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# PERRON'S METHOD FOR THE POROUS MEDIUM EQUATION

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ABSTRACT. This work extends Perron's method for the porous medium equation in the slow diffusion case. The main result shows that nonnegative continuous boundary functions are resolutive in a general cylindrical domain.

## 1. INTRODUCTION

The porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0$$

is an important prototype of a nonlinear parabolic equation and it is by now well understood. See the monographs [5], [11] and [14] for more on this topic. However, little is known about the boundary behaviour of solutions in irregular domains and with general boundary values, except for the case  $m = 1$ , when we have the classical heat equation [12]. We shall consider this challenging question. Our main objective is to apply the method introduced by Perron [8] for harmonic functions to this fascinating nonlinear equation. We focus on the slow diffusion case  $m > 1$  in cylindrical domains. For simplicity, we only consider nonnegative and bounded boundary functions, in which case the solutions are nonnegative and bounded as well, by the comparison principle. However, it is of utmost importance to allow solutions to attain the value zero, so that moving boundaries, such as those exhibited by the Barenblatt solution, are not excluded.

We consider the boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial\Omega \times [0, T), \\ u(x, 0) = g(x, 0), \end{cases}$$

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in a bounded open space-time cylinder  $\Omega_T = \Omega \times (0, T)$  in  $\mathbb{R}^{N+1}$ . The precise definitions of the solution and the boundary conditions will be given later. For a given boundary value function  $g$ , Perron's method produces two functions: the upper solution  $\overline{H}_g$  and the lower solution  $\underline{H}_g$  with  $\underline{H}_g \leq \overline{H}_g$ . Our first main result is that the upper and lower Perron solutions are indeed weak solutions of the porous medium equation. However, the upper and lower solutions may still take the wrong boundary values. The construction can be performed not only for space-time cylinders but also for more general domains in  $\mathbb{R}^{N+1}$ .

A central question in this theory is to determine when the upper and lower solutions are the same function. A classical result in this direction is *Wiener's resolutive theorem* for harmonic functions: if the boundary value function is continuous, the upper and lower Perron solutions coincide, see [13]. Our second main result extends this result to the porous medium equation. More precisely, nonnegative continuous boundary functions are resolutive for the porous medium equation in general cylindrical domains in the slow diffusion case  $m > 1$ . No regularity assumptions on the base of the space-time cylinder are needed. As far as we know, the corresponding result for more general domains in  $\mathbb{R}^{N+1}$  remains open.

Perron's method requires a parabolic comparison principle so that the upper and lower Perron solutions can be defined consistently. Our first step is to establish a comparison principle in general space-time cylinders. To prove the resolutive theorem we first reduce the situation to smooth boundary values by approximation. The key step in the proof for smooth boundary values is constructing super- and subsolutions which are sufficiently regular in the time direction. We use a penalized problem related to the obstacle problem for the porous medium equation for this purpose, see [3]. Delicate approximation results and energy estimates play a pivotal role in the argument. We hope that these results will have other applications as well.

## 2. WEAK SOLUTIONS AND WEAK SUPERSOLUTIONS

In this section, we discuss notion on which the construction of Perron solutions will be based. First, we introduce some notation.

Let  $\Omega$  be an open and bounded subset of  $\mathbb{R}^N$ , and let  $0 < t_1 < t_2 < T$ . We denote space-time cylinders by  $\Omega_T = \Omega \times (0, T)$  and  $U_{t_1, t_2} = U \times (t_1, t_2)$ , where  $U \subset \Omega$  is an open set. We call a cylinder  $U_{t_1, t_2}$  *regular* if the boundary of the base set  $U$  is smooth. The *parabolic boundary* of a space-time cylinder  $U_{t_1, t_2}$  is the set

$$\partial_p U_{t_1, t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times [t_1, t_2]),$$

i.e. only the initial and lateral boundaries are taken into account.

We use the notation  $H^1(\Omega)$  for the Sobolev space consisting of functions  $u$  in  $L^2(\Omega)$  such that the weak gradient exists and also belongs

to  $L^2(\Omega)$ . The Sobolev space with zero boundary values  $H_0^1(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  in  $H^1(\Omega)$ . The parabolic Sobolev space  $L^2(0, T; H^1(\Omega))$  consists of measurable functions  $u : \Omega_T \rightarrow [-\infty, \infty]$  such that  $x \mapsto u(x, t)$  belongs to  $H^1(\Omega)$  for almost all  $t \in (0, T)$ , and

$$\iint_{\Omega_T} (|u|^2 + |\nabla u|^2) \, dx \, dt < \infty.$$

The definition of the space  $L^2(0, T; H_0^1(\Omega))$  is similar. We say that  $u \in L_{loc}^2(0, T; H_{loc}^1(\Omega))$  if  $u$  belongs to the parabolic Sobolev space for all  $U_{t_1, t_2} \Subset \Omega_T$ .

**Definition 2.1.** Assume that  $m > 1$ . A nonnegative function  $u : \Omega_T \rightarrow \mathbb{R}$  is a *weak solution* of the porous medium equation

$$(2.2) \quad \frac{\partial u}{\partial t} - \Delta u^m = 0$$

in  $\Omega_T$ , if  $u^m \in L_{loc}^2(0, T; H_{loc}^1(\Omega))$  and

$$(2.3) \quad \iint_{\Omega_T} \left( -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \right) \, dx \, dt = 0$$

for all smooth test functions  $\varphi$  compactly supported in  $\Omega_T$ . We define *weak supersolutions* by requiring that the integral in (2.3) is nonnegative for nonnegative test functions  $\varphi$ .

Throughout the work we assume that  $m > 1$ , and consider only nonnegative solutions. However, it is important to allow solutions to attain the value zero. We refer to [11] for more on the porous medium equation, and numerous further references. Weak solutions are locally Hölder continuous after a possible redefinition on a set of  $(N + 1)$ -dimensional measure zero; see [4, 6, 14] or [11, Chapter 7]. Thus, without loss of generality, we may assume that solutions are continuous. Moreover, weak supersolutions are lower semicontinuous after a redefinition on a set of  $(N + 1)$ -dimensional measure zero, see [1].

Besides a local notion of weak solutions, we need a concept of weak solutions to the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial\Omega \times [0, T), \\ u(x, 0) = g(x, 0), \end{cases}$$

where  $g$  is a positive, continuous function defined on  $\bar{\Omega}_T$  with  $g^m \in L^2(0, T; H^1(\Omega))$ . The lateral boundary condition is interpreted in the Sobolev sense, meaning that  $u^m - g^m \in L^2(0, T; H_0^1(\Omega))$ . The initial condition is incorporated into the weak formulation by requiring that

$$(2.4) \quad \iint_{\Omega_T} \left( -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \right) \, dx \, dt = \int_{\Omega} g(x, 0) \varphi(x, 0) \, dx$$

for smooth test functions  $\varphi$  vanishing at the time  $T$  and with compact support in space. With this definition, solutions to the initial-boundary value problem are unique. This follows by an application of Oleĭnik's test function, see the proof of Lemma 3.2 below. It is straightforward to check that the initial values are attained in this sense if and only if

$$(2.5) \quad \lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \eta(x) \, dx = \int_{\Omega} g(x, 0) \eta(x) \, dx$$

for all  $\eta \in C_0^\infty(\Omega)$ .

### 3. VISCOSITY SUPERSOLUTIONS

We will employ the notion of *viscosity supersolutions* to (2.2), following [7]. The term ‘‘viscosity’’ is used here purely as a label. In the case  $m = 1$  this definition gives supertemperatures, see [12].

**Definition 3.1.** A function  $u : \Omega_T \rightarrow [0, \infty]$  is a *viscosity supersolution*, if

- (1)  $u$  is lower semicontinuous,
- (2)  $u$  is finite in a dense subset of  $\Omega_T$ , and
- (3) the following comparison principle holds: Let  $U_{t_1, t_2} \Subset \Omega$ , and let  $h$  be a solution to (2.2) which is continuous in  $\overline{U_{t_1, t_2}}$ . If  $h \leq u$  on  $\partial_p U_{t_1, t_2}$ , the  $h \leq u$  in  $U_{t_1, t_2}$ .

The definition of *viscosity subsolutions* is similar; they are upper semicontinuous, and the inequalities in the comparison principle are reversed.

Observe that these functions are defined at *every* point. A similar definition was introduced by F. Riesz [9] for the Laplacian. The fundamental example of a viscosity supersolution in the sense of Definition 3.1 is the Barenblatt solution [2, 15], which is given by the formula

$$\mathcal{B}_m(x, t) = \begin{cases} t^{-\lambda} \left( C - \frac{\lambda(m-1)}{2mn} \frac{|x|^2}{t^{2\lambda/n}} \right)_+^{1/(m-1)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where

$$\lambda = \frac{n}{n(m-1) + 2}.$$

The constant  $C$  is usually chosen so that

$$\int_{\Omega} \mathcal{B}_m(x, t) \, dx = 1$$

for all  $t > 0$ . It is a viscosity supersolution, but not a weak supersolution. This is due to the lack of integrability of the gradient. For other examples, see [11]. However, *bounded* viscosity supersolutions are weak supersolutions. In particular, their  $m$ th power belongs to  $L_{loc}^2(0, T; H_{loc}^1(\Omega))$ . Weak supersolutions are viscosity supersolutions, provided that they are lower semicontinuous, see [1] and [7]. In the

present work, it is enough for the reader to consider lower semicontinuous weak supersolutions that are defined at each point in their domain.

Some properties are immediate consequences of the definition. The pointwise minimum of a finite number of viscosity supersolutions is a viscosity supersolution. In particular, the truncations  $\min\{u, k\}$ ,  $k = 1, 2, \dots$ , of a viscosity supersolution  $u$  are viscosity supersolutions. The fact that an increasing limit of viscosity supersolutions is a viscosity supersolution, provided that the limit is finite in a dense subset, also follows directly from the definition.

Our main interest is Perron solutions with continuous boundary values in irregular domains. In this context, the situation does not change if one only considers bounded viscosity super- and subsolutions, and we shall do so from now on.

We begin with the definition of the *Poisson modification* of a viscosity supersolution. Let  $u$  be a bounded viscosity supersolution and  $U_{t_1, t_2} \Subset \Omega_T$  be a regular space-time cylinder. We define

$$P(u, U_{t_1, t_2}) = \begin{cases} u & \text{in } \Omega_T \setminus U_{t_1, t_2}, \\ h & \text{in } U_{t_1, t_2}, \end{cases}$$

where  $h$  is the solution in  $U_{t_1, t_2}$  with boundary values  $u$ . The function  $h$  is constructed as follows: by semicontinuity, we find an increasing sequence  $(\varphi_k)$  of smooth functions converging to  $u$  pointwise in  $U_{t_1, t_2}$  as  $k \rightarrow \infty$ . Let  $h_k$  be the solution with values  $\varphi_k$  on the parabolic boundary of  $U_{t_1, t_2}$ . Then  $h_k \leq u$  and the sequence  $(h_k)$  is increasing by the comparison principle. It follows that  $h = \lim h_k$  is a solution in  $U_{t_1, t_2}$ . Further, it is easy to verify that  $P(u, U_{t_1, t_2})$  is a viscosity supersolution in  $\Omega_T$ , see [7, pp. 157–158].

We need an auxiliary result to bypass the fact that we may not add constants to solutions.

**Lemma 3.2.** *Assume that  $g \in C(\overline{\Omega_T})$  is a function such that  $g^m \in L^2(0, T; H^1(\Omega))$  and  $0 \leq g \leq M$ . Define the function  $g_\varepsilon$  by*

$$g_\varepsilon = (g^m + \varepsilon^m)^{1/m},$$

where  $0 \leq \varepsilon \leq 1$ . Let  $u$  and  $u_\varepsilon$  be the unique solutions to the boundary value problems

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = 0 & \text{in } \Omega_T, \\ u^m - g^m \in L^2(0, T; H_0^1(\Omega)), \\ u(x, 0) = g(x, 0), \end{cases}$$

and

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon^m = 0 & \text{in } \Omega_T, \\ u_\varepsilon^m - g_\varepsilon^m \in L^2(0, T; H_0^1(\Omega)), \\ u_\varepsilon(x, 0) = g_\varepsilon(x, 0), \end{cases}$$

respectively. Then we have

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m) \, dx \, dt \leq \varepsilon^m |\Omega_T| (M + 1) + \varepsilon |\Omega_T| (M + 1)^m.$$

*Proof.* We use the so-called Oleĭnik's test function. The function  $u_\varepsilon^m - u^m - \varepsilon^m$  has zero boundary values on the lateral boundary in Sobolev's sense. Thus

$$\eta(x, t) = \begin{cases} \int_t^T (u_\varepsilon^m - u^m - \varepsilon^m) \, ds, & 0 < t < T, \\ 0, & t \geq T, \end{cases}$$

is an admissible test function for the equations satisfied by  $u$  and  $u_\varepsilon$ . This gives

$$\iint_{\Omega_T} \left( -u \frac{\partial \eta}{\partial t} + \nabla u^m \cdot \nabla \eta \right) \, dx \, dt = \int_{\Omega} u(x, 0) \eta(x, 0) \, dx,$$

and

$$\iint_{\Omega_T} \left( -u_\varepsilon \frac{\partial \eta}{\partial t} + \nabla u_\varepsilon^m \cdot \nabla \eta \right) \, dx \, dt = \int_{\Omega} u_\varepsilon(x, 0) \eta(x, 0) \, dx.$$

Since we have

$$\eta_t = -(u_\varepsilon^m - u^m) + \varepsilon^m \quad \text{and} \quad \nabla \eta = \int_t^T \nabla (u_\varepsilon^m - u^m) \, ds,$$

we obtain

$$\begin{aligned} & \iint_{\Omega_T} \left( (u_\varepsilon - u)(u_\varepsilon^m - u^m - \varepsilon^m) \right. \\ & \quad \left. + \nabla (u_\varepsilon^m - u^m) \cdot \int_t^T \nabla (u_\varepsilon^m - u^m) \, ds \right) \, dx \, dt \\ & = \int_{\Omega} (g_\varepsilon(x, 0) - g(x, 0)) \left( \int_0^T (u_\varepsilon^m - u^m - \varepsilon^m) \, ds \right) \, dx \end{aligned}$$

by subtracting the equations. Integration with respect to the variable  $t$  shows that the triple integral equals to

$$\frac{1}{2} \int_{\Omega} \left( \int_0^T (\nabla u_\varepsilon^m - \nabla u^m) \, ds \right)^2 \, dx,$$

which is a positive quantity. Thus we get the estimate

$$\begin{aligned} & \iint_{\Omega_T} (u - u_\varepsilon)(u^m - u_\varepsilon^m) \, dx \, dt \leq \varepsilon^m \iint_{\Omega_T} (u_\varepsilon - u) \, dx \, dt \\ & \quad + \int_{\Omega} (g_\varepsilon(x, 0) - g(x, 0)) \left( \int_0^T (u_\varepsilon^m - u^m) \, ds \right) \, dx \\ & \quad - \varepsilon^m T \int_{\Omega} (g_\varepsilon(x, 0) - g(x, 0)) \, dx. \end{aligned}$$

The last term on the right-hand side is negative, since  $g_\varepsilon \geq g$ , and we simply discard it. Furthermore, by the definition of  $g_\varepsilon$ , we have

$$g_\varepsilon - g = (g^m + \varepsilon^m)^{1/m} - g \leq \varepsilon$$

and, by the maximum principle, we conclude that  $u \leq M$  and  $u_\varepsilon \leq M + 1$ . The required estimate follows, since

$$u_\varepsilon - u \leq M + 1 \quad \text{and} \quad u_\varepsilon^m - u^m \leq (M + 1)^m. \quad \square$$

We conclude by a comparison principle between viscosity sub- and supersolutions. The essential feature here is that the base  $\Omega$  of the space-time cylinder  $\Omega_T$  may be an arbitrary bounded open set. For the comparison principle in regular cylinders, see [5, pp. 10-12] or [11, pp. 132-134].

**Theorem 3.3** (Comparison Principle). *Let  $u$  be a bounded viscosity subsolution and  $v$  a bounded viscosity supersolution such that*

$$\limsup_{z \rightarrow z_0} u \leq \liminf_{z \rightarrow z_0} v$$

for all  $z_0 \in \partial_p \Omega_T$ . Then  $u \leq v$  in  $\Omega_T$ .

*Proof.* Let  $\varepsilon_j = 1/j$ ,  $j = 1, 2, 3, \dots$ . By semicontinuity, we can find regular cylinders  $Q_j = U_j \times (t_j, T)$ , with  $U_j \Subset \Omega$ , such that

$$u^m \leq v^m + \varepsilon_j^m \quad \text{in} \quad \Omega_T \setminus Q_j.$$

Let  $w_j$  be the solution in  $Q_j$  with boundary values given by  $v$  on  $\partial_p Q_j$ , and let  $\tilde{w}_j$  be the solution with boundary values  $(v^m + \varepsilon_j^m)^{1/m}$  on  $\partial_p Q_j$ . Define the functions  $h_j$  and  $\tilde{h}_j$  by

$$h_j = \begin{cases} v & \text{in } \Omega_T \setminus Q_j, \\ w_j & \text{in } Q_j, \end{cases}$$

and

$$\tilde{h}_j = \begin{cases} (v^m + \varepsilon_j^m)^{1/m} & \text{in } \Omega_T \setminus Q_j, \\ \tilde{w}_j & \text{in } Q_j. \end{cases}$$

Recall that  $Q_j$  is a regular cylinder. An application of the comparison principle on  $Q_j$  shows that

$$h_j \leq v \quad \text{and} \quad u \leq \tilde{h}_j \quad \text{in } \Omega_T.$$

Now

$$0 \leq (u - v)_+ (u^m - v^m)_+ \leq \begin{cases} (\tilde{h}_j - h_j)(\tilde{h}_j^m - h_j^m) & \text{in } Q_j, \\ \varepsilon_j^m (u - v)_+ & \text{in } \Omega_T \setminus Q_j. \end{cases}$$

We integrate this estimate, apply Lemma 3.2 and let  $j \rightarrow \infty$  to get

$$\iint_{\Omega_T} (u - v)_+ (u^m - v^m)_+ \, dx \, dt = 0.$$

The claim follows.  $\square$



## 4. PERRON SOLUTIONS

The following definition of Perron solutions is based on the comparison principle.

**Definition 4.1.** Let  $g : \partial_p \Omega_T \rightarrow \mathbb{R}$  be given. The *upper class*  $\mathfrak{U}_g$  consists of the viscosity supersolutions  $v$  which are locally bounded from below, and satisfy

$$\liminf_{z \rightarrow \xi} v(z) \geq g(\xi)$$

for all  $\xi \in \partial_p \Omega_T$ . The *upper Perron solution* is defined as

$$\overline{H}_g(z) = \inf_{v \in \mathfrak{U}_g} v(z).$$

The *lower class*  $\mathfrak{L}_g$  consists of all viscosity subsolutions  $u$  that are locally bounded from above, and satisfy

$$\limsup_{z \rightarrow \xi} u(z) \leq g(\xi)$$

for all  $\xi \in \partial_p \Omega_T$ . The *lower Perron solution* is

$$\underline{H}_g(z) = \sup_{u \in \mathfrak{L}_g} u(z).$$

If there exists a function  $h \in C(\overline{\Omega}_T)$  solving the boundary value problem in the classical sense, then

$$h = \overline{H}_g = \underline{H}_g.$$

To see this, simply note that the function  $h$  belongs to both the upper class and the lower class. As we will see, both  $\overline{H}_g$  and  $\underline{H}_g$  are solutions to the equation. It should be noticed that even when the solutions coincide, they may attain wrong boundary values.

An immediate consequence of the comparison principle (Theorem 3.3) is that if  $u \in \mathfrak{L}_g$  and  $v \in \mathfrak{U}_g$ , then  $u \leq v$ . Thus

$$(4.2) \quad \underline{H}_g \leq \overline{H}_g$$

for bounded boundary functions  $g$ . A central issue in this theory is the question about when  $\underline{H}_g = \overline{H}_g$ . If this happens, the boundary function  $g$  is called *resolutive* and we denote the common function by  $H_g$ .

So far, the domain was a space-time cylinder  $\Omega_T$ . The definition of upper and lower Perron solutions given above makes sense in an arbitrary bounded open set  $\Upsilon$  in  $\mathbb{R}^{N+1}$ . Further, Lemma 4.3 and Theorem 4.6 below continue to hold, since their proofs are purely local. However, a comparison principle with the boundary values taken over the whole topological boundary of  $\Upsilon$  is not known for the porous medium equation. In particular, we do not know whether (4.2) remains true in this generality.

Before addressing the resolutivity question in the next section, we establish some basic properties of the lower and upper Perron solutions.

**Lemma 4.3.** *If  $g$  is bounded, then  $\overline{H}_g$  and  $\underline{H}_g$  are continuous in  $\Omega_T$ .*

*Proof.* We prove the claim for  $\overline{H}_g$ , the other case being similar. Take cylinders  $U_{t_1, t_2} \Subset V_{\sigma_1, \sigma_2} \Subset \Omega_T$ , and points  $z_1, z_2 \in U_{t_1, t_2}$ . Given a positive number  $\varepsilon$ , we will show that

$$\overline{H}_g(z_1) - \overline{H}_g(z_2) < 2\varepsilon,$$

provided that  $U_{t_1, t_2}$  is sufficiently small. We can find functions  $v_i^1$  and  $v_i^2$  from the upper class such that

$$\lim_{i \rightarrow \infty} v_i^1(z_1) = \overline{H}_g(z_1) \quad \text{and} \quad \lim_{i \rightarrow \infty} v_i^2(z_2) = \overline{H}_g(z_2).$$

Then also  $v_i = \min\{v_i^1, v_i^2\}$  is in the upper class, and we have

$$\lim_{i \rightarrow \infty} v_i(z_1) = \overline{H}_g(z_1) \quad \text{and} \quad \lim_{i \rightarrow \infty} v_i(z_2) = \overline{H}_g(z_2).$$

Let

$$w_i = P(v_i, V_{\sigma_1, \sigma_2}) \in \mathfrak{U}_g.$$

Then  $\overline{H}_g \leq w_i \leq v_i$ , and we have

$$v_i(z_1) < \overline{H}_g(z_1) + \varepsilon \quad \text{and} \quad v_i(z_2) < \overline{H}_g(z_2) + \varepsilon$$

for sufficiently large  $i$ . From the above facts and the local Hölder continuity of  $w_i$ , it follows that

$$\begin{aligned} \overline{H}_g(z_1) - \overline{H}_g(z_2) &\leq w_i(z_1) - w_i(z_2) + \varepsilon \\ &\leq \operatorname{osc}_{U_{t_1, t_2}} w_i + \varepsilon \leq 2\varepsilon \end{aligned}$$

by choosing  $U_{t_1, t_2}$  in a suitable way. Observe that the assumption on the boundedness of  $g$  implies that the modulus of continuity of  $w_i$  is independent of  $i$ . By exchanging the roles of  $z_1$  and  $z_2$ , we have

$$|\overline{H}_g(z_1) - \overline{H}_g(z_2)| \leq 2\varepsilon,$$

which completes the proof.  $\square$

To prove that Perron solutions are indeed weak solutions to the porous medium equation, we need some auxiliary results. The first of them is a Caccioppoli estimate. The proof can be found in [7, Lemma 2.15].

**Lemma 4.4.** *Let  $u$  be a weak supersolution in  $\Omega_T$  such that  $u^m \in L^2(0, T; W^{1,2}(\Omega))$  and  $0 \leq u \leq M$ . Then*

$$\iint_{\Omega_T} \eta^2 |\nabla u^m|^2 \, dx \, dt \leq 16M^{2m} T \int_{\Omega} |\nabla \eta|^2 \, dx + 6M^{m+1} \int_{\Omega} \eta^2 \, dx$$

for all nonnegative functions  $\eta \in C_0^\infty(\Omega)$ .

The preceding lemma implies a convergence result in a straightforward manner, see for example [7, Proof of Theorem 3.2].

**Proposition 4.5.** *Let  $0 \leq u_j \leq M$ ,  $j = 1, 2, \dots$ , be weak solutions that converge pointwise almost everywhere to a function  $u$ . Then  $u$  is also a weak solution.*

**Theorem 4.6.** *If  $g$  is bounded,  $\overline{H}_g$  and  $\underline{H}_g$  are weak solutions in  $\Omega_T$ .*

*Proof.* We give the proof for  $\overline{H}_g$ , the case of  $\underline{H}_g$  being again symmetrical. Let  $q_n$ ,  $n = 1, 2, 3, \dots$ , be an enumeration of the points in  $\Omega_T$  with rational coordinates. The first aim is to construct functions in the upper class converging to  $\overline{H}_g$  at the points  $q_n$ . To accomplish this, let  $v_i^n \in \mathfrak{U}_g$  be such that

$$\overline{H}_g(q_n) \leq v_i^n(q_n) < \overline{H}_g(q_n) + \frac{1}{i}, \quad i = 1, 2, 3, \dots,$$

and define

$$w_i = \min\{v_1^1, v_2^1, \dots, v_i^1, v_1^2, v_2^2, \dots, v_i^2, \dots, v_1^i, v_2^i, \dots, v_i^i\}.$$

Then  $w_i \in \mathfrak{U}_g$ ,  $w_1 \geq w_2 \geq w_3 \dots$ , and

$$\overline{H}(q_n) \leq w_i(q_n) \leq v_i^n(q_n),$$

for  $i \geq n$ . It follows that

$$\lim_{i \rightarrow \infty} w_i(q_n) = \overline{H}(q_n)$$

at each point  $q_n$ . Let  $U_{t_1, t_2} \Subset \Omega_T$  be an arbitrary regular cylinder and denote

$$W_i = P(w_i, U_{t_1, t_2}).$$

Then  $\overline{H}_g \leq W_i \leq w_i$ , the sequence  $(W_i)$  is decreasing, and its limit  $W$  is a solution in  $U_{t_1, t_2}$  by Proposition 4.5. At every point  $q_n$  we have

$$W(q_n) = \lim_{i \rightarrow \infty} W_i(q_n) = \overline{H}(q_n).$$

Both  $W$  and  $\overline{H}_g$  are continuous in  $U_{t_1, t_2}$ , and they coincide on a dense subset; hence they must coincide everywhere. Since  $W$  is a solution in  $U_{t_1, t_2}$ , so is  $\overline{H}_g$ . The property of being a solution is local, so the proof is complete.  $\square$

## 5. RESOLUTIVITY

The following theorem is our main result. It states that *continuous functions are resolute*.

**Theorem 5.1** (Resolutivity). *If  $g : \partial_p \Omega_T \rightarrow \mathbb{R}$  is continuous, then*

$$\overline{H}_g = \underline{H}_g.$$

To prove the resolutivity theorem, by approximation we first reduce the situation to smooth boundary values. For smooth boundary values, we need to construct functions belonging to the upper class  $\mathfrak{U}_g$  that are sufficiently smooth in time and attain the correct boundary and initial values in the weak sense. We do this by solving a penalized

equation. For this purpose, assume the function  $g$  to be continuously differentiable in  $\overline{\Omega}_T$  and such that  $g^m \in C^2(\overline{\Omega}_T)$ . Then

$$\Psi = \frac{\partial g}{\partial t} - \Delta g^m$$

is bounded. We will use the positive part  $\Psi_+ = \max\{\Psi, 0\}$  below. Choose a number  $\delta > 0$ , and let  $\zeta_\delta : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz function such that  $0 \leq \zeta_\delta(s) \leq 1$ ,  $\zeta_\delta(s) = 1$  for all  $s \geq 0$ ,  $\zeta_\delta(s) = 0$  for all  $x \leq -\delta$ , and  $|\zeta'_\delta(s)| \leq 2/\delta$ . We have the following existence result, see [3].

**Proposition 5.2.** *Let  $g$  be continuously differentiable in  $\overline{\Omega}_T$  and such that  $g^m \in C^2(\overline{\Omega}_T)$ . Then there exists a bounded weak solution  $u$  such that  $u^m \in L^2(0, T; H^1(\Omega))$  to the boundary value problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = \zeta_\delta(g^m - u^m)\Psi_+ & \text{in } \Omega_T, \\ u^m - g^m \in L^2(0, T; H_0^1(\Omega)), \\ u(x, 0) = g(x, 0), \end{cases}$$

satisfying the inequality  $u \geq g$  in  $\Omega_T$ .

*Remark 5.3.* In the proof of Theorem 5.1 below, we need to choose approximations of a given continuous function  $g$  so that the smoothness assumptions of Proposition 5.2 hold. This is accomplished by approximating a suitable smaller power  $g^\alpha$ ,  $\alpha \leq 1$ , of the function rather than the function  $g$  itself. Indeed, we may express the derivatives of the powers one and  $m$  in terms of the derivatives of the power  $\alpha$ . Some simple calculations show that the choice  $\alpha = \min\{1, m/2\}$  will do.

*Remark 5.4.* Due to our assumption that the boundary values are positive, the roles of upper and lower solutions in the proof of Theorem 5.1 are not quite symmetric. For subsolutions, we need a version of Proposition 5.2 where the solutions can change sign. See pp. 97-100 in [11] for the modifications needed to the arguments in [3].

We need an energy estimate for the time derivative of a solution to the the above equation. For similar results, see [10, Proposition 13] and [11, Section 3.2.5].

**Theorem 5.5.** *Assume that  $f \in L^\infty(\Omega_T)$ . Let  $u$  be a bounded weak solution to the boundary value problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = f & \text{in } \Omega_T, \\ u^m - g^m \in L^2(0, T; H_0^1(\Omega)), \\ u(x, 0) = g(x, 0). \end{cases}$$

Then

$$\frac{\partial u^{(m+1)/2}}{\partial t} \in L^2(\Omega_T),$$

with the estimate

$$\begin{aligned}
& \iint_{\Omega_T} \left| \frac{\partial u^{(m+1)/2}}{\partial t} \right|^2 dx dt + \int_{\Omega} |\nabla(u^m - g^m)|^2(x, T) dx \\
& \leq c \left( \int_{\Omega} |(g^m)_t(x, T)u(x, T) - (g^m)_t(x, 0)g(x, 0)| dx \right. \\
& \quad + \iint_{\Omega_T} (|u|^2 + u^{m-1}(|f|^2 + |\Delta g^m|^2)) dx dt \\
& \quad \left. + \iint_{\Omega_T} (|(g^m)_t|^2 + |\Delta g^m|^2 + |(g^m)_{tt}|^2 + |f|^2) dx dt \right).
\end{aligned}$$

Furthermore, we have

$$\frac{\partial u^q}{\partial t} \in L^2(\Omega_T)$$

for any  $q \geq (m+1)/2$ .

*Proof.* First we assume that  $u$  is smooth in  $\bar{\Omega}_T$ . This assumption may be removed by a standard approximation argument, see [11, Proof of Theorem 5.5] and [14, Section 1.3.2]. Denote  $w = u^{(m+1)/2}$ . Then

$$\begin{aligned}
\left| \frac{\partial w}{\partial t} \right|^2 &= \frac{(m+1)^2}{4} u^{m-1} |u_t|^2 \\
&= \frac{(m+1)^2}{4m} (u^m)_t u_t = \frac{(m+1)^2}{4m} ((u^m - g^m)_t u_t + (g^m)_t u_t) \\
&= \frac{(m+1)^2}{4m} ((u^m - g^m)_t (\Delta u^m - \Delta g^m) \\
& \quad + (g^m)_t u_t + \Delta g^m (u^m - g^m)_t + f(u^m - g^m)_t).
\end{aligned}$$

We focus on the first term after the last equality. To this end, we note that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u^m - g^m|^2 dx &= \int_{\Omega} \nabla(u^m - g^m) \cdot \nabla(u^m - g^m)_t dx \\
&= - \int_{\Omega} \Delta(u^m - g^m)(u^m - g^m)_t dx,
\end{aligned}$$

since  $u^m - g^m$  has zero boundary values on the lateral boundary. Thus an integration gives

$$\begin{aligned}
 \iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 dx dt &= -\frac{(m+1)^2}{8m} \int_0^T \frac{d}{dt} \int_{\Omega} |\nabla(u^m - g^m)|^2 dx dt \\
 &+ \frac{(m+1)^2}{4m} \iint_{\Omega_T} ((g^m)_t u_t + (u^m - g^m)_t \Delta g^m + f(u^m - g^m)_t) dx dt \\
 &= -\frac{(m+1)^2}{8m} \int_{\Omega} |\nabla(u^m - g^m)|^2 dx \Big|_0^T \\
 &+ \frac{(m+1)^2}{4m} \left( \iint_{\Omega_T} -(g^m)_{tt} u dx dt + \int_{\Omega} (g^m)_t u dx \Big|_0^T \right) \\
 &+ \frac{(m+1)^2}{4m} \iint_{\Omega_T} ((u^m)_t \Delta g^m - (g^m)_t \Delta g^m - f(g^m)_t + (u^m)_t f) dx dt.
 \end{aligned}$$

To proceed, we compute

$$\frac{\partial u^m}{\partial t} = u^{m-(m+1)/2} \frac{\partial w}{\partial t},$$

and apply Young's inequality to the two terms containing the time derivative of  $u^m$  to get

$$\iint_{\Omega_T} |(u^m)_t \Delta g^m| dx dt \leq \varepsilon \iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 dx dt + c_{\varepsilon} \iint_{\Omega_T} u^{m-1} |\Delta g^m|^2 dx dt$$

and

$$\iint_{\Omega_T} |(u^m)_t f| dx dt \leq \varepsilon \iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 dx dt + c_{\varepsilon} \iint_{\Omega_T} u^{m-1} |f|^2 dx dt.$$

We insert these inequalities into the estimate above, choose a sufficiently small  $\varepsilon$ , and absorb the matching terms to get

$$\begin{aligned}
 &\iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 dx dt + \int_{\Omega} |\nabla(u^m - g^m)|^2(x, T) dx \\
 &\leq c \left( \int_{\Omega} |\nabla(u^m - g^m)|^2(x, 0) dx + \iint_{\Omega_T} |(g_{tt})^m u| dx dt \right. \\
 &\quad + \int_{\Omega} |(g^m)_t(x, T)u(x, T) - (g^m)_t(x, 0)u(x, 0)| dx \\
 &\quad + \iint_{\Omega_T} u^{m-1} |\Delta g^m|^2 dx dt + \iint_{\Omega_T} |(g^m)_t \Delta g^m| dx dt \\
 &\quad \left. + \iint_{\Omega_T} |f(g^m)_t| dx dt + \iint_{\Omega_T} u^{m-1} |f|^2 dx dt \right).
 \end{aligned}$$

We recall that  $u^m = g^m$  at the initial time, so the required estimate follows from an application of Cauchy's inequality.  $\square$

**Lemma 5.6.** *Let  $g$  satisfy the smoothness assumptions of Proposition 5.2, and let  $v$  be the solution to the boundary value problem of Proposition 5.2. Let  $u = P(v, D_{t_1, T})$  be the Poisson modification of  $v$  with respect to a regular space-time cylinder  $D_{t_1, T}$  with  $D \Subset \Omega$  and  $0 < t_1 < T$ . Then*

$$\begin{aligned} & \iint_{D_{t_1, T}} |\nabla u^m|^2 \, dx \, dt + \sup_{t_1 < t < T} \int_D u^{m+1}(x, t) \, dx \\ & \leq c \left( \iint_{D_{t_1, T}} \left( |\nabla v^m|^2 + |v|^2 + \left| \frac{\partial v^m}{\partial t} \right|^2 \right) \, dx \, dt \right. \\ & \quad \left. + \sup_{t_1 < t < T} \int_D v^{m+1}(x, t) \, dx \right). \end{aligned}$$

*Proof.* The formal computations below are justified rigorously by a standard application of a suitable mollification in the time direction. Since  $u$  is a solution in  $D_{t_1, T}$  with boundary values given by  $v^m$ , we may use  $u^m - v^m$  as a test function and have

$$(5.7) \quad \iint_{D_{t_1, T}} \left( \frac{\partial u}{\partial t} (u^m - v^m) + \nabla u^m \cdot \nabla (u^m - v^m) \right) \, dx \, dt = 0.$$

The next goal is to eliminate the time derivative of  $u$ . We use the fact that  $(u^{m+1})_t = (m+1)u_t u^m$  and integrate by parts in the other term to get

$$\begin{aligned} & \iint_{D_{t_1, T}} \frac{\partial u}{\partial t} (u^m - v^m) \, dx \, dt \\ & = \frac{1}{m+1} \left( \int_D u^{m+1}(x, T) \, dx - \int_D u^{m+1}(x, t_1) \, dx \right) \\ & \quad + \iint_{D_{t_1, T}} u \frac{\partial v^m}{\partial t} \, dx \, dt \\ & \quad - \int_D u(x, T) v^m(x, T) \, dx + \int_D u(x, t_1) v^m(x, t_1) \, dx. \end{aligned}$$

This leads to the estimate

$$\begin{aligned} 0 & \leq \iint_{D_{t_1, T}} |\nabla u^m|^2 \, dx \, dt + \frac{1}{m+1} \int_D u^{m+1}(x, T) \, dx \\ & \leq \frac{1}{m+1} \int_D v^{m+1}(x, t_1) \, dx + \int_D v^{m+1}(x, T) \, dx - \iint_{D_{t_1, T}} u \frac{\partial v^m}{\partial t} \, dx \, dt \\ & \quad + \iint_{D_{t_1, T}} \nabla u^m \cdot \nabla v^m \, dx \, dt, \end{aligned}$$

since  $u \leq v$ . By Young's inequality, we obtain

$$\begin{aligned} & \iint_{D_{t_1, T}} \nabla u^m \cdot \nabla v^m \, dx \, dt \\ & \leq \varepsilon \iint_{D_{t_1, T}} |\nabla u^m|^2 \, dx \, dt + c_\varepsilon \iint_{D_{t_1, T}} |\nabla v^m|^2 \, dx \, dt. \end{aligned}$$

We insert this into the previous estimate, and absorb the matching terms. We arrive at

$$\begin{aligned} & \int_{D_{t_1, T}} |\nabla u^m|^2 \, dx \, dt + \frac{1}{m+1} \int_D u^{m+1}(x, T) \, dx \\ & \leq c \left( \int_D v^{m+1}(x, t_1) \, dx + \int_D v^{m+1}(x, T) \, dx \right. \\ & \quad \left. + \int_{D_{t_1, T}} |v| \left| \frac{\partial v^m}{\partial t} \right| + \iint_{D_{t_1, T}} |\nabla v^m|^2 \, dx \, dt \right). \end{aligned}$$

The proof is then completed by estimating the term with  $v^{m+1}(x, T)$  by a supremum over time, and by replacing  $T$  by  $t_1 < \tau < T$  such that

$$\int_D v^{m+1}(x, \tau) \, dx \geq \sup_{t_1 < t < T} \frac{1}{2} \int_D v^{m+1}(x, t) \, dx,$$

and applying Young's inequality.  $\square$

*Proof of Theorem 5.1.* By extension, we may assume that  $g$  is defined in the whole of  $\mathbb{R}^{N+1}$ . We first show that it suffices to prove that for smooth boundary values  $g$  that both the upper and lower Perron solution agree with the unique weak solution with boundary and initial values  $g$ , in the sense of (2.4). Let us denote  $\varepsilon_j = 1/j$ ,  $j = 1, 2, \dots$ . There exist functions  $\varphi_j$  satisfying the smoothness assumptions of Proposition 5.2, see Remark 5.3, converging uniformly to  $g$  and such that

$$\varphi_j^m \leq g^m \leq \varphi_j^m + \varepsilon_j^m.$$

Assuming the above conclusion for smooth functions, we get

$$H_{\varphi_j} \leq \underline{H}_g \leq \overline{H}_g \leq H_{(\varphi_j^m + \varepsilon_j^m)^{1/m}}.$$

Since

$$|H_{\varphi_j} - H_{(\varphi_j^m + \varepsilon_j^m)^{1/m}}| \rightarrow 0$$

as  $j \rightarrow \infty$  by Lemma 3.2, it follows that  $\underline{H}_g = \overline{H}_g$  almost everywhere. The conclusion that  $\underline{H}_g = \overline{H}_g$  everywhere follows by continuity of the Perron solutions.

Let us then assume that  $g$  is smooth, and let  $h$  be the unique weak solution with initial and boundary values given by  $g$ , i.e.  $h^m - g^m \in L^2(0, T; H_0^1(\Omega))$  and (2.4) holds. We need to show that  $h \geq \overline{H}_g$ ; the problem is that we do not know whether  $h$  belongs to the upper class



or not. To deal with this, let  $v$  be the solution of the the penalized boundary value problem of Proposition 5.2. Then also

$$v^m - g^m \in L^2(0, T; H_0^1(\Omega)).$$

Exhaust  $\Omega_T$  by an increasing sequence of regular cylinders  $Q_j = U_j \times (t_j, T)$ , and let  $w_j = P(v, Q_j)$ ,  $j = 1, 2, \dots$ . Then  $w_j \in \mathfrak{U}_g$ , the sequence  $(w_j)$  is decreasing, and  $\overline{H}_g \leq w_j$ . The limit function

$$w = \lim_{j \rightarrow \infty} w_j$$

is a solution in  $\Omega_T$ , and

$$w \geq \overline{H}_g$$

since  $w$  is a pointwise limit of functions in the upper class. It remains to show that  $w$  has the boundary and initial values given by  $g$ , since then by the uniqueness of weak solutions, we have

$$h = w \geq \overline{H}_g.$$

To check the lateral boundary values, note that the sequence  $(w_j^m - g^m)$  is bounded in  $L^2(0, T; H_0^1(\Omega))$  by Lemma 5.6. It follows that

$$w^m - g^m \in L^2(0, T; H_0^1(\Omega)),$$

since  $L^2(0, T; H_0^1(\Omega))$  is a closed subspace of  $L^2(0, T; H^1(\Omega))$  and weak limits must agree with pointwise limits.

We use the criterion (2.5) to show that the initial values of the limit function  $w$  are given by the function  $g(x, 0)$ . Let  $\eta \in C_0^\infty(\Omega)$  be arbitrary. Choose a time instant  $0 < t < T$ , and  $j$  large enough, so that  $t_j < t$  and so that the support of  $\eta$  is contained in  $U_j$ . We have

$$\begin{aligned} \left| \int_{\Omega} (w(x, t) - g(x, 0))\eta(x) dx \right| &\leq \left| \int_{\Omega} (w(x, t) - w_j(x, t))\eta(x) dx \right| \\ &+ \left| \int_{\Omega} (w_j(x, t) - w_j(x, t_j))\eta(x) dx \right| + \left| \int_{\Omega} (v(x, t_j) - g(x, 0))\eta(x) dx \right| \end{aligned}$$

by adding and subtracting suitable terms, using the triangle inequality, and the fact that  $w_j(x, t_j) = v(x, t_j)$  on the support of  $\eta$ . The first and third terms on the right tend to zero as  $j \rightarrow \infty$ . To deal with the second term, we formally test the equation satisfied by  $w_j$  with  $\varphi = \eta\chi_{(t_j, t)}$ , where  $\chi_{(t_j, t)}$  is the characteristic function of the interval  $(t_j, t)$ . This can be justified by an approximation argument. We get

$$\left| \int_{\Omega} (w_j(x, t) - w_j(x, t_j))\eta dx \right| = \left| \iint_{U_j \times (t_j, t)} \nabla w_j^m \cdot \nabla \eta dx dt \right|.$$

We estimate the right hand side by Hölder's inequality to get

$$\left| \iint_{U_j \times (t_j, t)} \nabla w_j^m \cdot \nabla \eta dx dt \right| \leq |\Omega|^{1/2} |t - t_j|^{1/2} \|\nabla w_j^m\|_{L^2(\Omega_T)} \|\nabla \eta\|_{L^\infty(\Omega)},$$

where we also used the fact that  $|U_j \times (t_j, t)| \leq |\Omega||t - t_j|$ . Since the norm of  $\nabla w_j^m$  can be controlled independently of  $j$  by applying Lemma 5.6, we may use this estimate for the second term to get

$$\left| \int_{\Omega} (w(x, t) - g(x, 0))\eta(x) \, dx \right| \leq ct \|\nabla \eta\|_{L^\infty(\Omega)}$$

after letting  $j \rightarrow \infty$ . Since  $\eta$  was arbitrary, letting  $t \rightarrow 0$  shows that (2.5) holds for the function  $w$ , as desired.

By a similar argument using the variant of Proposition 5.2 described in Remark 5.4, we see that  $h \leq \underline{H}_g$ , so that

$$h \leq \underline{H}_g \leq \overline{H}_g \leq h,$$

which completes the proof.  $\square$

#### REFERENCES

- [1] B. Avelin and T. Lukkari. Lower semicontinuity of weak supersolutions to the porous medium equation. Preprint. Available at <http://arxiv.org/abs/1312.2820>.
- [2] G. I. Barenblatt. On self-similar motions of a compressible fluid in a porous medium. *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 16:679–698, 1952.
- [3] V. Bögelein, T. Lukkari, and C. Scheven. The obstacle problem for the porous medium equation. In preparation.
- [4] B. E. J. Dahlberg and C. E. Kenig. Nonnegative solutions of the porous medium equation. *Comm. Partial Differential Equations*, 9(5):409–437, 1984.
- [5] P. Daskalopoulos and C. E. Kenig. *Degenerate diffusions – Initial value problems and local regularity theory*, volume 1 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2007.
- [6] E. DiBenedetto and A. Friedman. Hölder estimates for nonlinear degenerate parabolic systems. *J. Reine Angew. Math.*, 357:1–22, 1985.
- [7] J. Kinnunen and P. Lindqvist. Definition and properties of supersolutions to the porous medium equation. *J. Reine Angew. Math.*, 618:135–168, 2008.
- [8] O. Perron. Eine neue Behandlung der ersten Randwertaufgabe für  $\Delta u = 0$ . *Math. Z.*, 18(1):42–54, 1923.
- [9] F. Riesz. Sur les Fonctions Subharmoniques et Leur Rapport à la Théorie du Potentiel. *Acta Math.*, 48(3-4):329–343, 1926.
- [10] J. L. Vázquez. An introduction to the mathematical theory of the porous medium equation. In *Shape optimization and free boundaries (Montreal, PQ, 1990)*, volume 380 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 347–389. Kluwer Acad. Publ., Dordrecht, 1992.
- [11] J. L. Vázquez. *The Porous Medium Equation – Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007.
- [12] N. A. Watson. *Introduction to heat potential theory*, volume 182 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2012.
- [13] N. Wiener. Note on a paper of O. Perron. *J. Math. Phys.*, 4:21–32, 1925.
- [14] Z. Wu, J. Zhao, J. Yin, and H. Li. *Nonlinear diffusion equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. Translated from the 1996 Chinese original and revised by the authors.

- [15] Ya. B. Zel'dovič and A. S. Kompaneec. On the theory of propagation of heat with the heat conductivity depending upon the temperature. In *Collection in honor of the seventieth birthday of academician A. F. Ioffe*, pages 61–71. Izdat. Akad. Nauk SSSR, Moscow, 1950.

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