B}$ is continuous by assumption, the argument by Hildebrandt & Kaul [22] gives $u_j \in C^0(\overline{B}, \mathbb{R}^3)$, cf. also [11, Lemma 8.2]. Next, we extend $u_j - g$ by reflection across ∂B, i.e. we define

$$v_j(x) := \begin{cases} (u_j - g)(x), & \text{if } x \in B, \\ -(u_j - g)(\frac{x}{|x|^2}) & \text{if } x \in B_2 \setminus B, \end{cases}$$

for any $j \in \mathbb{N}_0$. From what we know so far about u_j, we conclude that these maps are continuous on B_2. Next, we recall that on the unit disk B, the maps v_j weakly solve

$$\frac{u_j - u_{j-1}}{h} - \Delta v_j = -2(H \circ u_j) D_1 u_j \times D_2 u_j + \Delta g.$$

By the conformal covariance of the Laplacian we have $\Delta v_j(x) = -\frac{1}{|x|^4} \Delta(u_j - g)(\frac{x}{|x|^2})$ on $B_2 \setminus B$ and therefore we can re-write the preceding equation in the form

$$-\Delta v_j = F_1 + F_2 \qquad \text{weakly on } B_2 \setminus \overline{B} \text{ and weakly on } B, \tag{4.2}$$

with $|F_1| \le C|Dv_j|^2$ and $F_2 \in L^2(B_2)$. More precisely, it holds

$$\|F_2\|_{L^2} \le C\left[\left\|\frac{u_j - u_{j-1}}{h}\right\|_{L^2} + \|Dg\|^2_{L^4} + \|\Delta g\|_{L^2}\right]$$

and $F_i(x) = -\frac{1}{|x|^4} F_i(\frac{x}{|x|^2})$ for $x \in B_2 \setminus \overline{B}$ and $i \in \{1, 2\}$. We claim that (4.2) is satisfied on all of B_2. To this end, we choose an arbitrary testing function $\varphi \in C^\infty_0(B_2 \setminus \overline{B_{1/2}}, \mathbb{R}^3)$ and decompose φ into its antisymmetric and its symmetric part with respect to reflection across ∂B, i.e. $\varphi = \varphi_- + \varphi_+$, where

$$\varphi_-(\tfrac{x}{|x|^2}) = -\varphi_-(x) \qquad \text{and} \qquad \varphi_+(\tfrac{x}{|x|^2}) = \varphi_+(x)$$

for all $x \in B_2 \setminus \overline{B_{1/2}}$. This implies in particular $\varphi_- = 0$ on ∂B so that φ_- is an admissible testing function in (4.2); note that $\varphi_- \in W^{1,2}_0 \cap L^\infty$ on $B_2 \setminus \overline{B}$ and B. We thereby deduce that

$$\int_{B_2} Dv_j \cdot D\varphi_- \, dx = \int_{B_2} (F_1 + F_2) \cdot \varphi_- \, dx \tag{4.3}$$

holds true. Next, we use the symmetry of φ_+ and of v_j to calculate

$$\begin{aligned}\int_{B_2\setminus B} Dv_j(x)\cdot D\varphi_+(x)\,dx &= -\int_{B_2\setminus B} D[v_j(\tfrac{x}{|x|^2})]\cdot D[\varphi_+(\tfrac{x}{|x|^2})]\,dx\\ &= -\int_{B_2\setminus B} Dv_j(\tfrac{x}{|x|^2})\cdot D\varphi_+(\tfrac{x}{|x|^2})\tfrac{1}{|x|^4}\,dx\\ &= -\int_{B\setminus B_{1/2}} Dv_j(x)\cdot D\varphi_+(x)\,dx.\end{aligned}$$

Here, we transformed in the second last line the integral by the transformation $\psi(x)=\frac{x}{|x|^2}$, which satisfies $|\det D\psi(x)|=\frac{1}{|x|^4}$. This means that

$$\begin{aligned}&\int_{B_2} Dv_j(x)\cdot D\varphi_+(x)\,dx\\ &\quad=\int_{B_2\setminus B} Dv_j(x)\cdot D\varphi_+(x)\,dx+\int_{B\setminus B_{1/2}} Dv_j(x)\cdot D\varphi_+(x)\,dx=0.\end{aligned}$$

A similar computation shows that

$$\int_{B_2}(F_1+F_2)\cdot\varphi_+\,dx=0.$$

Joining these results with (4.3), we arrive at

$$\int_{B_2} Dv_j\cdot D\varphi\,dx=\int_{B_2}(F_1+F_2)\cdot\varphi\,dx,$$

which proves the claim that v_j weakly solves

$$-\Delta v_j=F_1+F_2\qquad\text{on } B_2.$$

We recall that $|F_1|\le C|Dv_j|^2$ and $F_2\in L^2(B_2)$ so that this is exactly the type of system considered in [11]. Since we already know that v_j is continuous, we can apply the Calderón-Zygmund estimates from [11, Theorem 7.5] to deduce $v_j\in W^{1,4}_{\rm loc}(B_2,\mathbb{R}^3)$. At this stage one should mention that we only need to apply [11, Theorem 7.5] in the case of an interior disk. This implies that $F_1+F_2\in L^2_{\rm loc}(B_2,\mathbb{R}^3)$, so that a standard application of the difference quotient technique implies $v_j\in W^{2,2}_{\rm loc}(B_2,\mathbb{R}^3)$. In particular, we have $u_j-g=v_j|_B\in W^{2,2}(B,\mathbb{R}^3)$, which implies the claim. □

In [3, Lemma 7.3], we established the corresponding quantitative estimates on the time slices, see also [11, Lemma 8.1] for a version at the Plateau boundary. However, these estimates are not suitable for our purposes since they depend on the discrete time derivative. Here, we need to establish a parabolic counterpart of these estimates that exploits the fact that the sequence $(u_{j,h})_{j\in\mathbb{N}_o}$ solves a time-discrete parabolic system.

Theorem 4.4. *There are constants $\varepsilon_o=\varepsilon_o(\|H\|_{L^\infty})>0$ and $C_2=C_2(\|H\|_{L^\infty})$ such that the following holds: Assume that the maps $u_{j,h}\in W^{1,2}(B,\mathbb{R}^3)$ constructed in Section 3 satisfy*

$$\sup_{j_1+1\le j\le j_2}\int_{B_R(x_o)}|Du_{j,h}|^2\,dx\le\varepsilon_o \tag{4.4}$$

for some $j_1<j_2$ in $\mathbb{N}_0$ and some disk $B_R(x_o)\subset B$. Then, for any $r\in[\frac{R}{2},R)$ we can conclude

$$\begin{aligned}&\int_{B_r(x_o)}|Du_{j_2,h}|^2\,dx+h\sum_{j=j_1+1}^{j_2}\int_{B_r(x_o)}|D^2u_{j,h}|^2\,dx\\ &\qquad\le C_2\int_{B_R(x_o)}|Du_{j_1,h}|^2\,dx+\frac{C_2h}{(R-r)^2}\sum_{j=j_1+1}^{j_2}\int_{B_R(x_o)}|Du_{j,h}|^2\,dx\,.\end{aligned}$$

Proof. For the sake of convenience, we omit the step size h in the notation and write u_j instead of $u_{j,h}$. Moreover, all appearing disks will have center in x_o and therefore we omit in the notation also the center. From Lemma 4.3 we know $u_j \in W^{2,2}(B,\mathbb{R}^3)$ for every $j \in \mathbb{N}$. Consequently, we can test equation (3.6) with the testing functions $D_\alpha(\eta^2 D_\alpha u_j)$ for $\alpha \in \{1,2\}$, whenever $\eta \in C_0^\infty(B_R(x_o))$ is a smooth cut-off function. After three integrations by parts, this leads us to

$$\tfrac{1}{h}\int_B \eta^2 D_\alpha u_j \cdot (D_\alpha u_j - D_\alpha u_{j-1})\,dx + \int_B D_\alpha D u_j \cdot \big[\eta^2 D D_\alpha u_j + 2\eta D\eta \otimes D_\alpha u_j\big]\,dx \\ \le C\int_B |Du_j|^2 \big|D_\alpha(\eta^2 D_\alpha u_j)\big|\,dx.$$

We multiply this inequality by h, sum over $\alpha = 1,2$ and apply Young's inequality several times. In this way we deduce

$$\int_B \eta^2 |Du_j|^2\,dx + h\int_B \eta^2 |D^2 u_j|^2\,dx \\ \le \tfrac12 \int_B \eta^2 |Du_j|^2\,dx + \tfrac12\int_B \eta^2|Du_{j-1}|^2\,dx + \tfrac12 h\int_B \eta^2 |D^2u_j|^2\,dx \\ + Ch\int_B |D\eta|^2\,|Du_j|^2\,dx + Ch\int_B \eta^2 |Du_j|^4\,dx.$$

The first and third integral on the right-hand side can be re-absorbed into the left-hand side. Now, we fix radii $r \le \varrho < \sigma \le R$ and choose a cut-off function $\eta \in C_0^\infty(B_\sigma,[0,1])$ with $\eta \equiv 1$ on B_ϱ and $\|D\eta\|_{L^\infty} \le 2/(\sigma-\varrho)$. Then, from the preceding estimate we infer

$$\int_B \eta^2|Du_j|^2\,dx + h\int_{B_\varrho} |D^2u_j|^2\,dx \\ \le \int_B \eta^2 |Du_{j-1}|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2}\int_{B_\sigma} |Du_j|^2\,dx + Ch\int_{B_\sigma} |Du_j|^4\,dx.$$

Summing this estimate over $j = j_1+1,\dots,j_2$ and taking advantage of cancellations, we deduce

$$\int_{B_\varrho} |Du_{j_2}|^2\,dx + h\sum_{j=j_1+1}^{j_2}\int_{B_\varrho} |D^2u_j|^2\,dx \\ \le \int_{B_R} |Du_{j_1}|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2}\sum_{j=j_1+1}^{j_2}\int_{B_\sigma}|Du_j|^2\,dx + Ch\sum_{j=j_1+1}^{j_2}\int_{B_\sigma}|Du_j|^4\,dx.$$

For the estimate of the last term, we combine the Gagliardo-Nirenberg inequality (2.1) with the assumption (4.4), with the result

$$\int_{B_\sigma} |Du_j|^4\,dx \le C\varepsilon_o \int_{B_\sigma} |D^2u_j|^2 + \Big|\frac{Du_j}{\sigma}\Big|^2\,dx$$

for all $j \in \mathbb{N}$ with $j_1+1 \le j \le j_2$. Joining the two preceding estimates and keeping in mind $\sigma - \varrho \le \sigma$, we arrive at

$$f(\varrho) \le C\varepsilon_o f(\sigma) + \int_{B_R} |Du_{j_1}|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2}\sum_{j=j_1+1}^{j_2}\int_{B_r}|Du_j|^2\,dx,$$

where we have set

$$f(s) := \int_{B_s} |Du_{j_2}|^2\,dx + h\sum_{j=j_1+1}^{j_2}\int_{B_s}|D^2u_j|^2\,dx.$$

At this stage we choose $\varepsilon_o > 0$ so small that $C\varepsilon_o \le \frac12$. Then we can apply Lemma 2.1 to the function $f(s)$. This implies the asserted estimate. □

Corollary 4.5. *There are constants $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$ and $C_2 = C_2(\|H\|_{L^\infty})$ such that the following holds: Assume that the maps $u_{j,h} \in W^{1,2}(B,\mathbb{R}^3)$ constructed in Section 3 satisfy (4.4) for some $j_1 < j_2$ in $\mathbb{N}_0$ and some disk $B_R(x_o) \subset B$. Then, for any radius $r \in [\frac{R}{2}, R)$ and $j^* \in \mathbb{N}_0$ with $j_1 + 1 \le j^* < j_2$ we have*

$$\int_{B_r(x_o)} |Du_{j_2,h}|^2\,dx + h\sum_{j=j^*}^{j_2}\int_{B_r(x_o)} |D^2u_{j,h}|^2\,dx \le \left[\frac{C_2 h}{(R-r)^2} + \frac{C_2}{j^*-j_1}\right]\sum_{j=j_1}^{j_2}\int_{B_R(x_o)} |Du_{j,h}|^2\,dx\,. \tag{4.5}$$

Proof. We apply Theorem 4.4 with $j_1, j_1+1, \dots, j^*-1 < j_2$ instead of j_1 and sum up the resulting inequalities. The resulting inequality we divide by $j^* - j_1$. This yields the result. □

Corollary 4.6. *There are constants $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$ and $C_3 = C_3(\|H\|_{L^\infty})$ such that the following holds: Assume that the maps $u_{h_i} \in L^\infty(0,\infty; W^{1,2}(B,\mathbb{R}^3))$ constructed in Section 3 satisfy*

$$\liminf_{i\to\infty}\sup_{t\in(t_o-R^2,t_o)}\int_{B_R(x_o)\times\{t\}} |Du_{h_i}|^2\,dx \le \tfrac12\varepsilon_o$$

for some disk $B_R(x_o) \subset B$ and $t_o \in [R^2,\infty)$, then we can conclude that

$$\int_{Q_r(z_o)} |D^2u|^2\,dz \le \frac{C_3}{(R-r)^2}\int_{Q_R(z_o)} |Du|^2\,dx$$

holds true for any $0 < r < R$.

Proof. By passing to a subsequence if necessary, we can assume that

$$\sup_{t\in(t_o-R^2,t_o)}\int_{B_R(x_o)\times\{t\}} |Du_{h_i}|^2\,dx \le \varepsilon_o$$

holds for any $i \in \mathbb{N}$. Now we consider radii $0 < r < \varrho \le R$. Recalling the definition of u_{h_i} in (3.2), for a fixed value of $i \in \mathbb{N}$ we choose j_1 such that $j_1h_i \le t_o - \varrho^2 < (j_1+1)h_i$ and j_2 with $j_2h_i < t_o \le (j_2+1)h_i$. This choice implies $j_1 < j_2$ as long as h_i is sufficiently small. Finally the integer j^* is chosen to satisfy $j^*h_i < t_o - r^2 \le (j^*+1)h_i$. Note that on the intervall $(j-1)h_i < t \le jh_i$ with $j \in \{j_1+1, \dots, j_2\}$ we have $u_{h_i} \equiv u_{j,h_i}$, so that the preceding choices imply that the smallness assumption (4.4) in Corollary 4.5 holds true for large values of $i \in \mathbb{N}$. We calculate

$$(j^* - j_1)h_i \ge \varrho^2 - r^2 - h_i \ge (\varrho - r)^2 - h_i.$$

Applying (4.5) and recalling the definition of u_{h_i} we find

$$\begin{aligned}\int_{t_o-r^2}^{t_o-h_i}\int_{B_r(x_o)} |D^2u_{h_i}|^2\,dx\,dt &\le \int_{j^*h_i}^{j_2h_i}\int_{B_r(x_o)} |D^2u_{h_i}|^2\,dx\,dt\\ &\le \left[\frac{C_2}{(\varrho-r)^2} + \frac{C_2}{(j^*-j_1)h_i}\right]\int_{(j_1-1)h_i}^{j_2h_i}\int_{B_\varrho(x_o)} |Du_{h_i}|^2\,dx\,dt\\ &\le \frac{2C_2}{(\varrho-r)^2 - h_i}\int_{t_o-\varrho^2-2h_i}^{t_o}\int_{B_\varrho(x_o)} |Du_{h_i}|^2\,dx\,dt\,. \end{aligned} \tag{4.6}$$

The next step is to show the convergence of the right-hand side. To this end, we observe that by choosing $(r,\varrho) = (S,R)$, we infer from the above estimate that

$$\limsup_{i\to\infty}\int_{t_o-S^2}^{t_o-\delta}\int_{B_S(x_o)} |D^2u_{h_i}|^2\,dx\,dt < \infty$$

holds for every $S < R$ and $\delta > 0$. Therefore, we can apply [29, Thm. 5] with the choices $(X, B, Y) = (W^{2,2}(B_S(x_o)), W^{1,2}(B_S(x_o)), L^2(B_S(x_o)))$ and $p = 2$. Arguing similarly as after [3, (8.11)] we get $Du_{h_i} \to Du$ strongly in $L^2(B_S(x_o)\times(t_o-S^2, t_o-\delta))$, for every $S < R$ and $\delta > 0$. Now we consider the estimate (4.6) for the choice $(r, \varrho) = (r, \frac{1}{2}(r + R))$. Passing to the limit $h_i \downarrow 0$, we use the strong convergence $Du_{h_i} \to Du$ on the right-hand side and the lower semicontinuity of the L^2-norm with respect to weak convergence on the left-hand side, with the result

$$\int_{t_o-r^2}^{t_o-\delta}\int_{B_r(x_o)} |D^2u|^2\,dz \le \frac{8C_2}{(R-r)^2}\int_{t_o-R^2}^{t_o}\int_{B_R(x_o)} |Du|^2\,dx\,dt\,.$$

Since $\delta > 0$ is arbitrary, this implies the asserted estimate. □

In the course of the proof of the main regularity result, we also need the analogue of Theorem 4.4 up to the boundary.

Theorem 4.7. *There are constants $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$ and $C_2 = C_2(\|H\|_{L^\infty})$ such that the following holds true: Assume that the maps $u_{j,h} \in W^{1,2}(B,\mathbb{R}^3)$ constructed in Section 3 satisfy*

$$\sup_{j_1+1\le j\le j_2}\int_{B_R(x_o)\cap B} |Du_{j,h}|^2\,dx \le \varepsilon_o \tag{4.7}$$

for some $j_1 < j_2$ in $\mathbb{N}_0$ and some disk $B_R(x_o) \subset \mathbb{R}^2 \setminus B_{1/2}$ with center $x_o \in \overline{B}$. Then for any $r \in [\frac{R}{2}, R)$ we have

$$\begin{aligned}
&\int_{B_r(x_o)\cap B} |Du_{j_2,h}|^2\,dx + h\sum_{j=j_1+1}^{j_2}\int_{B_r(x_o)\cap B} |D^2u_{j,h}|^2\,dx\\
&\quad\le C_2\int_{B_R(x_o)\cap B} |Du_{j_1,h}|^2\,dx + \frac{C_2h}{(R-r)^2}\sum_{j=j_1+1}^{j_2}\int_{B_R(x_o)\cap B} |Du_{j,h}|^2\,dx\\
&\qquad + C_2(j_2-j_1)h\int_{B_R(x_o)\cap B} |D^2g|^2 + \frac{|Dg|^2}{(R-r)^2}dx + C_2\int_{B_R(x_o)\cap B} |Dg|^2\,dx.
\end{aligned}$$

Proof. Again, we omit the step size h and the center x_o in the notation. The mappings $v_j := u_j - g$ are weak solutions of the system

$$\frac{v_j - v_{j-1}}{h} - \Delta v_j = -2(H \circ u_j)D_1u_j \times D_2u_j + \Delta g \quad \text{on } B. \tag{4.8}$$

From Lemma 4.3 we know $u_j \in W^{2,2}(B,\mathbb{R}^3)$. Therefore, we may multiply the preceding system with the testing function $D_\vartheta(\eta^2 D_\vartheta v_j) \in L^2(B,\mathbb{R}^3)$, and afterwards integrate over B. Here $\eta \in C_0^\infty(B_R)$ and D_ϑ denotes the angular derivative. In the following we write B_R^+ instead of $B_R \cap B$. Since $D_\vartheta v_j \in W_0^{1,2}(B_R^+,\mathbb{R}^3)$, we may integrate by parts to get, similarly as in the proof of Theorem 4.4

$$\begin{aligned}
&\int_B \eta^2|D_\vartheta v_j|^2\,dx + h\int_B \eta^2|D_\vartheta Dv_j|^2\,dx\\
&\quad\le \int_B \eta^2|D_\vartheta v_{j-1}|^2\,dx + Ch\int_B |D\eta|^2|Dv_j|^2\,dx\\
&\qquad + Ch\int_B \eta^2|Du_j|^4\,dx + Ch\int_{B_R^+} |\Delta g|^2\,dx.
\end{aligned}$$

As in the proof of Theorem 4.4 for radii $r \le \varrho < \sigma \le R$ we choose a standard cut-off function $\eta \in C_0^\infty(B_\sigma,[0,1])$ with $\eta \equiv 1$ on B_ϱ and $\|D\eta\|_{L^\infty} \le 2/(\sigma - \varrho)$. Summing the preceding estimate over $j = j_1+1,\dots,j_2$ we thereby deduce

$$
\begin{aligned}
(4.9)\qquad & \int_B \eta^2 |D_\vartheta v_{j_2}|^2\,dx + h\sum_{j=j_1+1}^{j_2}\int_{B_\varrho^+}|D_\vartheta D v_j|^2\,dx \\
&\le \int_B \eta^2|D_\vartheta v_{j_1}|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2}\sum_{j=j_1+1}^{j_2}\int_{B_R^+}|Dv_j|^2\,dx \\
&\quad + Ch\sum_{j=j_1+1}^{j_2}\int_{B_\sigma^+}|Du_j|^4\,dx + Ch(j_2-j_1)\int_{B_R^+}|\Delta g|^2\,dx.
\end{aligned}
$$

To bound the full second derivative, we use the equation (4.8) for v_j in order to express the second radial derivative $D_r^2 v_j$ as follows:

$$
\begin{aligned}
D_r^2 v_j &= \Delta v_j - \tfrac{1}{|x|}D_r v_j - \tfrac{1}{|x|^2}D_\vartheta^2 v_j \\
&= \frac{u_j-u_{j-1}}{h} + 2(H\circ u_j)D_1u_j\times D_2u_j - \Delta g - \tfrac{1}{|x|}D_r v_j - \tfrac{1}{|x|^2}D_\vartheta^2 v_j.
\end{aligned}
$$

Keeping in mind that $|x|\ge\frac12$ on B_R we thereby deduce from (4.9) that

$$
\begin{aligned}
(4.10)\qquad & h\sum_{j=j_1+1}^{j_2}\int_{B_\varrho^+}|D^2 v_j|^2\,dx \\
&\le C\sum_{j=j_1+1}^{j_2}\int_{B_\varrho^+}\frac{|u_j-u_{j-1}|^2}{h}\,dx + \int_{B_R^+}|Dv_{j_1}|^2\,dx \\
&\quad + \frac{Ch}{(\sigma-\varrho)^2}\sum_{j=j_1+1}^{j_2}\int_{B_R^+}|Dv_j|^2\,dx + Ch\sum_{j=j_1+1}^{j_2}\int_{B_\sigma^+}|Du_j|^4\,dx \\
&\quad + Ch(j_2-j_1)\int_{B_R^+}|\Delta g|^2\,dx.
\end{aligned}
$$

In order to bound the first integral on the right-hand side, we test the equation (3.6) for u_j with the testing function $\eta^2(u_j-u_{j-1})$ for $\eta\in C_0^\infty(B_R)$ as above, which vanishes on ∂B because u_j and u_{j-1} have the same traces. An integration by parts and Young's inequality lead us to

$$
\begin{aligned}
&\int_B \eta^2\frac{|u_j-u_{j-1}|^2}{h}\,dx + \int_B \eta^2|Du_j|^2\,dx \\
&\le \int_B \eta^2 Du_j\cdot Du_{j-1} + 2\eta Du_j\cdot D\eta\otimes(u_j-u_{j-1})\,dx \\
&\quad + C\int_B \eta^2|Du_j|^2|u_j-u_{j-1}|\,dx \\
&\le \tfrac12\int_B\eta^2\frac{|u_j-u_{j-1}|^2}{h}\,dx + \tfrac12\int_B\eta^2|Du_j|^2\,dx + \tfrac12\int_B\eta^2|Du_{j-1}|^2\,dx \\
&\quad + Ch\int_B |D\eta|^2|Du_j|^2 + \eta^2|Du_j|^4\,dx.
\end{aligned}
$$

Re-absorbing the first two integrals on the right-hand side into the left-hand side we deduce

$$
\begin{aligned}
&\int_B \eta^2\frac{|u_j-u_{j-1}|^2}{h}\,dx + \int_B\eta^2|Du_j|^2\,dx \\
&\quad\le \int_B\eta^2|Du_{j-1}|^2\,dx + Ch\int_B|D\eta|^2|Du_j|^2 + \eta^2|Du_j|^4\,dx.
\end{aligned}
$$

Summing up over $j = j_1+1, \dots, j_2$, we arrive at

$$\int_{B_\varrho^+} |Du_{j_2}|^2\,dx + \sum_{j=j_1+1}^{j_2} \int_{B_\varrho^+} \frac{|u_j - u_{j-1}|^2}{h}\,dx$$
$$\le \int_{B_R^+} |Du_{j_1}|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2} \sum_{j=j_1+1}^{j_2} \int_{B_R^+} |Du_j|^2\,dx + Ch \sum_{j=j_1+1}^{j_2} \int_{B_\sigma^+} |Du_j|^4\,dx.$$

We use the preceding inequality in (4.10) to bound the integral over $\frac{1}{h}|u_j - u_{j-1}|^2$. Recalling also the definition $v_j = u_j - g$, we arrive at

$$h \sum_{j=j_1+1}^{j_2} \int_{B_\varrho^+} |D^2 u_j|^2\,dx$$
$$\le C \int_{B_R^+} |Du_{j_1}|^2 + |Dg|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2} \sum_{j=j_1+1}^{j_2} \int_{B_R^+} |Du_j|^2\,dx$$
$$+ Ch \sum_{j=j_1+1}^{j_2} \int_{B_\sigma^+} |Du_j|^4\,dx + Ch(j_2 - j_1) \int_{B_R^+} |D^2 g|^2 + \frac{|Dg|^2}{(\sigma-\varrho)^2}\,dx.$$

Adding the two preceding inequalities gives

$$\int_{B_\varrho^+} |Du_{j_2}|^2\,dx + h \sum_{j=j_1+1}^{j_2} \int_{B_\varrho^+} |D^2 u_j|^2 + \frac{|u_j - u_{j-1}|^2}{h^2}\,dx$$
$$\le C \int_{B_R^+} |Du_{j_1}|^2 + |Dg|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2} \sum_{j=j_1+1}^{j_2} \int_{B_R^+} |Du_j|^2\,dx$$
$$+ Ch \sum_{j=j_1+1}^{j_2} \int_{B_\sigma^+} |Du_j|^4\,dx + Ch(j_2 - j_1) \int_{B_R^+} |D^2 g|^2 + \frac{|Dg|^2}{(\sigma-\varrho)^2}\,dx.$$

We bound the second last term by the Gagliardo-Nirenberg inequality (2.1). By means of (4.7), we can estimate

$$Ch \int_{B_\sigma^+} |Du_j|^4\,dx \le Ch\varepsilon_o \int_{B_\sigma^+} |D^2 u_j|^2 + \Big|\frac{Du_j}{\sigma}\Big|^2\,dx$$

for all $j \in \mathbb{N}$ with $j_1 + 1 \le j \le j_2$. Choosing ε_o in dependence on $\|H\|_{L^\infty}$ small enough we can achieve $C\varepsilon_o \le \frac{1}{2}$. We thereby arrive at

$$\int_{B_\varrho^+} |Du_{j_2}|^2\,dx + h \sum_{j=j_1+1}^{j_2} \int_{B_\varrho^+} |D^2 u_j|^2 + \frac{|u_j - u_{j-1}|^2}{h^2}\,dx$$
$$\le \tfrac{1}{2} h \sum_{j=j_1+1}^{j_2} \int_{B_\sigma^+} |D^2 u_j|^2\,dx + \frac{Ch}{(\sigma-\varrho)^2} \sum_{j=j_1+1}^{j_2} \int_{B_R^+} |Du_j|^2\,dx$$
$$+ C \int_{B_R^+} |Du_{j_1}|^2 + |Dg|^2\,dx + Ch(j_2 - j_1) \int_{B_R^+} |D^2 g|^2 + \frac{|Dg|^2}{(\sigma-\varrho)^2}\,dx.$$

Applying the iteration Lemma 2.1, we can absorb the first integral on the right-hand side, with the result

$$\int_{B_r^+} |Du_{j_2}|^2\,dx + h \sum_{j=j_1+1}^{j_2} \int_{B_r^+} |D^2 u_j|^2 + \frac{|u_j - u_{j-1}|^2}{h^2}\,dx$$

$$\leq \frac{Ch}{(R-r)^2}\sum_{j=j_1+1}^{j_2}\int_{B_R^+}|Du_j|^2\,dx + C\int_{B_R^+}|Du_{j_1}|^2+|Dg|^2\,dx$$
$$+ Ch(j_2-j_1)\int_{B_R^+}|D^2g|^2+\frac{|Dg|^2}{(R-r)^2}\,dx.$$

This implies the asserted estimate. □

The preceding theorem implies the following two corollaries in the same way as in the interior case Theorem 4.4 implies Corollaries 4.5 and 4.6. Therefore, we state them without proof.

Corollary 4.8. *There are constants $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$ and $C_2 = C_2(\|H\|_{L^\infty})$ such that the following holds: Assume that the maps $u_{j,h} \in W^{1,2}(B,\mathbb{R}^3)$ constructed in Section 3 satisfy (4.4) for some $j_1 < j_2$ in $\mathbb{N}_0$ and $B_R(x_o) \subset \mathbb{R}^2 \setminus B_{1/2}$ with $x_o \in \overline{B}$. Then, for any $r \in [\frac{R}{2}, R)$ and $j^* \in \mathbb{N}_0$ with $j_1 + 1 \leq j^* < j_2$ we have*

$$\int_{B_r(x_o)\cap B}|Du_{j_2,h}|^2\,dx + h\sum_{j=j^*}^{j_2}\int_{B_r(x_o)\cap B}|D^2u_{j,h}|^2\,dx$$
$$\leq \left[\frac{C_2h}{(R-r)^2}+\frac{C_2}{j^*-j_1}\right]\sum_{j=j_1}^{j_2}\int_{B_R(x_o)\cap B}|Du_{j,h}|^2\,dx$$
$$+ C_2h(j_2-j_1)\int_{B_R(x_o)\cap B}|D^2g|^2\,dx$$
$$+\left[\frac{C_2h}{(R-r)^2}+\frac{C_2}{j^*-j_1}\right](j_2-j_1)\int_{B_R(x_o)\cap B}|Dg|^2\,dx\,.$$

Corollary 4.9. *There are constants $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$ and $C_3 = C_3(\|H\|_{L^\infty})$ such that the following holds: Assume that the maps $u_{h_i} \in L^\infty(0,\infty;W^{1,2}(B,\mathbb{R}^3))$ constructed in Section 3 satisfy*

$$\liminf_{i\to\infty}\sup_{t\in(t_o-R^2,t_o)}\int_{B_R^+(x_o)\times\{t\}}|Du_{h_i}|^2\,dx \leq \tfrac12\varepsilon_o$$

for some $B_R(x_o) \subset \mathbb{R}^2 \setminus B_{1/2}$ with $x_o \in \overline{B}$ and $t_o \in [R^2,\infty)$, then we can conclude that for any $0 < r < R$ the following quantitative $W^{2,2}$-estimate holds true:

$$\int_{Q_r^+(z_o)}|D^2u|^2\,dz \leq \frac{C_3}{(R-r)^2}\int_{Q_R^+(z_o)}|Du|^2\,dx + C_3R^2\int_{B_R^+(x_o)}|D^2g|^2+\frac{|Dg|^2}{(R-r)^2}\,dx.$$

Here, we have abbreviated $B_R^+(x_o) := B_R(x_o)\cap B$ and $Q_R^+(z_o) := Q_R(z_o)\cap B\times(0,\infty)$.

4.2. **Pushing the energy bounds one step further.**

In this chapter we show that the energy distribution of the minimizers $u_{j,h}$ is controlled in terms of the energy distribution of $u_{j-1,h}$ up to an error of size h. This means that the energy cannot increase too much if we pass from $j-1$ to j. The precise statement is as follows.

Lemma 4.10. *For every step size $h > 0$, radii $0 < \frac{R}{2} \leq r < R$, $x_o \in \overline{B}$ and $j \in \mathbb{N}$ we have*

$$\int_{B_r(x_o)\cap B}|Du_{j,h}|^2\,dx \leq C_3\int_{B_R(x_o)\cap B}|Du_{j-1,h}|^2\,dx + \frac{C_3h}{(R-r)^2}\mathbf{D}(u_o)$$

with a constant C_3 depending only on the constant $c \in (0,1)$ from the isoperimetric condition.

Proof. For the sake of notational convenience, we omit again the step size h in the proof and write e.g. $u_j = u_{j,h}$. Furthermore, we abbreviate $B_r^+ := B_r(x_o) \cap B$. Next, we fix radii ϱ, σ with $r \le \varrho < \sigma \le R$ and a cut-off function $\eta \in C_0^\infty(B_\sigma, [0,1])$ with $\eta \equiv 1$ on B_ϱ and $\|D\eta\|_{L^\infty} \le 2/(\sigma - \varrho)$. As comparison map for u_j, we define

$$v := u_j + \eta(u_{j-1} - u_j) = u_{j-1} + (1-\eta)(u_j - u_{j-1}).$$

By construction we have $v = u_j$ on ∂B, so that v is admissible as competitor for u_j. From the minimality property of u_j we thus deduce

$$\begin{aligned} &\tfrac{1}{2}\int_{B_\sigma^+} |Du_j|^2\,dx + 2\mathbf{V}_H(u_j, u_o) + \tfrac{1}{2h}\int_{B_\sigma^+} |u_j - u_{j-1}|^2\,dx \\ &\qquad \le \tfrac{1}{2}\int_{B_\sigma^+} |Dv|^2\,dx + 2\mathbf{V}_H(v, u_o) + \tfrac{1}{2h}\int_{B_\sigma^+} |v - u_{j-1}|^2\,dx. \end{aligned}$$

The volume terms can be estimated by an application of the isoperimetric condition (2.4). We obtain the bound

$$2\big|\mathbf{V}_H(v,u_o) - \mathbf{V}_H(u_j,u_o)\big| = 2|\mathbf{V}_H(v,u_j)| \le \tfrac{1}{2}c\int_{B_\sigma^+} |Du_j|^2 + |Dv|^2\,dx,$$

where we used that $v = u_j$ in $B \setminus B_\sigma$. Combining the two preceding estimates and taking into account that $|v - u_{j-1}| = (1-\eta)|u_j - u_{j-1}|$, we infer

$$\tfrac{1-c}{2}\int_{B_\sigma^+} |Du_j|^2\,dx + \tfrac{1}{2h}\int_{B_\sigma^+}\big[1-(1-\eta)^2\big]|u_j - u_{j-1}|^2\,dx \le \tfrac{1+c}{2}\int_{B_\sigma^+} |Dv|^2\,dx.$$

Since $\eta \equiv 1$ on B_ϱ^+, this implies

$$\int_{B_\varrho^+} |Du_j|^2\,dx + \tfrac{1}{h}\int_{B_\varrho^+} |u_j - u_{j-1}|^2\,dx \le C_1\int_{B_\sigma^+} |Dv|^2\,dx, \tag{4.11}$$

where we have set $C_1 := \frac{1+c}{1-c}$. Next, we estimate

$$\begin{aligned} |Dv|^2 &\le 2|\eta Du_{j-1} + (1-\eta)Du_j|^2 + 2|D\eta|^2|u_{j-1} - u_j|^2 \\ &\le 2\eta|Du_{j-1}|^2 + 2(1-\eta)|Du_j|^2 + \frac{8}{(\sigma-\varrho)^2}|u_{j-1} - u_j|^2, \end{aligned}$$

where we used the convexity of $|\cdot|^2$ in the last step. Plugging this estimate into (4.11) and keeping in mind that $\eta \equiv 1$ on B_ϱ, we obtain

$$\begin{aligned} &\int_{B_\varrho^+} |Du_j|^2\,dx + \tfrac{1}{h}\int_{B_\varrho^+} |u_j - u_{j-1}|^2\,dx \\ &\qquad \le 2C_1\int_{B_\sigma^+\setminus B_\varrho^+} |Du_j|^2\,dx + 2C_1\int_{B_R^+} |Du_{j-1}|^2\,dx \\ &\qquad\quad + \frac{8C_1}{(\sigma-\varrho)^2}\int_{B_R^+} |u_{j-1} - u_j|^2\,dx. \end{aligned}$$

At this stage we apply the hole-filling trick and add $2C_1\int_{B_\varrho^+} |Du_j|^2\,dx$ to both sides of the preceding inequality. Writing $\vartheta := \frac{2C_1}{2C_1+1} \in (0,1)$, we thereby get

$$\begin{aligned} &\int_{B_\varrho^+} |Du_j|^2\,dx \\ &\qquad \le \vartheta\int_{B_\sigma^+} |Du_j|^2\,dx + \int_{B_R^+} |Du_{j-1}|^2\,dx + \frac{4}{(\sigma-\varrho)^2}\int_{B_R^+} |u_{j-1} - u_j|^2\,dx. \end{aligned}$$

Since $\vartheta \in (0,1)$, we can apply the iteration Lemma 2.1 in order to get rid of the first integral on the right-hand side. This leads us to the estimate

$$\int_{B_r^+} |Du_j|^2\,dx \le C\int_{B_R^+} |Du_{j-1}|^2\,dx + \frac{C}{(R-r)^2}\int_{B_R^+} |u_{j-1} - u_j|^2\,dx$$

with a constant $C = C(\vartheta) = C(c)$. This implies the asserted estimate by bounding the last integral with the help of (3.5). □

4.3. **A local energy inequality.** In this section we show, that if the energy at a discrete time step is uniformly small (equi-distributed) at a certain scale r, then for a short time the energy at future discrete time steps cannot concentrate too much. Roughly speaking, if the energy is uniformly small at a certain discrete time scale then energy concentration cannot occur immediately on the following discrete future time scales.

Theorem 4.11. *There are positive constants $\delta = \delta(\|H\|_{L^\infty}, c, \mathbf{D}(u_o), \|Dg\|_{W^{1,2}}) > 0$ and $\varepsilon_1 = \varepsilon_1(\|H\|_{L^\infty}, c) > 0$, and a radius $r_o = r_o(\|Dg\|_{L^4}) \in (0, \frac{1}{4}]$ such that the following holds true: Whenever $j_1 \in \mathbb{N}_0$ is a fixed index such that the smallness condition*

$$\varepsilon := \sup_{x_o \in \overline{B}} \int_{B_r(x_o)\cap B} |Du_{j_1,h}|^2\,dx \le \varepsilon_1 \tag{4.12}$$

holds true for a fixed radius $r \in (0, r_o]$, then for any index $j_2 \in \mathbb{N}$ satisfying $j_1 < j_2 \le j_1 + \delta r^2/h$ and any $x_o \in \overline{B}$ we have

$$\begin{aligned} \int_{B_{r/2}(x_o)\cap B} &|Du_{j_2,h}|^2\,dx + h\sum_{j=j_1+1}^{j_2}\int_{B_{r/2}(x_o)\cap B}|D^2u_{j,h}|^2\,dx \\ &\le C_2\int_{B_r(x_o)\cap B}|Du_{j_1,h}|^2\,dx + \frac{C_2h}{r^2}\sum_{j=j_1+1}^{j_2}\int_{B_r(x_o)\cap B}|Du_{j,h}|^2\,dx \\ &\quad + C_2h(j_2-j_1)\int_{B_r(x_o)\cap B}|D^2g|^2 + \Big|\frac{Dg}{r}\Big|^2 dx + C_2\int_{B_r(x_o)\cap B}|Dg|^2dx, \end{aligned} \tag{4.13}$$

where $C_2 = C_2(\|H\|_{L^\infty}) \ge 1$ stands for the constant from Theorem 4.4. Here, the last two integrals can be omitted in the case of an interior disk $B_r(x_o) \subset B$. Moreover, we have

$$\begin{aligned} \int_{B_{r/2}(x_o)\cap B}|Du_{j_2,h}|^2\,dx &\le C_2\Big[\varepsilon + \frac{(j_2-j_1)h}{r^2}C_1\mathbf{D}(u_o)\Big] + C_2\int_{B_r(x_o)\cap B}|Dg|^2dx \\ &\quad + C_2h(j_2-j_1)\int_{B_r(x_o)\cap B}|D^2g|^2 + \Big|\frac{Dg}{r}\Big|^2dx\,. \end{aligned} \tag{4.14}$$

Proof. Again, we will drop the step size h from the subscript and write $u_j = u_{j,h}$, etc. We note that for $h > \delta r^2$, there is nothing to prove because then there is no $j_2 \in \mathbb{N}$ with $j_1 < j_2 \le j_1 + \delta r^2/h$. Consequently, we can restrict ourselves to the case $h \le \delta r^2$. Then we define:

$$j_* := \max\big\{j \in \mathbb{N}_{\ge j_1}:\ (4.13) \text{ holds } \forall j_2 \in (j_1, j]\cap\mathbb{N},\ \forall B_r(x_o)\subset\mathbb{R}^2\big\}.$$

Here, r is the fixed radius from the smallness assumption (4.12). Note that (4.13) holds trivially at level j_1. For the proof of (4.13), we need to show $j_* + 1 > j_1 + \delta r^2/h$. We assume for contradiction that $j_* + 1 \le j_1 + \delta r^2/h$. Since (4.13) holds true for any index $j \in (j_1, j_*]\cap\mathbb{N}$, we know

$$\int_{B_{r/2}(y)\cap B}|Du_j|^2\,dx \le C_2\varepsilon_1 + \frac{C_2h}{r^2}(j-j_1)\Big[C_1\mathbf{D}(u_o) + \|Dg\|^2_{W^{1,2}}\Big] + C_2\sqrt{\pi}\,r\|Dg\|^2_{L^4}$$

for all balls $B_r(y)$ with $y \in \overline{B}$. In turn we used the global energy bound (3.5), the fact $r \le 1$ and Hölder's inequality. Since we have assumed $jh \le j_*h < j_1h + \delta r^2$ and $r \le r_o$, this implies

$$\int_{B_{r/2}(y)\cap B}|Du_j|^2\,dx \le C_2\Big[\varepsilon_1 + \delta C_1\mathbf{D}(u_o) + \delta\|Dg\|^2_{W^{1,2}} + \sqrt{\pi}\,r_o\|Dg\|^2_{L^4}\Big],$$

for all indices $j \in (j_1, j_*] \cap \mathbb{N}$. Using this for $j = j_*$ and applying Lemma 4.10 to the pair u_{j_*+1}, u_{j_*} with $\frac{1}{4}r, \frac{1}{2}r$ instead of r, R, we deduce furthermore

$$\begin{aligned}\int_{B_{r/4}(y)\cap B} &|Du_{j_*+1}|^2\,dx \\ &\le C_3 \int_{B_{r/2}(y)\cap B} |Du_{j_*}|^2\,dx + \frac{16C_3 h}{r^2}\mathbf{D}(u_o) \\ &\le C_2C_3\Big[\varepsilon_1 + \delta(C_1+16)\mathbf{D}(u_o) + \delta\|Dg\|^2_{W^{1,2}} + \sqrt{\pi}\,r_o\|Dg\|^2_{L^4}\Big]\,.\end{aligned}$$

In the last step, we used the assumption $h \le \delta r^2$. Next, we cover a given ball $B_r(x_o)$ with center $x_o \in \overline{B}$ by balls $B_{r/4}(y)$. This can be done with a constant number C of such balls that does not depend on the radius r. Applying the preceding estimate on each of the balls $B_{r/4}(y)$ leads us to

$$\begin{aligned}\sup_{j_1<j\le j_*+1} &\int_{B_r(y)\cap B} |Du_j|^2\,dx \\ &\le CC_2C_3\Big[\varepsilon_1 + \delta(C_1+16)\mathbf{D}(u_o) + \delta\|Dg\|^2_{W^{1,2}} + \sqrt{\pi}\,r_o\|Dg\|^2_{L^4}\Big].\end{aligned}$$

At this stage, we fix the constants $\varepsilon_1, \delta > 0$, and the radius $r_o \in (0, \frac{1}{4}]$ so small that the right-hand side is bounded by $\varepsilon_o = \varepsilon_o(\|H\|_{L^\infty}) > 0$, by which we denote the minimum of the corresponding constants from Theorem 4.4 and Theorem 4.7. Since $r < r_o \le \frac{1}{4}$, we either have $B_r(x_o) \subset B$, in which case we can apply Theorem 4.4, or we have $B_r(x_o) \subset \mathbb{R}^2 \setminus B_{1/2}$, so that Theorem 4.7 is applicable. Hence, we obtain the bound

$$\begin{aligned}\int_{B_{r/2}(x_o)\cap B} &|Du_{j_*+1}|^2\,dx + h\sum_{j=j_1+1}^{j_*+1}\int_{B_{r/2}(x_o)\cap B}|D^2u_j|^2\,dx \\ &\le C_2\int_{B_r(x_o)\cap B}|Du_{j_1}|^2\,dx + \frac{C_2h}{r^2}\sum_{j=j_1+1}^{j_*+1}\int_{B_r(x_o)\cap B}|Du_j|^2\,dx \\ &\quad + C_2h(j_*+1-j_1)\int_{B_r(x_o)\cap B}|D^2g|^2 + \Big|\frac{Dg}{r}\Big|^2\,dx \\ &\quad + C_2\int_{B_r(x_o)\cap B}|Dg|^2\,dx\,,\end{aligned}$$

which means that (4.13) also holds for $j_2 = j_* + 1$, in contradiction to the definition of j_*. This yields the asserted bound (4.13). For the second claim, we simply use the smallness assumption (4.12) and the global energy bound (3.5) in order to bound the right-hand side of (4.13) from above. This completes the proof. □

At this point, Theorem 4.1 is an immediate consequence of Theorem 4.11.

5. AN ε-REGULARITY RESULT

and proof of Theorem 1.1

5.1. Interior regularity.

We start with the case of interior regularity.

Theorem 5.1. *There exists a constant $\varepsilon_2 = \varepsilon_2(\|H\|_{L^\infty}) > 0$ such that the following holds true: Assume that the maps $u_h \in L^\infty(0,\infty; W^{1,2}(B,\mathbb{R}^3))$ constructed in Section 3 satisfy for some $Q_R(z_o) \subset B \times (0,\infty)$ the smallness assumption*

$$\liminf_{i\to\infty}\sup_{t\in(t_o-R^2,t_o)}\int_{B_R(x_o)\times\{t\}}|Du_{h_i}|^2\,dx \le \tfrac{1}{2}\varepsilon_2 \tag{5.1}$$

and that $u \in L^\infty\big(0,\infty; W^{1,2}(B,\mathbb{R}^3)\big)$ is the limit map of the sequence (u_{h_i}) in the sense of (3.7). Then $Du \in C^{\beta,\beta/2}(Q_{R/2}(z_o), \mathbb{R}^{2\cdot 3})$ for any $\beta \in (0,1)$.

Proof. We first note that Corollary 4.6 and the smallness assumption (5.1) ensure that $D^2u \in L^2_{\rm loc}(Q_R(z_o))$, provided the constant ε_2 is smaller than the corresponding constant ε_o from Corollary 4.6. Therefore the Gagliardo-Nirenberg inequality can be applied to deduce that $Du \in L^4_{\rm loc}(Q_R(z_o))$. The proof is divided into several steps. We start with

5.1.1. *Caccioppoli inequality.* We consider $\frac{R}{2} \le \varrho < \tau \le R$ and set $\sigma := \frac{\varrho+\tau}{2}$. Next, we choose two cut-off functions $\eta \in C^\infty_0(B_\sigma(x_o), [0,1])$ with $\eta \equiv 1$ on $B_\varrho(x_o)$ and $|D\eta| \le \frac{2}{\sigma-\varrho} = \frac{4}{\tau-\varrho}$, and moreover a cut-off function $\zeta \in W^{1,\infty}(\mathbb{R}, [0,1])$ satisfying $\zeta \equiv 0$ on $(-\infty, t_o-\sigma^2)$, $\zeta(t) = \frac{1}{\sigma^2-\varrho^2}(t-t_o+\sigma^2)$ on $[t_o-\sigma^2, t_o-\varrho^2]$ and $\zeta \equiv 1$ on $(t_o-\varrho^2, \infty)$. In the weak form of the parabolic system (3.8) we then use the testing-function $\varphi(x,t) := \eta^2(x)\zeta(t)(u(x,t)-(u)_{z_o;R})$. Exploiting the fact $Du \in L^4_{\rm loc}(Q_R(z_o))$, we use Young's inequality in order to estimate

$$
\begin{aligned}
\int_{Q_R(z_o)} &\eta^2\zeta\partial_t u\cdot(u-(u)_{z_o;R}) + \zeta\eta^2|Du|^2\,dz \\
&\le 2\int_{Q_R(z_o)} \eta\zeta|Du||D\eta||u-(u)_{z_o;R}|\,dz \\
&\qquad + \|H\|_\infty \int_{Q_R(z_o)} \eta^2\zeta|Du|^2|u-(u)_{z_o;R}|\,dz \\
&\le \tfrac12\int_{Q_R(z_o)} \zeta\eta^2|Du|^2\,dz + \frac{C}{(\tau-\varrho)^2}\int_{Q_R(z_o)} |u-(u)_{z_o;R}|^2\,dz \\
&\qquad + (\tau-\varrho)^2\int_{Q_R(z_o)} \eta^2\zeta|Du|^4\,dz,
\end{aligned}
$$

for a constant $C = C(\|H\|_{L^\infty})$. The first term on the right-hand side can be absorbed into the left-hand side, and the first term on the left-hand side can be bounded from below by

$$
\begin{aligned}
\int_{Q_R(z_o)} \eta^2\zeta\partial_t u\cdot(u-(u)_{z_o;R})\,dz &= \tfrac12\int_{Q_R(z_o)} \eta^2\zeta\partial_t|u-(u)_{z_o;R}|^2\,dz \\
&\ge -\tfrac12\int_{Q_R(z_o)} \eta^2\zeta_t|u-(u)_{z_o;R}|^2\,dz.
\end{aligned}
$$

Joining the two preceding inequalities, using the bound $|\zeta_t| \le \frac{1}{\sigma^2-\varrho^2} \le \frac{1}{(\sigma-\varrho)^2} = \frac{4}{(\tau-\varrho)^2}$ and the fact that η has its support in $B_\sigma(x_o)$ we get

$$
\begin{aligned}
\int_{Q_R(z_o)} &\zeta\eta^2|Du|^2\,dz \\
&\le \frac{C}{(\tau-\varrho)^2}\int_{Q_R(z_o)} |u-(u)_{z_o;R}|^2\,dz + 2(\tau-\varrho)^2\int_{Q_\sigma(z_o)} |Du|^4\,dz.
\end{aligned}
$$

The second integral on the right-hand side is now estimated by an application of Gagliardo-Nirenberg's inequality (2.2), the smallness assumption (5.1) and Corollary 4.6. Note that the smallness condition continues to hold for u by lower semicontinuity. This leads us to

$$
\begin{aligned}
\int_{Q_\sigma(z_o)} |Du|^4\,dz &\le C \sup_{t_o-\sigma^2\le t\le t_o}\int_{B_\sigma(x_o)} |Du|^2\,dx \int_{Q_\sigma(z_o)} \Big[|D^2u|^2 + \Big|\frac{Du}{\sigma}\Big|^2\Big]\,dz \\
&\le C\,\varepsilon_2\int_{Q_\sigma(z_o)} \Big[|D^2u|^2 + \Big|\frac{Du}{\sigma}\Big|^2\Big]\,dz \\
&\le \frac{C\,\varepsilon_2}{(\tau-\varrho)^2}\int_{Q_\tau(z_o)} |Du|^2\,dz.
\end{aligned}
$$

Here, the constant C only depends on $\|H\|_{L^\infty}$. Inserting this above yields

$$
\int_{Q_\varrho(z_o)} |Du|^2\,dz \le C\,\varepsilon_2\int_{Q_\tau(z_o)} |Du|^2\,dz + \frac{C}{(\tau-\varrho)^2}\int_{Q_R(z_o)} |u-(u)_{z_o;R}|^2\,dz.
$$

At this point we may assume that $C(\|H\|_{L^\infty})\varepsilon_2 \le \frac{1}{2}$, so that the Iteration Lemma 2.1 is applicable. From this we infer the Caccioppoli inequality

$$\int_{Q_{R/2}(z_o)} |Du|^2\,dz \le \frac{C_c}{R^2}\int_{Q_R(z_o)} |u-(u)_{z_o;R}|^2\,dz, \tag{5.2}$$

where $C_c \equiv C(\|H\|_\infty) \ge 1$.

5.1.2. *Hölder continuity.* Having established the Caccioppoli inequality, we can proceed with a parabolic version of the harmonic approximation lemma that has been used in the elliptic setting in [31], see also [9]. To this end, we define the usual Campanato-type excess functional

$$E(u;z_o,R) := R^{-4}\int_{Q_R(z_o)} |u-(u)_{z_o;R}|^2\,dz.$$

Then, the parabolic system (3.8) and Caccioppoli's inequality (5.2) yield for any testing function $\varphi \in C_0^\infty(Q_{R/2}(z_o),\mathbb{R}^3)$ that

$$\begin{aligned}\bigg|\Big(\frac{R}{2}\Big)^{-2}&\int_{Q_{R/2}(z_o)} \big[u\cdot\partial_t\varphi - Du\cdot D\varphi\big]\,dz\bigg|\\ &\le \|H\|_\infty\,\|\varphi\|_{L^\infty}\Big(\frac{R}{2}\Big)^{-2}\int_{Q_{R/2}(z_o)}|Du|^2\,dz\\ &\le C_H C_c\,R\,E(u;z_o,R)\|D\varphi\|_{L^\infty},\end{aligned}$$

for a constant $C_H = C(\|H\|_\infty)\ge 1$. We could for example take $C_H = \max\{2\|H\|_\infty,1\}$. Therefore, the re-scaled function

$$v(x,t) := \frac{u(x,t)-(u)_{z_o;R/2}}{2^3 C_H C_c E(u;z_o,R)^{\frac{1}{2}}}$$

is an almost caloric function, in the sense that

$$\bigg|\Big(\frac{R}{2}\Big)^{-2}\int_{Q_{R/2}(z_o)}\big[v\cdot\partial_t\varphi - Dv\cdot D\varphi\big]\,dz\bigg| \le \tfrac{1}{2}R\,E(u;z_o,R)^{\frac{1}{2}}\|D\varphi\|_{L^\infty}$$

holds true for any $\varphi\in C_0^\infty(Q_{R/2}(z_o),\mathbb{R}^3)$. Moreover, by the definition of v and Caccioppoli's inequality (5.2), we have

$$\int_{Q_{R/2}(z_o)}\Big[\big(\tfrac{R}{2}\big)^{-4}|v|^2 + \big(\tfrac{R}{2}\big)^{-2}|Dv|^2\Big]\,dz \le 1.$$

Now, for $\varepsilon>0$ we let $\delta=\delta(\varepsilon)\in(0,1]$ be the constant from the caloric approximation Lemma [10, Lemma 4.3]. If the smallness condition

$$E(u;z_o,R)\le\delta^2$$

holds, we can apply the caloric approximation Lemma to infer the existence of a smooth caloric function $h\in L^2\big(t_o-(R/2)^2,t_o;W^{1,2}(B_{R/2}(x_o),\mathbb{R}^3)\big)$ satisfying

$$\int_{Q_{R/2}(z_o)}\Big[\big(\tfrac{R}{2}\big)^{-4}|h|^2 + \big(\tfrac{R}{2}\big)^{-2}|Dh|^2\Big]\,dz \le 1$$

and moreover

$$\Big(\frac{R}{2}\Big)^{-4}\int_{Q_{R/2}(z_o)}|v-h|^2\,dz\le\varepsilon.$$

Using the a priori estimate [5, Lemma 5.I] for the caloric function h we obtain for any $\theta\in(0,\frac{1}{4}]$ that

$$\begin{aligned}(\theta R)^{-4}&\int_{Q_{\theta R}(z_o)}|v-h(z_o)|^2\,dz\\ &\le 2(\theta R)^{-4}\int_{Q_{\theta R}(z_o)}\big[|v-h|^2+|h-h(z_o)|^2\big]\,dz\end{aligned}$$

$$\leq 2^{-3}\theta^{-4}\varepsilon + C\,\theta^2\Big(\frac{R}{2}\Big)^{-4}\int_{Q_{R/2}(z_o)}|h|^2\,dz$$
$$\leq C\big(\theta^{-4}\varepsilon+\theta^2\big).$$

Here we used in the last line also the L^2-bound for h from above. Scaling back from v to u and using the minimizing property of the mean value $(u)_{z_o;\theta R}$ in the mapping $\xi \mapsto \int_{Q_{\theta R}(z_o)}|u-\xi|^2\,dz$ we deduce that

$$(\theta R)^{-4}\int_{Q_{\theta R}(z_o)}|u-(u)_{z_o;\theta R}|^2\,dz \leq C\big(\theta^{-4}\varepsilon+\theta^2\big)E(u;z_o,R),$$

where the constant C depends only on $\|H\|_\infty$. Now, we fix the parameters ε and θ. For a given $\alpha\in(0,1)$ we first choose $\theta\in(0,\frac14]$ according to $C\theta^2\leq\frac12\theta^{2\alpha}$ and then $\varepsilon>0$ in such a way that also $C\theta^{-4}\varepsilon\leq\frac12\theta^{2\alpha}$. This determines the constants θ and ε in dependence on $\|H\|_\infty$ and α and consequently δ is fixed in dependence on the same data. Summarizing, we have shown so far: Given $\alpha\in(0,1)$ there exists $\delta=\delta(\|H\|_\infty,\alpha)>0$ such that if

$$E_o := R^{-4}\int_{Q_R(z_o)}|u-(u)_{z_o;R}|^2\,dz\leq\delta^2, \tag{5.3}$$

then

$$(\theta R)^{-4}\int_{Q_{\theta R}(z_o)}|u-(u)_{z_o;\theta R}|^2\,dz\leq\theta^{2\alpha}R^{-4}\int_{Q_R(z_o)}|u-(u)_{z_o;R}|^2\,dz,$$

where $\theta=\theta(\|H\|_\infty,\alpha)\in(0,\frac14]$. As usual, the iteration of the preceding inequality yields that

$$\varrho^{-4}\int_{Q_\varrho(z_o)}|u-(u)_{z_o;\varrho}|^2\,dz\leq C\Big(\frac{\varrho}{R}\Big)^{2\alpha}E_o$$

holds true for any $0<\varrho\leq R/2$ with a constant $C\equiv C(\|H\|_\infty,\alpha)$, provided that the smallness assumption (5.3) is satisfied. As usual, a slight modification of the argument yields that the preceding inequality does not hold only for the center z_o, but for all $w_o\in Q_{R/2}(z_o)$. By [7, Teorema 3.1], this implies the Hölder continuity of u with Hölder exponent α with respect to the parabolic metric together with the estimate

$$|u(z_1)-u(z_2)|\leq C\,E_o^{\frac12}\Big(\frac{d_{\mathcal P}(z_1,z_2)}{R}\Big)^{\alpha},\quad \text{for any } z_1,z_2\in Q_{R/2}(z_o), \tag{5.4}$$

for a constant $C\equiv C(\|H\|_\infty,\alpha)$.

Finally, we have to ensure that condition (5.3) is satisfied. This is a consequence of the standard Poincaré type inequality

$$\int_{Q_R(z_o)}|u-(u)_{z_o;R}|^2\,dz\leq C\,R^2\int_{Q_R(z_o)}|Du|^2\,dz+C(\|H\|_{L^\infty})\Big(\int_{Q_R(z_o)}|Du|^2\,dz\Big)^2$$

which holds true for solutions to (3.8). This inequality can be retrieved for example from [1, Lemma 8]. In view of the smallness assumption (5.1) – which is preserved in the limit $h_i\downarrow 0$ by lower semi-continuity – we find that $\int_{Q_R(z_o)}|Du|^2\,dz\leq R^2\varepsilon_2$. Using this estimate in the Poincaré inequality from above, we deduce

$$\int_{Q_R(z_o)}|u-(u)_{z_o;R}|^2\,dz\leq CR^4\varepsilon_2+CR^4\varepsilon_2^2\leq C\,R^4\varepsilon_2$$

with a constant $C\equiv C(\|H\|_{L^\infty})$. We can therefore fix $\varepsilon_2>0$ in dependence on $\|H\|_{L^\infty}$ and α in such a way that $C\varepsilon_2\leq\delta^2$. This implies that the smallness condition (5.3) holds true and thus yields the Hölder continuity of u in the interior.

5.1.3. *Hölder continuity of the spatial gradient.* Our next aim is to prove that the spatial gradient Du is Hölder continuous for any Hölder exponent less than 1. Let $w_o = (y_o, s_o) \in Q_{R/2}(z_o)$ and $\varrho < R/4$ such that the parabolic cylinder $Q_{2\varrho}(w_o)$ is contained in $Q_{R/2}(z_o)$. By $h \in C^0([s_o - \varrho^2, s_o]; L^2(B_\varrho(y_o))) \cap L^2(s_o - \varrho^2, s_o; W^{1,2}(B_\varrho(y_o)))$ we denote the solution to the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t h - \Delta h = 0 & \text{in } Q_\varrho(w_o) \\ h = u & \text{on } \partial_{\mathcal{P}} Q_\varrho(w_o). \end{cases}$$

Taking the difference of the weak formulation of the system (3.8) for u and the heat equation for h, we obtain

$$\int_{Q_\varrho(w_o)} \big[\partial_t(u-h)\cdot\varphi + D(u-h)\cdot D\varphi\big]\,dz = -2\int_{Q_\varrho(w_o)} (H\circ u)Du_1 \times Du_2 \cdot \varphi\,dz$$

for any testing function $\varphi \in L^2(s_o - \varrho^2, s_o; W^{1,2}_0(B_\varrho(y_o), \mathbb{R}^3))$. Here, we note that $\partial_t u, \partial_t h \in L^2(Q_\varrho(w_o), \mathbb{R}^3)$ by the construction in Section 3, respectively by standard regularity for the heat equation. Choosing $\varphi = u - h$, we get

$$\int_{Q_\varrho(w_o)} \big[\tfrac12 \partial_t |u-h|^2 + |Du - Dh|^2\big]\,dz \le \|H\|_{L^\infty} \sup_{Q_\varrho(w_o)} |u-h| \int_{Q_\varrho(w_o)} |Du|^2\,dz\,.$$

The first term on the left-hand side is non-negative, since $h = u$ on $\partial_{\mathcal{P}} Q_\varrho(w_o)$. Now, we fix $\alpha \in (0,1)$ and choose some boundary point $\bar z \in \partial_{\mathcal{P}} Q_\varrho(w_o)$. By the maximum principle we have

$$|h(\bar z) - h(z)| \le \max_{w \in \partial_{\mathcal{P}} Q_\varrho(w_o)} |u(\bar z) - u(w)| \le C[u]_{C^{0,\alpha}}\varrho^\alpha.$$

Hence, by (5.4) we have for any $z \in Q_\varrho(z_o)$ that

$$|u(z) - h(z)| \le |u(z) - u(\bar z)| + |h(\bar z) - h(z)| \le C\,E_o^{\frac12}\Big(\frac{\varrho}{R}\Big)^\alpha,$$

for a constant $C \equiv C(\|H\|_\infty, \alpha)$. Moreover, using the Caccioppoli inequality from (5.2) and (5.4), we find that

$$⨍_{Q_\varrho(w_o)} |Du|^2\,dz \le \frac{C}{\varrho^2} ⨍_{Q_{2\varrho}(w_o)} |u - (u)_{w_o;\varrho}|^2\,dz \le C\,\varrho^{2\alpha-2}R^{-2\alpha}E_o\,.$$

Inserting the last two estimates above and taking mean values yields the following *comparison estimate*

$$⨍_{Q_\varrho(w_o)} |Du - Dh|^2\,dz \le C\,\varrho^{3\alpha-2}R^{-3\alpha}E_o^{\frac32}\,.$$

We define the excess functional by

$$\Phi(u; w_o, \varrho) := ⨍_{Q_\varrho(w_o)} |Du - (Du)_{w_o;\varrho}|^2\,dz\,.$$

Now, let $0 < \tau \le \frac12$. Then, by the a priori estimates for h from [26, Lemma 4.13] we get

$$\begin{aligned} \Phi(u; w_o, \tau\varrho) &\le ⨍_{Q_{\tau\varrho}(w_o)} |Du - (Dh)_{w_o;\tau\varrho}|^2\,dz \\ &\le 2\Phi(h; w_o, \tau\varrho) + 2⨍_{Q_{\tau\varrho}(w_o)} |Du - Dh|^2\,dz \\ &\le C\Big[\tau^2\Phi(h; w_o, \varrho) + \tau^{-4}\varrho^{3\alpha-2}R^{-3\alpha}E_o^{\frac32}\Big] \\ &\le C\Big[\tau^2\Phi(u; w_o, \varrho) + \tau^{-4}\varrho^{3\alpha-2}R^{-3\alpha}E_o^{\frac32}\Big], \end{aligned} \tag{5.5}$$

where $C \equiv C(\|H\|_\infty, \alpha)$. At this point, a standard iteration argument, cf. [9, estimate (65)–(68)] shows that Du is Hölder continuous with respect to the parabolic metric with

Hölder exponent $\frac{3}{2}\alpha - 1$, provided $\alpha > \frac{2}{3}$. In order to eliminate the dependence of the constants on α, we fix some $\alpha \in (\frac{2}{3}, 1)$. Then, Du is Hölder continuous with Hölder exponent $\frac{3}{2}\alpha - 1 > 0$, and in particular the right-hand side of (3.8) is bounded. Therefore, we conclude by classical Schauder theory that Du is Hölder continuous with respect to the parabolic metric with any Hölder exponent $\beta < 1$. □

5.2. **Boundary regularity.** In this section we establish the Hölder continuity of Du up to the boundary.

Theorem 5.2. *There exists a constant $\varepsilon_2 = \varepsilon_2(\|H\|_{L^\infty}) > 0$ such that the following holds true: Assume that $g \in C^{1,\gamma}(B)$ for some $\gamma \in (0,1)$ and that the maps $u_h \in L^\infty(0,\infty; W^{1,2}(B,\mathbb{R}^3))$ constructed in Section 3 satisfy for some $x_o \in \partial B$, $t_o \in [R^2, \infty)$ and $0 < R \le \frac{1}{2}$ the smallness assumption*

$$\liminf_{i\to\infty} \sup_{t\in(t_o-R^2,t_o)} \int_{B_R^+(x_o)\times\{t\}} |Du_{h_i}|^2\,dx \le \tfrac{1}{2}\varepsilon_2 \tag{5.6}$$

and that $u \in L^\infty\big(0,\infty; W^{1,2}(B,\mathbb{R}^3)\big)$ is the limit map in the sense of (3.7), then $Du \in C^{\gamma,\gamma/2}(Q_{R/2}^+(z_o), \mathbb{R}^{2\cdot 3})$. Here, we have abbreviated $B_R^+(x_o) := B_R(x_o) \cap B$ and $Q_{R/2}^+(z_o) := Q_{R/2}(z_o) \cap B \times (0,\infty)$.

Proof. The starting point for the proof of Theorem 5.2 is the conclusion of Corollary 4.9: Due to our smallness assumption (5.6) we know that

$$\begin{aligned} &\int_{Q_r^+(z_o)} |D^2 u|^2\,dz \\ &\quad\le \frac{C_3}{(R-r)^2}\int_{Q_R^+(z_o)} |Du|^2\,dx + C_3 R^2 \int_{B_R^+(x_o)} |D^2 g|^2 + \frac{|Dg|^2}{(R-r)^2}\,dx \end{aligned} \tag{5.7}$$

holds true for any $0 < r < R$. Here, the constant $\varepsilon_2 > 0$ is still at our disposal. We only require at this stage that $\varepsilon_2 < \varepsilon_o$, where ε_o is the corresponding constant from Corollary 4.9. This allows us to prove a Caccioppoli-type inequality at the boundary, and this will be performed in the next subsection

5.2.1. *A Caccioppoli-type inequality at the boundary.* As in the interior situation we consider $\frac{R}{2} \le \varrho < \tau \le R$ and set $\sigma := \frac{\varrho+\tau}{2}$. As in Section 5.1.1, we choose the two cut-off functions $\eta \in C_0^\infty(B_\sigma(x_o), [0,1])$ and $\zeta \in W^{1,\infty}(\mathbb{R}, [0,1])$. Now, in the weak form of the parabolic system (3.8) we take the testing-function $\varphi(x,t) := \eta^2(x)\zeta(t)(u(x,t) - g(x))$. We note that this choice is admissible since $u(\cdot,t) = g$ on ∂B for a.e. $t > 0$ in the sense of traces. In turn we use Young's inequality and recall that g is independent of t, to conclude that

$$\begin{aligned} &\int_{Q_R^+(z_o)} \eta^2\zeta\partial_t u\cdot(u-g) + \eta^2\zeta|Du|^2\,dz \\ &\qquad\le \int_{Q_R^+(z_o)} \eta^2\zeta|Du||Dg|\,dz + 2\int_{Q_R^+(z_o)} \eta\zeta|Du||D\eta||u-g|\,dz \\ &\qquad\quad + \|H\|_\infty \int_{Q_R^+(z_o)} \eta^2\zeta|Du|^2|u-g|\,dz \\ &\qquad\le \tfrac{1}{2}\int_{Q_R^+(z_o)} \eta^2\zeta|Du|^2\,dz + R^2\int_{B_R^+(x_o)} |Dg|^2\,dx \\ &\qquad\quad + \frac{C}{(\tau-\varrho)^2}\int_{Q_R^+(z_o)} |u-g|^2\,dz + (\tau-\varrho)^2\int_{Q_R^+(z_o)} \eta^2\zeta|Du|^4\,dz, \end{aligned}$$

for a constant $C = C(\|H\|_{L^\infty})$. Using again that g is independent of t, the first term on the left-hand side can be re-written and estimated in the form

$$\int_{Q_R^+(z_o)} \eta^2\zeta\partial_t u\cdot(u-g)\,dz = \tfrac12\int_{Q_R^+(z_o)} \eta^2\zeta\partial_t|u-g|^2\,dz \geq -\tfrac12\int_{Q_R^+(z_o)} \eta^2\zeta_t|u-g|^2\,dz.$$

The following arguments follow closely the corresponding ones from the interior case. We use the estimate $|\zeta_t| \le \frac{1}{\sigma^2-\varrho^2} \le \frac{1}{(\sigma-\varrho)^2} = \frac{4}{(\tau-\varrho)^2}$ and the fact that η has its support in $B_\sigma(x_o)$ to deduce that

$$\int_{Q_R^+(z_o)} \zeta\eta^2|Du|^2\,dz \le 2R^2\int_{B_R^+(x_o)}|Dg|^2\,dz + \frac{C}{(\tau-\varrho)^2}\int_{Q_R^+(z_o)}|u-g|^2\,dz + 2(\tau-\varrho)^2\int_{Q_\sigma^+(z_o)}|Du|^4\,dz$$

holds true. We estimate the last integral by an application of Gagliardo-Nirenberg's inequality (2.2), the smallness assumption (5.1) – which continues to hold for the weak limit u by lower semi-continuity – and Corollary 4.9 with (r,R) replaced by (σ,τ). This leads us to

$$\begin{aligned}\int_{Q_\sigma^+(z_o)}|Du|^4\,dz &\le C\sup_{t_o-\sigma^2\le t\le t_o}\int_{B_\sigma^+(x_o)}|Du|^2\,dx\int_{Q_\sigma^+(z_o)}\Big[|D^2u|^2+\Big|\frac{Du}{\sigma}\Big|^2\Big]\,dz\\ &\le C\,\varepsilon_2\int_{Q_\sigma^+(z_o)}\Big[|D^2u|^2+\Big|\frac{Du}{\sigma}\Big|^2\Big]\,dz\\ &\le C\,\varepsilon_2\Big[\int_{Q_\tau^+(z_o)}\frac{|Du|^2}{(\tau-\varrho)^2}\,dz+\tau^2\int_{B_\tau^+(x_o)}|D^2g|^2+\frac{|Dg|^2}{(\tau-\varrho)^2}\,dx\Big].\end{aligned}$$

Here, the constant C only depends on $\|H\|_{L^\infty}$. We insert this above and obtain

$$\int_{Q_\varrho^+(z_o)}|Du|^2\,dz \le \frac{C}{(\tau-\varrho)^2}\int_{Q_R^+(z_o)}|u-g|^2\,dz + C\,\varepsilon_2\int_{Q_\tau^+(z_o)}|Du|^2\,dz + C\int_{B_R^+(x_o)}R^4|D^2g|^2+R^2|Dg|^2\,dx.$$

Since $\varepsilon_2 > 0$ is at our disposal, we can assume that $C(\|H\|_{L^\infty})\varepsilon_2 \le \frac12$. In this case the Iteration Lemma 2.1 can be applied and yields – using also Hölder's inequality – the following Caccioppoli-type inequality up to the boundary:

$$\int_{Q_{R/2}^+(z_o)}|Du|^2\,dz \le \frac{C_c}{R^2}\int_{Q_R^+(z_o)}|u-g|^2\,dz + C_cR^4\int_{B_R^+(x_o)}|D^2g|^2\,dx + C_cR^{4(1-\frac1p)}\Big[\int_{B_R^+(x_o)}|Dg|^p\,dx\Big]^{\frac2p}, \tag{5.8}$$

for any $p\in[2,\infty)$, with a constant $C_c \equiv C(\|H\|_\infty) \ge 1$.

5.2.2. *Flattening of the boundary.* In this section we describe the flattening of the boundary procedure which we use to transform to a situation with a flat boundary. In this special case, we will prove the regularity result with quantitative estimates and then transform these estimates back to the orginal set-up.

Without loss of generality we consider instead of the unit disk B the disk $D := B_1(e)$ centered at $e = (0,1)$. Moreover, as generic boundary point x_o we take the origin $0\in\partial D$. In this specific situation the boundary $\partial D\cap\{x_2<1\}$ is given by the graph of the function $(-1,1)\ni x_1\mapsto h(x_1) := 1-\sqrt{1-x_1^2}$. Note that $h(0)=h'(0)=0$. Further, we have that $|h'(x_1)|\le\kappa$ whenever $|x_1|\le\sigma(\kappa)$, where we have defined $\sigma(\kappa) := \kappa/\sqrt{1+\kappa^2}$.

Now, for $(x_1,x_2)\in(-1,1)\times\mathbb{R}$ we define $\Phi(x_1,x_2):=(x_1,x_2-h(x_1))$. Note that Φ maps $\partial D\cap\{x_2<1\}$ onto the line segment $(-1,1)\times\{0\}$. The inverse of Φ takes the form $\Psi(\zeta_1,\zeta_2)=\Phi^{-1}(\zeta_1,\zeta_2)=(\zeta_1,\zeta_2+h(\zeta_1))$. Of course, Ψ maps the line segment $(-1,1)\times\{0\}$ back to $\partial D\cap\{x_2<1\}$. An elementary computation shows that

$$D\Phi(x_1,x_2)=\begin{pmatrix}1 & 0\\ -h'(x_1) & 1\end{pmatrix}\quad\text{and}\quad D\Psi(\zeta_1,\zeta_2)=\begin{pmatrix}1 & 0\\ h'(\zeta_1) & 1\end{pmatrix}$$

holds true. But this implies that

$$\|D\Phi(x_1,x_2)\|\in\big[1-|h'(x_1)|,1+|h'(x_1)|\big]$$

and

$$\|D\Psi(\zeta_1,\zeta_2)\|\in\big[1-|h'(\zeta_1)|,1+|h'(\zeta_1)|\big]$$

holds true, where $\|\cdot\|$ denotes the operator norm. In particular, whenever $(x_1,x_2)\in(-\sigma(\kappa),\sigma(\kappa))\times\mathbb{R}$ we have

$$1-\kappa\le\|D\Phi(x_1,x_2)\|\le1+\kappa\,.$$

For $0<\varrho<1$ we have $D\cap B_\varrho=\{(x_1,x_2)\in B_\varrho : x_2>h(x_1)\}$. From this point on we consider only radii $0<\varrho\le\sigma(\kappa)$, where $\kappa>0$ is fixed in advance. Then, for points $x=(x_1,x_2)\in D\cap B_\varrho$ we have $|\Phi(x)|\le\varrho$ so that $\Phi(D\cap B_\varrho)\subset B_\varrho^+$. Here, we should indicate to the reader a slight abuse of notation: Since in most of the remaining proof we will work in the situation of a flat boundary, we will write $B_\varrho^+:=B_\varrho\cap\{\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^2:\zeta_2>0\}$ and $Q_\varrho^+:=Q_\varrho\cap[\{\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^2:\zeta_2>0\}\times(0,\infty)]$. However, this does not coincide with the notation in the statement of the Theorem. On the other hand, if $\zeta\in B^+_{\varrho/(1+\kappa)}$, then $|\Psi(\zeta)|\le\varrho$, so that $B^+_{\varrho/(1+\kappa)}\subset\Phi(D\cap B_\varrho)$. Therefore, for any radius $0<\varrho\le\sigma(\kappa)$ the following inclusions hold true:

$$B^+_{\varrho/(1+\kappa)}\subset\Phi\big(D\cap B_\varrho\big)\subset B^+_\varrho\,. \tag{5.9}$$

At this stage we have all prerequisites at hand in order to perform the flattening of the boundary in the parabolic system. We abbreviate $F:=-2(H\circ u)D_1u\times D_2u$ and consider a solution u of

$$\partial_t u-\Delta u=F\qquad\text{on } D\cap B_\varrho.$$

For the transformation of the parabolic system we start with a smooth testing function $\widetilde\varphi\in C_0^\infty\big(B^+_{\varrho/(1+\kappa)}\times(t_1,t_2),\mathbb{R}^3\big)$ and define $\varphi(x,t):=\widetilde\varphi(\Phi(x),t)$; extending $\widetilde\varphi$ by zero, we can assume $\varphi\in C_0^\infty\big((D\cap B_\varrho)\times(t_1,t_2);\mathbb{R}^3\big)$. Moreover, we define $\tilde u(\zeta,t):=u(\Psi(\zeta),t)$ and $\widetilde F(\zeta,t):=F(\Psi(\zeta),t)$. We perform a change of variables, exploit the fact $\det D\Phi\equiv1$ and use $\operatorname{spt}\widetilde\varphi\Subset B^+_{\varrho/(1+\kappa)}\times(t_1,t_2)$. In this way we obtain

$$\begin{aligned}0&=\int_{t_1}^{t_2}\int_{B_1(e)\cap B_\varrho}\big[u\cdot\varphi_t-Du\cdot D\varphi+F\cdot\varphi\big]\,dxdt\\&=\int_{t_1}^{t_2}\int_{\Phi(B_1(e)\cap B_\varrho)}\Big[\tilde u\cdot\widetilde\varphi_t-Du(\Psi,t)\cdot D\varphi(\Psi,t)+\widetilde F\cdot\widetilde\varphi\Big]\,d\zeta dt\\&=\int_{t_1}^{t_2}\int_{B^+_{\varrho/(1+\kappa)}}\Big[\tilde u\cdot\widetilde\varphi_t-D\tilde u[D\Psi]^{-1}\cdot D\widetilde\varphi[D\Psi]^{-1}+\widetilde F\cdot\widetilde\varphi\Big]\,d\zeta dt\\&=\int_{t_1}^{t_2}\int_{B^+_{\varrho/(1+\kappa)}}\Big[\tilde u\cdot\widetilde\varphi_t-a^{ij}D_i\tilde uD_j\widetilde\varphi+\widetilde F\cdot\widetilde\varphi\Big]\,d\zeta dt\,,\end{aligned}$$

where the coefficients a^{ij} are defined via

$$\big(a^{ij}(\zeta)\big)=\begin{pmatrix}1 & -h'(\zeta_1)\\ -h'(\zeta_1) & 1+h'(\zeta_1)^2\end{pmatrix}.$$

Note that $a^{ij}(\zeta) = a^{ji}(\zeta)$ is symmetric and equal to the identity at the origin, that is $a^{ij}(0) = \delta^{ij}$. Moreover, the coefficients are bounded and elliptic, in the sense that

$$(1-\kappa)^2|\xi|^2 \le a^{ij}(\zeta)\xi_i\xi_j \le (1+\kappa)^2|\xi|^2$$

for any $\xi \in \mathbb{R}^2$ and $\zeta \in (-\sigma(\kappa), \sigma(\kappa)) \times \mathbb{R}$. Now, a direct computation shows that

$$\big\|a^{ij}(\zeta) - \delta^{ij}\big\| \le |h'(\zeta_1)|\big(1 + |h'(\zeta_1)|\big) \le \frac{(1+\kappa)|\zeta_1|}{\sqrt{1-\zeta_1^2}} \le (1+\kappa)^2|\zeta_1|$$

holds true for ζ as before. Next, using the ellipticity bound for the coefficients a^{ij} we obtain the following estimate for the transformed inhomogeneity $\widetilde{F}$:

$$\big|\widetilde{F}\big| \le C_F\big|D\tilde{u}[D\Psi]^{-1}\big|^2 = C_F a^{ij} D_i\tilde{u} D_j\tilde{u} \le C_F(1+\kappa)^2|D\tilde{u}|^2.$$

A similar argument, using again the ellipticity of the coefficients a^{ij}, yields

$$\sup_{t_1<t<t_2} \int_{B^+_{\varrho/(1+\kappa)}\times\{t\}} |D\tilde{u}|^2\, d\zeta \le (1+\kappa)^2 \sup_{t_1<t<t_2} \int_{(D\cap B_\varrho)\times\{t\}} |Du|^2\, dx\,.$$

Next, it is easy to check that the transformed Dirichlet data $\tilde{g}(\zeta) := g(\Psi(\zeta))$ are of class $W^{2,q}\big(B^+_{\varrho/(1+\kappa)};\mathbb{R}^3\big)$. The corresponding $W^{2,q}$-norm can be estimated as follows:

$$\|\tilde{g}\|_{W^{2,q}(B^+_{\varrho/(1+\kappa)})} \le C(1+\kappa)^3\|g\|_{W^{2,q}(D\cap B_\varrho)}.$$

Here, we used that $h''(\zeta_1)$ can be bounded by $[1+\kappa^2]^{\frac{3}{2}}$ for $|\zeta_1| \le \varrho \le \sigma(\kappa)$. Finally, we need to transform the Caccioppoli-type inequality (5.8) to the flat situation. Using the transformation formula and the inclusions (5.9) we easily obtain for $0 < r \le \varrho$ that

$$\begin{aligned}
\int_{Q^+_{r/[2(1+\kappa)]}} |D\tilde{u}|^2\, dz &\le C \int_{-(r/2)^2}^0 \int_{D\cap B_{r/2}} |Du|^2\, dz \\
&\le C\bigg[\frac{1}{r^2}\int_{-r^2}^0\int_{D\cap B_r} |u-g|^2\, dz \\
&\qquad + r^4\int_{D\cap B_r} |D^2 g|^2\, dx + r^{4(1-\frac{1}{p})}\bigg(\int_{D\cap B_r} |Dg|^p\, dx\bigg)^{\frac{2}{p}}\bigg] \\
&\le C\bigg[\frac{1}{r^2}\int_{Q^+_r} |\tilde{u}-\tilde{g}|^2\, dz + r^4\int_{B^+_r} |D^2\tilde{g}|^2\, dx + r^{4(1-\frac{1}{p})}\bigg(\int_{B^+_r} |D\tilde{g}|^p\, dx\bigg)^{\frac{2}{p}}\bigg] \qquad (5.10)
\end{aligned}$$

holds true with a constant $C = C(\|H\|_\infty) \ge 1$. From here on, we do not anymore distinguish between x and ζ in the notation, simply writing x instead of ζ.

5.2.3. *Regularity up to the boundary.* We recall that we now use the notation $B^+_\varrho := B_\varrho(0) \cap \Omega$ in two slightly different meanings, depending on whether the domain of definition is $\Omega = B$ or $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. Summarizing the preceding argumentation, we may assume that the following *general assumptions* hold true: On a parabolic half cylinder $Q^+_\varrho := B^+_\varrho \times (-\varrho^2, 0)$ the mapping $\tilde{u} \in L^2\big(-\varrho^2, 0; W^{2,2}(B^+_\varrho, \mathbb{R}^3)\big)$ with $\partial_t\tilde{u} \in L^2(Q^+_\varrho;\mathbb{R}^3)$ is a weak solution of

$$\partial_t\tilde{u} - \operatorname{div}(AD\tilde{u}) = \widetilde{F} \qquad \text{in } Q^+_\varrho, \tag{5.11}$$

where $A = (a^{ij})_{1\le i,j\le 2}$ satisfies

$$\|A(x) - id\| \le 2|x_1| \tag{5.12}$$

for any $x = (x_1, x_2) \in B^+_\varrho$, and $\widetilde{F}\colon Q^+_\varrho \to \mathbb{R}^3$ is a measurable map satisfying

$$|\widetilde{F}| \le C_F|D\tilde{u}|^2 \quad \text{a.e. on } Q^+_\varrho. \tag{5.13}$$

Moreover, on the flat part $I_\varrho\times(-\varrho^2, 0)$ of the lateral boundary of Q^+_ϱ the mapping $\tilde{u}$ fulfills the Dirichlet-boundary condition $\tilde{u}(\cdot, t) = \tilde{g}$ in the sense of traces for a.e. $t \in (-\varrho^2, 0)$, for some Dirichlet datum $\tilde{g} \in W^{2,2}(B_\varrho^+, \mathbb{R}^3)$. Moreover, we assume that the following Caccioppoli-type estimate

$$\int_{Q_{r/4}^+} |D\tilde{u}|^2 \, dz \le C_c \left[\frac{1}{r^2} \int_{Q_r^+} |\tilde{u} - \tilde{g}|^2 \, dz + \Gamma^2(r) r^{4(1-\frac{1}{p})} \right] \tag{5.14}$$

holds true for any $0 < r \le \varrho$. The term $\Gamma(r)$ is defined by

$$\Gamma(r)^2 := \int_{B_r^+} |D^2\tilde{g}|^2 \, dx + \left(\int_{B_r^+} |D\tilde{g}|^p \, dx \right)^{\frac{2}{p}}. \tag{5.15}$$

Finally, we assume that the smallness condition

$$\sup_{-\varrho^2 < t < 0} \int_{B_\varrho^+ \times \{t\}} |D\tilde{u}|^2 \, dx \le \tilde{\varepsilon}_2 \tag{5.16}$$

holds true, where $\tilde{\varepsilon}_2 > 0$ is a fixed small parameter at our disposal. The precise value of $\tilde{\varepsilon}_2$ will be determined later in the course of the proof. Instead of (5.14) we work with the following version of the Caccioppoli inequality. Since $\tilde{u}(\cdot, t) = \tilde{g}$ on I_ϱ we replace in (5.14) the Dirichlet datum $\tilde{g}$ by its mean value $(\tilde{g})_{I_r}$. The appearing error term can be estimated with Poincaré's inequality as follows:

$$\begin{aligned} \frac{1}{r^2} \int_{Q_r^+} \big|\tilde{g} - (\tilde{g})_{I_r}\big|^2 \, dz &= \int_{B_r^+} \big|\tilde{g} - (\tilde{g})_{I_r}\big|^2 \, dx \\ &\le C\, r^2 \int_{B_r^+} |D\tilde{g}|^2 \, dx \le C\, \Gamma^2(r) r^{4(1-\frac{1}{p})}. \end{aligned} \tag{5.17}$$

Hence, (5.14) implies the following Caccioppoli-type estimate

$$\int_{Q_{r/4}^+} |D\tilde{u}|^2 \, dz \le C_c \left[\frac{1}{r^2} \int_{Q_r^+} |\tilde{u} - (\tilde{g})_{I_r}|^2 \, dz + \Gamma^2(r) r^{4(1-\frac{1}{p})} \right]. \tag{5.18}$$

We now define a *boundary excess functional* by letting

$$\mathbf{E}(\tilde{u}; r) := r^{-4} \int_{Q_r^+} |\tilde{u} - (\tilde{g})_{I_r}|^2 \, dz$$

for $0 < r \le \varrho$. Since $\tilde{g}$ is independent of t, we obtain for any $\varphi \in C_0^\infty(Q_{\varrho/4}^+, \mathbb{R}^3)$ that

$$\begin{aligned} &\left| \left(\frac{\varrho}{4}\right)^{-2} \int_{Q_{\varrho/4}^+} \big[(\tilde{u} - \tilde{g}) \cdot \partial_t \varphi - D(\tilde{u} - \tilde{g}) \cdot D\varphi \big] \, dz \right| \\ &\quad \le C_F \|\varphi\|_{L^\infty} \left(\frac{\varrho}{4}\right)^{-2} \int_{Q_{\varrho/4}^+} |D\tilde{u}|^2 \, dz + C\varrho^{-1} \|D\varphi\|_{L^\infty} \int_{Q_{\varrho/4}^+} |D\tilde{u}| \, dz \\ &\qquad + C\varrho^{2(1-\frac{1}{p})} \|D\varphi\|_{L^\infty} \left(\int_{B_\varrho^+} |D\tilde{g}|^p \, dx \right)^{\frac{1}{p}} \\ &\quad \le C \left[\varrho^{-1} \int_{Q_{\varrho/4}^+} |D\tilde{u}|^2 \, dz + \varrho^3 + \Gamma \varrho^{2(1-\frac{1}{p})} \right] \|D\varphi\|_{L^\infty} \\ &\quad \le C\varrho \Big[C_c \big(\mathbf{E}(\tilde{u}; \varrho) + \Gamma^2 \varrho^{2-\frac{4}{p}} \big) + \varrho^2 + \Gamma \varrho^{1-\frac{2}{p}} \Big] \|D\varphi\|_{L^\infty} \\ &\quad \le \tfrac{1}{4} \boldsymbol{\gamma} \varrho \Big[\mathbf{E}(\tilde{u}; \varrho) + (\Gamma + 1)^2 \varrho^{1-\frac{2}{p}} \Big] \|D\varphi\|_{L^\infty}, \end{aligned}$$

where the constant $\boldsymbol{\gamma}$ depends only on C_c, C_F, and $\Gamma := \Gamma(\varrho)$. Here we used in turn (5.11), (5.12), (5.13), Young's inequality and (5.18). Now, for $\varepsilon > 0$ to be determined later we take $\delta \equiv \delta(\varepsilon) \in (0, 1]$ to be the corresponding constant from the caloric approximation lemma [2, Lemma 4.1] and define the re-scaled function

$$v(x,t) := \frac{\tilde{u} - \tilde{g}}{\boldsymbol{\gamma} \big[\mathbf{E}(\tilde{u}; \varrho) + 4\delta^{-2}(\Gamma + 1)^4 \varrho^{2-\frac{4}{p}} \big]^{\frac{1}{2}}}.$$

Here $\boldsymbol{\gamma} \geq 1$ will be chosen later in a universal way not smaller than the corresponding constant from the last inequality. Then v is an almost caloric function on $Q^+_{\varrho/4}$, in the sense that

$$\left| \left(\frac{\varrho}{4}\right)^{-2} \int_{Q^+_{\varrho/4}} \big[v \cdot \partial_t \varphi - Dv \cdot D\varphi\big]\, dz \right| \leq \tfrac{1}{4}\varrho \left[\sqrt{\mathbf{E}(\tilde{u};\varrho)} + \tfrac{1}{2}\delta\right] \|D\varphi\|_{L^\infty}.$$

We now assume that

$$\mathbf{E}(\tilde{u};\varrho) \leq \tfrac{1}{4}\delta^2. \tag{5.19}$$

Then, for any $\varphi \in C_0^\infty(Q^+_{\varrho/4}, \mathbb{R}^3)$ we have

$$\left| \left(\frac{\varrho}{4}\right)^{-2} \int_{Q^+_{\varrho/4}} \big[v \cdot \partial_t \varphi - Dv \cdot D\varphi\big]\, dz \right| \leq \tfrac{1}{4}\varrho\, \delta \|D\varphi\|_{L^\infty}.$$

Moreover, by the definition of v and the Caccioppoli-type inequality (5.18), we have

$$\begin{aligned}
&\left(\frac{\varrho}{4}\right)^{-2} \int_{Q^+_{\varrho/4}} |Dv|^2\, dz \\
&\quad = \frac{2^4 \varrho^{-2}}{\boldsymbol{\gamma}^2 \big[\mathbf{E}(\tilde{u};\varrho) + 4\delta^{-2}(\Gamma+1)^4 \varrho^{2-\frac{4}{p}}\big]} \int_{Q^+_{\varrho/4}} |D\tilde{u} - D\tilde{g}|^2\, dz \\
&\quad \leq \frac{2^5 \varrho^{-2}}{\boldsymbol{\gamma}^2 \big[\mathbf{E}(\tilde{u};\varrho) + 4\delta^{-2}(\Gamma+1)^4 \varrho^{2-\frac{4}{p}}\big]} \left[\int_{Q^+_{\varrho/4}} |D\tilde{u}|^2\, dz + \int_{Q^+_{\varrho/4}} |D\tilde{g}|^2\, dz\right] \\
&\quad \leq \frac{2^5}{\boldsymbol{\gamma}^2 \big[\mathbf{E}(\tilde{u};\varrho) + 4\delta^{-2}(\Gamma+1)^4 \varrho^{2-\frac{4}{p}}\big]} \Big[C_c\Big(\mathbf{E}(\tilde{u};\varrho) + \Gamma^2 \varrho^{2-\frac{4}{p}}\Big) + C\Gamma^2 \varrho^{2-\frac{4}{p}}\Big] \\
&\quad \leq \frac{C}{\boldsymbol{\gamma}^2 \big[\mathbf{E}(\tilde{u};\varrho) + 4\delta^{-2}(\Gamma+1)^4 \varrho^{2-\frac{4}{p}}\big]} \Big[\mathbf{E}(\tilde{u};\varrho) + \Gamma^2 \varrho^{2-\frac{4}{p}}\Big] \leq 1
\end{aligned}$$

provided we choose $\boldsymbol{\gamma} \geq 1$ in dependence on C_c and C_F large enough. In conclusion, we have established the energy bound

$$\left(\frac{\varrho}{4}\right)^{-2} \int_{Q^+_{\varrho/4}} |Dv|^2\, dz \leq 1.$$

At this point we can apply the caloric approximation Lemma [2, Lemma 4.1] to infer the existence of a caloric function $h \in L^2\big(-(\varrho/8)^2, 0; W^{1,2}(B^+_{\varrho/8}, \mathbb{R}^3)\big)$ with $h = 0$ on the flat part of the lateral boundary $I_{\varrho/8} \times (-(\varrho/8)^2, 0)$, where $I_{\varrho/8} := B_{\varrho/8} \cap \{x_2 = 0\}$, satisfying

$$\left(\frac{\varrho}{8}\right)^{-2} \int_{Q^+_{\varrho/8}} |Dh|^2\, dz \leq 2^5$$

and

$$\left(\frac{\varrho}{8}\right)^{-4} \int_{Q^+_{\varrho/8}} |v - h|^2\, dz \leq \varepsilon.$$

Using a standard a priori estimate for the caloric function h, see e.g. [26, Lemma 4.13], we obtain for $\theta \in (0, \frac{1}{16}]$ that

$$\begin{aligned}
(\theta\varrho)^{-4} \int_{Q^+_{\theta\varrho}} |v|^2\, dz &\leq 2(\theta\varrho)^{-4} \int_{Q^+_{\theta\varrho}} \big[|v-h|^2 + |h|^2\big]\, dz \\
&\leq 2^{-11}\theta^{-4}\varepsilon + C\,\theta^2 \left(\frac{\varrho}{8}\right)^{-4} \int_{Q^+_{\varrho/8}} |h|^2\, dz \\
&\leq 2^{-11}\theta^{-4}\varepsilon + C\,\theta^2 \left(\frac{\varrho}{8}\right)^{-2} \int_{Q^+_{\varrho/8}} |Dh|^2\, dz \\
&\leq C\big(\theta^{-4}\varepsilon + \theta^2\big).
\end{aligned}$$

In the second last line we used the Poincaré inequality slicewise, which is possible since $h(\cdot,t)=0$ on $I_{\varrho/8}$ for a.e. $t\in(-(\varrho/8)^2,0)$. Scaling back from v to $\tilde u$ we obtain that

$$(\theta\varrho)^{-4}\int_{Q^+_{\theta\varrho}}|\tilde u-\tilde g|^2\,dz\le C\big(\theta^{-4}\varepsilon+\theta^2\big)\Big[\mathbf{E}(\tilde u;\varrho)+4\delta^{-2}(\Gamma+1)^4\varrho^{2-\frac{4}{p}}\Big],$$

where the constant C depends only on C_F and C_c. We choose $\alpha\in(0,1)$ and set

$$p:=\frac{2}{1-\alpha}>2,$$

so that $2-\frac{4}{p}=2\alpha$. Now, we fix the parameters ε and θ. For $\beta:=\frac12(1+\alpha)$ we choose $\theta\in(0,\frac{1}{16}]$ according to $C\theta^2\le\frac12\theta^{2\beta}$. Then, θ depends on C_c,C_F and α. Next, we fix $\varepsilon>0$ in such a way that $C\theta^{-4}\varepsilon\le\frac12\theta^{2\beta}$. This determines ε also in dependence on C_c,C_F and α and consequently δ is fixed in dependence on the same parameters. i.e. C_c,C_F,α. Thus, we have shown so far: There exists $\delta=\delta(C_c,C_F,\alpha)>0$ such that if (5.19) holds true then

$$(\theta\varrho)^{-4}\int_{Q^+_{\theta\varrho}}|\tilde u-\tilde g|^2\,dz\le\theta^{2\beta}\mathbf{E}(\tilde u;\varrho)+C(\Gamma+1)^4(\theta\varrho)^{2\alpha},$$

where $C=C(C_c,C_F,\alpha)$. In the left-hand side we replace the function $\tilde g$ by its mean value $(\tilde g)_{I_{\theta\varrho}}$. This can be achieved by (5.17) and leads to

$$\mathbf{E}(\tilde u;\theta\varrho)\le\theta^{2\beta}\mathbf{E}(\tilde u;\varrho)+C(\Gamma+1)^4(\theta\varrho)^{2\alpha},$$

with $\beta>\alpha$. As usual, the iteration of the preceding inequality yields that

$$\mathbf{E}(\tilde u;r)\le C\Big(\frac{r}{\varrho}\Big)^{2\alpha}\Big[\mathbf{E}(\tilde u;\varrho)+(\Gamma+1)^4\varrho^{2\alpha}\Big]$$

holds true for any $0<r\le\varrho/16$ with a constant $C\equiv C(C_c,C_F,\alpha)$, provided the smallness condition (5.19) is satisfied. Since $u=\tilde g$ on $I_\varrho\times(-\varrho^2,0)$, the Poincaré inequality (5.17) implies

$$\mathbf{E}(\tilde u;\varrho)=\frac{1}{\varrho^4}\int_{Q^+_\varrho}\big|\tilde u-(\tilde u)_{I_\varrho}\big|^2\,dz\le\frac{C}{\varrho^2}\int_{Q^+_\varrho}|D\tilde u|^2\,dz. \tag{5.20}$$

Plugging this into the preceding inequality, we infer

$$\frac{1}{r^4}\int_{Q^+_r}\big|\tilde u-(\tilde u)_{I_r}\big|^2\,dz\le C\Big(\frac{r}{\varrho}\Big)^{2\alpha}\bigg[\frac{1}{\varrho^2}\int_{Q^+_\varrho}|D\tilde u|^2\,dz+(\Gamma+1)^4\varrho^{2\alpha}\bigg] \tag{5.21}$$

whenever $r\in(0,\frac{\varrho}{16})$, provided the smallness condition (5.19) holds true. Scaling back from $\tilde u$ to u and using the minimality of the mean value $(u)_{Q^+_r}$ in the mapping $\xi\mapsto\int_{Q^+_r}|u-\xi|^2\,dz$, we arrive at

$$\begin{aligned}\frac{1}{r^4}\int_{Q^+_r(z_o)}&\big|u-(u)_{Q^+_r(z_o)}\big|^2\,dz\\&\le C\Big(\frac{r}{\varrho}\Big)^{2\alpha}\bigg[\frac{1}{\varrho^2}\int_{Q^+_{\varrho(1+\kappa)}(z_o)}|Du|^2\,dz+(\Gamma+1)^4\varrho^{2\alpha}\bigg].\end{aligned} \tag{5.22}$$

Finally, we perform the choice of $\tilde\varepsilon_2$ from (5.16). From (5.20) and (5.16) we have

$$\mathbf{E}(\tilde u;\varrho)\le\frac{C}{\varrho^2}\int_{Q^+_\varrho}|D\tilde u|^2\,dz\le C\sup_{t\in(-\varrho^2,0)}\int_{B^+_\varrho\times\{t\}}|D\tilde u|^2\,dx\le C\tilde\varepsilon_2.$$

Therefore, choosing $\tilde\varepsilon_2$ in dependence on C_c,C_F,α small enough to satisfy $C\tilde\varepsilon_2\le\frac14\delta^2$, this ensures that the smallness condition (5.19) is satisfied. In turn, the choice of $\tilde\varepsilon_2$ fixes also the constant ε_2 from (5.6) in dependence on $\|H\|_\infty$ and α. In what follows, we consider a fixed exponent $\alpha\in(\frac23,1)$, say $\alpha=\frac56$, which elminates the dependence on α.

At this point, we recall that we started the transformation argument in Section 5.2.2 with a generic boundary point $x_o\in\partial B$. Therefore, the inequality (5.22) holds true for any $z_o = (x_o, t_o)$ with $x_o \in \partial B$ and $\varrho > 0$ such that (5.6) is satisfied on $Q^+_{\varrho(1+\kappa)}(z_o)$. This, together with the analogous estimate for the interior case from Section 5.1 and the integral characterization of Hölder continuous functions from [7, Teorema 3.1] yields the Hölder continuity of u with Hölder exponent α with respect to the parabolic metric. More precisely, under the smallness assumption (5.6), we have established the Hölder continuity of u in $Q^+_\sigma(z_o)$ with Hölder exponent $\alpha \in (0,1)$ for any $0 < \sigma < R$. This also implies the Hölder continuity of $\tilde u$ in $Q^+_\sigma(z_o)$ with Hölder exponent $\alpha \in (0,1)$ for any $0 < \sigma < \varrho$.

5.2.4. *Hölder continuity of Du at the lateral boundary.* In this section we establish the Hölder continuity of the spatial gradient $D\tilde u$. We fix a radius $0 < r < \varrho/8$, and denote by $h \in C^0([-r^2, 0]; L^2(B^+_r, \mathbb{R}^3)) \cap L^2(-r^2, 0; W^{1,2}(B^+_r, \mathbb{R}^3))$ the unique solution to the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t h - \Delta h = 0 & \text{in } Q^+_r \\ h = \tilde u - \tilde g & \text{on } \partial_{\mathcal P} Q^+_r. \end{cases}$$

Taking the difference of the weak formulation of the system (5.11) for $\tilde u$ and the heat system for h, we obtain

$$\begin{aligned} &\int_{Q^+_r} \big[\partial_t(\tilde u - h)\cdot\varphi + D(\tilde u - h - \tilde g)\cdot D\varphi\big]\,dz \\ &\qquad = \int_{Q^+_r} \big[\widetilde F\cdot\varphi - (a^{ij} - \delta^{ij}) D_i\tilde u D_j\varphi - D\tilde g\cdot D\varphi\big]\,dz \\ &\qquad = \int_{Q^+_r} \big[\widetilde F\cdot\varphi - (a^{ij} - \delta^{ij}) D_i\tilde u D_j\varphi - \big(D\tilde g - (D\tilde g)^+_r\big)\cdot D\varphi\big]\,dz \end{aligned}$$

for any $\varphi \in L^2(-r^2, 0; W^{1,2}_0(B_r, \mathbb{R}^3))$. Here, $(D\tilde g)^+_r$ stands for the mean value of $D\tilde g$ over B^+_r. Furthermore, we note that $\partial_t\tilde u, \partial_t h \in L^2(Q^+_r, \mathbb{R}^3)$ by (5.11), respectively by standard regularity for the heat equation. Choosing $\varphi = \tilde u - h - \tilde g$ in the last identity and taking into account that $\partial_t\tilde g = 0$, we get

$$\begin{aligned} &\int_{Q^+_r} \big[\tfrac12\partial_t|\tilde u - h - \tilde g|^2 + |D(\tilde u - h - \tilde g)|^2\big]\,dz \\ &\quad\le C_F \sup_{Q^+_r}|\tilde u - h - \tilde g| \int_{Q^+_r}|D\tilde u|^2\,dz + \sup_{Q^+_r}\|A - id\| \int_{Q^+_r}|D\tilde u||D(\tilde u - h - \tilde g)|\,dz \\ &\qquad + \int_{Q^+_r}|D(\tilde u - h - \tilde g)|\big|D\tilde g - (D\tilde g)^+_r\big|\,dz \\ &\quad\le \Big[C_F \sup_{Q^+_r}|\tilde u - h - \tilde g| + \sup_{Q^+_r}\|A - id\|^2\Big] \int_{Q^+_r}|D\tilde u|^2\,dz \\ &\qquad + \tfrac12 \int_{Q^+_r}|D(\tilde u - h - \tilde g)|^2\,dz + r^2 \int_{B^+_r}\big|D\tilde g - (D\tilde g)^+_r\big|^2\,dz. \end{aligned}$$

The first term on the left-hand side is non-negative, since $h = \tilde u - \tilde g$ on $\partial_{\mathcal P} Q_r$. This together with Poincaré's and Hölder's inequality leads us to

$$\begin{aligned} &\int_{Q^+_r}|D(\tilde u - h - \tilde g)|^2\,dz \\ &\quad\le 2\Big[C_F \sup_{Q^+_r}|\tilde u - h - \tilde g| + \sup_{Q^+_r}\|A - id\|^2\Big] \int_{Q^+_r}|D\tilde u|^2\,dz + Cr^4 \int_{B^+_r}|D^2\tilde g|^2\,dz \\ &\quad\le 2\Big[C_F \sup_{Q^+_r}|\tilde u - h - \tilde g| + \sup_{Q^+_r}\|A - id\|^2\Big] \int_{Q^+_r}|D\tilde u|^2\,dz \\ &\qquad + C\,r^{6-\frac4q}\bigg(\int_{B^+_r}|D^2\tilde g|^q\,dz\bigg)^{\frac2q}. \end{aligned}$$

Here, $q > 2$ is the exponent from the assumption (1.8). Now, we fix $\alpha \in (0,1)$ and choose some boundary point $\bar z \in \partial_{\mathcal P} Q_r^+$. By the maximum principle for the heat equation we have

$$|h(\bar z) - h(z)| \le \max_{z' \in \partial_{\mathcal P} Q_r^+} |(\tilde u - \tilde g)(\bar z) - (\tilde u - \tilde g)(z')| \le C\big[[\tilde u]_{C^{0,\alpha}} + [\tilde g]_{C^{0,\alpha}}\big] r^\alpha .$$

Hence, by the Hölder continuity of u and g we have for any $z \in Q_r^+$ that

$$|(\tilde u - h - \tilde g)(z)| \le |(\tilde u - \tilde g)(z) - (\tilde u - \tilde g)(\bar z)| + |h(\bar z) - h(z)| \le C\, r^\alpha ,$$

where C depends on $[\tilde u]_{C^{0,\alpha}}$ and $[\tilde g]_{C^{0,\alpha}}$. Moreover, from (5.12) we know that $\|A - id\|^2 \le (2r)^2 \le 4r^\alpha$ on Q_r^+. Finally, the Caccioppoli inequality (5.10) (note that we can use the Caccioppoli inequality with $\kappa \le 1$) yields that

$$\fint_{Q_r^+} |D\tilde u|^2\, dz \le C_c \bigg[\frac{1}{r^2} \fint_{Q_{4r}^+} |\tilde u - \tilde g|^2\, dz + \Gamma_q^2\bigg] \le C r^{2\alpha - 2}$$

for any $0 < r \le \frac14 \varrho$, where we have abbreviated

$$\Gamma_q^2 := \bigg(\int_{B_\varrho^+} |D^2 \tilde g|^q\, dx\bigg)^{\frac{2}{q}} + \|D\tilde g\|^2_{L^\infty(B_\varrho^+)} . \tag{5.23}$$

We note that the constant C depends on $[\tilde u]_{C^{0,\alpha}}$, $[\tilde g]_{C^{0,\alpha}}$, Γ_q and the constant $C_{(5.10)}$ from (5.10). Inserting the preceding estimates above and taking mean values, we deduce

$$\fint_{Q_r^+} |D(\tilde u - h - \tilde g)|^2\, dz \le C\, r^\alpha \fint_{Q_r^+} |D\tilde u|^2\, dz + C\, \Gamma_q^2 r^{2 - \frac4q} \le C\, r^{2\beta},$$

where we defined

$$\beta := \min\big\{\tfrac32 \alpha - 1, 1 - \tfrac2q\big\} \in (0,1).$$

The constant in the preceding inequality depends only on $[\tilde u]_{C^{0,\alpha}}$, $[\tilde g]_{C^{0,\alpha}}$, Γ_q and $C_{(5.10)}$. We define the excess functional by

$$\Phi(\tilde u; r) := \fint_{Q_r^+} |D\tilde u - (D\tilde u)_r^+|^2\, dz ,$$

where $(D\tilde u)_r^+$ stands for the mean value of $D\tilde u$ over Q_r^+. Now, let $0 < \tau \le \frac12$. Then, by the a priori estimates for h from [26, Lemmas 4.13, 4.14] we get the following excess-estimate

$$\begin{aligned}
\Phi(\tilde u; \tau r) &\le \fint_{Q_{\tau r}^+} |D\tilde u - (Dh)_{\tau r}^+ - (D\tilde g)_{\tau r}^+|^2\, dz \\
&\le 3\Phi(h; \tau r) + 3\fint_{B_{\tau r}^+} |D\tilde g - (D\tilde g)_{\tau r}^+|^2\, dz + 3\fint_{Q_{\tau r}^+} |D(\tilde u - h - \tilde g)|^2\, dz \\
&\le C\bigg[\tau^2 \Phi(h; r) + \int_{B_{\tau r}^+} |D^2 \tilde g|^2\, dx + \tau^{-4} r^{2\beta}\bigg] \\
&\le C\Big[\tau^2 \Phi(\tilde u; r) + (\tau r)^{2(1 - \frac2q)} \Gamma_q^2 + \tau^{-4} r^{2\beta}\Big] \\
&\le C\big[\tau^2 \Phi(\tilde u; r) + \tau^{-4} r^{2\beta}\big],
\end{aligned}$$

where we used the definition of β in the last line. Transforming back from $\tilde u$ to u, the preceding excess-estimate for $D\tilde u$ can be transferred to Du, by slightly enlarging the constant and changing the involved radii. Moreover, we recall that the excess-estimate for Du does not depend on the center of the cylinder and therefore it holds true for any boundary point $z_o = (x_o, t_o)$ with $x_o \in \partial B$. Joining the boundary excess-estimates with the interior excess-estimates from (5.5) and then arguing as in the interior case, we conclude that Du is Hölder continuous with Hölder exponent $\beta \in (0,1)$ with respect to the parabolic metric. In particular, this ensures that the right-hand side of (3.8) is bounded. If we know moreover that $g \in C^{1,\gamma}(B)$ for some $\gamma \in (0,1)$, we can therefore conclude by classical Schauder theory that Du is Hölder continuous with respect to the parabolic metric for the same Hölder exponent γ. □

Now, we have the prerequisites to prove our first main theorem which ensures the short-time regularity of our weak solution.

Proof of Theorem 1.1. By $\delta = \delta(\|H\|_{L^\infty}, c, \mathbf{D}(u_o), \|Dg\|_{W^{1,2}}) > 0$, and $\varepsilon_1 = \varepsilon_1(\|H\|_{L^\infty}, c) > 0$, and $r_o = r_o(\|Dg\|_{L^4}) \in (0, \frac{1}{4}]$ we denote the corresponding constants from Corollary 4.2. We choose $r \in (0, r_o]$ small enough to satisfy

$$\varepsilon := \sup_{x_o \in \overline{B}} \int_{B_r(x_o)\cap B} |Du_o|^2\, dx \le \varepsilon_1.$$

The assertion of the L^2-$W^{2,2}$-regularity then follows from Corollary 4.2 by choosing $t_o = \delta r^2$. This corollary furthermore implies a local energy inequality that is valid up to the time t_o. By possibly diminishing the values of r and t_o we therefore can ensure that for every $x_o \in \overline{B}$ either the assumption (5.1) or (5.6) of Theorem 5.1, respectively of Theorem 5.2 is satisfied. At this point, the Hölder continuity of the gradient follows from Theorem 5.1 in the interior case, respectively from Theorem 5.2 in the boundary case. □

6. Global solutions with only finitely many singular times

We begin by recalling a result from [8] that will ensure a lower semicontinuity property for the volume functional $\mathbf{D} + 2\mathbf{V}_H(\cdot, u_o)$.

Lemma 6.1. *Consider a sequence of maps $u_i \in W^{1,2}(B, \mathbb{R}^3)$ with $u_i \rightharpoonup u$ weakly in $W^{1,2}(B, \mathbb{R}^3)$ and $u_i|_{\partial B} \to u|_{\partial B}$ uniformly in $L^\infty(\partial B, \mathbb{R}^3)$. For every $\varepsilon > 0$ there are a constant $R > 0$, a measurable set $G \subseteq B$, and a new sequence $\tilde u_i \in W^{1,2}(B, \mathbb{R}^3)$, such that the following holds after passing to a subsequence.*

(i) *$\tilde u_i = u$ on $B \setminus G$ with $\mathcal{L}^2(G) < \varepsilon$;*
(ii) *$\tilde u_i|_{\partial B} = u|_{\partial B}$ on ∂B;*
(iii) *$\tilde u_i(x) = u_i(x)$ in points $x \in B$ with $|u_i(x)| \ge R$ or $|u_i(x) - u(x)| \ge 1$;*
(iv) *$\lim_{i\to\infty} \|\tilde u_i - u_i\|_{L^\infty(B,\mathbb{R}^3)} = 0$;*
(v) *$\tilde u_i \rightharpoonup u$ weakly in $W^{1,2}(B, \mathbb{R}^3)$ as $i \to \infty$;*
(vi) *$\limsup_{i\to\infty}[\mathbf{D}_G(\tilde u_i) + \mathbf{D}_G(u)] \le \varepsilon + \liminf_{i\to\infty}[\mathbf{D}(u_i) - \mathbf{D}(u)]$;*
(vii) *If $u_i \in W^{1,2}(B, A)$ for a closed convex set $A \subset \mathbb{R}^3$, then the maps $\tilde u_i$ can be chosen with $\tilde u_i \in W^{1,2}(B, A)$.*

For the proof we refer to [8, Lemma 4.1]. We note that the property (vii) is clear from the construction in [8] since the maps $\tilde u_i$ are chosen as convex combinations of u_i and u, whose images are contained in the convex set A.

Corollary 6.2. *Assume that $H\colon A \to \mathbb{R}$ satisfies a spherical isoperimetric condition of type c. Then for every sequence $u_i \in W^{1,2}(B, A)$ with $u_i \rightharpoonup u$ weakly in $W^{1,2}(B, A)$ and $u_i|_{\partial B} \to u|_{\partial B}$ uniformly in $L^\infty(\partial B, A)$, we have the lower semicontinuity property*

$$\mathbf{D}(u) + 2\mathbf{V}_H(u, u_o) \le \liminf_{i\to\infty} [\mathbf{D}(u_i) + 2\mathbf{V}_H(u_i, u_o)].$$

Proof. For a given $0 < \varepsilon < \frac{1}{2}\mathbf{D}(u)$, we apply Lemma 6.1 to the sequence u_i, which provides us with a measurable set $G \subseteq B$ with $\mathcal{L}^2(G) < \varepsilon$ and a sequence $\tilde u_i \in W^{1,2}(B, A)$ with $\tilde u_i \in u_o + W^{1,2}_0(B, \mathbb{R}^3)$ with the properties listed in the lemma. From Lemma 2.5 we infer

$$|\mathbf{V}_H(\tilde u_i, u_i)| \le \|H\|_{L^\infty} \|\tilde u_i - u_i\|_{L^\infty} [\mathbf{D}(\tilde u_i) + \mathbf{D}(u_i)] \to 0$$

as $i \to \infty$. Because of this and Lemma 6.1 (vi), we can choose $i_o = i_o(\varepsilon)$ sufficiently large, to achieve that for $i \ge i_o$ there holds

$$|\mathbf{V}_H(\tilde u_i, u_i)| < \varepsilon \qquad \text{and} \qquad \mathbf{D}_G(\tilde u_i) + \mathbf{D}_G(u) \le 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u). \tag{6.1}$$

Next, we apply the isoperimetric condition (2.4) to get

$$2|\mathbf{V}_H(u, \tilde u_i)| \le c\,[\mathbf{D}_G(u) + \mathbf{D}_G(\tilde u_i)] \le 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u) \tag{6.2}$$

for all $i \geq i_o$. We combine (6.1) and (6.2) to the estimate

$$\begin{aligned}\mathbf{D}(u) + 2\mathbf{V}_H(u, u_o) &= \mathbf{D}(u) + 2\mathbf{V}_H(u, \tilde{u}_i) + 2\mathbf{V}_H(\tilde{u}_i, u_i) + 2\mathbf{V}_H(u_i, u_o) \\ &\leq \mathbf{D}(u_i) + 2\mathbf{V}_H(u_i, u_o) + 4\varepsilon\end{aligned}$$

for all $i \geq i_o(\varepsilon)$, where $\varepsilon > 0$ can be chosen arbitrarily small. Letting $i \to \infty$ implies the assertion of the lemma. □

From Theorem 1.1 we know that our weak solution u is regular for a short time. We now analyze the first singular time.

Lemma 6.3. *Let $T > 0$ denote the first singular time of the H-surface flow. Then there are sequences $h_i \downarrow 0$ and $t_i \uparrow T$ as well as a point $x_o \in \overline{B}$ such that*

$$\mathcal{L}^2 \llcorner \big|Du_{h_i}(\cdot, t_i)\big|^2 \overset{*}{\rightharpoondown} \mu \qquad \text{weakly* in the sense of Radon measures} \tag{6.3}$$

and $\mu(\{x_o\}) > \frac{1}{2}\varepsilon_2$ with the constant $\varepsilon_2 > 0$ being the minimum of the corresponding constants from Theorems 5.1 and 5.2.

Proof. Let (x_o, T) be a singular point of the flow, where $x_o \in \overline{B}$. Then there holds

$$\liminf_{i\to\infty} \sup_{t\in(T-R^2,T)} \int_{B_R(x_o)\cap B} |Du_{h_i}(x,t)|^2\, dx > \tfrac{1}{2}\varepsilon_2 \tag{6.4}$$

for every radius $R \in (0, 1)$, since otherwise, we could apply either Theorem 5.1 or Theorem 5.2 to prove regularity on $Q_{R/2}(x_o, T)$. Now, we consider a sequence $R_i \downarrow 0$. For each $i \in \mathbb{N}$, we exploit condition (6.4) to find a parameter $0 < h_i < \frac{1}{i}$ and a time $t_i \in (T - R_i^2, T)$ such that

$$\int_{B_{R_i}(x_o)\cap B} |Du_{h_i}(x, t_i)|^2\, dx > \tfrac{1}{2}\varepsilon_2. \tag{6.5}$$

The Radon measures $\mu_i := \mathcal{L}^2 \llcorner |Du_{h_i}(\cdot, t_i)|^2$ satisfy $\sup_{i\in\mathbb{N}} \mu_i(B) \leq 2C_1\mathbf{D}(u_o)$ by the energy inequality (3.5). Passing to a not relabeled subsequence we can therefore achieve the convergence $\mu_i \overset{*}{\rightharpoondown} \mu$ as $i \to \infty$ for a limiting Radon measure μ on $\overline{B}$. Now for any $R > 0$ we have $R_i < R$ for sufficiently large values of $i \in \mathbb{N}$. This implies

$$\mu\big(\overline{B_R(x_o) \cap B}\big) \geq \limsup_{i\to\infty} \mu_i(B_R(x_o) \cap B) \geq \limsup_{i\to\infty} \mu_i(B_{R_i}(x_o) \cap B) > \tfrac{1}{2}\varepsilon_2,$$

where the last estimate follows from the definition of μ_i and (6.5). Letting $R \downarrow 0$, we deduce $\mu(\{x_o\}) > \frac{1}{2}\varepsilon_2$, which completes the proof of the Lemma. □

Remark 6.4. We note that the choice of the sequence t_i in the preceding proof may depend on the singular point x_o. If this would not be the case, we could apply the following cut-off argument simultaneously at several singular points and deduce that at each singularity, a certain amount of energy is lost during the flow. Unfortunately, we can not argue in this way, and therefore (at this stage) we are not able to exclude the occurrence of infinitely many singularities at each singular time.

Lemma 6.5. *For the solution $u_T := u(\cdot, T)$ at the first singular time T we have the energy estimate*

$$\mathbf{D}(u_T) + 2\mathbf{V}_H(u_T, u_o) \leq \mathbf{D}(u_o) - \tfrac{1}{8}(1 - c)\,\varepsilon_2.$$

Proof. We abbreviate $u_i := u_{h_i}(\cdot, t_i) \in W^{1,2}(B, A)$ for $i \in \mathbb{N}$ with the sequences h_i, t_i from Lemma 6.3, so that

$$\mathcal{L}^2 \llcorner |Du_i|^2 \overset{*}{\rightharpoondown} \mu \quad \text{and} \quad \mu(\{x_o\}) > \tfrac{1}{2}\varepsilon_2 \text{ for some } x_o \in \overline{B}. \tag{6.6}$$

Moreover, we know that

$$\|u_i - u_T\|_{L^2} \leq \|u_{h_i}(\cdot, t_i) - u(\cdot, t_i)\|_{L^2} + \|u(\cdot, t_i) - u(\cdot, T)\|_{L^2} \to 0$$

holds true as $i \to \infty$ by the convergence $u_{h_i} \to u$ in $L^\infty(0,T;L^2(B,\mathbb{R}^3))$, cf. (3.7), and the fact $u \in C^0([0,T];L^2(B,\mathbb{R}^3))$. This means $u_i \to u_T$ in $L^2(B,\mathbb{R}^3)$ as $i \to \infty$. By passing to a subsequence if necessary, we can achieve moreover $u_i \rightharpoondown u_T$ weakly in $W^{1,2}(B,\mathbb{R}^3)$, as $i \to \infty$. For a parameter $\delta > 0$ to be chosen small later, we choose a radius $R \in (0,1)$ so small that

$$\mu\big(\overline{B_{2R}(x_o)} \setminus \{x_o\}\big) \le \delta \qquad \text{and} \qquad \int_{B_{2R}(x_o)\cap B} |Du_T|^2\,dx \le \delta. \tag{6.7}$$

We choose a cut-off function $\zeta \in C^\infty(\mathbb{R}^2,[0,1])$ with $\zeta \equiv 0$ on $B_1(0)$, $\zeta \equiv 1$ on $\mathbb{R}^2\setminus B_2(0)$ and $|D\zeta| \le 2$ on $\mathbb{R}^2$. Then we consider the rescaled versions $\zeta_R(x) := \zeta(\frac{x-x_o}{R})$, which satisfy $\zeta_R \equiv 0$ on $B_R(x_o)$, $\zeta_R \equiv 1$ on $\mathbb{R}^2 \setminus B_{2R}(x_o)$ and $|D\zeta_R| \le \frac{2}{R}$ on $\mathbb{R}^2$. For the mean values

$$\bar u_{i,R} := \fint_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B} u_i(x)\,dx$$

we define

$$\tilde u_i := \zeta_R(u_i - \bar u_{i,R}) + \bar u_{i,R},$$

which fulfills $\tilde u_i = u_i$ outside of $B_{2R}(x_o) \cap B$. Analogously, we write

$$\tilde u_T := \zeta_R(u_T - \bar u_{T,R}) + \bar u_{T,R}.$$

Next, we use the properties of the cut-off function ζ_R and Poincaré's inequality on the domain $(B_{2R}(x_o) \setminus B_R(x_o)) \cap B$ in order to estimate

$$\begin{aligned}
\int_{B_{2R}(x_o)\cap B} &|D\tilde u_i|^2\,dx \\
&\le 2\int_{B_{2R}(x_o)\cap B} \zeta_R^2|Du_i|^2\,dx + 2\int_{B_{2R}(x_o)\cap B} |D\zeta_R|^2|u_i - \bar u_{i,R}|^2\,dx \\
&\le 2\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B} |Du_i|^2\,dx + \frac{8}{R^2}\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B} |u_i - \bar u_{i,R}|^2\,dx \\
&\le C\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B} |Du_i|^2\,dx, \qquad (6.8)
\end{aligned}$$

for a universal constant C. Letting $i \to \infty$ and keeping in mind (6.6) and (6.7), we conclude

$$\limsup_{i\to\infty} \int_{B_{2R}(x_o)\cap B} |D\tilde u_i|^2\,dx \le C\mu\big(\overline{B_{2R}(x_o) \setminus B_R(x_o)}\big) \le C\delta. \tag{6.9}$$

From this we deduce that

$$\mathbf{D}_{B_{2R}(x_o)}(\tilde u_i) \le C\delta \tag{6.10}$$

holds true for all sufficiently large $i \in \mathbb{N}$. On the other hand by the lower semicontinuity $\liminf_{i\to\infty} \int_{B_{2R}(x_o)} |Du_i|^2\,dx \ge \mu(B_{2R}(x_o)) > \frac12\varepsilon_2$, we have

$$\mathbf{D}_{B_{2R}(x_o)}(u_i) \ge \tfrac14\,\varepsilon_2 \tag{6.11}$$

for large values of $i \in \mathbb{N}$. In what follows, we will only consider values of $i \in \mathbb{N}$ for which (6.10) and (6.11) are both satisfied. Since $\tilde u_i \rightharpoondown \tilde u_T$ weakly in $W^{1,2}(B,A)$ and also $\tilde u_i \to \tilde u_T$ in $L^\infty(\partial B,\mathbb{R}^3)$, we can apply Corollary 6.2. This ensures the lower semicontinuity of the energy functional in the sense

$$\mathbf{D}(\tilde u_T) + 2\mathbf{V}_H(\tilde u_T,u_o) \le \liminf_{i\to\infty}\big[\mathbf{D}(\tilde u_i) + 2\mathbf{V}_H(\tilde u_i,u_o)\big]. \tag{6.12}$$

In order to bound the right-hand side, we use that $\tilde u_i = u_i$ outside of $B_{2R}(x_o)$, the isoperimetric condition (2.4) and the bounds (6.10) and (6.11) in order to estimate

$$\begin{aligned}
(6.13)\qquad \mathbf{D}(\tilde u_i) &+ 2\mathbf{V}_H(\tilde u_i,u_o) - \mathbf{D}(u_i) - 2\mathbf{V}_H(u_i,u_o) \\
&= \mathbf{D}_{B_{2R}(x_o)}(\tilde u_i) - \mathbf{D}_{B_{2R}(x_o)}(u_i) + 2\mathbf{V}_H(\tilde u_i,u_i)
\end{aligned}$$

$$\begin{aligned}&\le \mathbf{D}_{B_{2R}(x_o)}(\tilde u_i) - \mathbf{D}_{B_{2R}(x_o)}(u_i) + c\big[\mathbf{D}_{B_{2R}(x_o)}(\tilde u_i) + \mathbf{D}_{B_{2R}(x_o)}(u_i)\big]\\ &= -(1-c)\mathbf{D}_{B_{2R}(x_o)}(u_i) + (1+c)\mathbf{D}_{B_{2R}(x_o)}(\tilde u_i)\\ &\le -\tfrac14(1-c)\,\varepsilon_2 + 2C\delta.\end{aligned}$$

Joining (6.12) and (6.13), we arrive at

$$\mathbf{D}(\tilde u_T) + 2\mathbf{V}_H(\tilde u_T, u_o) \le \liminf_{i\to\infty}\,\big[\mathbf{D}(u_i) + 2\mathbf{V}_H(u_i,u_o)\big] - \tfrac14(1-c)\,\varepsilon_2 + 2C\delta.$$

In order to bound the right-hand side, we use the monotonicity (3.4), with the result

$$\begin{aligned}\mathbf{D}(u_i) + 2\mathbf{V}_H(u_i,u_o) &= \mathbf{D}\big(u_{h_i}(\cdot,t_i)\big) + 2\mathbf{V}_H\big(u_{h_i}(\cdot,t_i),u_o\big)\\ &\le \mathbf{D}\big(u_{h_i}(\cdot,0)\big) + 2\mathbf{V}_H\big(u_{h_i}(\cdot,0),u_o\big) = \mathbf{D}(u_o).\end{aligned}$$

The combination of the two preceding estimates leads us to

$$\mathbf{D}(\tilde u_T) + 2\mathbf{V}_H(\tilde u_T,u_o) \le \mathbf{D}(u_o) - \tfrac14(1-c)\,\varepsilon_2 + 2C\delta. \tag{6.14}$$

It remains to replace $\tilde u_T$ by u_T on the left-hand side of the estimate. To this end, we estimate, similarly as in (6.13)

$$\begin{aligned}&\mathbf{D}(u_T) + 2\mathbf{V}_H(u_T,u_o) - \mathbf{D}(\tilde u_T) - 2\mathbf{V}_H(\tilde u_T,u_o)\\ &\quad= \mathbf{D}_{B_{2R}(x_o)}(u_T) - \mathbf{D}_{B_{2R}(x_o)}(\tilde u_T) + 2\mathbf{V}_H(u_T,\tilde u_T)\\ &\quad\le \mathbf{D}_{B_{2R}(x_o)}(u_T) - \mathbf{D}_{B_{2R}(x_o)}(\tilde u_T) + c[\mathbf{D}_{B_{2R}(x_o)}(u_T) + \mathbf{D}_{B_{2R}(x_o)}(\tilde u_T)]\\ &\quad\le (1+c)\mathbf{D}_{B_{2R}(x_o)}(u_T) \le (1+c)\tfrac12\delta \le \delta,\end{aligned} \tag{6.15}$$

where we used (6.7). Joining the bounds (6.14) and (6.15), we conclude

$$\mathbf{D}(u_T) + 2\mathbf{V}_H(u_T,u_o) \le \mathbf{D}(u_o) - \tfrac14(1-c)\,\varepsilon_2 + (2C+1)\delta.$$

Choosing $\delta>0$ small enough, we can achieve $(2C+1)\delta \le \frac18(1-c)\varepsilon_2$. This implies the asserted estimate and completes the proof. □

Proof of Theorem 1.2. Now, we define a global solution $u\in L^\infty(0,\infty;W^{1,2}(B,A))$ with time derivative $\partial_t u\in L^2(B\times(0,\infty))$ in the following way. We start with the global weak solution $\tilde u_1\colon B\times(0,\infty)\to A$ constructed in [3] for the prescribed initial values u_o. Writing $T_1>0$ for the first singular time of $\tilde u_1$, we let $u:=\tilde u_1$ on $B\times(0,T_1]$. Then we choose $u_{T_1}:=u(\cdot,T_1)$ as new initial values and construct a global weak solution $\tilde u_2\colon B\times(0,\infty)\to A$ using the method from [3]. Then, we define $u(x,t):=\tilde u_2(x,t-T_1)$ for $(x,t)\in B\times(T_1,T_2]$, where T_2-T_1 denotes the first singular time of the solution $\tilde u_2$. As in the first step we use u_{T_2} as new initial values for the solution on the time interval $(T_2,T_3]$. We continue this procedure inductively to obtain a solution $u\colon B\times(0,\infty)\to A$ which is regular except from the singular times $T_1,T_2,T_3,\dots$, and where $\tilde u_k$ is defined as the solution of the flow, where the initial values are chosen as the values of $\tilde u_{k-1}$ at the first singular time T_k-T_{k-1}. It remains to show, that there can be only finitely many singular times $T_1,\dots,T_k$. More precisely, we even can bound the possible number $k\in\mathbb{N}$ of singular times. To this end, we apply the estimate from the preceding lemma to the map $\tilde u_k$, which is a solution to the flow with initial values $u_{T_{k-1}}$ and attains the values u_{T_k} at the first singular time. This leads us to

$$\begin{aligned}\mathbf{D}(u_{T_k}) + 2\mathbf{V}_H(u_{T_k},u_o) &= \mathbf{D}(u_{T_k}) + 2\mathbf{V}_H(u_{T_k},u_{T_{k-1}}) + 2\mathbf{V}_H(u_{T_{k-1}},u_o)\\ &\le \mathbf{D}(u_{T_{k-1}}) + 2\mathbf{V}_H(u_{T_{k-1}},u_o) - \tfrac18(1-c)\varepsilon_2\end{aligned}$$

holds true. This inequality can be iterated to yield that

$$\begin{aligned}\mathbf{D}(u_{T_k}) + 2\mathbf{V}_H(u_{T_k},u_o) &\le \mathbf{D}(u_{T_1}) + 2\mathbf{V}_H(u_{T_1},u_o) - \tfrac18(k-1)(1-c)\varepsilon_2\\ &\le \mathbf{D}(u_o) - \tfrac18 k(1-c)\varepsilon_2.\end{aligned}$$

Here we can bound the left-hand side from below using the isoperimetric condition. This implies

$$\begin{aligned}\mathbf{D}(u_{T_k}) + 2\mathbf{V}_H(u_{T_k}, u_o) &\geq \mathbf{D}(u_{T_k}) - 2|\mathbf{V}_H(u_{T_k}, u_o)| \\ &\geq \mathbf{D}(u_{T_k}) - c\big[\mathbf{D}(u_{T_k}) + \mathbf{D}(u_o)\big] \\ &\geq -\mathbf{D}(u_o).\end{aligned}$$

Joining the two preceding estimates, we arrive at

$$\tfrac{1}{8}k(1-c)\varepsilon_2 \leq 2\mathbf{D}(u_o),$$

which implies that the number of singular times is bounded by

$$k \leq \frac{16\mathbf{D}(u_o)}{(1-c)\varepsilon_2}.$$

This finishes the proof of Theorem 1.2. □

REFERENCES

[1] V. Bögelein, Higher integrability for weak solutions of higher order degenerate parabolic systems. *Ann. Acad. Sci. Fenn. Math.* 33:387–412 (2008).

[2] V. Bögelein, F. Duzaar and G. Mingione, The boundary regularity of non-linear parabolic systems. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27(1):201–255 (2010).

[3] V. Bögelein, F. Duzaar and C. Scheven, Weak solutions to the heat flow for surfaces of prescribed mean curvature. *Trans. Amer. Math. Soc.* 365(9):4633–4677 (2013).

[4] V. Bögelein, F. Duzaar and C. Scheven, Global solutions to the heat flow for m-harmonic maps and regularity. *Indiana Univ. Math. J.* 61(6): 2157–2210 (2012).

[5] S. Campanato, Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega,\delta)$, *Ann. Mat. Pura Appl.*(IV) 73:55–102, 1966.

[6] Y. Chen and S. Levine, The existence of the heat flow of H-systems. *Discrete Contin. Dyn. Syst.* 8(1):219–236, 2002.

[7] G. Da Prato, Spazi $\mathcal{L}^{p,\vartheta}(\Omega,\delta)$ e loro proprietà. *Ann. Mat. Pura Appl.*(IV) 69:383–392, 1965.

[8] F. Duzaar and J. Grotowski, Existence and regularity for higher dimensional H-systems. *Duke Mathematical J.* 101(3):459–485, 2000.

[9] F. Duzaar, G. Mingione, The p-harmonic approximation and the regularity of p-harmonic maps. *Calc. Var. PDE* 20(3):235–256 (2004).

[10] F. Duzaar and G. Mingione. Second order parabolic systems, optimal regularity, and singular sets of solutions. *Ann. Inst. Henri Poincaré Anal. Non Linéaire*, 22:705–751, 2005.

[11] F. Duzaar and C. Scheven, The evolution of H-surfaces with a Plateau boundary condition. To appear in Ann. Inst. H. Poincaré, Anal. Non Linéaire. DOI: 10.1016/j.anihpc.2013.10.003

[12] F. Duzaar and K. Steffen, Existence of hypersurfaces with prescribed mean curvature in Riemannian manifolds. *Indiana Univ. Math. J.* 45:1045–1093, 1996.

[13] F. Duzaar and K. Steffen, Parametric surfaces of least H-energy in a Riemannian manifold. *Math. Ann.* 314:197–244, 1999.

[14] L. Evans and R. Gariepy, Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[15] H. Federer, Geometric Measure Theory. Springer, Berlin, 1969.

[16] E. Giusti, Direct Methods in the Calculus of Variations. World Scientific Publishing Company, Tuck Link, Singapore, 2003.

[17] R. Gulliver and J. Spruck, The Plateau problem for surfaces of prescribed mean curvature in a cylinder. *Invent. Math.* 13:169–178, 1971.

[18] R. Gulliver and J. Spruck, Existence theorems for parametric surfaces of prescribed mean curvature. *Indiana Univ. Math. J.* 22:445–472, 1972.

[19] E. Heinz, Über die Existenz einer Fläche konstanter mittlerer Krümmung mit gegebener Berandung. *Math. Ann.* 137:258–287, 1954.

[20] S. Hildebrandt, Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Anwendungen auf die Kapillaritätstheorie I. *Math. Z.* 112:205–213, 1969.

[21] S. Hildebrandt, Einige Bemerkungen über Flächen vorgeschriebener mittlerer Krümmung. *Math. Z.* 115:169–178, 1970.

[22] S. Hildebrandt and H. Kaul, Two-dimensional variational problems with obstructions, and Plateau's problem for H-surfaces in a Riemannian manifold. *Comm. Pure Appl. Math.* 25:187–223, 1972.

[23] M. Hong and D. Hsu, The heat flow for H-systems on higher dimensional manifolds. *Indiana Univ. Math. J.* 59(3):761–789, 2010.

[24] C. Leone, M. Misawa and A. Verde, A global existence result for the heat flow of higher dimensional H-systems. *J. Math. Pures Appl.* 97(3):282–294, 2012.
[25] C. Leone, M. Misawa and A. Verde, The regularity for nonlinear parabolic systems of p-Laplacian type with critical growth. preprint.
[26] G. Lieberman, Second Order Parabolic Differential Equations. World Scientific, Singapore, 1996.
[27] L. Nirenberg, An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa* (3) 20:733–737, 1966.
[28] O. Rey, Heat flow for the equation of surfaces with prescribed mean curvature. *Math. Ann.* 293:123–146, 1991.
[29] J. Simon, Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (IV) 146:65–96, 1987.
[30] L. Simon, Lectures on Geometric Measure Theory. Proc. of Centre for Math. Anal. 3, Australian National Univ., 1983.
[31] L. Simon, Theorems on regularity and singularity of energy minimizing maps. ETH Lecture Notes, Birkhäuser, Basel, 1996.
[32] K. Steffen, Isoperimetric inequalities and the problem of Plateau. *Math. Ann.* 222:97–144, 1976.
[33] K. Steffen, On the existence of surfaces with prescribed mean curvature and boundary. *Math. Z.* 146:113–135, 1976.
[34] M. Struwe, On the evolution of harmonic mappings of Riemannian surfaces. *Comm. Math. Helv.* 60:558–581, 1985.
[35] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.* 160:19–64, 1988.
[36] M. Struwe, Plateau's Problem and the Calculus of Variations. Mathematical Notes, 35. Princeton University Press, Princeton, NJ, 1988.
[37] H. Wente, An existence theorem for surfaces of constant mean curvature. *J. Math. Appl.* 26:318–344, 1969.

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