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# Stability and continuity of functions of least gradients *

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**Abstract**

In this note we prove that functions of least gradients (after modification on a set of measure zero) are continuous everywhere outside of their jump sets. As a tool, we also develop some stability properties of sequences of least gradient functions. We also apply these tools to show a maximum principle for functions of least gradients that arise as solutions to a Dirichlet problem.

## 1 Introduction

The theory of minimal surfaces in the Euclidean setting was studied extensively, for example, in [1], [26], [7], [11], [28], [29], [30] from the point of view of regularity. The literature on this subject is extensive, and it is impossible to list them all; only a small sampling is given here. Much of this

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study had been in the direction of understanding the regularity of minimal surfaces obtained (locally) as graphs of functions. However, the work of [7], [30] and [32] and others study a more general "least gradient" functions and their regularity, and it is shown in [30] that if the boundary data is Lipschitz continuous and the (Euclidean) boundary of the domain has positive mean curvature, then the least gradient solution to the corresponding Dirichlet problem is locally Lipschitz continuous on the domain. However, such Lipschitz regularity has been shown to fail even in some simple weighted Euclidean setting, see [12], where an example of a solution to the Dirichlet problem with Lipschitz boundary data with jump discontinuities in the domain is given. Therefore, in a more general setting, it is natural to ask whether the functions of least gradient are continuous outside of their jump sets. The principal goal of this note is to show that every function of least gradient is necessarily continuous outside of its jump set, even when the boundary data is not continuous.

The setting we consider here is that of a complete metric measure space $X = (X, d, \mu)$ equipped with a doubling Borel regular measure $\mu$ supporting a 1-Poincaré inequality. We consider functions of bounded variation in the sense of [3], [23], and [5], and functions of *least gradient* in a domain $\Omega \subset X$.

In considering such regularity properties of least gradients, we needed some tools related to stability properties of least gradient families. Therefore we extend the study also to include questions related to stability properties of least gradient functions (minimizers) and quasiminimizers. We show that being a function of least gradient and, more generally, being a BV-quasiminimizer are properties preserved under $L^1_{loc}(\Omega)$-convergence. We then obtain partial regularity results for functions of least gradients. Namely, we show that such minimizers are continuous at the points of approximate continuity, that is, away from the jump discontinuities of the function. Observe that by the results of [5], the jump set of a BV function has $\sigma$-finite codimension 1 Hausdorff measure; hence there is a plenitude of points where the least gradient function is continuous.

As a further application of the tools developed to study the above regularity, we obtain a maximum principle for least gradient functions obtained as solutions to a Dirichlet problem.

In tandem with the development of least gradient theory in the metric setting, the papers [12] and [13] develop the existence and trace theory of minimizers of functionals of linear growth in the metric setting (analogously to the problem considered by Giusti in [11]). In the case that the function $f$

of linear growth satisfies $f(0) = 0$ or $\liminf_{t\to 0^+} f(t)/t > 0$, then the results of this note can be easily adapted to study the regularity and maximum principle properties of the associated minimizers. We leave the interested reader to verify this.

This paper is organized as follows. In Section 2 we introduce the concepts and background needed for the study conducted in this note, and in Section 3 we consider the stability problem for least gradient functions on a given domain. In Section 4 we use the tools developed in Section 3 to show that least gradient functions are continuous everywhere outside of their jump sets. In the final section of this paper we give a further application of the tools developed in Section 3 to prove the maximum principle for functions of least gradient that arise as solutions to a Dirichlet problem.

# 2 Notation and background

In this section we introduce the notation and the problem we consider. A good discussion of BV functions in the Euclidean setting can be found in [31] and [4]. The theory of BV functions in the metric setting was first studied by Miranda Jr. in [23], and further developed in [2], [3], [5], and [19].

A function $f \in L^1(X)$ is said to be in the BV-class $BV(X)$ if there is a sequence $\{f_k\}_k$ of functions in $N^{1,1}(X)$ such that $f_k \to f$ in $L^1(X)$ and the sequence is bounded in $N^{1,1}(X)$. The *BV energy* of $f$ is given by

$$\|Df\|(X) = \inf_{\{f_k\}_k} \liminf_{k\to\infty} \left( \|f_k\|_{N^{1,1}(X)} - \|f_k\|_{L^1(X)} \right),$$

where the infimum is over all sequences of functions in $N^{1,1}(X)$ (or, equivalently, Lipschitz functions in $X$) that converge to $f$ in $L^1(X)$. See [23] and [5] for more information on functions in the BV-class as well as for the fact that $X \supset O \mapsto \|Df\|(O)$ is an outer measure.

**Definition 2.1.** We denote by $BV_c(\Omega)$ the functions $g \in BV(X)$ with compact support in $\Omega$, and by $BV_0(\Omega)$ the functions $g \in BV(X)$ such that $g = 0$ $\mu$-a.e. in $X \setminus \Omega$.

We next recall the various notions of minimizers; see e.g. [7].

**Definition 2.2.** A function $u \in BV_{loc}(\Omega)$ is of *least gradient* in $\Omega$ if whenever $g \in BV_c(\Omega)$ with $K = \text{supt}(g)$, we have that

$$\|Du\|(K) \le \|D(u+g)\|(K).$$

Following [18], we say that a set $E \subset X$ is said to be *of minimal surface in* $\Omega$ if $\chi_E \in BV_{loc}(\Omega)$ and $\chi_E$ is of least gradient in $\Omega$. Similarly, a function $u \in BV_{loc}(\Omega)$ is of *Q-quasi least gradient* in $\Omega$ for some $Q \geq 1$ if

$$\|Du\|(K) \leq Q\,\|D(u+g)\|(K)$$

for all $g \in BV_c(\Omega)$ and $K = \operatorname{supt}(g)$.

We also consider the corresponding Dirichlet problem in metric setting. Recall that in the Euclidean setting with a Lipschitz domain $\Omega$, the least gradient problem with a given boundary datum $f$ can be stated as

$$\min\left\{\|Du\|(\Omega) : \ u \in \mathrm{BV}(\Omega),\ u = f \ \text{ on } \ \partial\Omega\right\}.$$

If we do not require, for instance, the continuity of the boundary data, then the boundary value has to be understood in a suitable sense, e.g. a trace of a BV function. One possibility is to consider a relaxed problem with a penalization term

$$\|Du\|(\Omega) + \int_{\partial\Omega} |Tu - Tf| \,\mathrm{d}\mathcal{H}^{n-1},$$

where $Tu$ and $Tf$ are traces of $u$ and $f$ on $\partial\Omega$, provided such traces exist. In order to avoid working directly with traces of BV functions, we consider an equivalent problem formulated in a larger domain, namely in $X$.

**Definition 2.3.** Given $f \in BV(X)$, we say that $u \in BV(X)$ is a *solution to the Dirichlet problem for least gradients with boundary data* $f$, if $u - f \in BV_0(\Omega)$ and whenever $g \in BV_0(\Omega)$ we have

$$\|Du\|(\overline{\Omega}) \leq \|D(u+g)\|(\overline{\Omega}).$$

Note that such a solution is a function of least gradient and that this problem is the same as minimizing $\|Du\|(\overline{\Omega})$ over all $u \in BV(X)$ that satisfy $u - f \in BV_0(\Omega)$. Furthermore, since we have that $\|Du\|(X) = \|Du\|(\overline{\Omega}) + \|Df\|(X \setminus \overline{\Omega})$, the problem is equivalent to, and the minimizers are the same as, when minimizing $\|Du\|(X)$ over all $u \in BV(X)$ that satisfy $u - f \in BV_0(\Omega)$. Thus the problem we consider here is the same as in [12], [13]. We also introduce the following notion of a solution to the Dirichlet problem. Note that in the following definition, instead of working with the class $BV_0(\Omega)$, the perturbations $g$ are required to be in $BV_c(\Omega)$.

**Definition 2.4.** We say that $u \in BV(X)$ is a *very weak solution to the Dirichlet problem for least gradients with boundary data* $f$ if $u$ is of least gradient in $\Omega$, $u-f \in BV_0(\Omega)$, and whenever $h \in BV(X)$ such that $\text{supt}(h-f) \Subset \Omega$, we have

$$\|Du\|(\overline{\Omega}) \leq \|Dh\|(\overline{\Omega}).$$

Observe that solutions to the Dirichlet problem for least gradients with boundary data $f$ are necessarily very weak solutions. The converse is not true in general. For example, if $\Omega$ is a bounded Lipschitz domain in $X = \mathbb{R}^n$ and $f = \chi_{\mathbb{R}^n \setminus \Omega}$, then for each $\alpha \in [-1, 1]$ the function $u = 1 + \alpha\, \chi_\Omega$ is a very weak solution to the Dirichlet problem for least gradients with boundary data $f$, but the only solution in the sense of Definition 2.3 is the constant function 1 (corresponding to $\alpha = 0$). Hence the category of very weak solutions is a wider class than the category of solutions.

Note that if $\Omega$ is the unit ball in $\mathbb{R}^n$ and $f$ is the constant function 1 on $\mathbb{R}^n$, then both $u = f$ and $u = \chi_{\mathbb{R}^n \setminus \Omega}$ would be of least gradient in $\Omega$ and satisfy the requirement $u - f \in BV_0(\Omega)$, but while $u = f$ is a very weak solution, $u = \chi_{\mathbb{R}^n \setminus \Omega}$ is not. Thus while the notion of very weak solutions is stable under compact (in $\Omega$) perturbation of the boundary data, it is not stable under perturbation up to the boundary of $\Omega$.

# 3 Stability of least gradient families

To answer questions regarding continuity properties of functions of least gradient (off of their jump sets), it turns out that tools related to stability of least gradient families is needed. In this section we therefore study such stability properties.

The following stability result is key in the study of continuity properties of functions of least gradient. In the Euclidean setting this result is [22, Theorema 3]. The proof of this result (in the Euclidean setting) found in [22] is based on the knowledge of trace theorems for BV functions. Given the lack of trace theorems in the metric setting, the proof given here is different from that of [22], but the philosophy underlying the proof is the same. We thank Michele Miranda for explaining the proof in [22] (which is in Italian) and for suggesting a way to modify that proof.

**Proposition 3.1.** *Let $\Omega$ be a domain in $X$ and $\{u_k\}_k$ be a sequence of functions in $BV_{loc}(\Omega)$ that are of least gradient in $\Omega$, and suppose that $u$ is* *a function on $\Omega$ such that $u_k \to u$ in $L^1_{loc}(\Omega)$. Then $u$ is a function of least gradient in $\Omega$ with $u \in BV_{loc}(\Omega)$.*

An analogous statement holds for sequences of $Q$-quasi least gradient functions; the $L^1_{loc}(\Omega)$-limits of such sequences are also $Q$-quasi least gradients. The proof is mutatis mutandis the same as the proof of the above proposition; we leave the interested reader to verify this.

To prove the above proposition, we need the following version of the product rule (Leibniz rule) for functions of bounded variation, which is from [19]. If $u \in BV(X)$ and $v$ is a compactly supported Lipschitz function on $X$, then $vu \in BV(X)$ with

$$d\|D(vu)\| \le v\, d\|Du\| + |u|^\vee \|g_v\|_{L^\infty} \chi_{\mathrm{supt}(v)}\, d\mu.$$

See (4.1) for the definition of the upper approximate limit $|u|^\vee$ of $|u|$. We also need the following inequality from [18, Inequality (4.3)] for functions of least gradient in $\Omega$. Whenever $B = B(x,R) \Subset \Omega$, we have the De Giorgi inequality for each $0 < r < R$:

$$\|Du\|(B(x,r)) \le \frac{C}{R-r} \int_{B(x,R)} |u|\, d\mu.$$

*Proof of Proposition 3.1.* Because $u_k \to u$ in $L^1_{loc}(\Omega)$, we know that $u_k$ are bounded in $L^1(K)$ for compact sets $K \subset \Omega$. Hence by covering $K$ by balls of radius $r$ centered at points in $K$ so that concentric balls of radius $2r$ are relatively compact subsets of $\Omega$ and applying the above De Giorgi inequality we see that

$$\sup_k \|Du_k\|(K) \le C_K < \infty. \tag{3.1}$$

Thus by $u_k \to u$ in $L^1_{loc}(\Omega)$ we can conclude that $u \in BV_{loc}(\Omega)$. It now only remains to show that $u$ is of least gradient in $\Omega$. To do so, we fix a function $h \in BV(X)$ such that the support $K_0$ of $h$ is a compact subset of $\Omega$. We wish to show that

$$\|Du\|(K_0) \le \|D(u+h)\|(K_0).$$

By (3.1) we know that the sequence of Radon measures $\|Du_k\|$ is locally bounded on $\Omega$. Hence a diagonalization argument gives a subsequence, also denoted $\|Du_k\|$, and a Radon measure $\nu$ on $\Omega$, such that $\|Du_k\| \to \nu$ weakly in $\Omega$.

Let $K \subset \Omega$ be a compact set such that $K_0 \subset \mathrm{int}(K)$ and that

$$\|Du\|(\partial K) = \sup_k \|Du_k\|(\partial K) = \nu(\partial K) = 0,$$

and let $\varepsilon > 0$ such that $K_\varepsilon := \bigcup_{x \in K} B(x, \varepsilon) \Subset \Omega$. We choose a Lipschitz function $\eta$ on $X$ such that $0 \le \eta \le 1$ on $X$, $\eta = 1$ on $K_{\varepsilon/2}$, and $\eta = 0$ on $X \setminus K_\epsilon$, and for each positive integer $k$ we set

$$h_k := \eta(u + h) + (1 - \eta)u_k.$$

Note that $h_k = u_k$ on $\Omega \setminus K_\varepsilon$, and so by the minimality of $u_k$, the lower semicontinuity of the $BV$-energy in $L^1$, the above Leibniz rule, and by the fact that $u_k \to u$ in $L^1(K_\varepsilon)$, we have

$$\begin{aligned}
\|Du\|(K_\varepsilon) &\le \liminf_k \|Du_k\|(K_\varepsilon) \\
&\le \liminf_k \|Dh_k\|(K_\varepsilon) \\
&\le \|D(u+h)\|(K) + \liminf_k [C_\eta \int_{K_\varepsilon \setminus K_0} |u - u_k|^\vee \, d\mu \\
&\qquad\qquad + \|Du_k\|(K_\varepsilon \setminus K)] \ + \|Du\|(K_\varepsilon \setminus K) \\
&\le \|D(u+h)\|(K_\varepsilon) + \liminf_k \|Du_k\|(K_\varepsilon \setminus K) \\
&\le \|D(u+h)\|(K_\varepsilon) + \nu(\overline{K_\varepsilon \setminus K}).
\end{aligned}$$

Therefore

$$\|Du\|(K) \le \|D(u+h)\|(K_\varepsilon) + \nu(\overline{K_\varepsilon \setminus K}).$$

Now letting $\varepsilon \to 0$ we obtain

$$\|Du\|(K) \le \|D(u+h)\|(K) + \nu(\partial K) = \|D(u+h)\|(K).$$

Taking the infimum over all such $K \supset K_0$, we obtain

$$\|Du\|(K_0) \le \|D(u+h)\|(K_0).$$

Thus $u$ is of least gradient in $\Omega$. □

While the above proposition does not need the functions $u_k$ to be in the global space $BV(X)$, the next stability result considers what happens when $u_k \in BV(X)$ is a solution to the Dirichlet problem for least gradients with boundary data in $BV(X)$.

**Proposition 3.2.** *Let $\Omega$ be a bounded domain in $X$ such that $\mu(X \setminus \Omega) > 0$. Suppose $f_k \in BV(X)$ and that $u_k \in BV(X)$ be the least gradient solution of the Dirichlet problem on $\Omega$ with boundary data $f_k$. Suppose also that $f_k \to f$ in $BV(X)$ (that is, $\|f - f_k\|_{L^1(X)} + \|D(f_k - f)\|(X) \to 0$ as $k \to \infty$). Then there is a function $u \in BV(X)$ such that a subsequence of $u_k$ converges to $u$ in $L^1(\Omega)$ and $u$ is a very weak solution to the Dirichlet problem for least gradients in $\Omega$ for the boundary data $f$.*

*Proof.* Let $B \Supset \Omega$ be a ball such that $\mu(B \setminus \Omega) > 0$. By the 1-Poincaré inequality, if $v \in BV_0(\Omega)$, then

$$\int_B |v| \, d\mu \le C_0 \, \|Dv\|(\overline{\Omega}),$$

where $C_0$ depends only on the radius of $B$, the Poincaré inequality constants, the doubling constant of $\mu$, and the ratio $\mu(B \setminus \Omega)/\mu(B)$.

By definition, we know that $u_k - f_k \in BV_0(\Omega)$, and hence for each positive integer $k$,

$$\begin{aligned}\int_X |u_k - f_k| \, d\mu + \|D(u_k - f_k)\|(X) &= \int_\Omega |u_k - f_k| \, d\mu + \|D(u_k - f_k)\|(\overline{\Omega}) \\ &\le [C_0 + 1] \, \|D(u_k - f_k)\|(\overline{\Omega}) \\ &\le [C_0 + 1] \, \|Du_k\|(\overline{\Omega}) + [C_0 + 1] \, \|Df_k\|(\overline{\Omega}) \\ &\le 2[C_0 + 1] \, \|Df_k\|(\overline{\Omega}).\end{aligned}$$

The last inequality in the above series of inequalities follows from the minimality of $u_k$ for the Dirichlet problem with boundary data $f_k$. It follows that the sequence $\{u_k - f_k\}_k$ is a bounded sequence in $BV_0(\Omega)$, and hence by the compact embedding theorem for $BV(X)$ (see [23, Theorem 3.7]), there is a subsequence, also denoted by $\{u_k - f_k\}_k$, and a function $v \in BV_0(\Omega)$, such that $u_k - f_k \to v$ in $L^1_{loc}(X)$, and so by the compactness of $\overline{\Omega} \subset \overline{B}$, we know that $u_k - f_k \to v$ in $L^1(B)$. Hence $u_k \to f + v =: u$ in $L^1(B)$, and from Proposition 3.1 we know that $u$ is of least gradient in $\Omega$. Furthermore, $u - f = v \in BV_0(\Omega)$. If $h \in BV(X)$ with $\operatorname{supt}(f - h) \Subset \Omega$, then setting $g = f - h$ and $\widehat{f_k} = f_k + g$, we see that $\widehat{f_k} - h = f_k - f \to 0$ in $BV(X)$ as $k \to \infty$. Since $u_k$ is a solution in the sense of Definition 2.3 with boundary data $f_k$, we have $\|Du_k\|(\overline{\Omega}) \le \|D\widehat{f_k}\|(\overline{\Omega})$. By the lower semicontinuity of the

BV energy measure, we get

$$\begin{aligned}\|Du\|(\overline{\Omega}) + \|Df\|(X\setminus\overline{\Omega}) = \|Du\|(X) &\le \liminf_{k\to\infty}\|Du_k\|(X)\\ &= \liminf_{k\to\infty}\left[\|Du_k\|(\overline{\Omega}) + \|Df_k\|(X\setminus\overline{\Omega})\right]\\ &= \liminf_{k\to\infty}\|Du_k\|(\overline{\Omega}) + \|Df\|(X\setminus\overline{\Omega}).\end{aligned}$$

Therefore,

$$\|Du\|(\overline{\Omega}) \le \liminf_{k\to\infty}\|Du_k\|(\overline{\Omega}) \le \liminf_{k\to\infty}\|D\widehat{f_k}\|(\overline{\Omega}) = \|Dh\|(\overline{\Omega}).$$

Thus $u$ is a very weak solution to the Dirichlet problem with boundary data $f$ in the sense of Definition 2.4. This concludes the proof. □

The above two stability results required the sequence $u_k$ to converge in $L^1_{loc}(\Omega)$ (while the second stability result above did not explicitly require this, it was almost an immediate consequence of the hypothesis). The next proposition consider the weakest form of stability, namely, what happens when the sequence $u_k$ is known to converge pointwise almost everywhere to a function $u$ on $\Omega$. As to be expected, without additional restriction on the sequence $u_k$ itself, we cannot claim that the limit function $u$ is of least gradient in $\Omega$, see Example 3.5 below. Note that if $u$ is of least gradient in $\Omega$, then necessarily $u \in BV_{loc}(\Omega)$.

Recall that given an extended real-valued function $u$ on $\Omega$, its super-level sets are sets of the form $\{x \in \Omega : u(x) > t\}$ for some $t \in \mathbb{R}$.

**Proposition 3.3.** *Suppose that $u_k$, $k \in \mathbb{N}$, are functions of least gradient in a domain $\Omega$, and $u$ be a measurable function on $\Omega$, finite-valued $\mu$-almost everywhere in $\Omega$, such that $u_k \to u$ $\mu$-a.e. in $\Omega$. Then the super-level sets $\{x \in \Omega : u(x) > t\}$ are of minimal surfaces in $\Omega$ for almost every $t \in \mathbb{R}$. If in addition $\sup_k |u_k| \le M$ on $\Omega$, or if $\sup_k \|Du_k\|(K) = C_K < \infty$ for each $K \Subset \Omega$, then $u$ is of least gradient in $\Omega$ and there is a subsequence of $\{u_k\}$ that converges in $L^1_{loc}(\Omega)$ to $u$.*

To prove this proposition we need the following lemma, which is also quite useful in the study of continuity properties of minimizers undertaken in the next section.

**Lemma 3.4.** *Suppose that $u$ is a function of least gradient in $\Omega$. Then for each $t \in \mathbb{R}$, the set*

$$E_t := \{x \in \Omega : u(x) > t\}$$

*is a set of minimal surface in* $\Omega$*. Furthermore, with* $u_1 = \min\{u, t\}$ *and* $u_2 = (u - t)_+$*, we have that*

$$\|Du_1\| + \|Du_2\| = \|Du\|$$

*in the sense of measures.*

*Proof.* The argument is based on Bombieri–De Giorgi–Giusti [7]. We know from the coarea formula in [23] that for almost every $t \in \mathbb{R}$, $\chi_{E_t} \in BV_{loc}(\Omega)$ since $u \in BV_{loc}(\Omega)$. Fix $t \in \mathbb{R}$, and set

$$\begin{aligned} u_1 &= \min\{u, t\}, \\ u_2 &= (u - t)_+. \end{aligned}$$

Then $u_1, u_2 \in BV(\Omega)$ with $u = u_1 + u_2$.

By the coarea formula and by the fact that $P(\Omega : \Omega) = 0 = P(\emptyset : \Omega)$,

$$\begin{aligned} \|Du_1\|(\Omega) &= \int_{-\infty}^{\infty} P(\{x \in \Omega \,:\, u_1(x) < s\} : \Omega)\, ds \\ &= \int_{-\infty}^{t} P(\{x \in \Omega \,:\, u_1(x) < s\} : \Omega)\, ds \\ &= \int_{-\infty}^{t} P(\{x \in \Omega \,:\, u(x) < s\} : \Omega)\, ds. \end{aligned}$$

Again by the coarea formula, we have

$$\begin{aligned} \|Du_2\|(\Omega) &= \int_{-\infty}^{\infty} P(\{x \in \Omega \,:\, u_2(x) > s\} : \Omega)\, ds \\ &= \int_{0}^{\infty} P(\{x \in \Omega \,:\, u_2(x) > s\} : \Omega)\, ds \\ &= \int_{0}^{\infty} P(\{x \in \Omega \,:\, u(x) - t > s\} : \Omega)\, ds \\ &= \int_{t}^{\infty} P(\{x \in \Omega \,:\, u(x) > s\} : \Omega)\, ds \\ &= \int_{t}^{\infty} P(\{x \in \Omega \,:\, u(x) \le s\} : \Omega)\, ds \\ &= \int_{t}^{\infty} P(\{x \in \Omega \,:\, u(x) < s\} : \Omega)\, ds. \end{aligned}$$

The above computations hold even when $\Omega$ is replaced with any Borel set $K \subset \Omega$. Therefore we also have

$$\|Du_1\|(K) + \|Du_2\|(K) = \int_{-\infty}^{\infty} P(\{x \in \Omega \,:\, u(x) < s\} : K)\, ds = \|Du\|(K).$$

Now, for $h \in BV(\Omega)$ such that $K := \operatorname{supt}(h) \Subset \Omega$, we have by the minimality of $u$ and by the subadditivity of the BV-norm,

$$\begin{aligned}\|Du_1\|(K) + \|Du_2\|(K) \le \|D(u+h)\|(K) &= \|D(u_1 + u_2 + h)\|(K) \\ &\le \|D(u_1 + h)\|(K) + \|Du_2\|(K).\end{aligned}$$

It follows that

$$\|Du_1\|(K) \le \|D(u_1 + h)\|(K)$$

whenever $h \in BV(\Omega)$ has support in $K$. That is, $u_1$ is also of least gradient in $\Omega$. Mutatis mutandis we can show that $u_2$ is also of least gradient in $\Omega$. Hence we have that whenever $\varepsilon > 0$, the function

$$u_{t,\varepsilon} := \frac{1}{\varepsilon}\, \min\{\varepsilon,\, (u-t)_+\}$$

is of least gradient in $\Omega$. Note that by the Lebesgue dominated convergence theorem,

$$\int_\Omega |u_{t,\varepsilon} - \chi_{E_t}|\, d\mu = \int_{\{x \in \Omega \,:\, 0 < u(x) - t \le \varepsilon\}} \left(1 - \frac{u(x) - t}{\varepsilon}\right) d\mu(x) \to 0 \text{ as } \varepsilon \to 0.$$

Hence $u_{t,\varepsilon} \to \chi_{E_t}$ in $L^1_{loc}(\Omega)$ as $\varepsilon \to 0$, from which, together with Proposition 3.1, we conclude that $\chi_{E_t}$ is of least gradient in $\Omega$. This completes the proof of the lemma. $\square$

*Proof of Proposition 3.3.* Since $u_k \to u$ almost everywhere in $\Omega$, it follows that for almost every $t \in \mathbb{R}$ we have $\chi_{\{x \in \Omega \,:\, u_k(x) > t\}} \to \chi_{\{x \in \Omega \,:\, u(x) > t\}}$ almost everywhere in $\Omega$. Indeed, to see this, we set $N$ to be the collection of points $x \in \Omega$ such that $u_k(x)$ does not converge to $u(x)$. Then $\mu(N) = 0$. For $t \in \mathbb{R}$ we have that if $u_k(x) \le t$ for a subsequence of $k$ but $u(x) > t$, then $x \in N$. So, setting

$$K_t := \{x \in \Omega \setminus N \,:\, u_k(x) > t \text{ for a subsequence of } k \text{ but } u(x) \le t\},$$

for each $x \in \Omega \setminus (K_t \cup N)$ we see that $\chi_{\{y\in\Omega\,:\,u_k(y)>t\}}(x) \to \chi_{\{y\in\Omega\,:\,u(y)>t\}}(x)$. Note that for $x \in K_t$ we have $u(x) = t$. Therefore when $s \neq t$ we have $K_s \cap K_t$ is empty. So the family $\{K_t\}_{t\in\mathbb{R}}$ is pairwise disjoint, and hence by the local finiteness of $\mu$ there is at most a countable number of $t \in \mathbb{R}$ for which $\mu(K_t) > 0$. Thus we conclude that for almost every $t \in \mathbb{R}$, $\chi_{\{x\in\Omega\,:\,u_k(x)>t\}} \to \chi_{\{x\in\Omega\,:\,u(x)>t\}}$ almost everywhere in $\Omega$.

By Lemma 3.4 we know that $\chi_{\{x\in\Omega\,:\,u_k(x)>t\}}$ is of least gradient in $\Omega$ for each such $t \in \mathbb{R}$, and so by [18, Lemma 5.1] we know that whenever $B(x_0, 2r) \Subset \Omega$,

$$P(\{x \in \Omega \,:\, u_k(x) > t\} : B(x_0, r)) \le C\, \frac{\mu(B(x_0, r))}{r}.$$

It follows that whenever $K \Subset \Omega$,

$$\sup_k \|D\chi_{\{x\in\Omega\,:\,u_k(x)>t\}}\|(K) \le C_K < \infty,$$

and so by the compact embedding of $BV(K)$ (see [23]), there is a subsequence of $\{u_k\}$ such that $\chi_{\{x\in\Omega\,:\,u_k(x)>t\}}$ converges in $L^1(K)$ to $\chi_{\{x\in\Omega\,:\,u(x)>t\}}$. A diagonalization argument now yields a subsequence, also denoted $\{u_k\}$, such that

$$\chi_{\{x\in\Omega\,:\,u_k(x)>t\}} \to \chi_{\{x\in\Omega\,:\,u(x)>t\}}$$

in $L^1_{loc}(\Omega)$, and hence by Proposition 3.1 we know that $\{x \in \Omega \,:\, u(x) > t\}$ is of least gradient in $\Omega$. This concludes the proof of the first part of the proposition.

Now suppose that in addition,

$$M := \sup_k |u_k| < \infty.$$

Then by the coarea formula,

$$\begin{aligned}\|Du_k\|(B(x_0, r)) &= \int_{-M}^{M} \|D\chi_{\{x\in\Omega\,:\,u_k(x)>t\}}(B(x_0, r))\, dt \\ &\le 2CM\, \frac{\mu(B(x_0, r))}{r} < \infty,\end{aligned}$$

and the use again of compactness from [23] yields a subsequence of $\{u_k\}$ that converges to a function $v \in BV_{loc}(\Omega)$ in $L^1_{loc}(\Omega)$. Thus $u = v$ must be of least gradient in $\Omega$ by Proposition 3.1, completing the proof in the case that $\sup_k |u_k| \le M$.

Finally, if we know that whenever $K \Subset \Omega$ we have $C_K$ finite, then for a ball $B(x_0, 2\lambda r) \Subset \Omega$ we have by the 1-Poincaré inequality that

$$\sup_k \int_{B(x_0,r)} |u_k - u_{k,B(x_0,r)}|\, d\mu \le C\, r\, C_{\overline{B}(x_0,\lambda r)} < \infty,$$

and hence the sequence of $BV$-functions $\{u_k - u_{k,B(x_0,r)}\}_k$ has a convergent subsequence (convergent in $L^1(B(x_0,r))$) that converges to some function $v \in BV(B(x_0,r))$. It follows that the corresponding subsequence of the sequence $\{u_{k,B(x_0,r)}\}_k$ also must converge to some number $\alpha_{B(x_0,r)}$, and thus by the virtue of the identity $v = u + \alpha_{B(x_0,r)}$ we have that $u \in BV(B(x_0,r))$. Hence $u \in BV_{loc}(\Omega)$, and it now follows that $u$ is of least gradient in $\Omega$. $\square$

**Example 3.5.** In general, without further assumptions on the sequence $u_k$ (for example, as considered in the second part of Proposition 3.3 above), we cannot conclude that $u$ is of least gradient in $\Omega$. Here we illustrate this by considering the simple setting of Euclidean plane $\mathbb{R}^2$, with $\Omega = [-1,1] \times [0,1]$. We consider the function $f : \mathbb{R}^2 \to [0,\infty]$ given by $f(x,y) = 1/|x|$ for $x \ne 0$ and $f(0,y) = \infty$. Note that $f \notin L^1_{loc}(\Omega)$. For each positive integer $k$ we consider $f_k = \min\{f,k\}$. Then $f_k \in L^1(\Omega)$ for each $k$, and by the single-variable theory of functions of bounded variation, we now also that $f_k \in BV(\Omega)$ with $\|Df_k\|(\Omega) = 2k$. More specifically, if $A \subset [-1,1]$ and $F \subset [0,1]$ are Borel sets, then $\|Df_k\|(A \times F) = \mathcal{H}^1(F) \times \|D_1\psi_k\|(A)$, where $\psi_k : \mathbb{R} \to \mathbb{R}$ given by $\psi_k(x) = \min\{1/|x|, k\}$ and $\|D_1\psi_k\|$ is the one-dimensional $BV$-energy of $\psi_k$. When $A = [a,b] \subset [1/k, 1]$, we have $\|D_1\psi_k\|(A) = a^{-1} - b^{-1}$.

Again by the one-dimensional $BV$-theory, it can be seen that $f_k$ is of least gradient in $\Omega$. Taking $u_k = f_k$ now tells us that the pointwise limit function $u = f$ is not of least gradient in $\Omega$ because $\|Du\|(U) = \infty$ whenever $U \Subset \Omega$ is an open set intersecting the line segment $\{0\} \times (0,1)$. Recall that a part of the definition of functions of least gradients is that such functions must be in $BV_{loc}(\Omega)$.

# 4 Continuity of minimizers

An example in [12] shows that even when the boundary data $f$ is Lipschitz, in general one cannot show that there is a Lipschitz solution to the Dirichlet problem for least gradients in $\Omega$ with boundary data $f$. This is in contrast to the Euclidean situation, where it is known that if the boundary of the domain has strictly positive mean curvature (in a weak sense) and the boundary data is Lipschitz, then there is exactly one Lipschitz solution; see for example [30], [24], [25], [32]. The example in [12] is in a Euclidean convex Lipschitz domain, equipped with a 1-admissible weight in the sense of [15]. Hence even in the mildest setting modification of the Euclidean setting, things could go wrong.

We will show here that in quite a general setting (not merely for Dirichlet solutions of Lipschitz boundary data in a convex domain), functions of least gradient are continuous everywhere outside of their jump sets (after modification on a set of measure zero of course).

The approximate upper and lower limits of a function $u$ on $X$ are:

$$u^\vee(x) = \inf\left\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mu\left(\{u \le t\} \cap B(x,r)\right)}{\mu\left(B(x,r)\right)} = 1\right\}, \tag{4.1}$$
$$u^\wedge(x) = \sup\left\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mu\left(\{u \ge t\} \cap B(x,r)\right)}{\mu\left(B(x,r)\right)} = 1\right\}.$$

If $u^\vee(x) = u^\wedge(x)$ we denote by

$$\operatorname*{ap\,lim}_{y\to x} u(y) = u^\vee(x) = u^\wedge(x)$$

the *approximate limit* of function $u$ at $x \in \Omega$. The function $u$ is *approximately continuous* at $x \in \Omega$ if

$$\operatorname*{ap\,lim}_{y\to x} u(y) = u(x).$$

We know that given $u \in BV(X)$ we can redefine the function in a set of $\mu$-measure zero so that, outside the jump set $J_u = \{x \in X \,:\, u^\vee(x) > u^\wedge(x)\}$, $u$ is approximately continuous. Furthermore, we know that $J_u$ is of $\sigma$-finite $\mathcal{H}$-measure; see for example [19].

Recall that functions in $L^1_{loc}(X)$ are approximately continuous outside a set of measure zero. As the characteristic function of a cardioid shows, approximate continuity need not imply continuity, even under modification on a set of measure zero. Hence our claim that a function of least gradient, after modification on a null set, is continuous everywhere outside its jump set, is quite strong. Note that this function is in the BV class, but by Theorem 4.1 it cannot be a function of least gradient.

**Theorem 4.1.** *Let $u \in \mathrm{BV}(\Omega)$ be a function of least gradient. If $u$ is approximately continuous at point $x \in \Omega$, then $u$ is continuous at $x$.*

*Proof.* Lemma 3.4 implies that for every $t \in \mathbb{R}$ the set

$$E_t = \{x \in \Omega : \, u(x) > t\}$$

is a quasiminimal set in $\Omega$. Let $x \in \Omega$ be a point where $u$ is approximately continuous and let

$$t = u(x) = \operatorname*{ap\,lim}_{\Omega \ni y \to x} u(y).$$

We know that a function of least gradient is locally bounded, see [12, Theorem 4.2]. Thus $|t| < \infty$. Let $\varepsilon > 0$. We claim that $x \in \operatorname{int} E_{t-\varepsilon}$. The definition of approximate limit implies that $x \notin \operatorname{ext} E_{t-\varepsilon}$, thus it suffices to show that $x \notin \partial E_{t-\varepsilon}$. Let us assume, contrary to this that $x \in \partial E_{t-\varepsilon}$. The minimality and hence the quasiminimality of $E_{t-\varepsilon}$ in $\Omega$ implies that $\Omega \setminus E_{t-\varepsilon}$ is locally porous in $\Omega$, see [18, Theorem 5.2]. This means that there exists $r_x > 0$ and $C \geq 1$ such that whenever $0 < r < r_x$, there is a point $z \in B(x, r/2)$ such that

$$B(z, r/2C) \subset \Omega \setminus E_{t-\varepsilon},$$

where the constant $C$ is independent of $x$ and $r$. Now $B(z, r/2C) \subset B(x, r)$ and the doubling property of the measure gives that

$$\mu(B(z, r/2C)) \geq \gamma \mu(B(x, r)),$$

where $0 < \gamma < 1$ is independent of $x$ and $r$. Thus

$$\limsup_{r \to 0} \frac{\mu\left(\{u > t - \varepsilon\} \cap B(x, r)\right)}{\mu\left(B(x, r)\right)} \leq 1 - \gamma < 1.$$

This contradicts the fact that the approximate limit of $u$ at $x$ is $t$, since $u^\wedge(x) = t$ implies that

$$\lim_{r \to 0} \frac{\mu\left(\{u \geq t - \varepsilon/2\} \cap B(x, r)\right)}{\mu\left(B(x, r)\right)} = 1.$$

Therefore $x \notin \partial E_{t-\varepsilon}$ and hence $x \in \operatorname{int} E_{t-\varepsilon}$. By applying a similar argument to the sublevel sets $F_t = \{x \in \Omega : u \leq t\}$ (now $\Omega \setminus F_t = E_t$, which is also locally porous) gives that $x \in \operatorname{int} F_{t+\varepsilon}$. Thus for every $\varepsilon > 0$ there exist $r > 0$ such that $|u(x) - u(y)| \leq \varepsilon$ for all $y \in B(x, r)$. Hence $u$ is continuous at $x$. □

# 5 Maximum principle

In this section we prove a maximum principle for solutions to the Dirichlet problem for least gradients. Note that by considering truncations of the approximating sequences we have the following: if $f \in BV(X)$ with $M_1 \le f \le M_2$, then there is a solution $u \in BV(X)$ to the Dirichlet problem for least gradients with boundary data $f$ such that $M_1 \le u \le M_2$. Since we do not have a uniqueness result, this does not automatically imply that all solutions enjoy the property; one has to prove the maximum principle independently.

**Theorem 5.1.** *If $\psi \in BV(X)$ such that $\psi \le M$ on $X$, $\Omega \subset X$ is a bounded domain such that $\mu(X \setminus \Omega) > 0$, and if $u \in BV_0(\Omega) + \psi$ is a solution to the Dirichlet problem for least gradients with boundary data $\psi$, then we must have $u \le M$ on $\Omega$.*

*Proof.* Note that because $u \in BV_0(\Omega) + \psi$ and $\psi \le M$, we also have that $u_1 \in BV_0(\Omega) + \psi$. Hence by Lemma 3.4 with $t = M$,

$$\|Du_1\|(\overline{\Omega}) + \|Du_2\|(\overline{\Omega}) = \|Du\|(\overline{\Omega}) \le \|Du_1\|(\overline{\Omega}),$$

from which we see that $\|Du_2\|(\overline{\Omega}) = 0$. Therefore by the 1-Poincaré inequality it follows that $u_2$ is locally constant on $\Omega$, and because $\Omega$ is connected, it follows that $u_1$ is constant on $\Omega$. We denote this constant $L$.

As pointed out above, $u_1 \in BV_0(\Omega)+\psi$; it follows that since $u = u_1+u_2 \in BV_0(\Omega)+\psi$, we must have $u_2 \in BV_0(\Omega)$, that is, $L\chi_\Omega \in BV(X)$. Therefore if $L \ne 0$, we must have that $P(\Omega, X)$ is finite, and hence because $\|Du_2\|(\overline{\Omega}) = 0$, we also must have that $P(\Omega, X) = 0$.

On the other hand, since $\Omega$ is bounded and $\mu(X \setminus \Omega) > 0$, we can find a ball $B \supset \Omega$ with $\mu(B \setminus \Omega) > 0$. Now by the 1-Poincaré inequality, we arrive at the contradiction

$$0 < \min\{\mu(B \cap \Omega), \mu(B \setminus \Omega)\} \le C\,\mathrm{rad}(B)\, P(\Omega, \lambda B) = 0.$$

Hence it must be that $L = 0$, that is, $u \le M$ on $\Omega$. □

Unlike in the nonlinear potential theory associated with $p$-harmonic functions for $p > 1$, here we cannot replace the condition $\mu(X \setminus \Omega) > 0$ with the requirement of $\mathrm{Cap}_1(X \setminus \Omega) > 0$, since there are closed sets of measure zero but with positive 1-capacity, and their complements would then have zero perimeter in $X$.

**Corollary 5.2.** *Under the hypothesis of Theorem 5.1, if $\psi \in BV(X)$ has the property that $M_1 \leq \psi \leq M_2$, then every solution to the Dirichlet problem for least gradients with boundary data $\psi$ in $\Omega$ has the property that $M_1 \leq u \leq M_2$ in $\Omega$.*

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