THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Tug-of-war, market manipulation and option pricing

K. Nyström and M. Parviainen

REPORT No. 24, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -24-13/14- -SE+fall

TUG-OF-WAR, MARKET MANIPULATION AND OPTION PRICING

K. NYSTRÖM, M. PARVIAINEN

ABSTRACT. We develop an option pricing model based on a tug-of-war game involving the the issuer and holder of the option. This two-player zero-sum stochastic differential game is formulated in a multi-dimensional financial market and the agents try, respectively, to manipulate/control the drift and the volatility of the asset processes in order to minimize and maximize the expected discounted pay-off defined at the terminal date T. We prove that the game has a value and that the value function is the unique viscosity solution to a terminal value problem for a partial differential equation involving the non-linear and completely degenerate parabolic infinity Laplace operator.

1. INTRODUCTION

A feature of illiquid markets is that large transactions move prices. This is a disadvantage for traders that need to liquidate large portfolios or keep their stock holdings close to a pre specified target but there are also situations where investors may benefit from moving prices. For example, a large trader holding a large number of options may have an incentive to attempt to impact the dynamics of the underlying and to move the option value in a favorable direction if the increase in the option value outweighs the trading costs in the underlying. There are some empirical evidence, see [GS], [P01], [KS92], that in illiquid markets option traders are in fact able to increase a derivatives value by moving the price of the underlying.

In this paper we consider option pricing in the context of a two-player zero-sum stochastic differential game in a multi-dimensional financial market where the issuer and holder of the option try, respectively, to manipulate/control the drift and the volatility of the asset processes in order to minimize and maximize, respectively, the expected discounted pay-off defined at the terminal date T. An important contribution of this paper is that we are able to establish a connection between option

2010 *Mathematics Subject Classification.* 91G80, 91A15, 91A23, 60H15, 49L25, 35K59.

Key words and phrases. Infinity Laplace, non-linear parabolic partial differential equation, option pricing, stochastic differential game, tug-of-war.

Acknowledgements. Work partly done during a visit to the Institut Mittag-Leffler. MP is supported by the Academy of Finland.

pricing and games referred to as tug-of-wars. While in the prevailing model for option pricing, the governing partial differential equation, the Black-Scholes equation, is a linear second order parabolic equation, in our context the underlying partial differential equation becomes substantially more involved due to the presence of the non-linear and completely degenerate infinity and parabolic infinity Laplace operator.

1.1. **Price dynamics.** We here give a heuristic description of price formation process underlying our model. We let

$$S(t)=(S_1(t,\omega),...,S_n(t,\omega)):[0,T]\times\Omega\to\mathbb{R}^n_+$$

be the stochastic process which represents the prices of n assets at time $t\in[0,T]$. To keep mathematical tractability we formulate the dynamics of $S=S(t)$ as a system of stochastic differential equations for the vector of log-returns

$$\begin{aligned} X(t) &= (X_1(t,\omega),...,X_n(t,\omega)):[0,T]\times\Omega\to\mathbb{R}^n,\\ X_i(t) &= \log(S_i(t)) \quad \text{for} \quad i\in\{1,...,n\}. \end{aligned}$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space satisfying the standard assumptions and let $\{\xi_{i,k}\}$, for $i\in\{1,\dots,n,n+1\}$, $k\in\mathbb{N}$, be sequences of i.i.d random variables such that

$$\mathbb{P}(\xi_{i,k}=1)=1/2=\mathbb{P}(\xi_{i,k}=-1).$$

In particular, $\{\xi_{i,k}\}$, represent the outcomes of sequences of standard coin toss. We let $\mathcal{F}_k$ be a filtration of $\mathcal{F}$ to which $\{\xi_{i,k}\}$ are adapted.

Let $N\in\mathbb{N}$ denote our discretization parameter. We let $X^N_{i,k}$ denote the state of the log-returns of asset i after step k. Then, on this level the model is

$$\begin{aligned} X^N_{i,k}-X^N_{i,k-1} &= \text{a random walk increment with drift}\\ &\quad+\text{ an increment resulting from price manipulation}\\ &\qquad\text{modeled as a tug-of-war game.} \end{aligned}$$

To be more precise, at step k of the game a sample of $(\xi_{1,k},\dots,\xi_{n+1,k})$ is generated. For component $i\in\{1\dots,n\}$ contribution from the random walk with drift is modeled as

$$\frac{\mu_k}{N}+\frac{2}{\sqrt{N}}\sigma_k\xi_{i,k},$$

where μ_i is a real value giving a drift and σ_i represents the magnification of the step $2\xi_{i,k}$ (volatility can vary from the asset to asset).

To formulate the increment resulting from price manipulation modeled as a tug-of-war game we let $\{\theta^\pm_k\}=\{(\theta^\pm_{1,k},\dots,\theta^\pm_{n,k})\}$ be $\mathcal{F}_k$-adapted random variables such that $\theta^\pm_k\in\{x\in\mathbb{R}^n:\ |x|\le 1/\sqrt{N}\}$. The sequences $\{\theta^+_k\}$, $\{\theta^-_k\}$, correspond to the control actions of the maximizing and minimizing player in the game to be described. In the tug-of-war game the idea is that $\{\theta_k^+\}$, $\{\theta_k^-\}$, represent the displacement exercised by the maximizing and minimizing player, respectively. In particular, it is assumed that each of the two players can affect the price process and push it in a favorable direction, but the turns to do so are taken randomly. As the players are constantly competing against each other, this game is called a tug-of-war game. In this setting, the increment of component i, at step k, based on coin toss $\xi_{n+1,k}$, is

$$\begin{aligned}
&2\sigma_i\bigg(\theta_{i,k}^+\frac{(1+\xi_{n+1,k})}{2}+\theta_{i,k}^-\frac{(1-\xi_{n+1,k})}{2}\bigg)\\
&\qquad=2\sigma_i\bigg(\frac{(\theta_{i,k}^+-\theta_{i,k}^-)}{2}\xi_{n+1,k}+\frac{(\theta_{i,k}^++\theta_{i,k}^-)}{2}\bigg).
\end{aligned}$$

Again it is assumed that the actions of the players are magnified by the factors $\{\sigma_i\}$.

Put together, the position of the log-returns after j steps is $X_j^N = (X_{i,j}^N, ..., X_{i,j}^N)$, where

$$\begin{aligned}
X_{i,j}^N =&x_i+\mu_i\frac{j}{N}+\frac{2}{\sqrt{N}}\sigma_i\sum_{k=1}^{j}\xi_{i,k}\\
&+2\sigma_i\sum_{k=1}^{j}\bigg(\frac{(\theta_{i,k}^+-\theta_{i,k}^-)}{2}\xi_{n+1,k}+\frac{(\theta_{i,k}^++\theta_{i,k}^-)}{2}\bigg)
\end{aligned}$$

We define $\{W_i^N(t)\}_{t\geq 0}$, $i\in\{1,...,n+1\}$, by setting

$$(W_1^N(0),\dots,W_{n+1}^N(0))=0$$

and using the relations

$$W_i^N(t) \;=\; W_i^N((k-1)/N)+\bigg(t-\frac{k-1}{N}\bigg)\sqrt{N}\xi_{i,k},$$

whenever $t\in((k-1)/N,k/N],\ k\in\mathbb{N}$. Moreover, we define continuous time processes by setting

$$X^N(t)=X_{[Nt]}^N,\quad \theta^{\pm,N}(t)=\sqrt{N}\theta_{[Nt]}^{\pm}.$$

With this notation, the above dynamics becomes

$$X_i^N(t)=A_i^N(t)+B_i^N(t)+C_i^N(t),\qquad i\in\{1,\dots,n\}, \tag{1.1}$$

where

$$\begin{aligned}
A_i^N(t) &= x_i+\int_0^t\mu_i ds+\int_0^t\sigma_i dW_i^N(s),\\
B_i^N(t) &= \sigma_i\int_0^t(\theta_i^{+,N}(s)-\theta_i^{-,N}(s))dW_{n+1}^N(s),\\
C_i^N(t) &= \sigma_i\int_0^t\sqrt{N}(\theta_i^{+,N}(s)+\theta_i^{-,N}(s))ds.
\end{aligned}$$

Then by passing to the limit, using Donsker's invariance principle,

$$\begin{aligned} A_i^N(t) &\to x_i + \int_0^t \mu_i ds + \int_0^t \sigma_i dW_i(s), \\ B_i^N(t) &\to \sigma_i \int_0^t (\theta_i^+(s) - \theta_i^-(s)) dW_{n+1}(s), \end{aligned}$$

as $N \to \infty$ where W_i, W_{n+1}, are standard and independent Brownian motions. The key difficulty when attempting to understand the continuous time limit of the outlined price dynamics, as $N \to \infty$, is to understand the asymptotics of the term $C_i^N(t)$. A solution due to [AB10] in the context of time independent equations, is to replace $\sqrt{N}$ with dynamically controlled quantities d^+ and d^-. This approach is motivated by the connection of tug-of-war games to the infinity Laplace operator in [PSSW09]. Therefore, as described later, the core of the model is given by

$$\begin{aligned} dX_i(s) = &\Big(\mu_i + \sigma_i (d_i^+(s) + d_i^-(s))(\theta_i^+(s) + \theta_i^-(s))\Big) ds \\ &+ \sigma_i dW_i(s) + \sigma_i (\theta_i^+(s) - \theta_i^-(s)) dW_{n+1}(s), \end{aligned} \tag{1.2}$$

with sufficient assumptions on the controls $d^\pm$, $\theta^\pm$. This is to be considered as the continuous time limit of (1.1).

1.2. **Fair game value of options.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ denote a complete filtered probability space with a right-continuous filtration supporting an $(n+1)$-dimensional and $\{\mathcal{F}_t\}$-adapted Brownian motion $W = (W_1, ..., W_{n+1})$. We assume that all components are independent.

There are two competing players, one maximizing and one minimizing, which both attempt to control and manipulate the log-returns $X(t)$ of the underlying assets. Denote by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$. We let

$$\mathcal{H} := \mathbb{S}^{n-1} \times [0, \infty),$$

and

$$A^+ := A^+(t) := (\theta^+(t), d^+(t)), \qquad A^- := A^-(t) := (\theta^-(t), d^-(t)),$$

where

$$\theta^\pm(t) \in \mathbb{S}^{n-1},\ d^\pm(t) \in [0, \infty),\ t \in [0, T],$$

are $\{\mathcal{F}_t\}$-adapted stochastic processes representing the control actions of the maximizing and minimizing player. Heuristically, $\theta^\pm(t)$ denotes the directions and $d^\pm(t)$ the lengths of the steps taken by the players. We let $\mathcal{AC}$ denote the set of all admissible controls.

Each player also chooses a strategy $\rho^\pm$ which represents a respond to the actions of the opponent, i.e. the strategies $\rho^\pm$ are functions from the space of controls to the space of controls. In particular, given a control of the opponent, a strategy gives the corresponding control of the player, and we let $\mathcal{S}$ denote the set of all admissible strategies.

Detailed definitions of (admissible) controls ($\mathcal{AC}$) and strategies ($\mathcal{S}$) are given below in Definitions 2.1 and 2.2. Using this notation the dynamics of the log-returns is given by (1.2).

Note that in (1.2), the time-dependent controls of the players enter in the drift coefficient, and in the diffusion coefficient of the one-dimensional Brownian motion W_{n+1}. Hence, this part of the dynamics is degenerate in the sense that it is possible for the players to completely switch off the one-dimensional Brownian motion W_{n+1}.

Given $A^{\pm} = (\theta^{\pm}, d^{\pm})$ and a pay-off function g at T, we set

$$\begin{aligned} J^{(x,t)}(A^+, A^-) &:= \mathbb{E}[e^{-r(T-t)} g(X^{(x,t)}(T))] \\ &= \mathbb{E}[e^{-r(T-t)} g(X(T))] \end{aligned} \tag{1.3}$$

where the superscript (x, t) indicates that the game starts at position x at time t. The expectation $\mathbb{E}[\cdot]$ is taken with respect to the measure $\mathbb{P}$.

Definition 1.1. The upper and lower values of the stochastic dynamic game are denoted by $U^+(x, t)$ and $U^-(x, t)$, and defined as

$$\begin{aligned} U^+(x,t) &= \sup_{\rho^+ \in \mathcal{S}} \inf_{A^- \in \mathcal{AC}} J^{(x,t)}(\rho^+(A^-), A^-), \\ U^-(x,t) &= \inf_{\rho^- \in \mathcal{S}} \sup_{A^+ \in \mathcal{AC}} J^{(x,t)}(A^+, \rho^-(A^+)). \end{aligned}$$

The game is said to have a value at (x, t) if $U^+(x, t) = U^-(x, t)$. If $U^+(x, t) = U^-(x, t) =: U(x, t)$, then we say that $U(x, t)$ is the fair game value of the option.

1.3. **Statement of main results.** Our fair game value of the option is related to the degenerate partial differential operator F

$$\begin{aligned} F(u, Du, D^2u) :=& \frac{2}{|Du|^2} \Big(\sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j} \sigma_i \sigma_j \Big) \\ &+ \frac{1}{2} \Big(\sum_{i=1}^{n} \sigma_i^2 u_{x_i x_i} \Big) + \sum_{i=1}^{n} \mu_i u_{x_i} - ru, \end{aligned} \tag{1.4}$$

where $Du := (u_{x_1}, ..., u_{x_n})'$, and D^2u is the matrix consisting of the second order derivatives. We consider the terminal value problem

$$\begin{cases} \partial_t u + F(u, Du, D^2u) = 0, & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, T) = g(x), & \text{on } \mathbb{R}^n. \end{cases} \tag{1.5}$$

Solutions to (1.5) has to be understood in the sense of viscosity solutions as defined in the bulk of the paper. Concerning the pay-off g, we adopt the following convention: throughout the paper it is our standing assumption that the function g is a positive bounded Lipschitz function, i.e.

$$\sup_{x\in\mathbb{R}^n} g(x) + \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|g(x)-g(y)|}{|x-y|} \leq L, \tag{1.6}$$

for some $L < \infty$. The assumptions concerning boundedness and positivity of g are only imposed to minimize additional technical difficulties. The main result of the paper is the following.

Theorem 1.2. *Let g be as in* (1.6) *and let $U^\pm$ be the upper and lower values of the stochastic dynamic game as in Definition 1.1. Then*

$$U^+ \equiv U^- \quad on \quad \mathbb{R}^n \times [0,T]$$

and $U := U^+ \equiv U^-$ is the unique viscosity solution to (1.5). *In particular, $U(x,t)$ is the fair game value of the option.*

Theorem 1.2 can be compared to the well known fact that, after changing to log-returns, the arbitrage free price of a simple European contract in the original Black-Scholes model is the unique solution to the Cauchy problem for a second order uniformly parabolic equation (heat equation). We emphasize that our fair value makes no reference to, and is different from, classical concepts of arbitrage free pricing. It is an interesting area of future research to understand, for example, notions of arbitrage free pricing in the context of the stochastic dynamic games considered in this paper. Furthermore, while this paper mainly is of theoretical nature, an interesting future project is to study (1.5) from a numerical point of view.

Example 1.3. *Consider $g : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$g(x_1, ..., x_n) = \max\{K - w_1 e^{x_1} - .... - w_n e^{x_n}, 0\} \tag{1.7}$$

where $w_1 + ... + w_n = 1$, $w_i \geq 0$, and where K, the strike, is a positive real number. Then g represents the pay-off of a put option written on the index $w_1 S_1(T) + ... + w_n S_n(T) = w_1 e^{x_1} + ... + w_n e^{x_n}$ with strike K. Obviously g satisfies (1.6) *for some $L < \infty$. We note that in general the Lipschitz regularity is a feature of many commonly traded financial derivatives.*

1.4. **Brief outline of the proof of Theorem 1.2.** The complexity in proving Theorem 1.2 stems from the unboundedness of controls and strategies as well as from the potential degeneracy of the underlying dynamics. To overcome the unboundedness of the action sets we first approximate the original stochastic differential game by a sequence of games with bounded controls $\mathcal{AC}_m$ and bounded strategies $\mathcal{S}_m$ where the bounds tend to ∞ as $m \to \infty$. The upper and lower values of the associated stochastic dynamic games are defined analogously as for the unbounded controls game above, i.e.

$$U_m^+(x,t) = \sup_{\rho^+\in\mathcal{S}_m} \inf_{A^-\in\mathcal{AC}_m} J^{(x,t)}(\rho^+(A^-), A^-),$$

$$U_m^-(x,t) = \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)),$$

where $J^{(x,t)}$ is given in (1.3). The upper and lower values are unique. An important step is to connect the value functions to viscosity solutions to the following terminal value problems involving Bellman-Isaacs type equations:

$$\begin{aligned} \partial_t u - H_m^+(u, Du, D^2u) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) &= g(x) \quad \text{on } \mathbb{R}^n, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \partial_t u - H_m^-(u, Du, D^2u) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) &= g(x) \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{1.9}$$

The operators $H_m^\pm$ are introduced later and we here simply note that the equations in (1.8), (1.9), are non-linear parabolic equations and these equations are the relevant Bellman-Isaacs equations associated to our problem in the case of bounded controls. In Section 2 we briefly discuss comparison principles, existence and uniqueness of viscosity solutions to (1.8), (1.9). The comparison and uniqueness follows along the lines of Giga, Goto, Ishii and Sato, see [GGIS91], by using doubling of variables as well as the theorem of sums. Existence is established by the construction of appropriate barriers and by the use of Perron's method.

In Lemma 3.1, we prove that the unique solutions to (1.8), (1.9), $u_m^\pm$, satisfy

$$u_m^+ = U_m^+, \ u_m^- = U_m^-. \tag{1.10}$$

In other words, the unique solutions to stated terminal value problems produce the upper and lower values of the associated stochastic games. The proof uses Ito's formula and estimates for stochastic differential equations.

To continue, we prove in Lemma 5.1 that

$$H_m^\pm \to -F \qquad \text{as} \qquad m \to \infty.$$

To complete the proof of Theorem 1.2, a key step is to prove that there exists $m_0 \in \{1, 2, \ldots\}$ such that the families

$$\{u_m^\pm : \ m \geq m_0\}$$

are equicontinuous (Lemma 5.2). The proof of this fact is based on a barrier argument. These results enable us to conclude by the Arzelà-Ascoli theorem, see Lemma 5.3, that there exists a continuous function u such that

$$u_m^\pm(x,t) \to u(x,t), \tag{1.11}$$

and that the limit u is the unique solution to (1.5).

Finally, at the end of Section 5 we prove Theorem 1.2 by showing that when the bounds on the controls increase, then a subsequence of corresponding value functions converge to a value function for the game with unbounded controls. This together with (1.10) and (1.11) yields the result.

Our approach is influenced by the works of Swiech [Swi96] as well as Atar and Budhiraja [AB10]. A different approach to stochastic games is due to Fleming and Souganidis [FS89], see also [BL08]. Indeed, the approach in [FS89] is based on establishing a dynamic programming principle based on careful approximation arguments working directly with the value functions. Then using this the authors prove that the value functions also solve the associated Bellman-Isaacs equations. However, our model contains degenerate diffusion and unbounded controls and our approach relies on viscosity theory for non-linear and degenerate partial differential equations already from the beginning. In particular, instead of establishing a dynamic programming principle for the value function, we show, as explained above, that the unique viscosity solution to the corresponding partial differential equation satisfies a dynamic programming principle.

Our work is developed based on the recently established connections between discrete in time tug-of-war games and infinity harmonic functions [PSSW09], and tug-of-war with noise in the context of p-harmonic functions [PS08]. We here also mention the approach based on nonlinear mean value formulas developed in [MPR10] and [MPR12]. Continuous in time stochastic differential games and infinity harmonic functions were considered in [AB10], and [AB11]. The equation consider in this paper, modulo the presence of the model related constants and a change of the time direction, coincides with normalized p-Laplace operator considered in [MPR10] in connection with normalized p-parabolic equations and tug-of-war games, see also [BG] and [Doe11]. The parabolic equation involving a normalized infinity Laplacian is studied in [JK06].

2. Preliminaries

Recall that $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ denotes a complete filtered probability space with a right-continuous filtration supporting an $(n+1)$-dimensional and $\{\mathcal{F}_s\}$-adapted Brownian motion $W = (W_1, ..., W_{n+1})$. We assume that all components are standard and independent Brownian motions.

Definition 2.1 (Controls)**.** Let

$$A := A(s) := (\theta(s), d(s))$$

be a progressively measurable stochastic process on $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ taking values in $\mathcal{H} = \mathbb{S}^{n-1} \times [0, \infty)$. We set

$$\Lambda := \Lambda(A) := \sup_{\omega\in\Omega} \sup_{s\in[0,T]} d(s, \omega) \in [0, \infty].$$

The control is said to be admissible provided that $\Lambda < \infty$, and we denote the set of all admissible controls by $\mathcal{AC}$.

Definition 2.2 (Strategies)**.** A mapping

$$\rho : \mathcal{AC} \to \mathcal{AC}$$

is said to be a strategy if, for all $A, \tilde{A} \in \mathcal{AC}$, with the notation

$$A' := \rho(A), \qquad \tilde{A}' := \rho(\tilde{A}),$$

and for every $\tau \in [0, T]$, the following holds. If

$$\mathbb{P}(A(s) = \tilde{A}(s) \text{ for a.e. } s \in [0, \tau]) = 1 \quad \text{and} \quad \Lambda(A) = \Lambda(\tilde{A})$$

then

$$\mathbb{P}(A'(s) = \tilde{A}'(s) \text{ for a.e. } s \in [0, \tau]) = 1 \quad \text{and} \quad \Lambda(A') = \Lambda(\tilde{A}').$$

Given $\rho \in \mathcal{S}$, we set

$$\Lambda(\rho) := \sup_{A \in \mathcal{AC}} \Lambda(\rho(A)) \in [0, \infty].$$

The strategy is said to be admissible provided that $\Lambda(\rho) < \infty$, and we denote the set of all admissible strategies by $\mathcal{S}$.

In general, and to this end, we in the following by controls and strategies mean admissible controls and strategies in the sense of Definition 2.1 and Definition 2.2.

In the following we will approximate unbounded controls by bounded ones.

Definition 2.3. For $m \in \{1, 2, \ldots\}$, we define

$$\begin{aligned} \mathcal{AC}_m &:= \{A \in \mathcal{AC} : \ \Lambda(A) \le m\}, \\ \mathcal{S}_m &:= \{\rho \in \mathcal{S} : \ \Lambda(\rho) \le m\}. \end{aligned} \tag{2.1}$$

For convenience, we record the definitions of the game values already described in the previous section.

Definition 2.4. The upper and lower values of the underlying stochastic dynamic game, with controls in $\mathcal{AC}_m$ and strategies in $\mathcal{S}_m$, are defined as

$$\begin{aligned} U_m^+(x,t) &= \sup_{\rho^+ \in \mathcal{S}_m} \inf_{A^- \in \mathcal{AC}_m} J^{(x,t)}(\rho^+(A^-), A^-), \\ U_m^-(x,t) &= \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)). \end{aligned}$$

2.1. **Bellman-Isaacs equations with bounded action sets: viscosity solutions.** Let $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ and let $\mathcal{M}(n)$ denote the set of all symmetric $n \times n$-dimensional matrices. Given a matrix, or vector M, we let M' denote the transpose of M. We define $\Phi : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{M}(n) \to \mathbb{R}$ through

$$\begin{aligned}\Phi(\theta^+, \theta^-, d^+, d^-, p, M)\\ &= -\frac{1}{2}(\theta^+ - \theta^-)'\Sigma M \Sigma(\theta^+ - \theta^-)\\ &\quad -\frac{1}{2}\operatorname{trace}(\Sigma^2 M) - (d^+ + d^-)(\theta^+ + \theta^-) \cdot p - \mu \cdot p.\end{aligned}$$

In the following we denote by $\mathrm{LSC}(\mathbb{R}^n \times [0,T])$ the set of lower semi-continuous functions, i.e. all functions

$$f : (\mathbb{R}^n \times [0,T]) \to \mathbb{R} \cup \{\infty\}$$

such that

$$\liminf_{(y,s)\to(x,t)} f(y,s) \geq f(x,t).$$

Likewise, we denote by $\mathrm{USC}(\mathbb{R}^n \times [0,T])$ the set of upper semi-continuous functions, i.e. all functions

$$f : (\mathbb{R}^n \times [0,T]) \to \mathbb{R} \cup \{-\infty\}$$

such that

$$\limsup_{(y,s)\to(x,t)} f(y,s) \leq f(x,t).$$

We define $\mathrm{LSC}_l(\mathbb{R}^n \times [0,T])$ to consist of functions $h \in \mathrm{LSC}(\mathbb{R}^n \times [0,T])$ which satisfy the (linear) growth condition

$$|h(x,t)| \leq c(1 + |x|) \tag{2.2}$$

and for some $c \in [1, \infty)$. The space $\mathrm{USC}_l(\mathbb{R}^n \times [0,T])$ is defined analogously. Furthermore,

$$\mathrm{C}_l(\mathbb{R}^n \times [0,T]) = \mathrm{USC}_l(\mathbb{R}^n \times [0,T]) \cap \mathrm{LSC}_l(\mathbb{R}^n \times [0,T]).$$

In addition, ignoring t we define $\mathrm{C}_l(\mathbb{R}^n)$ by analogy. Finally, $C^{1,2}(\mathbb{R}^n \times [0,T])$ denotes the space of functions that are once continuously differentiable in time and twice continuously differentiable in space.

Given $m \in \{1, 2, \ldots\}$, we also introduce the notation

$$\mathcal{H}_m = \{(\theta, d) \in \mathcal{H} : d \leq m\},$$

and we define $\tilde{H}_m^+, \tilde{H}_m^- : \mathbb{R}^n \times \mathcal{M}(n) \to \mathbb{R}$ through

$$\begin{aligned}\tilde{H}_m^+(p, M) &= \sup_{(\theta^-,d^-)\in\mathcal{H}_m} \inf_{(\theta^+,d^+)\in\mathcal{H}_m} \Phi(\theta^+, \theta^-, d^+, d^-, p, M),\\ \tilde{H}_m^-(p, M) &= \inf_{(\theta^+,d^+)\in\mathcal{H}_m} \sup_{(\theta^-,d^-)\in\mathcal{H}_m} \Phi(\theta^+, \theta^-, d^+, d^-, p, M).\end{aligned}$$

Also we define $H_m^+, H_m^- : \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}(n) \to \mathbb{R}$ through

$$H_m^+(\xi, p, M) = \tilde{H}_m^+(p, M) + r\xi,$$

$$H_m^-(\xi, p, M) \quad = \quad \tilde{H}_m^-(p, M) + r\xi. \tag{2.3}$$

Next we introduce terminal value problems involving Bellman-Isaacs type equations associated to the game with bounded controls.

$$\begin{aligned} \partial_t u - H_m^+(u, Du, D^2u) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) &= g(x) \qquad \text{on } \mathbb{R}^n, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \partial_t u - H_m^-(u, Du, D^2u) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) &= g(x) \qquad \text{on } \mathbb{R}^n. \end{aligned} \tag{2.5}$$

A suitable concept of solution to the above equations is the viscosity solution. Also recall our standing assumption (1.6) for g.

Definition 2.5. (a) A function $\bar{u}_m^+ \in \mathrm{LSC}_l(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution to (2.4) if $\bar{u}_m^+(x,T) \geq g(x)$ for all $x \in \mathbb{R}^n$ and if the following holds. If $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and we have $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ such that

$$\begin{aligned} (i) \qquad & \bar{u}_m^+(x_0,t_0) = \phi(x_0,t_0), \\ (ii) \qquad & \bar{u}_m^+(x,t) > \phi(x,t) \text{ for } (x,t) \neq (x_0,t_0), \end{aligned}$$

then

$$\partial_t \phi(x_0,t_0) \leq H_m^+(\bar{u}_m^+(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)).$$

(b) A function $\underline{u}_m^+ \in \mathrm{USC}_l(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution to (2.4) if $\underline{u}_m^+(x,T) \leq g(x)$ for all $x \in \mathbb{R}^n$ and if the following holds. If $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and we have $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ such that

$$\begin{aligned} (i) \qquad & \underline{u}_m^+(x_0,t_0) = \phi(x_0,t_0), \\ (ii) \qquad & \underline{u}_m^+(x,t) < \phi(x,t) \text{ for } (x,t) \neq (x_0,t_0), \end{aligned}$$

then

$$\partial_t \phi(x_0,t_0) \geq H_m^+(\underline{u}_m^+(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)).$$

(c) If u_m is both a viscosity supersolution and a viscosity subsolution to (2.4), then u_m is a viscosity solution to (2.4).

(d) The definitions for the equation (2.5) are analogous with H_m^+, $\bar{u}_m^+$, $\underline{u}_m^+$ replaced by H_m^-, $\bar{u}_m^-$, $\underline{u}_m^-$.

Remark 2.6. *Note that $H_m^+(u,p,X)$ is continuous with respect to u,p,X also when $p = 0$. In addition, H_m^+ is degenerate elliptic in the sense that*

$$H_m^+(u,p,X) \leq H_m^+(u,p,Y) \tag{2.6}$$

for any $X \geq Y$. The analogous statements hold for $H_m^-(u,p,X)$.

2.2. **Bellman-Isaacs equation with bounded action sets: existence and uniqueness of viscosity solutions.**

Lemma 2.7. *Let* $\underline{u}_m^+, \bar{u}_m^+ \in C_l(\mathbb{R}^n \times [0,T])$ *and* $\underline{u}_m^-, \bar{u}_m^- \in C_l(\mathbb{R}^n \times [0,T])$, *be viscosity sub- and supersolutions to* (2.4) *and* (2.5)*, respectively. Then*

$$\underline{u}_m^- \leq \bar{u}_m^- \quad \text{and} \quad \underline{u}_m^+ \leq \bar{u}_m^+.$$

For the proof of the above comparison principle, see [GGIS91], in particular, the argument starting from page 27. Similarly it follows by a comparison with a large enough constant that solutions not only satisfy the linear growth conditions but are bounded.

Lemma 2.8. *Let* $y \in \mathbb{R}^n$ *and let* L *be the Lipschitz constant of* g*. Consider* $0 < \varepsilon \ll 1$*, and let*

$$\bar{w}(x,t) = g(y) + \frac{A}{\varepsilon^2}(T-t) + 2L(|x-y|^2+\varepsilon)^{1/2},$$
$$\underline{w}(x,t) = g(y) - \frac{A}{\varepsilon^2}(T-t) - 2L(|x-y|^2+\varepsilon)^{1/2}.$$

Then we can choose A*, independent of* y*,* ε *and* m*, so that* $\bar{w}$ *and* $\underline{w}$ *are viscosity super- and subsolutions to* (2.4) *as well as to* (2.5)*.*

Proof. We will only prove the result for (2.4) since the proof for (2.5) is analogous. First we immediately see that

$$\underline{w}(x,T) \leq g(x) \leq \bar{w}(x,T)$$

whenever $x \in \mathbb{R}^n$. To prove that $\bar{w}$ is a viscosity supersolution to (2.4) we verify that

$$\partial_t \bar{w}(x,t) - H_m^+(\bar{w}(x,t), D\bar{w}(x,t), D^2\bar{w}(x,t)) \leq 0$$

whenever $(x,t) \in \mathbb{R}^n \times [0,T]$. In the following, we split

$$\Phi := \Phi(\theta^+, \theta^-, d^+, d^-, D\bar{w}, D^2\bar{w})$$

into

$$\Phi = \Phi_1 + \Phi_2, \tag{2.7}$$

where

$$\Phi_1 := -\frac{1}{2}(\theta^+ - \theta^-)'\Sigma D^2\bar{w}\Sigma(\theta^+ - \theta^-)$$
$$- \frac{1}{2}\operatorname{trace}(\Sigma^2 D^2\bar{w}) - \mu \cdot D\bar{w},$$
$$\Phi_2 := -(d^+ + d^-)(\theta^+ + \theta^-) \cdot D\bar{w}.$$

Then, by a straightforward calculation we see that

$$|\Phi_1| \leq cL(|x-y|^2+\varepsilon)^{-1/2}, \tag{2.8}$$

for all $(\theta^-, d^-) \in \mathcal{H}_m$, $(\theta^+, d^+) \in \mathcal{H}_m$, and for some c independent of y, L, m and ε. Next, we focus on estimating

$$\sup_{(\theta^-,d^-)\in\mathcal{H}_m} \inf_{(\theta^+,d^+)\in\mathcal{H}_m} \Phi_2$$

$$= 2L \sup_{(\theta^-,d^-)\in\mathcal{H}_m} \inf_{(\theta^+,d^+)\in\mathcal{H}_m} \Big(-(d^+ + d^-)(\theta^+ + \theta^-) \cdot \frac{(x-y)}{(|x-y|^2+\varepsilon)^{1/2}} \Big).$$

We can without loss of generality assume that $x \neq y$. We then let $\theta^- = -(x-y)/|x-y|$ and note that

$$-(d^+ + d^-)(\theta^+ + \theta^-) \cdot \frac{(x-y)}{(|x-y|^2+\varepsilon)^{1/2}} \geq 0.$$

Hence, combining this estimate and (2.8) with (2.7) we see that

$$\begin{aligned} \bar{w}_t(x,t) - H_m^+(\bar{w}(x,t), D\bar{w}(x,t), D^2\bar{w}(x,t)) \\ \leq -\frac{A}{\varepsilon^2} + cL(|x-y|^2+\varepsilon)^{-1/2} \\ \leq -\frac{A}{\varepsilon^2} + cL\varepsilon^{-1/2} \end{aligned}$$

whenever $(x,t) \in \mathbb{R}^n \times [0,T]$. Hence, if we let $A = 2cL \geq cL\varepsilon^{3/2}$ then $-A\varepsilon^{-2} + cL\varepsilon^{-1/2} \leq 0$ and hence we can conclude that $\bar{w}$ is a supersolution to (2.4). The proof that $\underline{w}$ is a viscosity subsolution to (2.4) is analogous. ☐

From now on we fix A so that $\bar{w}$ and $\underline{w}$ are viscosity super- and subsolutions, as stated in Lemma 2.8, to (2.4) as well as to (2.5).

Lemma 2.9. *If u_m^+ and u_m^- are viscosity solutions to* (2.4) *and* (2.5)*, respectively, then*

$$\underline{w} \leq u_m^\pm \leq \bar{w}.$$

Proof. The lemma is an immediate consequence of Lemma 2.8 and Lemma 2.7. ☐

Lemma 2.10. *There exist unique viscosity solutions u_m^+ and u_m^- to* (2.4) *and* (2.5)*, respectively.*

The proof of this result can be found in [Gig06] and it is based on the Perron's method where one constructs the so called upper and lower Perron solution by taking inf / sup over the suitable super/subsolutions. That the constructed solutions assume the correct terminal data can then be proved by using the above barriers.

3. Solving the stochastic dynamic game with bounded action sets

The purpose of the section is to prove the following theorem.

Lemma 3.1. *Let u_m^+ and u_m^- be the unique solutions to* (2.4) *and* (2.5)*, respectively, ensured by Lemma 2.10. Then*

$$u_m^+(x,t) = U_m^+(x,t) := \sup_{\rho^+\in\mathcal{S}_m} \inf_{A^-\in\mathcal{AC}_m} J^{(x,t)}(\rho^+(A^-),A^-),$$
$$u_m^-(x,t) = U_m^-(x,t) := \inf_{\rho^-\in\mathcal{S}_m} \sup_{A^+\in\mathcal{AC}_m} J^{(x,t)}(A^+,\rho^-(A^+)),$$

whenever $(x,t)\in\mathbb{R}^n\times[0,T]$.

3.1. **Proof of Lemma 3.1 assuming additional regularity on $u_m^\pm$.** We here prove Lemma 3.1 assuming smoothness on u_m^+ and u_m^-.

Lemma 3.2. *Let u_m^+ and u_m^- be the unique solutions to* (2.4) *and* (2.5)*, respectively, ensured by Lemma 2.10. Assume, in addition, that $u_m^\pm, \partial_t u_m^\pm, Du_m^\pm, D^2u_m^\pm$ are Lipschitz continuous in $\mathbb{R}^n\times[0,T)$. Then Lemma 3.1 holds.*

Proof. The proof is based on the connection between solutions and value functions provided by the Ito formula, in connection with suitable discretized controls chosen based on the solution. The discretization error can then be estimated by utilizing the smoothness assumptions. At the end, we pass to a limit with the discretization parameter. We only supply the proof in the case of u_m^-, the proof for u_m^+ being analogous.

Given $k\in\{1,2,\ldots\}$ and $(x,t)\in\mathbb{R}^n\times[0,T]$, we can choose, since u_m^- is a solution to (2.5), a control $(\theta_0^+,d_0^+)\in\mathcal{H}_m$ such that

$$\begin{aligned}\sup_{(\theta^-,d^-)\in\mathcal{H}_m}&\left\{\Phi(\theta_0^+,\theta^-,d_0^+,d^-,Du_m^-(x,t),D^2u_m^-(x,t))+ru_m^-(x,t)\right\}\\ &\le \partial_t u_m^-(x,t)+k^{-1}.\end{aligned}\tag{3.1}$$

Based on $(\theta_0^+,d_0^+)=(\theta_{0,1}^+,....,\theta_{0,n}^+,d_0^+)$ and an arbitrary, but fixed, control $(\theta^-,d^-)\in\mathcal{AC}_m$, we let $X^0(s):=X^{0,(x,t)}(s)$ be defined as in (1.2) assuming also the initial condition $X^0(t)=x$. In the following we let

$$\Phi_0^X(s) := \Phi(\theta_0^+,\theta^-(s),d_0^+,d^-(s),Du_m^-(X^0(s),s),D^2u_m^-(X^0(s),s)),$$
$$\Phi_0^x(s) := \Phi(\theta_0^+,\theta^-(s),d_0^+,d^-(s),Du_m^-(x,s),D^2u_m^-(x,s)).$$

Since $u_m^-,\partial_t u_m^-,Du_m^-,D^2u_m^-$ are Lipschitz continuous in $\mathbb{R}^n\times[0,T)$ we can apply the Ito formula to $u_m^-(X^0(s),s)$ and we see that

$$\begin{aligned}du_m^-(X^0(s),s) &= \partial_t u_m^-(X^0(s),s)ds+\sum_{i=1}^n\partial_{x_i}u_m^-(X^0(s),s)dX_i^0(s)\\ &\quad+\frac12\sum_{i,j=1}^n\partial_{x_ix_j}u_m^-(X^0(s),s)\,dX_i^0(s)\,dX_j^0(s)\\ &=(\partial_t u_m^-(X^0(s),s)-\Phi_0^X(s))ds\\ &\quad+\sum_{i=1}^n\partial_{x_i}u_m^-(X^0(s),s)(\sigma_i dW_i(s)+\sigma_i(\theta_{0,i}^+-\theta_i^-(s))dW_{n+1}(s)).\end{aligned}$$

Using this we have

$$
\begin{aligned}
&d(e^{-rs}u_m^-(X^0(s),s))\\
=&e^{-rs}(\partial_t u_m^-(X^0(s),s)-\Phi_0^X(s)-ru_m^-(X^0(s),s))ds\\
&+e^{-rs}\sum_{i=1}^n \partial_{x_i}u_m^-(X^0(s),s)(\sigma_i dW_i(s)+\sigma_i(\theta_{0,i}^+-\theta_i^-(s))dW_{n+1}(s)).
\end{aligned}
$$

Hence, if we let $\Delta t=(T-t)/k$, then

$$
\begin{aligned}
&\mathbb{E}[e^{-r(t+\Delta t)}u_m^-(X^0(t+\Delta t),t+\Delta t))-e^{-rt}u_m^-(X^0(t),t)]\\
&\quad=\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-rs}(\partial_t u_m^-(X^0(s),s)-\Phi_0^X(s)-ru_m^-(X^0(s),s))ds\bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
u_m^-(x,t) &= \mathbb{E}[e^{-r\Delta t}u_m^-(X^0(t+\Delta t),t+\Delta t)] \\
&- \mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}(\partial_t u_m^-(X^0(s),s)-\Phi_0^X(s)-ru_m^-(X^0(s),s))ds\bigg].
\end{aligned}
\tag{3.2}
$$

We let

$$
\begin{aligned}
I_1&=-\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}(\partial_t u_m^-(X^0(s),s)-\partial_t u_m^-(x,t))ds\bigg],\\
I_2&=-\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}(\Phi_0^x(t)-\Phi_0^X(s))ds\bigg],\\
I_3&=-\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}r(u_m^-(X^0(t),t))-u_m^-(X^0(s),s))ds\bigg].
\end{aligned}
\tag{3.3}
$$

Then, using this notation we observe that

$$
\begin{aligned}
&u_m^-(x,t)\\
&=\mathbb{E}[e^{-r\Delta t}u_m^-(X^0(t+\Delta t),t+\Delta t)]+I_1+I_2+I_3\\
&\quad-\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}\Big(\partial_t u_m^-(x,t)-\Phi_0^x(t)-ru_m^-(x,t)\Big)ds\bigg].
\end{aligned}
\tag{3.4}
$$

Next, using (3.1) we see that

$$
\begin{aligned}
-&\mathbb{E}\bigg[\int_t^{t+\Delta t} e^{-r(s-t)}\Big(\partial_t u_m^-(x,t)-\Phi_0^x(t)-ru_m^-(x,t)\Big)ds\bigg]\\
&\le k^{-1}\Delta t.
\end{aligned}
\tag{3.5}
$$

Furthermore, by Lipschitz continuity of $u_m^-,\partial_t u_m^-,Du_m^-,D^2u_m^-$ we can conclude that

$$
\begin{aligned}
|I_1|+|I_2|+|I_3|&\le c\mathbb{E}\bigg[\int_t^{t+\Delta t}(|X^0(s)-x|+\Delta t)ds\bigg]\\
&\le c(\Delta t)^2+c\mathbb{E}\bigg[\int_t^{t+\Delta t}|X^0(s)-x|ds\bigg]
\end{aligned}
\tag{3.6}
$$

for some generic constant c. To estimate the expectation in the previous estimate we will have to use the equation satisfied by $X^0(s)$. Indeed, recall that

$$X_i^0(s) - x_i = \int_t^s \Big(\mu_i + (d_{1,i}^+ + d_i^-(\tau))(\theta_{0,i}^+ + \theta_i^-(\tau))\Big) d\tau$$
$$+ \int_t^s \sigma_i dW_i(\tau) + \int_t^s \sigma_i(\theta_{0,i}^+ - \theta_i^-(\tau)) dW_{n+1}(\tau).$$

Hence, simply using the Hölder inequality and the Ito isometry we see that

$$\begin{aligned}\mathbb{E}\Big[\int_t^{t+\Delta t} |X^0(s) - x| ds\Big] &= \Big[\int_t^{t+\Delta t} \mathbb{E}[|X^0(s) - x|] ds\Big] \\ &\leq \Big[\int_t^{t+\Delta t} (\mathbb{E}[|X^0(s) - x|^2])^{1/2} ds\Big] \\ &\leq c(\Delta t)^{3/2},\end{aligned}$$

and where c is allowed to depend on m defining the class $\mathcal{AC}_m$. Combining the above estimates, we conclude that

$$|I_1| + |I_2| + |I_3| \leq c((\Delta t)^2 + (\Delta t)^{3/2}) \leq c(\Delta t)^{3/2}.$$

Hence returning to (3.2), also recalling (3.3) and (3.4), we see that

$$\begin{aligned} u_m^-(x,t) \ \leq \ & \mathbb{E}[e^{-r\Delta t} u_m^-(X^{0,(x,t)}(t+\Delta t), t+\Delta t)] \\ & + c(\Delta t)^{3/2} + k^{-1}\Delta t, \end{aligned} \tag{3.7}$$

where $\Delta t = (T-t)/k$.

We will now use (3.7) in an iterative construction. Indeed, we let $t_j = t + j\Delta t$ for $j = 0, ..., k-1$ and we first note, using (3.7), that for $j = 0$ we have

$$\begin{aligned} u_m^-(x,t_0) \ \leq \ & \mathbb{E}[e^{-r\Delta t} u_m^-(X^{0,(x,t_0)}(t_1), t_1)] \\ & + c(\Delta t)^{3/2} + k^{-1}\Delta t. \end{aligned}$$

We next consider $j = 1$. Then, using that u_m^- is a solution to (2.5), that $\mathcal{H}_m$ is a separable metric space, and uniform continuity, it follows that there exist a sequence $\{(\theta_{1l}^+, d_{1l}^+)\}_{l=1}^\infty \subset \mathcal{H}_m$ and a covering $\{B(y_{1l}, r_{1l})\}_{l=1}^\infty$ of $\mathbb{R}^n$, such that

$$\sup_{(\theta^-,d^-)\in\mathcal{H}_m} \Big\{\Phi(\theta_{1l}^+, \theta^-, d_{1l}^+, d^-, Du_m^-(y,t_1), D^2u_m^-(y,t_1)) + ru_m^-(y,t_1)\Big\}$$
$$\leq \partial_t u_m^-(y,t_1) + k^{-1} \qquad \text{whenever} \quad y \in B(y_{1l}, r_{1l}).$$

We let $\psi_1 : \mathbb{R}^n \to \mathcal{H}_m$ be defined as

$$\psi_1(y) = (\psi_1^\theta(y), \psi_1^d(y)) := (\theta_{1l}^+, d_{1l}^+) \text{ if } y \in B(y_{1l}, r_{1l}) \setminus \cup_{i=1}^{l-1} B(y_{1i}, r_{1i}).$$

Furthermore, this time we let

$$\Phi_1^y(s) := \Phi(\psi_1^\theta(y), \theta^-, \psi_1^d(y), d^-, Du_m^-(y,t_1), D^2u_m^-(y,t_1)).$$

Then

$$\sup_{(\theta^-,d^-)\in\mathcal{H}_m}\left\{\Phi_1^y(s)+ru_m^-(y,t_1)\right\}\leq\ \partial_t u_m^-(y,t_1)+k^{-1}$$

whenever $y\in\mathbb{R}^n$. We now let

$$(\theta_1^+(s),d_1^+(s))=(\theta_0^+,d_0^+),$$

for $s\in[t_0,t_1)=[t,t+\Delta t)$, and

$$(\theta_1^+(s),d_1^+(s))=(\psi_1^\theta(X^0(t_1)),\psi_1^d((X^0(t_1))) \tag{3.8}$$

for $s\in[t_1,t_2)=[t+\Delta t,t+2\Delta t)$. In this way we have now constructed a new control $(\theta_1^+,d_1^+)\in\mathcal{AC}_m$. Next, with this $(\theta_1^+,d_1^+)\in\mathcal{AC}_m$ and an arbitrary, but fixed, control $(\theta^-,d^-)\in\mathcal{AC}_m$ we construct $X^1(s)=X^{1,(x,t)}(s)$ for $s\in[t_0,t_2)$ satisfying the initial condition $X^1(t)=x$ and the dynamics in (1.2). By construction it follows that $X^1(s)=X^0(s)$ for $s\in[t_0,t_1)$. We can now repeat the argument above to conclude that

$$\begin{aligned}u_m^-(X^0(t_1),t_1)\ &\leq\ \mathbb{E}[e^{-r\Delta t}u_m^-(X^1(t_2),t_2)]\\&\quad+c(\Delta t)^{3/2}+k^{-1}\Delta t.\end{aligned}$$

In particular, we see that

$$\begin{aligned}\mathbb{E}[e^{-r\Delta t}u_m^-(X^0(t_1),t_1)]\ &\leq\ \mathbb{E}[e^{-2r\Delta t}u_m^-(X^1(t_2),t_2)]\\&\quad+c(\Delta t)^{3/2}+k^{-1}\Delta t.\end{aligned} \tag{3.9}$$

Combining this with (3.7) we conclude

$$\begin{aligned}u_m^-(x,t)\ &\leq\ \mathbb{E}[e^{-2r\Delta t}u_m^-(X^1(t_2),t_2)]\\&\quad+c2(\Delta t)^{3/2}+2k^{-1}\Delta t.\end{aligned}$$

By carefully iterating the above argument we get a sequence of controls $(\theta_j^+,d_j^+)\in\mathcal{AC}_m$, for $j\in\{0,1,...,k-1\}$, and a sequence of processes $X^j(s)=X^{j,(x,t)}(s)$ based on the controls (θ_j^+,d_j^+) and an arbitrary, but fixed, $(\theta^-,d^-)\in\mathcal{AC}_m$. In particular, we have

$$\begin{aligned}u_m^-(x,t)\ &\leq\ \mathbb{E}[e^{-jr\Delta t}u_m^-(X^j(t_j),t_j)]\\&\quad+cj(\Delta t)^{3/2}+jk^{-1}\Delta t,\end{aligned}$$

for all $j\in\{1,...,k\}$. Now, applying this inequality with $j=k$ we conclude that

$$\begin{aligned}u_m^-(x,t)\ &\leq\ \mathbb{E}[e^{-r(T-t)}u_m^-(X^k(T),T))]+c(T-t)(\Delta t)^{1/2}+\Delta t\\&=\ \mathbb{E}[e^{-r(T-t)}g(X^k(T))]+c(T-t)(\Delta t)^{1/2}+\Delta t,\end{aligned} \tag{3.10}$$

where we also used $u_m^-(x,T)=g(x)$ in the last line.

Summing up, given $k\in\{1,2,\ldots\}$ and $(x,t)\in\mathbb{R}^n\times[0,T]$ we have constructed controls (θ_k^+,d_k^+) such that for an arbitrary, but fixed, $(\theta^-,d^-)\in\mathcal{AC}_m$, (3.10) holds with X^k defined as in (1.2) based on (θ_k^+,d_k^+), (θ^-,d^-), and $X^k(t)=x$.

Next, consider $\rho^- \in \mathcal{S}_m$. Based on the above argument we now construct (θ_j^+, d_j^+) and $(\theta^-, d^-)|_{[t_j,t_{j+1})}$, $j = 0, 1, ..., k-1$, using ρ^-. Indeed, given $(x,t) \in \mathbb{R}^n \times [0,T]$ we first let (θ_0^+, d_0^+) be as above and set

$$(\theta^-, d^-)|_{[t_0,t_1)} := \rho^-(\theta_0^+, d_0^+)|_{[t_0,t_1)}.$$

Then having defined $(\theta^-, d^-)|_{[t_0,t_1)}$, we may define (θ_1^+, d_1^+) on $[t_0, t_2)$ as in (3.8). Then repeating the above argument, we set

$$(\theta^-, d^-)|_{[t_1,t_2)} := \rho^-(\theta_1^+, d_1^+)|_{[t_1,t_2)}.$$

In particular, proceeding inductively, we can based on $\rho^- \in \mathcal{S}_m$ construct the controls (θ_k^+, d_k^+) and (θ^-, d^-) such that

$$\begin{aligned} u_m^-(x,t) &\leq \mathbb{E}[e^{-r(T-t)} g(X^k(T))] + c(T-t)(\Delta t)^{1/2} + \Delta t \\ &\leq \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)) \\ &\quad + c(T-t)(\Delta t)^{1/2} + \Delta t, \end{aligned}$$

for all k, $\Delta t = (T-t)/k$, and for all $\rho^- \in \mathcal{S}_m$. In particular, letting $k \to \infty$ we end up with

$$u_m^-(x,t) \leq \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)). \tag{3.11}$$

To prove the opposite inequality, we note, by analogy that given $k \in \{1, 2, \ldots\}$ and $(x,t) \in \mathbb{R}^n \times [0,T]$, we can choose, again since u_m^- is a solution to (2.5), a control $(\theta_0^-, d_0^-) \in \mathcal{H}_m$ such that

$$\begin{aligned} \inf_{(\theta^+, d^+) \in \mathcal{H}_m} &\left\{ \Phi(\theta^+, \theta_0^-, d^+, d_0^-, Du_m^-(x,t), D^2 u_m^-(x,t)) + r u_m^-(x,t) \right\} \\ &\geq \partial_t u_m^-(x,t) - k^{-1}. \end{aligned}$$

Then, by arguing similarly as above, we deduce that we can construct controls (θ_k^-, d_k^-) such that for an arbitrary but fixed $(\theta^+, d^+) \in \mathcal{AC}_m$,

$$\begin{aligned} u_m^-(x,t) &\geq \mathbb{E}[e^{-r(T-t)} u_m^-(X^k(T), T))] - c(T-t)(\Delta t)^{1/2} - \Delta t \\ &= \mathbb{E}[e^{-r(T-t)} g(X^k(T))] - c(T-t)(\Delta t)^{1/2} - \Delta t, \end{aligned} \tag{3.12}$$

with X^k defined as in (1.2) based on (θ_k^-, d_k^-), (θ^+, d^+), and $X^k(t) = x$.

Next we define a strategy $\rho_k^- \in \mathcal{S}_m$ as follows. Given $A^+ = (\theta^+, d^+) \in \mathcal{AC}_m$ we construct (θ_k^-, d_k^-) as above and then simply let

$$\rho_k^-(A^+) = \rho_k^-(\theta^+, d^+) := (\theta_k^-, d_k^-).$$

Then, using (3.12) we see that

$$u_m^-(x,t) \geq J^{(x,t)}(A^+, \rho_k^-(A^+)) - c(T-t)(\Delta t)^{1/2} - \Delta t,$$

and, hence

$$\begin{aligned} u_m^-(x,t) &\geq \inf_{\rho^- \in \mathcal{S}_m} J^{(x,t)}(A^+, \rho^-(A^+)) \\ &\quad - c(T-t)(\Delta t)^{1/2} - \Delta t, \end{aligned}$$

for all k, $\Delta t = (T-t)/k$, and for all $A^+ \in \mathcal{AC}_m$. In particular, letting $k \to \infty$ we can conclude that

$$u_m^-(x,t) \geq \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)).$$

Combining this with (3.11) we see that the proof of Lemma 3.2 for u_m^- is complete. □

3.2. **Proof of Lemma 3.1.** In the following we only supply the proof in the case of u_m^- in $\mathbb{R}^n \times [0,T]$, the proof for u_m^+ being analogous. Given a large non-negative integer j we let $T_j = T - \frac{1}{j}$ and $\mathbb{R}_j^n := \mathbb{R}^n \times [\frac{1}{j}, T_j]$. Given j fixed we in the following use, for $\epsilon > 0$ small, the sup-convolution

$$u_\varepsilon(x_0,t_0) := \sup_{(x,t)\in\mathbb{R}_j^n} \{u_m^-(x,t) - \frac{(t_0-t)^2 + (x_0-x)^2}{2\varepsilon}\}$$

whenever $(x_0,t_0) \in \mathbb{R}_j^n$. Then $u_m^-(x_0,t_0) \leq u_\varepsilon(x_0,t_0)$ whenever $(x_0,t_0) \in \mathbb{R}_j^n$ and u_ε is a semi-convex function, i.e. there exists a constant $c > 0$ such that $u_\varepsilon(x,t) + c(|x|^2 + t^2)$ is convex. Furthermore, provided ε is small enough,

$$H_m^-(x,t,Du_\varepsilon, D^2u_\varepsilon) \leq \partial_t u_\varepsilon(x,t) + \omega(\varepsilon)$$

for a.e. $(x,t) \in \mathbb{R}_j^n$ and where $\omega(\varepsilon)$ is a bounded modulus which depends on the continuity of u_m^-. For details and properties of sup-convolutions, see for example [CIL92], [Ish95] and [Lin12].

Next, using the standing assumptions on g, see (1.6), and the comparison principle we see that $0 \leq u_m^-(x,t) \leq L$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Furthermore, using the boundedness of u_m^- it follows that the supremum used in the definition of $u_\varepsilon(x_0,t_0)$ is obtained at some point (x^*,t^*). In particular,

$$\begin{aligned} 0 \leq u_m^-(x_0,t_0) \leq u_\varepsilon(x_0,t_0) &= u_m^-(x^*,t^*) - \frac{(t_0-t^*)^2 + (x_0-x^*)^2}{2\varepsilon} \\ \leq L - \frac{(t_0-t^*)^2 + (x_0-x^*)^2}{2\varepsilon} \end{aligned}$$

and hence

$$\sqrt{(t_0-t^*)^2 + (x_0-x^*)^2} \leq \sqrt{2L\varepsilon},$$

where we deduce a condition $\sqrt{2L\varepsilon} < 1/j$ for ε.

Given $\delta > 0$ and small we in the following let η_δ denote a standard mollifier in $\mathbb{R}^{n+1}$. Restricting $\delta \ll ((j-1)^{-1} - j^{-1})/2$ we see that $u_\varepsilon^\delta(x,t) := u_\varepsilon * \eta_\delta(x,t)$, the convolution of u_ε and η_δ, is well-defined whenever $(x,t) \in \mathbb{R}^n_{j-1}$. Then, in particular

$$\begin{aligned} u^\delta_\varepsilon &\to u_\varepsilon, \quad \text{uniformly on } \mathbb{R}^n_{j-1}, \\ Du^\delta_\varepsilon &\to Du_\varepsilon, \quad \text{a.e. in } \mathbb{R}^n_{j-1}, \\ \partial_t u^\delta_\varepsilon &\to \partial_t u_\varepsilon, \quad \text{a.e. in } \mathbb{R}^n_{j-1}, \\ D^2 u^\delta_\varepsilon &\to D^2 u_\varepsilon, \quad \text{a.e. in } \mathbb{R}^n_{j-1}. \end{aligned}$$

The last statement is based on Alexandrov's theorem, see for example Section 6 of [EG92] or [JJ12]. Furthermore,

$$H^-_m(x,t,Du^\delta_\varepsilon, D^2u^\delta_\varepsilon) \le \partial_t u^\delta_\varepsilon(x,t) + \omega(\varepsilon) + \gamma_\delta(x,t),$$

a.e. in $\mathbb{R}^n_{j-1}$ where $\gamma_\delta(x,t) \to 0$ as $\delta \to 0$, and γ_δ is uniformly continuous (since it is a result of the standard convolution; the modulus of continuity is not claimed to be uniform in δ) and bounded uniformly in δ (recall uniform semiconvexity and uniform Lipschitz continuity of u^δ_ε with respect to δ). Now, using that $u^\delta_\varepsilon, \partial_t u^\delta_\varepsilon, Du^\delta_\varepsilon, D^2u^\delta_\varepsilon$ are Lipschitz continuous in $\mathbb{R}^n_{j-1}$ we can argue as in the proof of Lemma 3.2 and conclude that

$$\begin{aligned} u^\delta_\varepsilon(x,t) \le \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} \mathbb{E}\bigg[\int_t^{T_{j-1}} & e^{-r(s-t)} h^\delta_\varepsilon(X(s),s)\,ds \\ & + e^{-r(T_{j-1}-t)} u^\delta_\varepsilon(X(T_{j-1}),T_{j-1})\bigg] \end{aligned} \tag{3.13}$$

whenever $(x,t) \in \mathbb{R}^n_{j-1}$ and where $h^\delta_\varepsilon := \omega(\varepsilon) + \gamma_\delta$. Indeed, (3.1) now reads as

$$\begin{aligned} \sup_{(\theta^-,d^-)\in\mathcal{H}_m} &\left\{\Phi(\theta^+_0,\theta^-,d^+_0,d^-,Du^\delta_\varepsilon(x,t),D^2u^\delta_\varepsilon(x,t)) + ru^\delta_\varepsilon(x,t)\right\} \\ &\le \partial_t u^\delta_\varepsilon(x,t) + h^\delta_\varepsilon(x,t) + k^{-1}, \end{aligned}$$

for $(x,t) \in \mathbb{R}^n_{j-1}$, and thus we may estimate the second term on the right hand side of (3.2) by using $h^\delta_\varepsilon(x,t)$. In particular, in (3.5), the expression $\partial_t u^-_m(x,t) - \Phi^x_0(t) - ru^-_m(x,t)$ may be replaced by $h^\delta_\varepsilon(x,t)$. Using these observations and arguing as in the proof of Lemma 3.2 we see that (3.9) now reads

$$\begin{aligned} \mathbb{E}[e^{-r\Delta t}u^\varepsilon_\delta(X^0(t_1),t_1)] &\le \mathbb{E}[e^{-2r\Delta t}u^\varepsilon_\delta(X^1(t_2),t_2)] \\ - \mathbb{E}[\int_{t_1}^{t_2} & e^{-r(s-t)}h^\delta_\varepsilon(X^1(s),s)\,ds] + c(\Delta t)^{3/2} + k^{-1}\Delta t + c\Delta t\rho(\Delta t), \end{aligned}$$

where $\Delta t = (T_{j-1}-t)/k$, the modulus of continuity ρ in the last error term depends on the modulus of continuity of h^δ_ε, and results from the calculations similar to those following (3.6) except using ρ instead of $|\cdot|$. Next, iterating the above reasoning in time, along the lines of the proof of Lemma 3.2, completes the argument. In particular, the last error term yields $c(T_{j-1}-t)\rho((T_{j-1}-t)/k)$. Then, letting $k\to\infty$ gives (3.13).

Next we want, for j fixed, to let $\delta\to 0$ and $\varepsilon\to 0$ in (3.13). To do this, recall that the underlying dynamics X is defined through the stochastic differential equation in (1.2) and based on (uniformly) bounded controls encoded through $\mathcal{S}_m$ and $\mathcal{AC}_m$. In particular, X solves an sde with (uniformly) bounded coefficients. Using this we first observe, using a standard martingale argument, that given $\theta>0$, there exists $R=R_\theta$ such that

$$\mathbb{P}(\sup_{t\leq s\leq T_{j-1}}|X(s)|\geq R)\leq\theta. \tag{3.14}$$

Furthermore, given $\theta>0$ and R as above we choose $\Omega_\theta\subset B_R:=B_R(0)$ such that $|\Omega_\theta|<\theta$ and such that

$$\gamma_\delta\to 0 \text{ uniformly in } (B_R(0)\setminus\Omega_\theta)\times[(j-1)^{-1},T_{j-1}] \text{ as } \delta\to 0. \tag{3.15}$$

Given $E\subset\mathbb{R}^n$ we in the following let χ_E denote the indicator function of E. Then, first using (3.14) we see that

$$\begin{aligned}
&\int_t^{T_{j-1}}\mathbb{E}[e^{-r(s-t)}h_\varepsilon^\delta(X(s),s)]\,ds\\
&\quad\leq\int_t^{T_j}\mathbb{E}[h_\varepsilon^\delta(X(s),s)\chi_{B_R}(X(s))]+\int_t^{T_j}\mathbb{E}[h_\varepsilon^\delta(X(s),s)\chi_{B_R^c}(X(s))]\,ds\\
&\quad\leq I_1^{\varepsilon,\delta}(\theta)+I_2^{\varepsilon,\delta}(\theta)+c(T_{j-1}-t)\theta
\end{aligned}$$

for some harmless constant independent of j, ε, δ, and θ. Here

$$\begin{aligned}
I_1^{\varepsilon,\delta}(\theta) &:= \int_t^{T_{j-1}}\mathbb{E}[h_\varepsilon^\delta(X(s),s)\chi_{\Omega_\theta}(X(s))],\\
I_2^{\varepsilon,\delta}(\theta) &:= \int_t^{T_{j-1}}\mathbb{E}[h_\varepsilon^\delta(X(s),s)\chi_{B_R\setminus\Omega_\theta}(X(s))]\,ds.
\end{aligned}$$

Now

$$I_1^{\varepsilon,\delta}(\theta)\leq c\int_t^{T_{j-1}}\mathbb{E}[\chi_{\Omega_\theta}(X(s))]\,ds\leq c(T_{j-1}-t)|\Omega_\theta| \tag{3.16}$$

and consequently

$$I_1^{\varepsilon,\delta}(\theta)\leq c(T_{j-1}-t)\theta$$

for some constant c independent of $\varepsilon,\delta,\theta$. Note that the estimate in (3.16) is far from straightforward. Indeed, (3.16) is a fundamental estimate by Krylov and stated as Theorem 4 on p.66 in [Kry09]. It is interesting to note that there is, at the core of Krylov's proof of this estimate, a Alexandrov-Bakelman-Pucci-type estimate for uniformly parabolic equation, see Krylov [Kry76]. In particular, a short calculation shows that our dynamics satisfies the sufficient assumptions stated in [Kry09] for the validity of the estimate in (3.16).

Next we note that

$$I_2^{\varepsilon,\delta}(\theta) \to 0 \text{ when we first let } \delta \to 0, \text{ and then } \varepsilon \to 0,$$

as we see from the fact that $h_\varepsilon^\delta = \omega(\varepsilon) + \gamma_\delta$, (3.15), and that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Taking the limits $\delta \to 0$ and $\varepsilon \to 0$ in (3.13), we see by the above that

$$u(x,t) \le \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} \mathbb{E}[e^{-r(T_{j-1}-t)} u(X(T_{j-1}), T_{j-1})],$$

whenever $(x,t) \in \mathbb{R}^n_{j-1}$. Finally, using the barriers given by Lemma 2.9, and by arguing as at the end of the proof of Lemma 5.2 stated below, we can then conclude by letting $j \to \infty$ that

$$u(x,t) \le \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} \mathbb{E}[e^{-r(T-t)} g(X(T))],$$

and this completes the proof of Lemma 3.1. □

4. The limit equation $\partial_t u + F(u, Du, D^2u) = 0$

We here start by introducing the relevant notion of viscosity super- and subsolutions to (1.5).

Definition 4.1. (*a*) A function $\bar{u} \in \mathrm{LSC}_l(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution to (1.5) if $\bar{u}(x,T) \ge g(x)$ for all $x \in \mathbb{R}^n$ and if the following holds. If $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and we have $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ such that

$$\begin{aligned} (i) \quad & \bar{u}(x_0,t_0) = \phi(x_0,t_0), \\ (ii) \quad & \bar{u}(x,t) > \phi(x,t) \text{ for } (x,t) \ne (x_0,t_0), \end{aligned}$$

then

$$0 \ge \partial_t \phi(x_0,t_0) + F(\bar{u}(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)), \tag{4.1}$$

whenever $D\phi(x_0,t_0) \ne 0$, and

$$0 \ge \partial_t \phi(x_0,t_0) + \liminf_{p \to 0} F(\bar{u}(x_0,t_0), p, D^2\phi(x_0,t_0)) \tag{4.2}$$

whenever $D\phi(x_0,t_0) = 0$.
(*b*) A function $\underline{u} \in \mathrm{USC}_l(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution to (1.5) if $\underline{u}(x,T) \le g(x)$ for all $x \in \mathbb{R}^n$ and if the following holds. If $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and we have $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ such that

$$\begin{aligned} (i) \quad & \underline{u}(x_0,t_0) = \phi(x_0,t_0), \\ (ii) \quad & \underline{u}(x,t) < \phi(x,t) \text{ for } (x,t) \ne (x_0,t_0), \end{aligned}$$

then

$$0 \le \partial_t \phi(x_0,t_0) + F(\underline{u}(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \tag{4.3}$$

whenever $D\phi(x_0,t_0) \ne 0$, and

$$0 \le \partial_t \phi(x_0,t_0) + \limsup_{p \to 0} F(\underline{u}(x_0,t_0), p, D^2\phi(x_0,t_0)), \tag{4.4}$$

whenever $D\phi(x_0, t_0) = 0$.
(c) If u is both a viscosity supersolution and a viscosity subsolution to (1.5), then u is a viscosity solution to (1.5).

We let

$$F^* := \limsup_{p\to 0} F \text{ and } F_* := \liminf_{p\to 0} F.$$

Using this notation we see that (4.1) and (4.2) can be written at once as

$$0 \geq \partial_t \phi(x_0, t_0) + F_*(\bar{u}(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)),$$

and (4.3) and (4.4) as

$$0 \leq \partial_t \phi(x_0, t_0) + F^*(\underline{u}(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)).$$

Similarly to Lemma 2.7 the following lemma also follows from [GGIS91].

Lemma 4.2. *Let $\underline{u}, \bar{u} \in C_l(\mathbb{R}^n \times [0, T])$ be viscosity sub- and supersolutions to* (1.5) *in the sense of Definition 4.1. Then*

$$\underline{u}(x, t) \leq \bar{u}(x, t),$$

whenever $(x, t) \in \mathbb{R}^n \times [0, T]$.

Furthermore, arguing as in Lemma 2.8 and Lemma 2.9 we see that we can construct barriers to (1.5), use them for comparison, and prove the following lemma.

Lemma 4.3. *Let $y \in \mathbb{R}^n$ and let L be the Lipschitz constant of g. Consider $0 < \varepsilon \ll 1$, and let*

$$\bar{w}(x, t) = g(y) + \frac{A}{\varepsilon^2}(T - t) + 2L(|x - y|^2 + \varepsilon)^{1/2},$$
$$\underline{w}(x, t) = g(y) - \frac{A}{\varepsilon^2}(T - t) - 2L(|x - y|^2 + \varepsilon)^{1/2}.$$

Then we can choose A, independent of y, ε and m, so that $\bar{w}$ and $\underline{w}$ are viscosity super- and subsolutions to (1.5)*. Consequently, for such A, and if u is a viscosity solution to* (1.5)*, we have*

$$\underline{w} \leq u \leq \bar{w}.$$

Below we always choose, when applying $\bar{w}$ and $\underline{w}$, A so that Lemma 4.3 holds.

Theorem 4.4. *There exists a unique viscosity solution u to* (1.5)*.*

The uniqueness part of Theorem 4.4 follows from Lemma 4.2. The existence part of Theorem 4.4 again follows from Perron's method also using Lemma 4.3 as discussed after Lemma 2.10.

Next, using a modification of the techniques [CGG91], [ES91], see also [JK06] and [KMP12], we prove the following lemma which states that the set of test functions used in Definition 4.1 can be reduced. We consider continuous sub-/supersolutions since later we only need this result for solutions.

Lemma 4.5. *Let $u \in C_l(\mathbb{R}^n \times [0,T])$. Then to test whether or not u is a viscosity super- or subsolution at (x_0,t_0) in the sense of Definition 4.1, it is enough to consider test functions $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ such that either*

$$\begin{aligned} &(i) \qquad D\phi(x_0,t_0) \neq 0 \ \textit{or} \\ &(ii) \qquad D\phi(x_0,t_0) = 0 \ \textit{and} \ D^2\phi(x_0,t_0) = 0. \end{aligned}$$

Proof. We here only prove the lemma in the context of subsolutions. The proof is by contradiction. Indeed, assume that there exists a function $u \in \mathrm{C}_l(\mathbb{R}^n \times [0,T])$, *which fails to be a subsolution at* (x_0,t_0) in the sense of Definition 4.1 even though the following holds. If $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and $\phi \in C^{1,2}(\mathbb{R}^n \times [0,T])$ are such that

$$\begin{aligned} &(i) \qquad u(x_0,t_0) = \phi(x_0,t_0), \\ &(ii) \qquad u(x,t) < \phi(x,t) \text{ for } (x,t) \neq (x_0,t_0), \end{aligned}$$

then

$$0 \leq \partial_t \phi(x_0,t_0) + F^*(u(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)), \tag{4.5}$$

whenever

$$\begin{aligned} &(i) \qquad D\phi(x_0,t_0) \neq 0 \text{ or} \\ &(ii) \qquad D\phi(x_0,t_0) = 0 \text{ and } D^2\phi(x_0,t_0) = 0. \end{aligned}$$

Now, since u is assumed to fail to be a subsolution we see that there must also exist a test function φ touching from above, and $\varepsilon > 0$, such that

$$0 > \partial_t \varphi(x_0,t_0) + F^*(u(x_0,t_0), D\varphi(x_0,t_0), D^2\varphi(x_0,t_0)) + \varepsilon \tag{4.6}$$

and such that

$$D\varphi(x_0,t_0) = 0 \text{ and } D^2\varphi(x_0,t_0) \neq 0.$$

In addition, we may assume that $u - \varphi$ has a strict global maximum at (x_0,t_0). Let

$$w(x,t,y,s) := w_j(x,t,y,s) = u(x,t) - \varphi(y,s) - \Psi_j(x,t,y,s), \tag{4.7}$$

where

$$\Psi(x,t,y,s) := \Psi_j(x,t,y,s) = \frac{j}{4}|x-y|^4 + \frac{j}{2}(t-s)^2.$$

By comparison and the structure of the barriers in Lemma 4.3, we see that there exists $(x_j,t_j,y_j,s_j) \in \mathbb{R}^n \times (0,T) \times \mathbb{R}^n \times (0,T)$ such that

$$w(x_j,t_j,y_j,s_j) = \sup_{(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,T]} w(x,t,y,s). \tag{4.8}$$

Furthermore,

$$(x_j,t_j,y_j,s_j) \to (x_0,t_0,x_0,t_0) \quad \text{as} \quad j \to \infty.$$

We now consider two cases.

Case 1: there exists an infinite sequence of j:s such that $x_j = y_j$ for each such j.

Case 2: there exists an $j_0 \in \{1, 2, \ldots\}$ such that $x_j \neq y_j$ for all j, $j > j_0$.

We first analyze Case 1 and we let $x_j = y_j$. Then, by construction,

$$\begin{aligned} u(x,t) \ \leq \ & u(x_j,t_j) + \varphi(y,s) - \varphi(y_j,s_j) \\ & + \Psi(x,t,y,s) - \Psi(x_j,t_j,y_j,s_j), \end{aligned}$$

whenever $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,T]$. In particular,

$$u(x,t) \leq u(x_j,t_j) + \Psi(x,t,y_j,s_j) - \Psi(x_j,t_j,y_j,s_j) \tag{4.9}$$

whenever $(x,t) \in \mathbb{R}^n \times (0,T)$. Moreover, since $x_j = y_j$, we observe that $D_x\Psi(x_j,t_j,y_j,s_j) = 0$, $D^2_{xx}\Psi(x_j,t_j,y_j,s_j) = 0$, and thus we can conclude that the function on the right in (4.9) is an admissible test function at (x_j,t_j) for the conclusion in (4.5). For brevity, we drop the arguments (x_j,t_j,y_j,s_j) in Ψ and its derivatives in the displays below. We have

$$0 \leq \Psi_t + F^*(u(x_j,t_j), D_x\Psi, D^2_{xx}\Psi).$$

Then observe that $0 = -D^2_{yy}\Psi \leq D^2\varphi(y_j,s_j)$, $0 = -D_y\Psi = D\varphi(y_j,s_j)$ and $-\Psi_s = \partial_t\varphi(y_j,s_j)$, since $(y,s) \mapsto \varphi(y,s) + \Psi(x_j,t_j,y,s)$ has a local minimum at (y_j,s_j) by (4.8). By these facts, ellipticity of F^*, and (4.6) it follows that

$$\Psi_s = -\partial_t\varphi(y_j,s_j) > F^*(u(y_j,s_j), -D_y\Psi, -D^2_{yy}\Psi) + \frac{\varepsilon}{2}.$$

Combining the previous two displays, we obtain

$$r(u(x_j,t_j) - u(y_j,s_j)) < \frac{(t_j - s_j) - (t_j - s_j)}{\varepsilon} - \frac{\varepsilon}{2} = -\frac{\varepsilon}{2}$$

which is a a contradiction for large enough j since u is continuous.

We next analyze Case 2. In this case, using the Theorem of sums, see [CIL92], we see that there exist $(\Psi_t, D_x\Psi, X) \in \overline{\mathcal{P}}^{2,+}u(x,t)$ and $(-\Psi_s, -D_y\Psi, -Y) \in \overline{\mathcal{P}}^{2,-}\varphi(y,t)$ such that $X \leq -Y$. In particular, since $D_x\Psi \neq 0$ it follows that

$$\begin{aligned} & 0 \leq \Psi_t + F^*(u(x_j,t_j), D_x\Psi, X) \\ & 0 > -\Psi_s + F^*(u(y_j,s_j), -D_y\Psi, -Y) + \frac{\varepsilon}{2} \end{aligned} \tag{4.10}$$

where the second inequality follows from (4.6) by continuity. Thus

$$\begin{aligned} \Psi_t + \Psi_s &> -F^*(u(x_j,t_j), D_x\Psi, X) + F^*(u(y_j,s_j), -D_y\Psi, -Y) + \frac{\varepsilon}{2} \\ &\geq \frac{\varepsilon}{2} + r(u(y_j,s_j) - u(x_j,t_j)). \end{aligned}$$

Finally, observing that $\Psi_t = -\Psi_s$ we see that the last display implies that

$$0 \geq \frac{\varepsilon}{2} + r(u(y_j, s_j) - u(x_j, t_j)).$$

and this now produces a contradiction for j large enough. The proof for a supersolution is similar. □

5. Going to the limit: general action sets as $m \to \infty$

In the following we will use the following lemma.

Lemma 5.1. *Let $\xi_m, \xi \in \mathbb{R}$, $p_m, p \in \mathbb{R}^n \setminus \{0\}$, $M_m, M \in \mathcal{M}(n)$, be such that*

$$\xi_m \to \xi, \quad p_m \to p, \quad \text{and} \quad M_m \to M,$$

as $m \to \infty$. Then

$$H_m^{\pm}(\xi_m, p_m, M_m) \to -F(\xi, p, M).$$

Proof. We here only prove the statement for H_m^-, the proof for H_m^+ being analogous. Furthermore, to prove that $H_m^-(\xi_m, p_m, M_m) \to -F(\xi, p, M)$ it is sufficient to consider those terms in H_m^- which are affected by $\inf_{(\theta^+, d^+) \in \mathcal{H}_m} \ \sup_{(\theta^-, d^-) \in \mathcal{H}_m}$. In particular, we focus on

$$\begin{aligned}\tilde{\Phi}(\theta^+, \theta^-, d^+, d^-, p_m, M_m) \ &:= \ -\frac{1}{2}(\theta^+ - \theta^-)'\Sigma M_m \Sigma(\theta^+ - \theta^-) \\ &\quad -(d^+ + d^-)(\theta^+ + \theta^-) \cdot p_m.\end{aligned}$$

Setting

$$\tilde{\Phi}_m := \inf_{(\theta^+, d^+) \in \mathcal{H}_m} \ \sup_{(\theta^-, d^-) \in \mathcal{H}_m} \tilde{\Phi},$$

we observe that

$$\begin{aligned}\tilde{\Phi}_m \leq \sup_{(\theta^-, d^-) \in \mathcal{H}_m} \Big(&-\frac{1}{2}(p_m/|p_m| - \theta^-)'\Sigma M_m \Sigma(p_m/|p_m| - \theta^-) \\ &- d^-(p_m/|p_m| + \theta^-) \cdot p_m\Big),\end{aligned}$$

and that $(p_m/|p_m| + \theta^-) \cdot p_m \geq 0$ whenever $\theta^- \in \mathbb{S}^{n-1}$. In particular, we conclude that $\tilde{\Phi}_m$ is bounded from above as $m \to \infty$.

Using that the set $\{(\theta^+, d^+) \in \mathcal{H}_m\}$ is compact, we see that there exists a (θ_m^+, d_m^+) realizing the infimum in the definition of $\tilde{\Phi}_m$ and we next prove that there exists, given $\varepsilon > 0$, a $m_0 = m_0(\varepsilon)$ such that

$$|p_m| - \varepsilon \leq \theta_m^+ \cdot p_m \leq |p_m| \qquad \text{whenever} \quad m \geq m_0. \tag{5.1}$$

Obviously we only have to establish the lower bound and to do this we assume, on the contrary, that there exists $\varepsilon > 0$ and $m_j \to \infty$, such that

$$\theta_{m_j}^+ \cdot p_{m_j} \leq |p_{m_j}| - \varepsilon \text{ as } j \to \infty. \tag{5.2}$$

If this is the case then

$$\begin{aligned}\tilde{\Phi}_{m_j} &= \sup_{(\theta^-,d^-)\in\mathcal{H}_{m_j}} \Big(-\frac{1}{2}(\theta^+_{m_j}-\theta^-)'\Sigma M_{m_j}\Sigma(\theta^+_{m_j}-\theta^-) \\ &\qquad - (d^+_{m_j}+d^-)(\theta^+_{m_j}+\theta^-)\cdot p_{m_j}\Big) \qquad (5.3)\\ &\geq -c-(d^+_{m_j}+m_j)(\theta^+_{m_j}-p_{m_j}/|p_{m_j}|)\cdot p_{m_j} \\ &\geq -c+(d^+_{m_j}+m_j)\varepsilon\end{aligned}$$

since $(-p_{m_j}/|p_{m_j}|, m_j)\in\mathcal{H}_{m_j}$ and for some harmless constant c. However, (5.3) contradicts the boundedness of $\tilde{\Phi}_{m_j}$ as $m_j\to\infty$, hence (5.2) must be false and (5.1) must hold.

Using (5.1) we see that

$$\theta^+_m \to p/|p| \text{ as } m\to\infty. \qquad (5.4)$$

Furthermore, using that

$$\begin{aligned}\tilde{\Phi}_m &\geq -\frac{1}{2}(\theta^+_m+p_m/|p_m|)'\Sigma M_m\Sigma(\theta^+_m+p_m/|p_m|)\\ &\quad -(d^+_m+m)(\theta^+_m-p_m/|p_m|)\cdot p_m\\ &\geq -\frac{1}{2}(\theta^+_m+p_m/|p_m|)'\Sigma M_m\Sigma(\theta^+_m+p_m/|p_m|),\end{aligned}$$

in combination with (5.4), we have that

$$\liminf_{m\to\infty}\tilde{\Phi}_m \geq -2(p/|p|)'\Sigma M\Sigma p/|p|. \qquad (5.5)$$

This yields, recalling the rest of the terms in the definition of H^-_m, that

$$\liminf_{m\to\infty} H^-_m(\xi_m,p_m,M_m)\geq -F(\xi,p,M).$$

To complete the proof it only remains to prove that

$$\limsup_{m\to\infty} H^-_m(\xi_m,p_m,M_m)\leq -F(\xi,p,M). \qquad (5.6)$$

To do this we first note, again using the definition of (θ^+_m, d^+_m), that

$$\begin{aligned}\tilde{\Phi}_m &= \sup_{(\theta^-,d^-)\in\mathcal{H}_m}\tilde{\Phi}(\theta^+_m,\theta^-,d^+_m,d^-,p_m,M_m)\\ &\leq \sup_{(\theta^-,d^-)\in\mathcal{H}_m}\tilde{\Phi}(p_m/|p_m|,\theta^-,m,d^-,p_m,M_m).\end{aligned}$$

Furthermore, again using compactness we see that we can choose (θ^-_m, d^-_m) realizing the supremum in the last display. Hence,

$$\begin{aligned}\tilde{\Phi}_m &\leq -\frac{1}{2}(p_m/|p_m|-\theta^-_m)'\Sigma M_m\Sigma(p_m/|p_m|-\theta^-_m)\\ &\quad -(m+d^-_m)(p_m/|p_m|+\theta^-_m)\cdot p_m.\end{aligned}$$

Using this we deduce that $\theta_m^- \to -p/|p|$, since otherwise the above estimate would imply $\liminf_{m\to\infty} \tilde\Phi_m = -\infty$ contradicting (5.5). Furthermore,

$$\begin{aligned}\tilde\Phi_m \leq & -\frac{1}{2}(p_m/|p_m| - \theta_m^-)'\Sigma M_m \Sigma(p_m/|p_m| - \theta_m^-)\\ & - (m + d_m^-)(p_m/|p_m| + \theta_m^-)\cdot p_m\\ \leq & -\frac{1}{2}(p_m/|p_m| - \theta_m^-)'\Sigma M_m \Sigma(p_m/|p_m| - \theta_m^-),\end{aligned}$$

and taking $\limsup_{m\to\infty}$ we see that (5.6) holds. This completes the proof of the lemma. □

Lemma 5.2. *Let u_m^+ and u_m^- be the unique solutions to* (2.4) *and* (2.5)*, respectively, ensured by Lemma 2.10. Then there exists $m_0 \in \{1, 2, \ldots\}$ such that the families*

$$\{u_m^\pm :\ m \geq m_0\}$$

are equicontinuous on $\mathbb{R}^n \times [0, T]$.

Proof. We here only prove that $\{u_m^+ :\ m \geq m_0\}$ is equicontinuous on $\mathbb{R}^n \times [0, T]$, the proof for $\{u_m^- :\ m \geq m_0\}$ being analogous. Using Lemma 3.1 we have that

$$u_m^+(x, t) = U_m^+(x, t) := \sup_{\rho^+\in\mathcal{S}_m} \inf_{A^-\in\mathcal{AC}_m} J^{(x,t)}(\rho^+(A^-), A^-),$$

whenever $(x, t) \in \mathbb{R}^n \times [0, T]$ and an observation is that we can use both a stochastic as well as a pde point of view to prove the lemma. Furthermore, the processes underlying the stochastic formulation, see (1.2), all end at T. Suppose that we consider two games, one starting from (x_1, t_1) and one starting from (x_2, t_2) with $t_1 < t_2$. We want to show, uniformly in m, that $|u_m^+(x_1, t_1) - u_m^+(x_2, t_2)|$ can be made arbitrary small by considering (x_1, t_1) and (x_2, t_2) sufficiently close. Since the controls and strategies can always be 'copied' for the processes starting from (x_1, t_1), (x_2, t_2), cf. p. 105 [PS08], and by considering same samples, this is possible since we use space and time independent Brownian motions, we see that it is enough to consider a situation (x_1, t_1) and (x_2, T) with $t_1 < T$. In particular, we now want to prove that given $\delta > 0$ there exists $\eta > 0$ such that

$$|u_m^+(x_1, t_1) - u_m^+(x_2, T)| = |u_m^+(x_1, t_1) - g(x_2)| \leq \delta$$

whenever $|x_1 - x_2| + T - t_1 \leq \eta$. Recall the barriers

$$\begin{aligned}\bar w(x, t) &= g(x_2) + \frac{A}{\varepsilon^2}(T - t) + 2M(|x - x_2|^2 + \varepsilon)^{1/2},\\ \underline{w}(x, t) &= g(x_2) - \frac{A}{\varepsilon^2}(T - t) - 2M(|x - x_2|^2 + \varepsilon)^{1/2}.\end{aligned}$$

Using Lemma 2.9 we have

$$\underline{w} \leq u_m^+ \leq \bar w.$$

In particular,

$$|u_m^+(x_1,t) - g(x_2)| \le \frac{A}{\varepsilon^2}(T-t) + 2M(|x_1-x_2|^2+\varepsilon)^{1/2}.$$

Let $|x_1 - x_2| + T - t_1 \le \varepsilon^{5/2}$. Then, for $\varepsilon < 1$

$$|u_m^+(x_1,t_1) - g(x_2)| \le A\varepsilon^{1/2} + 4M\varepsilon^{1/2} \le (A+4M)\varepsilon^{1/2},$$

and we conclude, by choosing ε small enough. This completes the proof. □

Lemma 5.3. *Let u_m^+ and u_m^- be the unique solutions to* (2.4) *and* (2.5), *respectively. Then*

$$u_m^\pm(x,t) \to u(x,t),$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where u is the continuous, unique viscosity solution to (1.5).

Proof. We again only prove the result for u_m^+, the proof for u_m^- being similar. We first recall that the existence of u_m^+ is ensured by Lemma 2.10. Furthermore, by comparison with a supersolution L, we see that the sequence $\{u_m^+\}$ is uniformly bounded in $\mathbb{R}^n \times [0,T]$. Using this and Lemma 5.2, we can first conclude, using the Arzelà-Ascoli theorem, that there exists u, continuous on $\mathbb{R}^n \times [0,T]$, such that

$$u_m^+(x,t) \to u(x,t) \quad \text{as } m \to \infty. \tag{5.7}$$

We next prove that u is a viscosity subsolution in $\mathbb{R}^n \times [0,T)$ to (1.5). To do this, let $\phi \in C^2$ touch u strictly from above at (x_0,t_0). Then, using the uniform convergence it follows that there exists $(x_m,t_m) \to (x_0,t_0)$ such that

$$u_m^+ - \phi$$

has a strict max at (x_m,t_m). Hence

$$\partial_t\phi(x_m,t_m) \ge H_m^+(u_m^+(x_m,t_m), D\phi(x_m,t_m), D^2\phi(x_m,t_m)). \tag{5.8}$$

Note that, as $m \to \infty$, $\partial_t\phi(x_m,t_m) \to \partial_t\phi(x_0,t_0)$, $u_m^+(x_m,t_m) \to u(x_0,t_0)$, $D\phi(x_m,t_m) \to D\phi(x_0,t_0)$, $D^2\phi(x_m,t_m) \to D^2\phi(x_0,t_0)$, and we want to pass to the limit in (5.8). Suppose first that $D\phi(x_0,t_0) \neq 0$. Then, using Lemma 5.1 we see that

$$\begin{aligned} H_m^+(u_m^+(x_m,t_m),&D\phi(x_m,t_m), D^2\phi(x_m,t_m)) \\ &\to -F(u(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \end{aligned}$$

as $m \to \infty$. Next, suppose that $D\phi(x_0,t_0) = 0$. In this case we can, by Lemma 4.5, also assume, without loss of generality, that $D^2\phi(x_0,t_0) = 0$. But in this case

$$\begin{aligned} H_m^+(u_m^+(x_m,t_m),&D\phi(x_m,t_m), D^2\phi(x_m,t_m)) \\ &\to -F^*(u(x_0,t_0),0,0). \end{aligned}$$

In particular, in either case we can conclude that

$$\partial_t \phi(x_0, t_0) \geq -F^*(u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)). \tag{5.9}$$

and hence u is a continuous viscosity subsolution to (1.5). The proof of the result that u is also a supersolution to (1.5) is similar. We omit further details. □

We are in position to prove the main result of the paper, Theorem 1.2, which states that the game with unbounded controls has the value and that the value function

$$\begin{aligned} u = U^+(x,t) &= \sup_{\rho^+ \in \mathcal{S}} \inf_{A^- \in \mathcal{AC}} J^{(x,t)}(\rho^+(A^-), A^-) \\ &= \inf_{\rho^- \in \mathcal{S}} \sup_{A^+ \in \mathcal{AC}} J^{(x,t)}(A^+, \rho^-(A^+)) = U^-(x,t) \end{aligned}$$

is the unique solution u to (1.5).

Proof of Theorem 1.2. We will here only prove that $u = U^-$ since the proof is analogously in the other case. Recall that Lemma 3.1 states that

$$u_m^-(x,t) = U_m^-(x,t) = \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)).$$

Furthermore, using Lemma 5.3 we have

$$u_m^-(x,t) \to u(x,t),$$

where u is the solution to (1.5). Thus it suffices to prove that

$$U_m^-(x,t) \to U^-(x,t) \text{ as } m \to \infty. \tag{5.10}$$

To prove (5.10) we first note that

$$\begin{aligned} U^-(x,t) &= \inf_{\rho^- \in \mathcal{S}} \sup_{A^+ \in \mathcal{AC}} J^{(x,t)}(A^+, \rho^-(A^+)) \\ &\geq \inf_{\rho^- \in \mathcal{S}} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)). \end{aligned}$$

In particular, given $\varepsilon > 0$, using that $U^-(x,t)$ is finite by our assumptions on g and that $\mathcal{S} = \cup_m \mathcal{S}_m$, we see that there exists $m_0 = m_0(\varepsilon)$ such that

$$U^-(x,t) \geq \inf_{\rho^- \in \mathcal{S}} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)) \geq U_m^-(x,t) - \varepsilon$$

whenever $m \geq m_0$. We can therefore conclude that

$$U^-(x,t) \geq \limsup_{m \to \infty} U_m^-(x,t).$$

To complete the proof of (5.10) it hence only remains to prove that

$$U^-(x,t) \leq \liminf_{m \to \infty} U_m^-(x,t). \tag{5.11}$$

To prove (5.11), we first fix a strategy which estimates the infimum when the supremum is taken over the controls $\mathcal{AC}_k$. Then by choosing k large enough, we can closely estimate the original supremum taken over $\mathcal{AC}$ by a supremum taken over $\mathcal{AC}_k$. To write down the details, we recall that

$$\mathcal{AC}_k := \{A \in \mathcal{AC} :\ \Lambda(A) \leq k\},$$
$$\mathcal{S}_m := \{\rho \in \mathcal{S} :\ \Lambda(\rho) \leq m\},$$

for $k = 1, 2, \ldots$. Fix $\varepsilon > 0$. For each k, we choose $\rho^-_{km} \in \mathcal{S}_m$ such that

$$\begin{aligned} &\sup_{A^+ \in \mathcal{AC}_k} J^{(x,t)}(A^+, \rho^-_{km}(A^+)) \\ &\leq \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_k} J^{(x,t)}(A^+, \rho^-(A^+)) + \varepsilon. \end{aligned} \tag{5.12}$$

Next we define

$$\rho^-_m(A^+) := \rho^-_{km}(A^+) \text{ whenever } A^+ \in \mathcal{AC}_k \setminus \mathcal{AC}_{k-1},$$

and we set $\mathcal{AC}_0 = \emptyset$ in order to get started. Now, using that $\mathcal{AC} = \cup_k \mathcal{AC}_k$ we have, for $k \geq k_\varepsilon$ sufficiently large, that

$$\begin{aligned} U^-(x,t) &\leq \sup_{A^+ \in \mathcal{AC}} J^{(x,t)}(A^+, \rho^-_m(A^+)) \\ &\leq \sup_{A^+ \in \mathcal{AC}_k} J^{(x,t)}(A^+, \rho^-_m(A^+)) + \varepsilon \\ &\leq \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_k} J^{(x,t)}(A^+, \rho^-(A^+)) + 2\varepsilon \end{aligned}$$

where we on the last line have used (5.12). Assuming $m \geq k_\varepsilon$, we may choose $k = m$ in the last display and hence we can conclude that given $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon)$ such that if $m \geq m_0$, then

$$\begin{aligned} U^-(x,t) &\leq \inf_{\rho^- \in \mathcal{S}_m} \sup_{A^+ \in \mathcal{AC}_m} J^{(x,t)}(A^+, \rho^-(A^+)) + 2\varepsilon \\ &= U^-_m(x,t) + 2\varepsilon, \end{aligned}$$

and this proves (5.11). □

References

[AB10] R. Atar and A. Budhiraja. A stochastic differential game for the inhomogeneous ∞-Laplace equation. *Ann. Probab.*, 38(2):498–531, 2010.

[AB11] R. Atar and A. Budhiraja. On near optimal trajectories for a game associated with the ∞-Laplacian. *Probab. Th. Rel. Fields*, 151(3–4):509–528, 2011.

[BG] A. Banerjee and N. Garofalo. Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations. *arXiv preprint arXiv:1202.5068*, 2012.

[BL08] R. Buckdahn and J. Li. Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations. *SIAM J. Control Optim.*, 47(1): 444–475, 2008.

[CGG91] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[CIL92] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.

[Doe11] K. Does. An evolution equation involving the normalized p-laplacian. *Commun. Pure Appl. Anal.*, 10(1):361–396, 2011.

[EG92] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* CRC Press, Boca Raton, FL, 1992.

[ES91] L. C. Evans and J. Spruck. Motion of level sets by mean curvature I. *J. Differential Geom.*, 33(3):635–681, 1991.

[FS89] W. H. Fleming and P. E. Souganidis. On the existence of value functions of two-player, zero-sum stochastic differential games. *Indiana Univ. Math. J.*, 38(2):293–314, 1989.

[GS] M. Gallmeyer and D. Seppi. Derivative security induced price manipulation. Working paper, Carnegie Mellon University, 2000.

[GGIS91] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. *Indiana Univ. Math. J.*, 40(2):443–470, 1991.

[Gig06] Y. Giga. *Surface evolution equations. A level set approach.*. Volume 99 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 2006.

[Ish95] H. Ishii. On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions. *Funkcial. Ekvac.* 38(1):101–120, 1995.

[JJ12] V. Julin and P. Juutinen. A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation. *Comm. Partial Differential Equations*, 37(5):934–946, 2012.

[JK06] P. Juutinen and B. Kawohl. On the evolution governed by the infinity Laplacian. *Math. Ann.*, 335(4):819–851, 2006.

[KMP12] B. Kawohl, J. J. Manfredi, and M. Parviainen. Solutions of nonlinear PDEs in the sense of averages. *J. Math. Pures Appl. (9)*, 97(2):173–188, 2012.

[Kry76] N. V. Krylov. Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation. *Siberian Math. J.*, 17(2):226–236, 1976. English translation.

[Kry09] N. V. Krylov. *Controlled diffusion processes*, volume 14 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2009.

[KS92] P. Kumar and D. Seppi. Futures manipulation with cash settlement. *J. Finance*, 47(4):1485–1502, 1992.

[Lin12] P. Lindqvist. Regularity of supersolutions. *Lecture Notes in Math.*, 2045:73–131, 2012.

[MPR10] J. J. Manfredi, M. Parviainen, and J.D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM J. Math. Anal.*, 42(5):2058–2081, 2010.

[MPR12] J. J. Manfredi, M. Parviainen, and J.D. Rossi. On the definition and properties of p-harmonious functions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 11(2):215–241, 2012.

[PSSW09] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.* 22:167–210, 2009.

[PS08] Y. Peres, S. Sheffield, Tug-of-war with noise: a game theoretic view of the p-Laplacian. *Duke Math. J.*, 145(1):91–120, 2008.

[P01] C. Pirrong. Manipulation of cash-settled futures contracts. *J. Bus.*, 74(2): 221–244, 2001.

[Swi96] A. Swiech. Another approach to the existence of value functions of stochastic differential games. *J. Math. Anal. Appl.*, 204(3):884–897, 1996.

Kaj Nyström, Department of Mathematics, Uppsala University, S-751 06 Uppsala, Sweden
E-mail address: kaj.nystrom@math.uu.se

Mikko Parviainen, Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland
E-mail address: mikko.j.parviainen@jyu.fi