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REPORT No. 31, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -31-13/14- -SE+fall

SHARP BOUNDEDNESS AND CONTINUITY RESULTS FOR THE SINGULAR POROUS MEDIUM EQUATION

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ABSTRACT. We consider non-homogeneous, singular ($0 < m < 1$) parabolic equations of porous medium type of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = \mu \qquad \text{in } E_T,$$

where E_T is a space time cylinder, and μ is a Radon-measure having finite total mass $\mu(E_T)$. In the range $\frac{(N-2)_+}{N} < m < 1$ we establish sufficient conditions for the boundedness and the continuity of u in terms of a natural Riesz potential of the right-hand side measure μ.

1. INTRODUCTION

In this paper we consider nonlinear parabolic equations whose prototype equation is given by the singular porous medium equation

$$u_t - \Delta u^m = \mu \qquad \text{in } E_T, \tag{1.1}$$

where μ stands in the most general case for a Radon measure with finite total mass $\mu(E_T) < \infty$, and $0 < m < 1$. Here, E is a bounded open set in $\mathbb{R}^N$, $T > 0$ and $N \geq 2$. Further, E_T denotes the space time cylinder $E \times (0,T)$. More generally, we deal with singular parabolic equations of the type

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = \mu \qquad \text{in } E_T, \tag{1.2}$$

where the vector-field $\mathbf{A}\colon E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable with respect to $(x,t) \in E_T$ for all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^N$, continuous with respect to (u,ξ) for a.e. $(x,t) \in E_T$, and moreover satisfies for some $0 < C_o \leq C_1 < \infty$ and $0 < m < 1$ the following growth and ellipticity conditions:

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi \geq mC_o|u|^{m-1}|\xi|^2 \\ |\mathbf{A}(x,t,u,\xi)| \leq mC_1|u|^{m-1}|\xi|, \end{cases} \tag{1.3}$$

whenever $(x,t) \in E_T$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. For the Radon measure μ we may assume without loss of generality that it is extended to $\mathbb{R}^{N+1}$ by letting $\mu\lfloor(\mathbb{R}^{n+1} \setminus E_T) = 0$. Of course, our assumptions cover the prototype equation (1.1) and moreover equations with coefficients of the type

$$u_t - \operatorname{div}\big(\mathbf{a}(x,t)Du^m\big) = \mu \qquad \text{in } E_T, \tag{1.4}$$

where $\mathbf{a} = \mathbf{a}(x,t)$ is a bounded, measurable, uniformly elliptic matrix.

As already mentioned before, throughout the paper we consider the *singular case* $0 < m < 1$. The main goal of our paper is to establish a certain kind of *linear potential*

Date: March 12, 2014.
2010 *Mathematics Subject Classification.* Primary 35K67; Secondary 31B15.
Key words and phrases. Porous medium equation, boundedness, continuity, Riesz potential.

theory for singular porous medium type equations, providing sharp regularity results for the solutions u, such as boundedness and continuity, in terms of a linear Riesz potential of the right-hand side measure μ. It will be established, that the same results, valid for solutions of the heat equation

$$u_t - \Delta u = \mu,$$

hold true also for the singular porous medium equation (1.1), and more generally for parabolic equations as considered in (1.2), with the vector field $\mathbf{A}$ satisfying the structure conditions (1.3).

To describe our results, we need to give the precise definition of weak solutions.

Definition 1.1 (weak energy solution)**.** *A non-negative function* $u\colon E_T \to \mathbb{R}$ *satisfying*

$$u \in C^0_{\rm loc}\big(0,T; L^{1+m}_{\rm loc}(E)\big), \quad u^m \in L^2_{\rm loc}\big(0,T; W^{1,2}_{\rm loc}(E)\big) \tag{1.5}$$

is termed weak energy solution of the singular porous medium type equation (1.2) *if and only if for every subset* $U \Subset E$ *and every subinterval* $[t_1,t_2] \subset (0,T)$ *the following equation*

$$\int_U u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_U \big[-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi\big]\,dxdt = \int_{U\times(t_1,t_2)} \varphi\,d\mu \tag{1.6}$$

holds true for any testing function

$$\varphi \in L^2_{\rm loc}\big(0,T; W^{1,2}_0(U)\big) \quad \textit{with} \quad \varphi, \varphi_t \in L^\infty(U_T).$$

In (1.6) *the symbol* Du *has to be understood in the sense of the following definition:*

$$Du := \tfrac{1}{m} u^{1-m} Du^m\,. \tag{1.7}$$

The assumption that the testing function φ and its time derivative φ_t must be bounded has to be imposed in order to guarantee that the terms involving the time derivative and the right-hand side of (1.6) are well defined. All other integrals appearing there are finite, due to the other assumptions on u and φ.

Even though in the literature (1.2) with $0 < m < 1$ is frequently referred to as *fast diffusion equation*, in the following we will prefer the term *singular porous medium equation*.

To formulate our first main result, we define the localized (or truncated) *parabolic Riesz-potential*. For a point $z_o \in E_T$ and $r, \theta > 0$ such that $Q_{r,\theta}(z_o) \subset E_T$ we set

$$\mathbf{I}^\mu_\beta(z_o, r, \theta) := \int_0^r \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2}(z_o))}{\varrho^{N+2-\beta}}\frac{d\varrho}{\varrho}\,, \qquad \beta \in (0, N+2]\,.$$

Here, $Q_{\varrho,\theta}(z_o) := B_\varrho(x_o) \times (t_o - \theta, t_o)$ stands for a general parabolic cylinder in $\mathbb{R}^{N+1}$; see § 2.1. In the case that $\theta = r^2$ the potential $\mathbf{I}^\mu_\beta$ reduces to the standard localized parabolic Riesz potential

$$\mathbf{I}^\mu_\beta(z_o, r) := \mathbf{I}^\mu_\beta(z_o, r, r^2) \equiv \int_0^r \frac{\mu(Q_{\varrho,\varrho^2}(z_o))}{\varrho^{N+2-\beta}}\frac{d\varrho}{\varrho}\,.$$

With the definition of $\mathbf{I}^\mu_\beta$ at hand, we can state our first main result, a linear potential estimate for weak solutions of the porous medium equation. The precise statement is as follows:

Theorem 1.1 (Riesz potential bound for weak energy solutions)**.** *Let* u *be a non-negative weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field* $\mathbf{A}$ *fulfills the growth and ellipticity conditions* (1.3) *with*

$$\frac{(N-2)_+}{N} < m < 1. \tag{1.8}$$

Then, for any given $\lambda \in \big(0,\ \frac{\beta}{N(1+m)}\big]$, *where* $\beta := 2 - N(1-m)$, *almost every* $z_o \in E_T$, *and every parabolic cylinder* $Q_{4r,\theta}(z_o) \subset E_T$, *the following potential estimate*

$$
\begin{aligned}
u(z_o) \le \boldsymbol{\gamma}\bigg(\frac{1}{\theta^{1+\frac{N}{2}}}\iint_{Q_{r,\theta}} u^{1+\lambda}\,dxdt\bigg)^{\frac{2}{\beta+2\lambda}} \\
+ \boldsymbol{\gamma}\Big(\frac{\theta}{r^2}\Big)^{\frac{1}{1-m}} + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}}
\end{aligned}
\tag{1.9}
$$

holds true with a universal constant $\boldsymbol{\gamma}$ *depending only on* N, m, C_o, C_1 *and* λ.

Some words on Theorem 1.1 are in order. First of all, in the case of the homogeneous porous medium equation, i.e. when $\mu \equiv 0$, from (1.9) one almost recovers the well known sup-estimate for weak energy solutions, as given for example in [10, Proposition B.4.1] (with $r = 1$ and $\lambda_1 = \beta \equiv 2 - N(1-m)$), that is

$$
\sup_{\frac{1}{2}Q_{r,\theta}(z_o)} u \le \boldsymbol{\gamma}\Big(\frac{r^2}{\theta}\Big)^{\frac{N}{\beta}}\bigg(\frac{1}{|Q_{r,\theta}|}\iint_{Q_{r,\theta}} u\,dxdt\bigg)^{\frac{2}{\beta}} + \boldsymbol{\gamma}\Big(\frac{\theta}{r^2}\Big)^{\frac{1}{1-m}}.
$$

We should mention that our estimate differs from the one in [10, Proposition B.4.1] due to the presence of the parameter λ, which in our case can be chosen arbitrarily small, but not equal to zero, since the constant blows up in the limit $\lambda \downarrow 0$. On the other hand, in the homogeneous case $m = 1$, that is, when the equation does not admit a singular structure anymore and $\mu \neq 0$, we arrive almost at the zero order linear Riesz potential estimate from [12] for nonlinear parabolic equations with linear growth in the gradient variable. Again, the only difference appears in the integral

$$
\boldsymbol{\gamma}\bigg[\frac{1}{|Q_r|}\iint_{Q_r(z_o)} u^{1+\lambda}\,dxdt\bigg]^{\frac{1}{1+\lambda}}.
$$

Exactly as before, here the parameter λ can be chosen arbitrarily small, but not equal to zero, since the constant $\boldsymbol{\gamma}$ might blow up as $\lambda \downarrow 0$. However, this phenomenon is in perfect accordance with the zero order nonlinear Wolff potential estimates for the elliptic, respectively parabolic p-Laplacian equation from the seminal paper [16], respectively the parabolic analogue from [24, 26]. In the elliptic case the right-hand side of the potential estimate contains an integral of the form

$$
\boldsymbol{\gamma}\bigg[\frac{1}{|B_r|}\int_{B_r(x_o)} u^{p-1+\lambda}\,dx\bigg]^{\frac{1}{1+\lambda}}
$$

with $\lambda > 0$ sufficiently small. Also here the classical case $p = 2$ cannot be completely recovered by letting $\lambda \downarrow 0$. With this respect, our result is in perfect accordance with the known results in [10, 12, 24, 26].

As an immediate consequence of the potential estimate (1.9) we obtain

Corollary 1.1. *Additionally to the hypotheses of Theorem* 1.1, *assume that*

$$
z \mapsto \mathbf{I}_2(z, r)
$$

is locally bounded in E_T *for some* $r > 0$. *Then,* $u \in L^\infty_{\rm loc}(E_T)$.

Note that the local boundedness of the truncated Riesz potential $\mathbf{I}_2^{\mu}(\cdot, r)$ for some $r > 0$, guarantees the local boundedness of u, and moreover yields $\lim_{\varrho\downarrow 0}\mathbf{I}_2^{\mu}(z, \varrho) = 0$ for a.e. $z \in E_T$. Now, the second main result of this paper is a sufficient criterion ensuring the continuity of solutions. The precise statement is as follows:

Theorem 1.2 (Continuity of weak energy solutions via Riesz potentials)**.** *Let u be a non-negative, locally bounded, weak energy solution of the porous medium equation* (1.2) *in the sense of Definition* 1.1*, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity conditions* (1.3) *and m is as in* (1.8)*. Furthermore, consider $E_o \Subset E_T$ and assume that*

$$\lim_{r\downarrow 0} \sup_{z_o\in E_o} \mathbf{I}_2^\mu(z_o, r) = 0 \tag{1.10}$$

holds true. Then, u is continuous in E_o.

The notable point in the previous results is the assertion that local boundedness of the truncated Riesz potential implies the local boundedness of weak energy solutions, while locally uniform convergence to zero of the truncated Riesz potential implies the continuity of weak energy solutions. With this respect, our result is of borderline type. It is surprising that the Riesz potential $\mathbf{I}_2^\mu$ plays the same role as in the linear setting, although such a fact is by now well known; in [13] a related result for stationary p-Laplacian type systems concerning the characterization of the continuity of the gradient in terms of a non-linear Wolff potential can be found. The results specialize in the case $p = 2$ to a characterization in terms of a linear Riesz potential, although the equations considered are non-linear. A similar phenomenon, again in terms of linear Riesz potentials, concerning evolutionary p-Laplacian type equations with respect to gradient continuity has been observed in [22].

In the present paper we state the results, i.e. Theorems 1.1 and 1.2, in form of apriori estimates. However, they also apply to the very weak solutions u of the prototype equation (1.1) constructed by Lukkari in [28]. In a forthcoming paper [4], we will show how the results of [28] can be generalized to porous medium type equations with coefficients, so that Theorem 1.1 applies to very weak solutions of these more general fast diffusion equations with measure data (and possibly Theorem 1.2 as well).

Finally, we test our results against the *Barenblatt fundamental solution*

$$\mathcal{B}_m(x,t) := \begin{cases} \dfrac{1}{t^{\frac{N}{\beta}}}\left[1 + b\left(\dfrac{|x|}{t^{\frac{1}{\beta}}}\right)^2\right]_+^{\frac{1}{m-1}}, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{1.11}$$

where $\frac{(N-2)_+}{N} < m < 1$ and

$$\beta = N(m-1) + 2 \quad \text{and} \quad b = b(N,m) = \frac{N(1-m)}{2Nm\beta}\,. \tag{1.12}$$

It is well known, that the Barenblatt fundamental solution is the very weak solution in $\mathbb{R}^N \times [-\varepsilon, +\infty)$, $\varepsilon > 0$ of the porous medium equation

$$\partial_t u - \Delta u^m = \delta_{(0,0)},$$

where $\delta_{(0,0)}$ is the delta-function at the origin. The uniqueness of the solution of the porous when the *initial datum* is a locally integrable function has been proved in [14], and it has been extended to measures in [8] (see also [9, Chapter 3]). We claim that the potential estimate (1.9) from Theorem 1.1 shows the correct decay at the origin. By its very definition, the Riesz potential can be bounded from above by

$$\mathbf{I}_2^\mu(z_o, r, \theta) \le \int_0^\infty \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2}(z_o))}{\varrho^N}\frac{d\varrho}{\varrho}.$$

For the sake of simplicity, in the following we will write u instead of $\mathcal{B}_m$. Consider a point $z_o = (0, t_o)$ with $t_o > 0$ (note that $u(0,t_o) > 0$), and let $\theta := [u(0,t_o)]^{1-m} r^2$, which is the correct intrinsic scaling for the porous medium equation. In order for the cylinder $Q_{\varrho,\varrho^2\theta/16r^2}(z_o)$ to contain the origin, we need to have that

$$\varrho^2 \frac{[u(0,t_o)]^{1-m} r^2}{16r^2} \geq t_o \qquad \Longleftrightarrow \qquad \varrho \geq 4\sqrt{t_o[u(0,t_o)]^{m-1}}.$$

Consequently, we have

$$\begin{aligned}\mathbf{I}_2^{\delta_o}(z_o,4r,\theta) &\leq \int_{4\sqrt{t_o[u(0,t_o)]^{m-1}}}^{\infty} \frac{\mu(Q_{\varrho,\varrho^2\theta/16r^2}(z_o))}{\varrho^N}\frac{d\varrho}{\varrho} \\ &= \int_{4\sqrt{t_o[u(0,t_o)]^{m-1}}}^{\infty} \frac{d\varrho}{\varrho^{N+1}} = \frac{1}{4^N N t_o^{\frac{N}{2}}[u(0,t_o)]^{\frac{(m-1)N}{2}}}.\end{aligned}$$

Now, if we limit ourselves to consider only the bound from above that comes from the Riesz potential, namely

$$u(z_o) \leq \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o,4r,\theta)^{\frac{2}{\beta}},$$

we obtain that

$$u(0,t_o) \leq \boldsymbol{\gamma}\left(\frac{r^2}{[u(0,t_o)]^{1-m}r^2}\right)^{\frac{N}{\beta}}\left(\frac{1}{t_o^{\frac{N}{2}}[u(0,t_o)]^{\frac{(m-1)N}{2}}}\right)^{\frac{2}{\beta}} = \boldsymbol{\gamma}\, t_o^{-\frac{N}{\beta}}$$

holds true. This corresponds exactly to the time decay of the Barenblatt solution $\mathcal{B}_m$, and shows that (1.9) is optimal under this point of view. A similar decay estimate is given in [18, § 1.3] for the gradient of the Barenblatt fundamental solution of the parabolic p–Laplacian.

From Theorem 1.2 one can easily derive two simple consequences. This has already been done in case of porous medium type equations when $m > 1$ in [3, § 5]. We only recall two possible consequences. The first is concerned with the case of measures satisfying a density condition. More precisely, if the measure μ satisfies

$$\mu\big(Q_{\varrho,\varrho^2}(z_o)\big) \leq C\varrho^{N+\varepsilon}$$

for any $Q_{\varrho,\varrho^2}(z_o) \Subset E_T$, then u is locally continuous on E_T. Another consequence is the following important assertion:

$$\mu \in L\big(\tfrac{N+2}{2},1\big) \quad \Longrightarrow \quad u \text{ is locally continuous in } E_T.$$

Here, $L(\frac{N+2}{2},1)$ denotes the Lorentz space for the parameters $(\frac{N+2}{2},1)$ (see [15] for more details). Note that the assumption $\mu \in L(\frac{N+2}{2},1)$ is independent of $m \geq 1$. Now, a few words on the applications are in order. We compare our results with those from the classical theory for parabolic equations of the form (1.2) with coefficients satisfying (1.3) in the case $m = 1$. It is well known that for this kind of equations the assumption $\mu \in L^{\frac{N+2}{2}+\varepsilon}(E_T)$, for some arbitrarily small $\varepsilon > 0$, leads to continuity of u: this can be retrieved for example from [11, Section IV]. Note that this integrability assumption falls into the range of the density condition from above. Further, in the threshold case $\mu \in L^{\frac{N+2}{2}}(E_T)$, solutions might be even unbounded. Note that $L^{\frac{N+2}{2}+\varepsilon} \subset L(\frac{N+2}{2},1)$ for any $\varepsilon > 0$.

A few words concerning the history of the problem are in order. As far as the regularity for equations with the same structure considered here, with $0 < m < 1$ and $\mu = 0$, is concerned, boundedness and continuity of solutions were and still are a major issue.

To our knowledge, the boundedness of solutions was first proved in [14, Theorem 2.2] for the prototype fast diffusion equation with $\frac{(N-2)_+}{N} < m < 1$, and later extended to a more general equation in [8, Corollary 3.20]. In [6, Theorem 2.1] a unified treatment to the boundedness of solutions of the prototype equation in the whole range $0 < m < 1$ is given; in [10, Appendix B] the same result is proved for solutions of general quasilinear parabolic equations of the form (1.2) with coefficients satisfying (1.3), once more in the full range $0 < m < 1$.

As for the continuity, Chen and DiBenedetto [7] proved that locally bounded solutions of the parabolic p-Laplacian with $1 < p < 2$ are locally Hölder continuous, and their methods extend to singular porous medium equations of the form (1.2). Such a result can also be obtained from the estimates proved in [29] for doubly nonlinear singular equations, setting $p = 2$. In [10, Appendix B] a complete proof of the Hölder continuity of solutions of general quasilinear parabolic equations of the form (1.2) with coefficients satisfying (1.3) is given.

Finally, we would like to comment on potential estimates for degenerate, respectively singular parabolic equations of p-Laplacian type. Non-linear Wolff potential estimates for the *solution* u were first established in the degenerate case $p \geq 2$ for time independent measures in [24]. Later on, the result was extended to the singular case $\frac{2N}{N+2} < p < 2$ in [25], while the case of general measures was treated in the degenerate case in [26]. These results can be viewed as the natural extension of the Kilpelainen and Maly results from [16] to parabolic p-Laplacian type equations. The first results concerning *gradient estimates* for degenerate quasi-linear parabolic equations in terms of nonlinear potentials can be found in [27, 18]. In these papers local boundedness of the gradient of weak solutions in terms of non-linear Wolff potentials of the right-hand side measure was established for the degenerate case $p \geq 2$. Starting with [19, 20] pointwise estimates for the gradient in terms of non-linear Wolff potentials in the degenerate case, and linear Riesz potentials in the singular case has been achieved; see [21] for the final step and the Riesz potential estimates for the gradient of the solution in the degenerate case. As already mentioned before, the role of linear Riesz potentials for gradient continuity results has been observed in [22].

Acknowledgement. We acknowledge the warm hospitality of the Institut Mittag-Leffler, where this paper was written, during the program "Evolutionary problems" in the Fall 2013.

2. Preliminaries

2.1. **Notations.** For a point $z \in \mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ we shall always write $z = (x,t)$. By $B_r(x_o) \equiv \{x \in \mathbb{R}^N : |x - x_o| < r\}$ we denote the open ball in $\mathbb{R}^N$ with center $x_o \in \mathbb{R}^N$ and radius $r > 0$. Moreover, we write

$$Q_{r,\theta}(z_o) := B_r(x_o) \times (t_o - \theta, t_o),$$

where $z_o = (x_o, t_o) \in \mathbb{R}^{N+1}$ and $r, \theta > 0$. Whenever writing $2Q$ for a cylinder $Q \equiv Q_{r,\theta}(z_o)$ we mean $2Q = Q_{2r,4\theta}(z_o)$.

2.2. **Auxiliary lemmas.** Throughout the paper we will frequently use the following parabolic Sobolev embedding; see [11, Prop. 3.7, p. 7].

Lemma 2.1. *Let $Q_{\varrho,\theta}(z_o)$ be a parabolic cylinder with $0 < \varrho, \theta \leq 1$ and $1 < p < \infty$, $0 < r < \infty$. Then there exists a constant $\boldsymbol{\gamma}$ depending only on N, p, r such that for every*

$$u \in L^\infty\big(t_o - \theta, t_o; L^r(B_\varrho(x_o))\big) \cap L^p\big(t_o - \theta, t_o; W^{1,p}(B_\varrho(x_o))\big)$$

there holds

$$\iint_{Q_{\varrho,\theta}(z_o)} |u|^q \,dxdt \le \boldsymbol{\gamma}\bigg(\sup_{t\in(t_o-\theta,t_o)}\int_{B_\varrho(x_o)\times\{t\}} |u|^r\,dx\bigg)^{\frac{p}{N}} \iint_{Q_{\varrho,\theta}(z_o)} \Big[\Big|\frac{u}{\varrho}\Big|^p + |Du|^p\Big]\,dxdt\,,$$

where

$$q=\frac{p(N+r)}{N}\,.$$

2.3. **Mollification in time.** For $v\in L^1(E_T)$ we define the following mollification in time

$$\llbracket v\rrbracket_h(\cdot,t) := \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} v(\cdot,s)\,ds, \qquad \text{for } h\in(0,T] \text{ and } t\in[0,T], \tag{2.1}$$

and its time reversed analogue

$$\llbracket v\rrbracket_{\bar h}(\cdot,t) := \frac{1}{h}\int_t^T e^{\frac{t-s}{h}} v(\cdot,s)\,ds, \qquad \text{for } h\in(0,T] \text{ and } t\in[0,T].$$

For the main properties of this mollification we refer to [17, Lemma 2.2], or [5, Appendix B] (note that $\llbracket\cdot\rrbracket_{\bar h}$ has similar properties as $\llbracket\cdot\rrbracket_h$). If u is a weak solution of the porous medium equation (1.2), then its time regularization $\llbracket u\rrbracket_h$ satisfies

$$\iint_{E_T} \partial_t\llbracket u\rrbracket_h\varphi + \llbracket\mathbf{A}(x,\tau,u,Du)\rrbracket_h\cdot D\varphi\,dxdt = \iint_{E_T}\llbracket\varphi\rrbracket_{\bar h}\,d\mu, \tag{2.2}$$

for any $\varphi\in L^2(0,T;W^{1,2}(E))\cap L^\infty(E_T)$ with compact support in E_T; cf. [2, §2.4].

2.4. **Auxiliary functions.** For $\lambda\in(0,1)$ and $s\ge 0$ we define the following auxiliary function which will show up in a natural way in the energy estimates:

$$V_\lambda(s) := (1+s)^{\frac{1-\lambda}{2}}-1.$$

We now state some helpful estimates for V_λ from [2, §2.2], which will be useful later on.

Lemma 2.2. *For any $\varepsilon\in(0,1]$ and $s\ge 0$ there holds*

$$V_\lambda(s)\le s^{\frac{1-\lambda}{2}}$$

and

$$s^{1+\lambda}\le \varepsilon^{1+\lambda}+\boldsymbol{\gamma}_\varepsilon V_\lambda(s)^{\frac{2(1+\lambda)}{1-\lambda}},$$

where the constant $\boldsymbol{\gamma}_\varepsilon$ depends on ε and λ, and blows up as $\varepsilon^{-\frac{(1+\lambda)^2}{1-\lambda}}$ in the limit $\varepsilon\downarrow 0$.

Furthermore, we shall need the following elementary estimates.

Lemma 2.3. *Assume that $m,\lambda\in(0,1)$. Then, for any $a,d>0$, $b\ge a$ and any $\varepsilon\in(0,1]$, there exists a constant $\boldsymbol{\gamma}=\boldsymbol{\gamma}(m,\lambda)$ such that there holds*

$$\Big(\frac{a}{d}\Big)^{1-m}\frac{b^m-a^m}{d^m}\le\varepsilon+\frac{\boldsymbol{\gamma}}{\varepsilon d}\int_a^b\bigg[1-\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]\,ds.$$

Proof. Using [2, Lemma 2.3], we can bound the integral on the right-hand side from below as follows:

$$\int_a^b\bigg[1-\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]\,ds$$

$$
\begin{aligned}
&\ge a^{1-m}\int_a^b s^{m-1}\bigg[1-\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]\,ds\\
&=\frac{d}{m}\bigg[\Big(\frac{a}{d}\Big)^{1-m}\frac{b^m-a^m}{d^m}-\frac{1}{1-\lambda}\bigg(\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{b^m-a^m}{d^m}\bigg)^{1-\lambda}-1\bigg)\bigg]\\
&\ge\frac{\varepsilon\, d}{\boldsymbol{\gamma} m}\bigg[\Big(\frac{a}{d}\Big)^{1-m}\frac{b^m-a^m}{d^m}-\varepsilon\bigg],
\end{aligned}
$$

where the constant $\boldsymbol{\gamma}$ depends only on λ. This yields the claim of the Lemma. □

Lemma 2.4. *Assume that* $m,\lambda\in(0,1)$. *Then, for any* $a,d>0$, $b\ge a$ *and any* $\varepsilon\in(0,1]$, *there exists a constant* $\boldsymbol{\gamma}_\varepsilon=\boldsymbol{\gamma}_\varepsilon(m,\lambda,\varepsilon)$ *such that there holds*

$$
\bigg(\frac{b^m-a^m}{d^m}\bigg)^{\frac1m}\le\varepsilon+\frac{\boldsymbol{\gamma}_\varepsilon}{d}\int_a^b\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds.
$$

Proof. In the integral on the right-hand side, we perform the following substitution:

$$
\xi=\frac{s^m-a^m}{d^m}\qquad\Longleftrightarrow\qquad s=\big(a^m+d^m\xi\big)^{\frac1m}.
$$

With $\xi_o:=\frac{b^m-a^m}{d^m}$ this gives

$$
\begin{aligned}
&\int_a^b\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]\,ds\\
&\qquad\ge\frac{d^m}{m}\int_{\xi_o/2}^{\xi_o}\big(a^m+d^m\xi\big)^{\frac1m-1}\big[1-(1+\xi)^{-\lambda}\big]\,d\xi\\
&\qquad\ge\frac{d^m}{m}\Big(\frac{b^m+a^m}{2}\Big)^{\frac1m-1}\int_{\xi_o/2}^{\xi_o}\big[1-(1+\xi)^{-\lambda}\big]\,d\xi\\
&\qquad=\frac{d^m}{m}\Big(\frac{b^m+a^m}{2}\Big)^{\frac1m-1}\Big[\tfrac12\xi_o-\tfrac{1}{1-\lambda}\big((1+\xi_o)^{1-\lambda}-(1+\tfrac12\xi_o)^{1-\lambda}\big)\Big]\\
&\qquad=\frac{d^m}{2m}\Big(\frac{b^m+a^m}{2}\Big)^{\frac1m-1}\xi_o\big[1-F_\lambda(\xi_o)\big],
\end{aligned}
$$

where we defined for $\xi>0$ the non-negative real-valued function

$$
F_\lambda(\xi):=\frac{2\big[(1+\xi)^{1-\lambda}-(1+\frac12\xi)^{1-\lambda}\big]}{(1-\lambda)\xi}\,.
$$

Note that F_λ can be continuously extended in $\xi=0$ by letting $F_\lambda(0)=1$. Moreover, F_λ is strictly decreasing, with $\lim_{\xi\to\infty}F_\lambda(\xi)=0$. Now, given $\varepsilon>0$ we distinguish two cases. If $\xi_o\ge\varepsilon^m$, we have $F_\lambda(\xi_o)\le F_\lambda(\varepsilon^m)$, and therefore

$$
\begin{aligned}
\int_a^b\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds&\ge\frac{d^m}{2m}\bigg(\frac{b^m+a^m}{2}\bigg)^{\frac1m-1}\xi_o\big[1-F_\lambda(\varepsilon^m)\big]\\
&=\frac{1-F_\lambda(\varepsilon^m)}{2^{\frac1m}m}\big(b^m+a^m\big)^{\frac1m-1}\big(b^m-a^m\big)\\
&\ge\frac{1-F_\lambda(\varepsilon^m)}{2^{\frac1m}m}\big(b^m-a^m\big)^{\frac1m},
\end{aligned}
$$

or equivalently

$$
\bigg(\frac{b^m-a^m}{d^m}\bigg)^{\frac1m}\le\frac{2^{\frac1m}m}{d[1-F_\lambda(\varepsilon^m)]}\int_a^b\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds.
$$

Otherwise, if $0 < \xi_o < \varepsilon^m$, we have

$$\left(\frac{b^m - a^m}{d^m}\right)^{\frac{1}{m}} - \varepsilon \le 0 \le \frac{1}{d}\int_a^b \left[1 - \left(1 + \frac{s^m - a^m}{d^m}\right)^{-\lambda}\right] ds.$$

Joining the two previous inequalities proves the claim. $\square$

Lemma 2.5. *Let $m \in (0,1)$. Then, there exists a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m)$ such that for any $0 \le a \le b$ there holds*

$$b - a \le \boldsymbol{\gamma}\big(b^m - a^m\big)^{\frac{1}{m}} + \boldsymbol{\gamma} a^{1-m}\big(b^m - a^m\big).$$

Proof. First, we rewrite

$$b^m - a^m = m(b-a)\int_0^1 (a + s(b-a))^{m-1}\, ds.$$

From [1, Lemma 2.1] we know that

$$\int_0^1 (a + s(b-a))^{m-1}\, ds \ge \big(a^2 + b^2\big)^{\frac{m-1}{2}} \ge \frac{1}{\boldsymbol{\gamma}(m)(a^{1-m} + b^{1-m})}.$$

Therefore, we have

$$\begin{aligned} b - a &\le \boldsymbol{\gamma}\big(b^m - a^m\big)\big(b^{1-m} + a^{1-m}\big) \\ &= \boldsymbol{\gamma}\big(b^m - a^m\big)\big((b^m - a^m + a^m)^{\frac{1-m}{m}} + a^{1-m}\big) \\ &\le \boldsymbol{\gamma}\big(b^m - a^m\big)\big((b^m - a^m)^{\frac{1-m}{m}} + a^{1-m}\big) \\ &= \boldsymbol{\gamma}\big(b^m - a^m\big)^{\frac{1}{m}} + \boldsymbol{\gamma} a^{1-m}\big(b^m - a^m\big)\,. \end{aligned}$$

This yields the assertion of the lemma. $\square$

2.5. **The logarithmic function.** For later purposes we introduce the *Logarithmic function* ψ as follows: For parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with $0 < \mathbf{c} < \mathbf{a}$ and $\mathbf{b} \ge 0$ we define for $s < \mathbf{a}+\mathbf{b}+\mathbf{c}$ the function

$$\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}(s) := \ln_+\left(\frac{\mathbf{a}}{\mathbf{a} - (s-\mathbf{b})_+ + \mathbf{c}}\right) := \begin{cases} \ln\left(\dfrac{\mathbf{a}}{\mathbf{a} - s + \mathbf{b} + \mathbf{c}}\right), & \text{if } \mathbf{b}+\mathbf{c} < s < \mathbf{a}+\mathbf{b}+\mathbf{c}, \\ 0, & \text{if } s \le \mathbf{b}+\mathbf{c}. \end{cases} \tag{2.3}$$

The first and second order derivatives of $\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}$ can be easily computed. For the first order derivative we have

$$0 \le \big(\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}\big)'(s) = \begin{cases} \dfrac{1}{\mathbf{a} - s + \mathbf{b} + \mathbf{c}}, & \text{if } \mathbf{b}+\mathbf{c} < s < \mathbf{a}+\mathbf{b}+\mathbf{c}, \\ 0, & \text{if } s < \mathbf{b}+\mathbf{c}. \end{cases}$$

Away from $s = b + c$, the second order derivative is given by

$$\big(\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}\big)''(s) = \big[\big(\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}\big)'(s)\big]^2 \ge 0.$$

3. ENERGY ESTIMATES

Let $\lambda \in (0,1)$, $a, \tilde a, d > 0$. We consider (1.6) on parabolic cylinders of the form

$$Q_\varrho^{(\tilde a)}(z_o) := B_\varrho^{(\tilde a)}(x_o) \times \big(t_o - \varrho^2, t_o\big), \quad \text{where } B_\varrho^{(\tilde a)}(x_o) := B_{\tilde a^{\frac{m-1}{2}}\varrho}(x_o).$$

These cylinders are the natural ones for the singular porous medium equation, since on the one hand they take into account the scaling of the equation, and on the other hand we have $Q_\varrho^{(\tilde a_2)}(z_o) \subset Q_\varrho^{(\tilde a_1)}(z_o)$ if $\tilde a_2 > \tilde a_1$. In the following we shall prove two different energy estimates. In both cases we will use cut-off functions $\eta \in C_0^1(B_\varrho^{(\tilde a)}(x_o), [0,1])$ satisfying $\eta \equiv 1$ on $B_{\varrho/2}^{(\tilde a)}(x_o)$ and $|D\eta| \le 4\tilde a^{\frac{1-m}{2}}/\varrho$ and $\zeta \in W_0^{1,\infty}(\mathbb{R}, [0,1])$ defined by

$$\zeta(t) := \begin{cases} 0 & \text{if } t \in \big(-\infty, t_o - \varrho^2\big] \cap \big[\tau, \infty\big) \\ \frac{4}{3\varrho^2}\big(t - (t_o - \varrho^2)\big) & \text{if } t \in \big(t_o - \varrho^2, t_o - (\frac{\varrho}{2})^2\big) \\ 1 & \text{if } t \in \big[t_o - (\frac{\varrho}{2})^2, \tau - \delta\big] \\ -\frac{1}{\delta}(t - \tau) & \text{if } t \in (\tau - \delta, \tau), \end{cases}$$

where $\tau \in (t_o - (\frac{\varrho}{2})^2, t_o]$ and $0 < \delta < t_o - (\frac{\varrho}{2})^2 - \tau$. In the sequel we omit the center z_o in our notation. Moreover, we denote by Q^+, respectively $B^+(t)$ the level sets $Q_\varrho^{(\tilde a)} \cap \{u > a\}$ and $B_\varrho^{(\tilde a)} \cap \{u(\cdot,t) > a\}$. To obtain a first energy estimate, we choose the testing function in the form

$$\varphi := \eta^2 \zeta v \quad \text{where} \quad v := 1 - \left(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{(u^m - a^m)_+}{d^m}\right)^{-\lambda}. \tag{3.1}$$

Since $\zeta(t_o) = 0 = \zeta(t_o - \varrho^2)$, the boundary integral in (1.6) vanishes. From now on, we proceed formally. However, the following computations can be made rigorous by use of the mollification (2.1). First, we compute the term involving the time derivative:

$$\begin{aligned}
&\iint_{Q^+} u_t \varphi \, dxdt \\
&= \iint_{Q^+} \eta^2 \zeta \frac{\partial}{\partial t}\bigg(\int_a^{u(\cdot,t)} \bigg[1 - \bigg(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{s^m - a^m}{d^m}\bigg)^{-\lambda}\bigg] ds\bigg) dxdt \\
&= -\iint_{Q^+} \eta^2 \zeta_t \int_a^{u(\cdot,t)} \bigg[1 - \bigg(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{s^m - a^m}{d^m}\bigg)^{-\lambda}\bigg] ds \, dxdt \\
&= -\frac{4}{3\varrho^2} \int_{t_o - \varrho^2}^{t_o - (\varrho/2)^2} \int_{B^+(t)} \eta^2 \int_a^{u(\cdot,t)} \bigg[1 - \bigg(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{s^m - a^m}{d^m}\bigg)^{-\lambda}\bigg] ds \, dxdt \\
&\quad + \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_{B^+(t)} \eta^2 \int_a^{u(\cdot,t)} \bigg[1 - \bigg(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{s^m - a^m}{d^m}\bigg)^{-\lambda}\bigg] ds \, dxdt \\
&= \mathrm{I} + \mathrm{II}(\delta),
\end{aligned}$$

with the obvious labeling of I and II(δ). We note that the integral I is independent of δ. In the limit $\delta \downarrow 0$ we obtain for a.e. $\tau \in (t_o - (\varrho/2)^2, t_o)$ that

$$\lim_{\varepsilon \downarrow 0} \mathrm{II}(\delta) = \int_{B^+(\tau)} \eta^2 \int_a^u \bigg[1 - \bigg(1 + \Big(\frac{a}{d}\Big)^{1-m} \frac{s^m - a^m}{d^m}\bigg)^{-\lambda}\bigg] ds \, dx$$

holds true. As usual, the right-hand side of the preceding identity gives the sup-term in the final energy estimate. In turn we use the fact that $1 - (\dots)^{-\lambda} \le 1$ and that $u > a$ on

$B^+(t)$ to conclude that

$$|\,\mathrm{I}\,| \le \frac{4d}{3\varrho^2}\int_{t_o-\varrho^2}^{\tau}\int_{B^+(t)}\eta^2\,\frac{u-a}{d}\;dxdt.$$

Next, we treat the diffusion term in the weak formulation (1.6). We obtain

$$\begin{aligned}&\iint_{Q^+}\mathbf{A}(x,t,u,Du)\cdot D\varphi\,dxdt\\&\quad=\iint_{Q^+}\eta^2\zeta\mathbf{A}(x,t,u,Du)\cdot Dv\,dxdt+2\iint_{Q^+}\eta\zeta v\,\mathbf{A}(x,t,u,Du)\cdot D\eta\,dxdt\\&\quad=:\mathrm{III}+\mathrm{IV}.\end{aligned}$$

Using the ellipticity condition $(1.3)_1$, the term III can be bounded from below by

$$\begin{aligned}\mathrm{III}&=\iint_{Q^+}\eta^2\zeta\mathbf{A}(x,t,u,Du)\cdot Dv\,dxdt\\&=\frac{\lambda}{d^m}\Big(\frac{a}{d}\Big)^{1-m}\iint_{Q^+}\eta^2\zeta\frac{\mathbf{A}(x,t,u,Du)\cdot Du^m}{\big(1+\big(\frac{a}{d}\big)^{1-m}\frac{u^m-a^m}{d^m}\big)^{1+\lambda}}\,dxdt\\&\ge\frac{\lambda C_o}{d^m}\Big(\frac{a}{d}\Big)^{1-m}\iint_{Q^+}\eta^2\zeta\,\frac{|Du^m|^2}{\big(1+\big(\frac{a}{d}\big)^{1-m}\frac{u^m-a^m}{d^m}\big)^{1+\lambda}}\,dxdt=:\mathcal{L}\,.\end{aligned}$$

To estimate the term IV, we use the growth condition $(1.3)_2$, the fact that $v\le 1$, Young's inequality, and finally $|D\eta|\le 4\tilde a^{\frac{1-m}{2}}/\varrho$, to conclude that

$$\begin{aligned}|\mathrm{IV}|&\le 2\iint_{Q^+}\eta\zeta v|\mathbf{A}(x,t,u,Du)||D\eta|\,dxdt\\&\le\frac{8C_1\tilde a^{\frac{1-m}{2}}}{\varrho}\iint_{Q^+}\eta\zeta|Du^m|\,dxdt\\&\le\tfrac12\mathcal{L}+\frac{32C_1^2d^m\tilde a^{1-m}}{\lambda C_o\varrho^2}\Big(\frac{d}{a}\Big)^{1-m}\iint_{Q^+}\zeta\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{u^m-a^m}{d^m}\bigg)^{1+\lambda}\,dxdt\,.\end{aligned}$$

Finally, it remains to estimate the right-hand side integral. This, however, is obvious, since

$$\bigg|\iint_{Q^+}\varphi\mu\,dxdt\bigg|\le\mu(Q^+).$$

Combining the estimates obtained so far, letting $\delta\downarrow 0$, and recalling that $\lambda\in(0,1)$ we arrive at

$$\begin{aligned}&\int_{B^+(\tau)}\eta^2\int_a^u\bigg[1-\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dx\\&\quad+\frac{d}{a^{1-m}}\int_{t_o-\varrho^2}^{\tau}\int_{B^+(t)}\eta^2\zeta\bigg|D\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{u^m-a^m}{d^m}\bigg)^{\frac{1-\lambda}{2}}\bigg|^2\,dxdt\\&\le\frac{\boldsymbol{\gamma} d}{\varrho^2}\iint_{Q^+}\bigg[\Big(\frac{\tilde a}{a}\Big)^{1-m}\bigg(1+\Big(\frac{a}{d}\Big)^{1-m}\frac{u^m-a^m}{d^m}\bigg)^{1+\lambda}+\frac{u-a}{d}\bigg]dxdt+\boldsymbol{\gamma}\mu(Q^+)\\&\le\frac{\boldsymbol{\gamma} d}{\varrho^2}\Big[1+\Big(\frac{\tilde a}{a}\Big)^{1-m}\Big]\iint_{Q^+}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dxdt+\boldsymbol{\gamma}\mu(Q^+),\end{aligned}$$

for a.e. $\tau\in(t_o-(\varrho/2)^2,t_o)$. In the last line we used $a^{1-m}(u^m-a^m)\le u-a$ on Q^+. Dividing both sides of the preceding inequality by d, and recalling that $\eta\equiv 1$ on $B^{(\tilde a)}_{\varrho/2}$ leads to our first *energy estimate*

$$
\begin{aligned}
&\sup_{t\in\Lambda_{\varrho/2}}\int_{B^{(\tilde a)}_{\varrho/2}\times\{t\}\cap\{u>a\}}\frac1d\int_a^u\bigg[1-\bigg(1+\Big(\frac ad\Big)^{1-m}\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dx\\
&\qquad+\frac{1}{a^{1-m}}\iint_{Q^{(\tilde a)}_{\varrho/2}\cap\{u>a\}}\bigg|DV_\lambda\bigg(\Big(\frac ad\Big)^{1-m}\frac{u^m-a^m}{d^m}\bigg)\bigg|^2dxdt\\
&\qquad\le\frac{\gamma}{\varrho^2}\Big[1+\Big(\frac{\tilde a}{a}\Big)^{1-m}\Big]\iint_{Q^{(\tilde a)}_{\varrho}\cap\{u>a\}}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}dxdt+\frac{\gamma\mu\big(Q^{(\tilde a)}_\varrho\big)}{d},
\end{aligned}\tag{3.2}
$$

for a constant $\boldsymbol{\gamma}$ depending only on m,C_o,C_1,λ.

In the remaining part of this section we prove a second energy estimate. For this we modify the choice of the testing function in (3.1). Instead of (3.1) we define

$$
\varphi:=\eta^2\zeta v\quad\text{where}\quad v:=1-\bigg(1+\frac{(u^m-a^m)_+}{d^m}\bigg)^{-\lambda}.\tag{3.3}
$$

The difference with respect to (3.1) is that we replace the factor $(a/d)^{1-m}$ by one. Again, since $\zeta(t_o)=0=\zeta(t_o-\varrho^2)$ the boundary integral in (1.6) vanishes. As in the first energy estimate, we proceed formally when treating the term involving the time derivative. We compute

$$
\begin{aligned}
&\iint_{Q^+}u_t\varphi\,dxdt\\
&\quad=\iint_{Q^+}\eta^2\zeta\,\frac{\partial}{\partial t}\bigg(\int_a^{u(\cdot,t)}\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\bigg)dxdt\\
&\quad=-\iint_{Q^+}\eta^2\zeta_t\int_a^{u(\cdot,t)}\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dxdt\\
&\quad=-\frac{4}{3\varrho^2}\int_{t_o-\varrho^2}^{t_o-(\varrho/2)^2}\int_{B^+(t)}\eta^2\int_a^{u(\cdot,t)}\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dxdt\\
&\qquad+\frac1\delta\int_{\tau-\delta}^{\tau}\int_{B^+(t)}\eta^2\int_a^{u(\cdot,t)}\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dxdt\\
&\quad=:\mathrm{I}+\mathrm{II}(\delta),
\end{aligned}
$$

with the obvious labeling of I and II(δ). As before, the integral I does not depend on δ. In the limit $\delta\downarrow 0$ we obtain

$$
\lim_{\delta\downarrow0}\mathrm{II}(\delta)=\int_{B^+(\tau)}\eta^2\int_a^u\bigg[1-\bigg(1+\frac{s^m-a^m}{d^m}\bigg)^{-\lambda}\bigg]ds\,dx
$$

for a.e. $\tau\in(t_o-(\varrho/2)^2,t_o)$. The right-hand side of the preceding identity yields the sup-term in the final energy estimate. Using in turn that $1-(\dots)^{-\lambda}\le1$, and that $u>a$ on the domain on $B^+(t)$ we find that

$$
|\,\mathrm{I}\,|\le\frac{4d}{3\varrho^2}\int_{t_o-\varrho^2}^{\tau}\int_{B^+(t)}\eta^2\,\frac{u-a}{d}\,dxdt
$$

holds true. Next, we treat the diffusion term in the weak formulation (1.6). We obtain

$$
\begin{aligned}
&\iint_{Q^+} \mathbf{A}(x,t,u,Du)\cdot D\varphi\,dxdt \\
&\qquad = \iint_{Q^+} \eta^2\zeta\mathbf{A}(x,t,u,Du)\cdot Dv\,dxdt + 2\iint_{Q^+} \eta\zeta v\,\mathbf{A}(x,t,u,Du)\cdot D\eta\,dxdt \\
&\qquad =: \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

Using the ellipticity bound $(1.3)_1$, the term III can be controlled from below by

$$
\begin{aligned}
\mathrm{III} &= \iint_{Q^+} \eta^2\zeta\mathbf{A}(x,t,u,Du)\cdot Dv\,dxdt \\
&= \frac{\lambda}{d^m}\iint_{Q^+} \eta^2\zeta\frac{\mathbf{A}(x,t,u,Du)\cdot Du^m}{\big(1+\frac{u^m-a^m}{d^m}\big)^{1+\lambda}}\,dxdt \\
&\ge \frac{\lambda C_o}{d^m}\iint_{Q^+} \eta^2\zeta\,\frac{|Du^m|^2}{\big(1+\frac{u^m-a^m}{d^m}\big)^{1+\lambda}}\,dxdt =: \mathcal{L}\,.
\end{aligned}
$$

To estimate the term IV, we use the growth condition $(1.3)_2$, the fact that $v\le 1$, Young's inequality, and finally $|D\eta|\le 4\tilde a^{\frac{1-m}{2}}/\varrho$, to conclude that

$$
\begin{aligned}
|\mathrm{IV}| &\le 2\iint_{Q^+} \eta\zeta v|\mathbf{A}(x,t,u,Du)||D\eta|\,dxdt \\
&\le \frac{8C_1\tilde a^{\frac{1-m}{2}}}{\varrho}\iint_{Q^+} \eta\zeta|Du^m|\,dxdt \\
&\le \tfrac12\mathcal{L} + \frac{32C_1^2 d^m\tilde a^{1-m}}{\lambda C_o\varrho^2}\iint_{Q^+} \zeta\Big(1+\frac{u^m-a^m}{d^m}\Big)^{1+\lambda}\,dxdt\,.
\end{aligned}
$$

Finally, the right-hand side integral can be estimated by

$$
\bigg|\iint_{Q^+} \varphi\mu\,dxdt\bigg| \le \mu(Q^+).
$$

Combining the preceding estimates, dividing the result by d, letting $\delta\downarrow 0$, and recalling that $\lambda\in(0,1)$, we arrive at

$$
\begin{aligned}
&\frac1d\int_{B^+(\tau)}\eta^2\int_a^u\bigg[1-\Big(1+\frac{s^m-a^m}{d^m}\Big)^{-\lambda}\bigg]\,ds\,dx \\
&\qquad + \frac{1}{d^{1-m}}\iint_{t_o-\varrho^2}^{\tau}\int_{B^+(\tau)}\eta^2\zeta\bigg|D\Big(1+\frac{u^m-a^m}{d^m}\Big)^{\frac{1-\lambda}{2}}\bigg|^2\,dxdt \\
&\qquad \le \frac{\gamma}{\varrho^2}\iint_{Q^+}\bigg[\Big(\frac{\tilde a}{d}\Big)^{1-m}\Big(1+\frac{u^m-a^m}{d^m}\Big)^{1+\lambda}+\frac{u-a}{d}\bigg]\,dxdt + \frac{\gamma\mu(Q^+)}{d}.
\end{aligned}
$$

The preceding estimate holds true for a.e. $\tau\in(t_o-(\varrho/2)^2,t_o)$. Now we recall that $\eta\equiv1$ on $B^{(\tilde a)}_{\varrho/2}$. This leads to the second *energy estimate*

$$
\begin{aligned}
&\sup_{t\in\Lambda_{\varrho/2}}\int_{B^{(\tilde a)}_{\varrho/2}\times\{t\}\cap\{u>a\}}\frac1d\int_a^u\bigg[1-\Big(1+\frac{s^m-a^m}{d^m}\Big)^{-\lambda}\bigg]ds\,dx \\
&\qquad + \frac{1}{d^{1-m}}\iint_{Q^{(\tilde a)}_{\varrho/2}\cap\{u>a\}}\bigg|DV_\lambda\Big(\frac{u^m-a^m}{d^m}\Big)\bigg|^2\,dxdt
\end{aligned}
$$

$$\leq \frac{\gamma}{\varrho^2}\iint_{Q_\varrho^{(\tilde a)}\cap\{u>a\}}\bigg[\Big(\frac{\tilde a}{d}\Big)^{1-m}\Big(1+\frac{u^m-a^m}{d^m}\Big)^{1+\lambda}+\frac{u-a}{d}\bigg]dxdt+\frac{\gamma\mu\big(Q_\varrho^{(\tilde a)}\big)}{d}, \tag{3.4}$$

for a constant γ depending on m, C_o, C_1, λ.

4. Proof of Theorem 1.1

In this section we give the proof of the potential estimate. We start with the

4.1. Choice of parameters. Let

$$\lambda \in \big(0, \tfrac{\beta}{N(1+m)}\big] \qquad \text{where } \beta := 2 - N(1-m). \tag{4.1}$$

We note that $\beta > 0$, since we assume that $m > \frac{N-2}{N}$. Therefore, the choice of λ is always possible. Furthermore, $\lambda \leq \frac{\beta}{N(1+m)} \leq \frac{1}{N}$, since $m \leq 1$. We are going to prove the result on some general cylinder of the form $Q_{r,\theta}(z_o) \Subset E_T$. To avoid an overburdened notation, from now on, we omit the center z_o. For $j \in \mathbb{N}_0$ we define sequences of radii r_j, parameters θ_j and cylinders Q_j as follows:

$$r_j := \frac{r}{2^j}, \quad \theta_j := \frac{\theta}{2^{2j}}, \quad Q_j := B_j \times \Lambda_j = B_{a_j^{\frac{m-1}{2}}\theta_j^{\frac12}} \times (-\theta_j, 0),$$

where the parameters a_j will be chosen inductively in the following. Moreover, for $a > a_j$ and $j \in \mathbb{N}_0$ we denote

$$\boldsymbol{K}_j(a) := \frac{1}{|Q_j|}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{a-a_j}\Big)^{1+\lambda} dxdt. \tag{4.2}$$

Now, we perform the choice of the parameters a_j. We let $\kappa \in (0,1)$ be a fixed parameter, which will be chosen later in the proof in a universal way in dependence on the structural parameters. We set $a_{-1} := 0$, and

$$a_o := \Big(\frac{\theta}{r^2}\Big)^{\frac{1}{1-m}} + \bigg(\frac{1}{\kappa\,\theta^{1+\frac{n}{2}}}\iint_{Q_{r,\theta}} u^{1+\lambda}\,dxdt\bigg)^{\frac{2}{\beta+2\lambda}}. \tag{4.3}$$

We now assume that for $j \geq 0$ the numbers $a_{-1}, a_o, \dots, a_j$ have already been selected. The numbers $d_\ell = a_{\ell+1} - a_\ell$ are then defined for $\ell = -1, 0, \dots, j-1$. In particular, we have $d_{-1} = a_o$. Now, we choose a_{j+1} according to the following procedure: First, we observe that $(a_j, \infty) \ni a \mapsto \boldsymbol{K}_j(a)$ is strictly decreasing, continuous, $\lim_{a\downarrow a_j}\boldsymbol{K}_j(a) = \infty$, and $\lim_{a\to\infty}\boldsymbol{K}_j(a) = 0$. In case of

$$\boldsymbol{K}_j\big(a_j + \tfrac12 d_{j-1}\big) \leq \kappa, \tag{4.4}$$

we define

$$a_{j+1} := a_j + \tfrac12 d_{j-1}. \tag{4.5}$$

Otherwise, i.e. when

$$\boldsymbol{K}_j\big(a_j + \tfrac12 d_{j-1}\big) > \kappa \tag{4.6}$$

holds true, we choose a_{j+1} according to (note that $K_j(\cdot)$ is a continuous function)

$$a_{j+1} := \sup\big\{a \in \big(a_j + \tfrac12 d_{j-1}, \infty\big) : \boldsymbol{K}_j(a) > \kappa\big\} \tag{4.7}$$

so that in this case we have

$$\boldsymbol{K}_j(a_{j+1}) = \kappa \quad \text{and} \quad a_{j+1} > a_j + \tfrac12 d_{j-1}. \tag{4.8}$$

Having chosen a_{j+1}, we let $d_j := a_{j+1} - a_j$ and define

$$\boldsymbol{k}_j := \boldsymbol{K}_j(a_{j+1}) \equiv \frac{1}{|Q_j|} \iint_{Q_j \cap \{u > a_j\}} \Big(\frac{u - a_j}{d_j} \Big)^{1+\lambda} dxdt \le \kappa\,. \tag{4.9}$$

We note that the bound $\boldsymbol{k}_j \le \kappa$ follows from the construction of a_{j+1}. Since $a_{j+1} > a_j$ (which holds true by construction), we have that the following inclusion

$$2Q_{j+1} \subseteq Q_j$$

for any $j \in \mathbb{N}_0$. Furthermore, since $a_j \ge a_o > (\theta/r^2)^{\frac{1}{1-m}}$ (the last inequality follows by the definition of a_o), we also have $a_j^{\frac{m-1}{2}} \theta_j^{\frac{1}{2}} < r_j$, so that

$$Q_j \subseteq Q_{r_j, \theta_j}\,. \tag{4.10}$$

4.2. **A first bound on a_j.** Here, we will ensure that the sequence $\{a_j\}$ does not grow too fast. More precisely, denoting by $\boldsymbol{\alpha}_N$ the volume of the unit ball $B_1 \subset \mathbb{R}^N$, we define

$$B := \max \bigg\{ 2\, \boldsymbol{\alpha}_N^{-\frac{1}{1+\lambda}},\ 2^{\frac{2(N+4)}{\beta + 2\lambda}} \bigg\} \ge 2$$

and prove that

$$a_j \le B\, a_{j-1} \qquad \text{for any } j \in \mathbb{N}. \tag{4.11}$$

We establish the claim (4.11) by induction. First, using $B \ge 2$ and the definitions of a_o and β, we compute

$$\begin{aligned}
\boldsymbol{K}_0(Ba_o) &= \frac{1}{|Q_o|} \iint_{Q_o \cap \{u > a_o\}} \bigg(\frac{u - a_o}{(B-1)a_o} \bigg)^{1+\lambda} dxdt \\
&\le \frac{2^{1+\lambda}}{B^{1+\lambda} a_o^{1+\lambda} |Q_o|} \iint_{Q_o} u^{1+\lambda}\, dxdt \\
&\le \frac{2^{1+\lambda} a_o^{\frac{N(1-m)}{2}}}{\boldsymbol{\alpha}_N B^{1+\lambda} a_o^{1+\lambda} \theta^{1+\frac{N}{2}}} \iint_{Q_{r,\theta}} u^{1+\lambda}\, dxdt \\
&= \frac{2^{1+\lambda} a_o^{-\frac{\beta+2\lambda}{2}}}{\boldsymbol{\alpha}_N B^{1+\lambda} \theta^{1+\frac{N}{2}}} \iint_{Q_{r,\theta}} u^{1+\lambda}\, dxdt \\
&\le \frac{2^{1+\lambda}\kappa}{\boldsymbol{\alpha}_N B^{1+\lambda}} \le \kappa.
\end{aligned}$$

Now, we distinguish two cases. By construction of a_1, we either have $a_1 = a_o + \frac{1}{2} d_{-1} = \frac{3}{2} a_o \le B a_o$, or $K_0(a_1) = \kappa \ge \boldsymbol{K}_0(Ba_o)$. Since $a \mapsto \boldsymbol{K}_0(a)$ is decreasing, we infer in this case that $a_1 \le Ba_o$. This implies that (4.11) holds for $j = 1$. Now, we consider $j \in \mathbb{N}$ and assume that (4.11) holds for any $\ell = 1, \ldots, j$. In this case we have the estimate

$$\begin{aligned}
\boldsymbol{K}_j(Ba_j) &= \frac{1}{|Q_j|} \iint_{Q_j \cap \{u > a_j\}} \bigg(\frac{u - a_j}{(B-1)a_j} \bigg)^{1+\lambda} dxdt \\
&\le \frac{4}{B^{1+\lambda} |Q_j|} \iint_{Q_j \cap \{u > a_j\}} \bigg(\frac{u - a_j}{a_j} \bigg)^{1+\lambda} dxdt \\
&\le \frac{4}{B^{1+\lambda} |Q_j|} \iint_{Q_{j-1} \cap \{u > a_{j-1}\}} \bigg(\frac{u - a_{j-1}}{a_j - a_{j-1}} \bigg)^{1+\lambda} dxdt
\end{aligned}$$

$$= \frac{4|Q_{j-1}|}{B^{1+\lambda}|Q_j|}\,\boldsymbol{k}_{j-1} \le \frac{2^{N+4}}{B^{1+\lambda}}\Big(\frac{a_j}{a_{j-1}}\Big)^{\frac{N(1-m)}{2}}\kappa \le 2^{N+4}B^{-\frac{\beta+2\lambda}{2}}\kappa \le \kappa$$

We now argue as before. Either we have $a_{j+1} = a_j + \frac{1}{2}d_{j-1} \le \frac{3}{2}a_j \le Ba_j$, or $K_j(a_{j+1}) = \kappa \ge \boldsymbol{K}_j(Ba_j)$, implying again $a_{j+1} \le Ba_j$. This proves that (4.11) holds also for $j+1$, and completes the induction argument. Hence, (4.11) holds.

As an immediate consequence of (4.11), we have for any $j \in \mathbb{N}$ that

$$\frac{|Q_{j-1}|}{|Q_j|} = 2^{N+2}\Big(\frac{a_j}{a_{j-1}}\Big)^{\frac{N(1-m)}{2}} \le 2^{N+2}B^{\frac{N(1-m)}{2}} \equiv \boldsymbol{\gamma}(N,m,\lambda). \tag{4.12}$$

4.3. **Recursive bounds for d_j.** We claim the existence of some universal constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,C_o,C_1,\lambda)$ such that there holds:

$$d_j \le \tfrac{1}{2}d_{j-1} + \frac{\boldsymbol{\gamma}\mu(2Q_{r_j,\theta_j})}{|B_j|} \qquad \forall j \in \mathbb{N}_0\,. \tag{4.13}$$

For the proof of (4.13) we can assume without loss of generality that $\boldsymbol{k}_j = \kappa$. Indeed, if we would have $\boldsymbol{k}_j < \kappa$, then a_{j+1} is defined via (4.5), which, by the definition of d_j is equivalent to $d_j = a_{j+1} - a_j = \frac{1}{2}d_{j-1}$, so that (4.13) holds trivially. Furthermore, we can assume $d_j > \frac{1}{2}d_{j-1}$ holds true. Otherwise, we would have $d_j \le \frac{1}{2}d_{j-1}$, which once again yields the claim (4.13). After this elementary reductions we start with the actual proof.

We begin with the derivation of some useful preliminary estimates which will be needed later on several times . By the definition of d_{j-1} we have

$$1 = \frac{a_j - a_{j-1}}{d_{j-1}} \le \frac{u - a_{j-1}}{d_{j-1}} \qquad \text{on } \{u > a_j\}. \tag{4.14}$$

Since we are assuming $d_j > \frac{1}{2}d_{j-1}$, we can conclude that

$$\frac{u-a_j}{d_j} \le \frac{u-a_{j-1}}{d_j} \le 2\frac{u-a_{j-1}}{d_{j-1}} \qquad \text{on } \{u > a_j\}. \tag{4.15}$$

Using in turn (4.14), the inclusion $Q_j \cap \{u > a_j\} \subset Q_{j-1} \cap \{u > a_{j-1}\}$, (4.12), and $\boldsymbol{k}_{j-1} \le \kappa$ we conclude that

$$\begin{aligned} |Q_j \cap \{u > a_j\}| &\le \iint_{Q_j\cap\{u>a_j\}} \Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda} dxdt \\ &\le |Q_{j-1}|\boldsymbol{k}_{j-1} \le |Q_{j-1}|\kappa \le \boldsymbol{\gamma}|Q_j|\kappa, \end{aligned} \tag{4.16}$$

with $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,\lambda)$. Next, we apply the energy estimate (3.2) on the cylinder $2Q_j$. We start with the application of (3.2) with the choice of (a_j, a_j, d_j) for $(a, \tilde{a}, d)$ and get

$$\begin{aligned} \sup_{t\in\Lambda_j} \int_{B_j\times\{t\}\cap\{u>a_j\}} &\frac{1}{d_j}\int_{a_j}^{u}\Bigg[1 - \bigg(1 + \Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{s^m - a_j^m}{d_j^m}\bigg)^{-\lambda}\Bigg] ds\,dx \\ &+ \frac{1}{a_j^{1-m}}\iint_{Q_j\cap\{u>a_j\}}\bigg|DV_\lambda\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m - a_j^m}{d_j^m}\bigg)\bigg|^2 dxdt \\ &\le \frac{\boldsymbol{\gamma}}{\theta_j}\iint_{2Q_j\cap\{u>a_j\}}\Big(1 + \frac{u-a_j}{d_j}\Big)^{1+\lambda} dxdt + \frac{\boldsymbol{\gamma}\mu(2Q_j)}{d_j}, \end{aligned}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m,C_o,C_1,\lambda)$. To estimate the right-hand side further, we combine (4.14) and (4.15), to conclude

$$1 + \frac{u-a_j}{d_j} \le 3\frac{u-a_{j-1}}{d_{j-1}} \qquad \text{on } \{u > a_j\}. \tag{4.17}$$

Inserting this in the energy estimate, using $2Q_j \cap \{u > a_j\} \subset Q_{j-1} \cap \{u > a_{j-1}\}$ and (4.12), the right-hand side integral can be bounded by

$$\begin{aligned}\frac{\boldsymbol{\gamma}}{\theta_j}\iint_{2Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}dxdt &\le \frac{\boldsymbol{\gamma}}{\theta_j}\iint_{Q_{j-1}\cap\{u>a_{j-1}\}}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}dxdt\\ &= \frac{\boldsymbol{\gamma}\,|Q_{j-1}|}{\theta_j}\,\boldsymbol{k}_{j-1} \le \frac{\boldsymbol{\gamma}\,|Q_{j-1}|}{\theta_j}\,\kappa \le \boldsymbol{\gamma}\,|B_j|\kappa,\end{aligned} \tag{4.18}$$

where the constant $\boldsymbol{\gamma}$ only depends on N, m, C_o, C_1 and λ. With this inequality the above energy estimate gives

$$\begin{aligned}\sup_{t\in\Lambda_j}\int_{B_j\times\{t\}\cap\{u>a_j\}}\frac{1}{d_j}\int_{a_j}^{u}\Big[1-\Big(1+\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{s^m-a_j^m}{d_j^m}\Big)^{-\lambda}\Big]ds\,dx&\\ +\frac{1}{a_j^{1-m}}\iint_{Q_j\cap\{u>a_j\}}\Big|DV_\lambda\Big(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\Big)\Big|^2dxdt&\\ \le \boldsymbol{\gamma}|B_j|\Big[\kappa+\frac{\mu(2Q_j)}{d_j|B_j|}\Big]&\end{aligned} \tag{4.19}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, \nu, L, \lambda)$. Next, we apply the second energy estimate (3.4) on the cylinder $2Q_j$ with (a_j, a_j, d_j) instead of $(a, \tilde{a}, d)$. In this way, we obtain the inequality

$$\begin{aligned}\sup_{t\in\Lambda_j}&\int_{B_j\times\{t\}\cap\{u>a_j\}}\frac{1}{d_j}\int_{a_j}^{u}\Big[1-\Big(1+\frac{s^m-a_j^m}{d_j^m}\Big)^{-\lambda}\Big]ds\,dx\\ &+\frac{1}{d_j^{1-m}}\iint_{Q_j\cap\{u>a_j\}}\Big|DV_\lambda\Big(\frac{u^m-a_j^m}{d_j^m}\Big)\Big|^2dxdt\\ \le \boldsymbol{\gamma}&\Big[\frac{1}{\theta_j}\iint_{2Q_j\cap\{u>a_j\}}\Big[\Big(\frac{a_j}{d_j}\Big)^{1-m}\Big(1+\frac{u^m-a_j^m}{d_j^m}\Big)^{1+\lambda}+\frac{u-a_j}{d_j}\Big]dxdt+\frac{\mu(2Q_j)}{d_j}\Big].\end{aligned}$$

Using $u^m - a_j^m \le (u - a_j)^m$, (4.17), $d_j > \frac{1}{2}d_{j-1}$ and (4.11), we find that

$$\begin{aligned}\Big(\frac{a_j}{d_j}\Big)^{1-m}\Big(1+\frac{u^m-a_j^m}{d_j^m}\Big)^{1+\lambda} &\le \boldsymbol{\gamma}(m)B^{1-m}\Big(\frac{a_{j-1}}{d_{j-1}}\Big)^{1-m}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{m(1+\lambda)}\\ &\le \boldsymbol{\gamma}(m)B^{1-m}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+m\lambda}\\ &\le \boldsymbol{\gamma}(m)B^{1-m}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\end{aligned}$$

holds true, as well as

$$\frac{u-a_j}{d_j} \le 2\frac{u-a_{j-1}}{d_{j-1}} \le 2\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}.$$

Inserting this above and using (4.18), we obtain that

$$\begin{aligned}\sup_{t\in\Lambda_j}\int_{B_j\times\{t\}\cap\{u>a_j\}}\frac{1}{d_j}\int_{a_j}^{u}\Big[1-\Big(1+\frac{s^m-a_j^m}{d_j^m}\Big)^{-\lambda}\Big]ds\,dx&\\ +\frac{1}{d_j^{1-m}}\iint_{Q_j\cap\{u>a_j\}}\Big|DV_\lambda\Big(\frac{u^m-a_j^m}{d_j^m}\Big)\Big|^2dxdt&\\ \le \boldsymbol{\gamma}\Big[\frac{1}{\theta_j}\iint_{2Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}dxdt+\frac{\mu(2Q_j)}{d_j}\Big]&\end{aligned}$$

$$\leq \boldsymbol{\gamma}|B_j|\bigg[\kappa+\frac{\mu(2Q_j)}{d_j|B_j|}\bigg], \tag{4.20}$$

holds true for a constant $\boldsymbol{\gamma}=\boldsymbol{\gamma}(N,m,C_o,C_1,\lambda)$.

To proceed further, we recall that the preliminary reduction allows us to assume $\kappa=\boldsymbol{k}_j=\boldsymbol{K}_j(a_{j+1})$. In the sequel we use this fact, Lemma 2.5 and Lemma 2.2 to estimate, with $\tilde\varepsilon\in(0,1)$ to be chosen later in the proof,

$$\begin{aligned}
\kappa&=\frac{1}{|Q_j|}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{1+\lambda}dxdt\\
&\leq\frac{\boldsymbol{\gamma}}{|Q_j|}\iint_{Q_j\cap\{u>a_j\}}\bigg(\frac{u^m-a_j^m}{d_j^m}\bigg)^{\frac{1+\lambda}{m}}dxdt\\
&\quad+\frac{\boldsymbol{\gamma}}{|Q_j|}\iint_{Q_j\cap\{u>a_j\}}\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\bigg)^{1+\lambda}dxdt\\
&\leq\boldsymbol{\gamma}\big[\tilde\varepsilon^{\frac{1+\lambda}{m}}+\tilde\varepsilon^{1+\lambda}\big]\frac{|Q_j\cap\{u>a_j\}|}{|Q_j|}+\frac{\boldsymbol{\gamma}_{\tilde\varepsilon}}{|Q_j|}\,(\mathrm{I}+\mathrm{II}),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}&:=\iint_{Q_j\cap\{u>a_j\}}\bigg|V_\lambda\bigg(\frac{u^m-a_j^m}{d_j^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{m(1-\lambda)}}dxdt,\\
\mathrm{II}&:=\iint_{Q_j\cap\{u>a_j\}}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{1-\lambda}}dxdt
\end{aligned}$$

and with constants $\boldsymbol{\gamma}=\boldsymbol{\gamma}(m)$ and $\boldsymbol{\gamma}_{\tilde\varepsilon}=\boldsymbol{\gamma}(m,\lambda,\tilde\varepsilon)$. Taking into account $\tilde\varepsilon^{\frac{1+\lambda}{m}}\leq\tilde\varepsilon^{1+\lambda}$ and the measure bound (4.16), we can further estimate the right-hand side and conclude that

$$\kappa\leq\boldsymbol{\gamma}\tilde\varepsilon^{1+\lambda}\kappa+\frac{\boldsymbol{\gamma}_{\tilde\varepsilon}}{|Q_j|}\,(\mathrm{I}+\mathrm{II}). \tag{4.21}$$

Note that $\boldsymbol{\gamma}=\boldsymbol{\gamma}(N,m,\lambda)$ and $\boldsymbol{\gamma}_{\tilde\varepsilon}=\boldsymbol{\gamma}_{\tilde\varepsilon}(m,\lambda,\tilde\varepsilon)$. In order to estimate the terms I and II, we shall use Gagliardo-Nirenberg's inequality and the energy estimates (3.2) and (3.4) from before. We start with the estimation of I. Using Gagliardo-Nirenberg's inequality from Lemma 2.1 with $p=2$ and $q=\frac{2(1+\lambda)}{m(1-\lambda)}$ and $r=N\big[\frac{1+\lambda}{m(1-\lambda)}-1\big]$, we find that

$$\begin{aligned}
\mathrm{I}&=\iint_{Q_j}\bigg|V_\lambda\bigg(\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{m(1-\lambda)}}dxdt\\
&\leq\boldsymbol{\gamma}\bigg[\sup_{t\in\Lambda_j}\int_{B_j\times\{t\}}\bigg|V_\lambda\bigg(\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^{N[\frac{1+\lambda}{m(1-\lambda)}-1]}dx\bigg]^{\frac{2}{N}}\\
&\quad\cdot\iint_{Q_j}\frac{a_j^{1-m}}{\theta_j}\bigg|V_\lambda\bigg(\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^2+\bigg|DV_\lambda\bigg(\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^2dxdt\\
&=:\boldsymbol{\gamma}\,\mathrm{I}_1(\mathrm{I}_2+\mathrm{I}_3),
\end{aligned}$$

with the obvious meaning of I_1, I_2 and I_3. For the estimate of I_1 we apply in turn Lemma 2.2, Hölder's inequality (note that $\frac{N}{2}[1+\lambda-m(1-\lambda)]\leq1$ by the choice of λ in (4.1)), Lemma 2.4 and the energy estimate (4.20). In the computation we also use the abbreviations $q := 1+\lambda - m(1-\lambda)$ and $B_j^+(t) := B_j \times \{t\} \cap \{u > a_j\}$. In this way, we get for any $\varepsilon \in (0,1)$ that

$$\begin{aligned}
\mathrm{I}_1 &\le \left[\sup_{t\in\Lambda_j} \int_{B_j^+(t)} \left(\frac{u^m - a_j^m}{d_j^m} \right)^{\frac{Nq}{2m}} dx \right]^{\frac{2}{N}} \\
&\le |B_j|^{\frac{2}{N}} \left[\sup_{t\in\Lambda_j} \frac{1}{|B_j|} \int_{B_j^+(t)} \left(\frac{u^m - a_j^m}{d_j^m} \right)^{\frac{1}{m}} dx \right]^{q} \\
&\le |B_j|^{\frac{2}{N}} \left[\varepsilon + \boldsymbol{\gamma}_\varepsilon \left[\sup_{t\in\Lambda_j} \frac{1}{|B_j|} \int_{B_j^+(t)} \frac{1}{d_j} \int_{a_j}^{u} 1 - \left(1 + \frac{s^m - a_j^m}{d_j^m} \right)^{-\lambda} ds dx \right]^{q} \right] \\
&\le |B_j|^{\frac{2}{N}} \left[\varepsilon + \boldsymbol{\gamma}_\varepsilon \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right]^{1+\lambda - m(1-\lambda)} \right],
\end{aligned}$$

where $\boldsymbol{\gamma}_\varepsilon = \boldsymbol{\gamma}_\varepsilon(m,\lambda,\varepsilon)$. To estimate the second term I_2 we again use Lemma 2.2, (4.15), (4.14), (4.9) and (4.12). This leads us to

$$\begin{aligned}
\mathrm{I}_2 &\le \frac{a_j^{1-m}}{\theta_j} \iint_{Q_j \cap \{u > a_j\}} \left(\frac{u^m - a_j^m}{d_j^m} \right)^{1-\lambda} dxdt \\
&\le \frac{\boldsymbol{\gamma} a_j^{1-m}}{\theta_j} \iint_{Q_j \cap \{u > a_j\}} \left(\frac{u - a_j}{d_j} \right)^{m(1-\lambda)} dxdt \\
&\le \frac{\boldsymbol{\gamma} a_j^{1-m}}{\theta_j} \iint_{Q_j \cap \{u > a_j\}} \left(\frac{u - a_{j-1}}{d_{j-1}} \right)^{1+\lambda} dxdt \\
&\le \frac{\boldsymbol{\gamma} a_j^{1-m} |Q_{j-1}| \kappa}{\theta_j} \le \boldsymbol{\gamma} a_j^{1-m} |B_j| \kappa,
\end{aligned}$$

for a constant $\boldsymbol{\gamma}$ depending on N, m and λ. Finally, by the energy estimate (4.20) and the fact that $d_j = a_{j+1} - a_j \le (B-1)a_j = \boldsymbol{\gamma}(N,m,\lambda)\, a_j$ which follows from (4.11), we have, this time with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,C_o,C_1,\lambda)$, that there holds

$$\mathrm{I}_3 \le \boldsymbol{\gamma}\, d_j^{1-m} |B_j| \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right] \le \boldsymbol{\gamma}\, a_j^{1-m} |B_j| \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right].$$

Inserting the estimates obtained for $\mathrm{I}_1 - \mathrm{I}_3$ above, leads us to the final estimate of I:

$$\begin{aligned}
\mathrm{I} &\le \boldsymbol{\gamma}\, |B_j|^{\frac{2}{N}} a_j^{1-m} |B_j| \left[\varepsilon + \boldsymbol{\gamma}_\varepsilon \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right]^{1+\lambda - m(1-\lambda)} \right] \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right] \\
&= \boldsymbol{\gamma}\, |Q_j| \left[\varepsilon + \boldsymbol{\gamma}_\varepsilon \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right]^{1+\lambda - m(1-\lambda)} \right] \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right] \\
&= \boldsymbol{\gamma} \varepsilon |Q_j| \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right] + \boldsymbol{\gamma}_\varepsilon |Q_j| \left[\kappa + \frac{\mu(2Q_j)}{d_j |B_j|} \right]^{2+\lambda - m(1-\lambda)},
\end{aligned} \tag{4.22}$$

for constants $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,m,C_o,C_1,\lambda)$ and $\boldsymbol{\gamma}_\varepsilon = \boldsymbol{\gamma}_\varepsilon(N,m,C_o,C_1,\lambda,\varepsilon)$. Now, we turn our attention to the estimate of the term II in (4.21). Using Gagliardo-Nirenberg's inequality from Lemma 2.1 with $p = 2$ and $q = \frac{2(1+\lambda)}{1-\lambda}$ and $r = \frac{2\lambda N}{1-\lambda}$ we find that

$$\mathrm{II} = \iint_{Q_j} \left| V_\lambda \left(\left(\frac{a_j}{d_j} \right)^{1-m} \frac{(u^m - a_j^m)_+}{d_j^m} \right) \right|^{\frac{2(1+\lambda)}{1-\lambda}} dxdt \le \boldsymbol{\gamma}\, \mathrm{II}_1 \big(\mathrm{II}_2 + \mathrm{II}_3 \big),$$

where we have set

$$\mathrm{II}_1 := \bigg[\sup_{t\in\Lambda_j}\int_{B_j\times\{t\}}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^{\frac{2\lambda N}{1-\lambda}}dx\bigg]^{\frac{2}{N}},$$

$$\mathrm{II}_2 := \frac{a_j^{1-m}}{\theta_j}\iint_{Q_j}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^2 dxdt,$$

$$\mathrm{II}_3 := \frac{a_j^{1-m}}{\theta_j}\iint_{Q_j}\bigg|DV_\lambda\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{d_j^m}\bigg)\bigg|^2 dxdt.$$

For the estimate of II_1 we in turn apply Lemma 2.2, Hölder's inequality (note that $\lambda\le\frac{1}{N}$ by (4.1)), Lemma 2.3 and the energy estimate (4.19). Again, to simplify the notation we define $B_j^+(t) := B_j\times\{t\}\cap\{u>a_j\}$. In this way, we obtain for any $\varepsilon\in(0,1]$ that

$$\begin{aligned}
\mathrm{II}_1 &\le \bigg[\sup_{t\in\Lambda_j}\int_{B_j^+(t)}\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\bigg)^{\lambda N}dx\bigg]^{\frac{2}{N}}\\
&\le |B_j|^{\frac{2}{N}}\bigg[\sup_{t\in\Lambda_j}\frac{1}{|B_j|}\int_{B_j^+(t)}\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\,dx\bigg]^{2\lambda}\\
&\le |B_j|^{\frac{2}{N}}\bigg[\varepsilon+\sup_{t\in\Lambda_j}\frac{\boldsymbol{\gamma}}{\varepsilon d_j|B_j|}\int_{B_j^+(t)}\int_a^u\bigg[1-\bigg(1+\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{s^m-a_j^m}{d_j^m}\bigg)^{-\lambda}\bigg]ds\,dx\bigg]^{2\lambda}\\
&\le |B_j|^{\frac{2}{N}}\bigg[\varepsilon+\frac{\boldsymbol{\gamma}}{\varepsilon}\bigg[\kappa+\frac{\mu(2Q_j)}{d_j|B_j|}\bigg]\bigg]^{2\lambda},
\end{aligned}$$

for a constant $\boldsymbol{\gamma}=\boldsymbol{\gamma}(m,\lambda)$. To estimate the second term II_2 we use Lemma 2.2 again, the elementary inequality $a_j^{1-m}(u^m-a_j^m)\le u-a_j$, which holds true on the set $\{u>a_j\}$, (4.14), (4.15), (4.9) and (4.12). This leads us to

$$\begin{aligned}
\mathrm{II}_2 &\le \frac{a_j^{1-m}}{\theta_j}\iint_{Q_j\cap\{u>a_j\}}\bigg(\Big(\frac{a_j}{d_j}\Big)^{1-m}\frac{u^m-a_j^m}{d_j^m}\bigg)^{1-\lambda}dxdt\\
&\le \frac{a_j^{1-m}}{\theta_j}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{1-\lambda}dxdt\\
&\le \frac{2a_j^{1-m}}{\theta_j}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}dxdt\\
&\le \frac{2a_j^{1-m}|Q_{j-1}|}{\theta_j}\frac{1}{|Q_{j-1}|}\iint_{Q_{j-1}\cap\{u>a_{j-1}\}}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}dxdt\\
&\le \frac{\boldsymbol{\gamma}a_j^{1-m}|Q_{j-1}|\kappa}{\theta_j}\le\boldsymbol{\gamma}a_j^{1-m}|B_j|\kappa,
\end{aligned}$$

where the constant $\boldsymbol{\gamma}$ depends only on N,m and λ. Finally, by the energy estimate (4.19) we immediately have

$$\mathrm{II}_3\le\boldsymbol{\gamma}\,a_j^{1-m}|B_j|\bigg[\kappa+\frac{\mu(2Q_j)}{d_j|B_j|}\bigg],$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, \nu, L, \lambda)$. Inserting the estimates for II_1 – II_3 into the inequality for II we find

$$\begin{aligned} \mathrm{II} &\le \boldsymbol{\gamma}\, |B_j|^{\frac{2}{N}} a_j^{1-m} |B_j| \bigg[\varepsilon + \varepsilon^{-1}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg]\bigg]^{2\lambda}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg] \\ &= \boldsymbol{\gamma}\, |Q_j| \bigg[\varepsilon + \varepsilon^{-1}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg]\bigg]^{2\lambda}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg] \\ &\le \boldsymbol{\gamma}\varepsilon^{2\lambda}|Q_j|\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg] + \frac{\boldsymbol{\gamma}|Q_j|}{\varepsilon^{2\lambda}}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg]^{1+2\lambda}, \end{aligned} \tag{4.23}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda)$. Joining (4.22) and (4.23) with (4.21), we obtain

$$\begin{aligned} \kappa &\le \boldsymbol{\gamma}\tilde{\varepsilon}^{1+\lambda}\kappa + \boldsymbol{\gamma}_{\tilde{\varepsilon}}\left(\varepsilon + \varepsilon^{2\lambda}\right)\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg] + \frac{\boldsymbol{\gamma}_{\tilde{\varepsilon}}}{\varepsilon^{2\lambda}}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg]^{1+2\lambda} \\ &\quad + \boldsymbol{\gamma}_{\varepsilon}\boldsymbol{\gamma}_{\tilde{\varepsilon}}\bigg[\kappa + \frac{\mu(2Q_j)}{d_j|B_j|}\bigg]^{2+\lambda-m(1-\lambda)} \\ &\le \bigg[\boldsymbol{\gamma}\,\tilde{\varepsilon}^{1+\lambda} + \boldsymbol{\gamma}_{\tilde{\varepsilon}}\big(\varepsilon + \varepsilon^{2\lambda}\big) + \frac{\boldsymbol{\gamma}_{\tilde{\varepsilon}}\,\kappa^{2\lambda}}{\varepsilon^{2\lambda}} + \boldsymbol{\gamma}_{\varepsilon}\boldsymbol{\gamma}_{\tilde{\varepsilon}}\kappa^{1+\lambda-m(1-\lambda)}\bigg]\kappa \\ &\quad + \boldsymbol{\gamma}_{\varepsilon}\boldsymbol{\gamma}_{\tilde{\varepsilon}}\bigg[\frac{\mu(2Q_j)}{d_j|B_j|} + \bigg(\frac{\mu(2Q_j)}{d_j|B_j|}\bigg)^{2+\lambda-m(1-\lambda)}\bigg] \end{aligned}$$

holds true with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda)$, and moreover $\boldsymbol{\gamma}_{\varepsilon}$ depending on the same quantities and additionally on ε, and finally $\boldsymbol{\gamma}_{\tilde{\varepsilon}}$ depending on the same quantities and additionally $\tilde{\varepsilon}$. We note, that we used $1 < 1 + 2\lambda \le 2 + \lambda - m(1-\lambda)$ in the last line. Note also that $\varepsilon, \tilde{\varepsilon}, \kappa \in (0,1)$ are still at our disposal. Now we perform the choices of these parameters. We first choose $\tilde{\varepsilon}$ to satisfy $\boldsymbol{\gamma}\tilde{\varepsilon}^{1+\lambda} = \frac{1}{6}$. This fixes $\boldsymbol{\gamma}_{\tilde{\varepsilon}}$ in dependence on n, m, C_o, C_1 and λ. Next, we choose ε such that $\boldsymbol{\gamma}_{\tilde{\varepsilon}}(\varepsilon + \varepsilon^{2\lambda}) = \frac{1}{6}$, which fixes $\boldsymbol{\gamma}_{\varepsilon}$ in dependence on the same parameters. Having fixed $\tilde{\varepsilon}$ and ε, we choose κ such that $\boldsymbol{\gamma}_{\varepsilon}\kappa^{2\lambda}/\varepsilon^{2\lambda} + \boldsymbol{\gamma}_{\varepsilon}\boldsymbol{\gamma}_{\tilde{\varepsilon}}\kappa^{1+\lambda-m(1-\lambda)} \le \frac{1}{6}$. Note that up to now ε and κ depend only on N, m, C_o, C_1 and λ. With these choices, the first term on the right-hand side can be re-absorbed into the left-hand side, so that the preceding inequality gives

$$\kappa \le \boldsymbol{\gamma}\bigg[\frac{\mu(2Q_j)}{d_j|B_j|} + \bigg(\frac{\mu(2Q_j)}{d_j|B_j|}\bigg)^{2+\lambda-m(1-\lambda)}\bigg]$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda)$. Now, we distinguish whether $\mu(2Q_j)/(d_j|B_j|) \le 1$ or not. If this holds true, we have

$$\kappa \le 2\boldsymbol{\gamma}\,\frac{\mu(2Q_j)}{d_j|B_j|} \quad\Longleftrightarrow\quad d_j \le \frac{2\boldsymbol{\gamma}}{\kappa}\,\frac{\mu(2Q_j)}{|B_j|}\,.$$

In the other case, we obtain

$$\kappa \le 2\boldsymbol{\gamma}\bigg(\frac{\mu(2Q_j)}{d_j|B_j|}\bigg)^{2+\lambda-m(1-\lambda)} \quad\Longleftrightarrow\quad d_j \le \Big(\frac{2\boldsymbol{\gamma}}{\kappa}\Big)^{\frac{1}{2+\lambda-m(1-\lambda)}}\,\frac{\mu(2Q_j)}{|B_j|}\,.$$

Joining the two cases, and taking into account the inclusion (4.10), we finally arrive at the claim (4.13); more precisely, we have shown that d_j is bounded by the second summand of the right-hand side of (4.13). This, finally proves the claim.

4.4. **Potential estimates.** In this section we provide the argument, which finally leads to the potential estimate (1.9). We assume that $z_o \in E_T$ is a Lebesgue point of u, in particular that $u(z_o) < \infty$. We define

$$\mathcal{S} := \{j \in \mathbb{N}_0 : a_j \le u(z_o)\},$$

and assume that there exists some $\ell \in \mathcal{S}$ with $\ell \ge 1$. The other case will be treated separately. Since the sequence $\{a_j\}$ is increasing, we know that $j \in \mathcal{S}$ for any index $j < \ell$. Summing the inequalities (4.13) for $j = 0, \dots, \ell$ and recalling the definition of $d_j = a_{j+1} - a_j$ we find that

$$\begin{aligned}
a_{\ell+1} - a_o &= \sum_{j=0}^{\ell} \big(a_{j+1} - a_j\big) \\
&\le \tfrac12 \sum_{j=0}^{\ell} \big(a_j - a_{j-1}\big) + \boldsymbol{\gamma} \sum_{j=0}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{|B_j|} \\
&= \tfrac12 a_\ell + \boldsymbol{\gamma} \sum_{j=0}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{|B_j|} \\
&\le \tfrac12 a_{\ell+1} + \boldsymbol{\gamma} \sum_{j=0}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{|B_j|}.
\end{aligned}$$

Recalling the definition of $|B_j| = \boldsymbol{\alpha}_N \theta_j^{\frac N2} a_j^{\frac{N(m-1)}{2}}$ and $\{0, \dots, \ell\} \subset \mathcal{S}$, and using Young's inequality we conclude that

$$\begin{aligned}
a_{\ell+1} &\le 2a_o + \boldsymbol{\gamma} \sum_{j=0}^{\ell} a_j^{\frac{N(1-m)}{2}} \frac{\mu(2Q_{r_j,\theta_j})}{\theta_j^{\frac N2}} \\
&\le 2a_o + \boldsymbol{\gamma}\, u(z_o)^{\frac{N(1-m)}{2}} \sum_{j=0}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{\theta_j^{\frac N2}} \\
&\le \tfrac12 u(z_o) + 2a_o + \boldsymbol{\gamma} \Bigg[\sum_{j=0}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{\theta_j^{\frac N2}}\Bigg]^{\frac{2}{\beta}}.
\end{aligned}$$

Next, we estimate the sum involving the measure μ in terms of a the truncated Riesz potential. Using the definitions of r_j and θ_j we find that

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{\mu(2Q_{r_j,\theta_j})}{\theta_j^{\frac n2}} &= \Big(\frac{r^2}{\theta}\Big)^{\frac N2} \sum_{j=0}^{\infty} \frac{\mu(Q_{r_{j-1}, r_{j-1}^2\theta/r^2})}{r_j^N} \\
&\le \Big(\frac{r^2}{\theta}\Big)^{\frac N2} \sum_{j=0}^{\infty} \int_{r_{j-1}}^{r_{j-2}} \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{r_j^N} \frac{d\varrho}{r_{j-2} - r_{j-1}} \\
&\le \Big(\frac{r^2}{\theta}\Big)^{\frac N2} 2^{2N+1} \sum_{j=0}^{\infty} \int_{r_{j-1}}^{r_{j-2}} \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{\varrho^N} \frac{d\varrho}{\varrho} \\
&= 2^{2N+1} \Big(\frac{r^2}{\theta}\Big)^{\frac N2} \int_0^{4r} \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{\varrho^n} \frac{d\varrho}{\varrho} \\
&= 2^{2N+1} \Big(\frac{r^2}{\theta}\Big)^{\frac N2} \mathbf{I}_2(z_o, 4r, \theta).
\end{aligned}$$

Therefore, we conclude that

$$a_{\ell+1} \le \tfrac{1}{2}u(z_o) + 2a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}} . \tag{4.24}$$

The preceding inequality holds true for any $\ell \in \mathcal{S}$ with $\ell \ge 1$. We now distinguish between the cases $\mathcal{S} = \mathbb{N}_0$ and $\mathcal{S} \neq \mathbb{N}_0$. In the case $\mathcal{S} = \mathbb{N}_0$, we know that (4.24) holds for any $\ell \in \mathbb{N}$ and therefore

$$a_\infty := \lim_{\ell\to\infty} a_\ell \le \tfrac{1}{2}u(z_o) + 2a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}} < \infty .$$

The definition of d_j implies that $d_j \to 0$ as $j \to \infty$. Recalling the definitions of r_j, θ_j, $\boldsymbol{k}_j$ and the uniform bounds (4.9), that is $\boldsymbol{k}_j \le \kappa$, we conclude that

$$\begin{aligned}
\big(u(z_o) - a_\infty\big)_+^{1+\lambda} &= \lim_{j\to\infty} \iint_{Q_j} (u - a_j)_+^{1+\lambda}\, dxdt \\
&= \lim_{j\to\infty} \frac{d_j^{1+\lambda}}{|Q_j|} \iint_{Q_j\cap\{u>a_j\}} \Big(\frac{u-a_j}{d_j}\Big)^{1+\lambda} dxdt \\
&\le \kappa \lim_{j\to\infty} d_j^{1+\lambda} = 0 .
\end{aligned}$$

This, however, implies

$$u(z_o) \le a_\infty \le \tfrac{1}{2}u(z_o) + 2a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}},$$

and since $u(z_o) < \infty$, we conclude that

$$u(z_o) \le 4a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}}. \tag{4.25}$$

holds true in the case $\mathcal{S} = \mathbb{N}_0$.

Now we deal with the other case, i.e. when $\mathcal{S} \neq \mathbb{N}_0$. In this case we define $\tilde\ell := \min(\mathbb{N}_0 \setminus \mathcal{S})$. This means that $\tilde\ell$ is the smallest integer for which $a_{\tilde\ell} > u(z_o)$. If $\tilde\ell \ge 2$ we can apply (4.24) with $\ell := \tilde\ell - 1$ and conclude that

$$u(z_o) \le a_{\tilde\ell} \le \tfrac{1}{2}u(z_o) + 2a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}}$$

and hence

$$u(z_o) \le 4a_o + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}}. \tag{4.26}$$

Otherwise, either $\tilde\ell = 1$ or $\tilde\ell = 0$. In the first case, by (4.11) we have

$$u(z_o) < a_1 \le B a_o, \tag{4.27}$$

while in the second case, i.e. when $\tilde\ell = 0$, we directly have

$$u(z_o) < a_o. \tag{4.28}$$

We recall that B depends only on N, m, λ. Combining the estimates (4.25) – (4.28), and recalling the definition of a_o we finally arrive at the following bound for $u(z_o)$:

$$u(z_o) \le \boldsymbol{\gamma}\Big(\frac{\theta}{r^2}\Big)^{\frac{1}{1-m}} + \boldsymbol{\gamma}\bigg(\frac{1}{\theta^{1+\frac{N}{2}}} \iint_{Q_{r,\theta}} u^{1+\lambda}\, dxdt\bigg)^{\frac{2}{\beta+2\lambda}} + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{N}{\beta}} \mathbf{I}_2(z_o, 4r, \theta)^{\frac{2}{\beta}}$$

with a constant $\boldsymbol{\gamma}$ depending only on N, m, C_o, C_1, and λ. This proves the claim.

5. Further Energy Estimates

For a real number k and a function $v \in L^1(E)$ we consider the *truncations* of v defined as follows:

$$(v-k)_+ \equiv \max\{v-k;0\}, \qquad (v-k)_- \equiv \max\{-(v-k);0\}.$$

The following energy estimates for $(u-k)_-$ and $(u-k)_+$ are quite standard and can for example be retrieved from [10]. The precise statement is as follows.

Proposition 5.1. *There exists a positive constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m)$, such that there holds: Whenever u is a non-negative weak energy solution of* (1.2) *in E_T, in the sense of Definition* 1.1*, then for every cylinder $Q_{\varrho,\theta}(z_o) \subset E_T$, every $k > 0$, and every cutoff function $\zeta \in W^{1,\infty}(Q_{\varrho,\theta}(z_o), [0,1])$ vanishing on $\partial B_\varrho(x_o) \times (t_o - \theta, t_o)$, we have*

$$\begin{aligned}
&\sup_{t_o-\theta<t\le t_o} \int_{B_\varrho(x_o)\times\{t\}} (u-k)_-^2 \zeta^2\,dx + C_o k^{m-1} \iint_{Q_{\varrho,\theta}(z_o)} \big|D[(u-k)_-\zeta]\big|^2\,dxdt \\
&\qquad \le \boldsymbol{\gamma} k \int_{B_\varrho(x_o)\times\{t_o-\theta\}} (u-k)_- \zeta^2\,dx + \boldsymbol{\gamma} k^2 \iint_{Q_{\varrho,\theta}(z_o)} \chi_{\{u<k\}} \zeta |\zeta_t|\,dxdt \\
&\qquad\quad + \boldsymbol{\gamma}_1 k^{m+1} \iint_{Q_{\varrho,\theta}(z_o)} \chi_{\{u<k\}} |D\zeta|^2\,dxdt
\end{aligned} \tag{5.1}$$

where $\boldsymbol{\gamma}_1 := \boldsymbol{\gamma}(m)(C_o + C_1^2/C_o)$. Moreover, we have

$$\begin{aligned}
&\sup_{t_o-\theta<t\le t_o} \int_{B_\varrho(x_o)\times\{t\}} (u-k)_+^2 \zeta^2\,dx - \int_{B_\varrho(x_o)\times\{t_o-\theta\}} (u-k)_+^2 \zeta^2\,dx \\
&\qquad\qquad + C_o \iint_{Q_{\varrho,\theta}(z_o)} |u|^{m-1} |D(u-k)_+|^2 \zeta^2\,dxdt \\
&\le \boldsymbol{\gamma} \iint_{Q_{\varrho,\theta}(z_o)} \big(u-k\big)_+^2 \zeta |\zeta_t|\,dxdt + \boldsymbol{\gamma}_1 \iint_{Q_{\varrho,\theta}(z_o)} |u|^{m-1} (u-k)_+^2 |D\zeta|^2\,dxdt \\
&\qquad\qquad + \boldsymbol{\gamma} \int_{Q_{\varrho,\theta}(z_o)} (u-k)_+\,d\mu.
\end{aligned} \tag{5.2}$$

Proof. After a translation we may assume $(x_o, t_o) = (0,0)$. The proof of (5.1) has been given in [10, Chapter 3, § 9], taking in (1.6) the testing function

$$\varphi_- = -\zeta^2 \,(u^m - k^m)_-$$

over $B_\varrho \times (-\theta, t]$, where $-\theta < t \le 0$. The use of $-(u^m - k^m)_-$ in this testing function is justified, modulo a mollification procedure with respect to t, as explained in detail in [2, § 2, 3]. We omit the details, since the procedure is quite standard. With this respect, all the further computations are done on a formal basis, when writing u_t. Moreover, we take into account that

$$-\int_{B_\varrho\times(-\theta,t]} (u^m - k^m)_- \zeta^2\,d\mu \le 0\,,$$

and therefore, it can be discarded.

The proof of (5.2) has been given in [10, Appendix B, § 2], by taking in (1.6) the testing function

$$\varphi_+ = \zeta^2\,(u-k)_+$$

over $B_\varrho \times (-\theta, t]$, where $-\theta < t \le 0$. The same remarks as above hold true for the mollification process with respect to t. However, now we have

$$\int_{B_\varrho \times (-\theta,t]} (u-k)_+ \zeta^2 \, d\mu \ge 0 \,,$$

and the term cannot be discarded in the computations. □

In the following Lemma we consider a weak energy solution of (1.2) in E_T and a general cylinder $Q_{\varrho,\theta}(z_o) \Subset E_T$. From our potential estimate in Theorem 1.1 we already know that $\sup_{Q_{\varrho,\theta}(z_o)} u < \infty$ and therefore also

$$H := \sup_{Q_{\varrho,\theta}(z_o)} (u-k)_+ < \infty$$

for any $k \ge 0$. Without loss of generality we can restrict our considerations to values $k < \sup_{Q_{\varrho,\theta}(z_o)} u$. Otherwise, we would have $H = 0$. Finally, let $0 < \mathbf{c} < \min\{1, H\}$. We recall the definition of the Logarithmic function $\psi_{(H,k,\mathbf{c})}$ from (2.3) and consider for $z \in Q_{\varrho,\theta}(z_o)$ the function

$$\psi(u)(z) := \big(\psi_{(H,k,\mathbf{c})} \circ u\big)(z) = \ln_+ \left[\frac{H}{H - (u(z)-k)_+ + \mathbf{c}} \right],$$

which will be used in the formulation of the following Lemma.

Proposition 5.2. *There exists a constant $\boldsymbol{\gamma}$, depending only on N, m, C_o, C_1, such that for any weak energy solution u of (1.2), in E_T in the sense of Definition 1.1, for every cylinder $Q_{\varrho,\theta}(z_o) \Subset E_T$, and for every level $k \ge 0$, there holds:*

$$\begin{aligned} \sup_{t_o-\theta<t<t_o} & \int_{B_\varrho(x_o)\times\{t\}} \psi^2(u)\zeta^2 \, dx \\ & \le \int_{B_\varrho(x_o)\times\{t_o-\theta\}} \psi^2(u)\zeta^2 \, dx + \boldsymbol{\gamma} \iint_{Q_{\varrho,\theta}(z_o)} u^{m-1}\psi(u)|D\zeta|^2 \, dxdt \\ & \quad + \frac{2}{\mathbf{c}} \ln\Big(\frac{H}{\mathbf{c}}\Big) \int_{Q_{\varrho,\theta}(z_o)} \chi_{\{u>k\}} \, d\mu. \end{aligned}$$

Here, $\zeta \in W_0^{1,\infty}(B_\varrho(x_o), [0,1])$ is a cutoff function independent of t. The constant $\boldsymbol{\gamma}$ is of the form $4mC_1^2/C_o$.

Proof. Without loss of generality we assume that $(x_o, t_o) = (0,0)$, and omit henceforth the center in our notation. We work within the cylinder $Q^\tau \equiv B_\varrho \times (-\theta, \tau)$, with $-\theta < \tau < 0$. In the weak formulation (1.6) we choose

$$\varphi = \zeta^2 \big[\psi^2\big]'(u) = 2\psi(u)\psi'(u)\zeta^2$$

as testing function. Then, a direct calculation shows $[\psi^2]'' = 2(1+\psi)\psi'^2$. Therefore, we have

$$\big[\psi^2\big]''(u) = \big[2(1+\psi)\psi'^2\big](u) \in L^\infty(Q_{\varrho,\theta}(z_o)).$$

This implies that φ is an admissible testing function, modulo a mollification procedure with respect to time. Note that $\psi(u) \ne 0$ implies that $u > k + \mathbf{c} > 0$ and therefore $|D\varphi| \in L^2(Q_{\varrho,\theta}(z_o))$. Now, since $\psi(u)$ vanishes on the set where $(u-k)_+ = 0$, we find

$$\iint_{Q^\tau} u_t [\psi^2]'(u)\zeta^2 \, dxdt = \int_{B_\varrho\times\{\tau\}} \psi^2(u)\zeta^2 \, dx - \int_{B_\varrho\times\{-\theta\}} \psi^2(u)\zeta^2 \, dx.$$

The term involving the vector-field $\mathbf{A}(x,t,u,Du)$ is estimated with the help of the lower bound $(1.3)_1$ and Young's inequality as follows:

$$\begin{aligned}\iint_{Q^\tau}\mathbf{A}(x,t,u,Du)\cdot D\varphi\,dxdt &\ge 2mC_o\iint_{Q^\tau}\big[(1+\psi)\psi'^2\big](u)u^{m-1}|Du|^2\zeta^2\,dxdt\\ &\quad-4mC_1\iint_{Q^\tau}u^{m-1}|Du|\big[\psi\psi'\big](u)\zeta|D\zeta|\,dxdt\\ &\ge mC_o\iint_{Q^\tau}\big[(1+\psi)\psi'^2\big](u)u^{m-1}|Du|^2\zeta^2\,dxdt\\ &\quad-\frac{4mC_1^2}{C_o}\iint_{Q^\tau}\psi(u)u^{m-1}|D\zeta|^2\,dxd\tau\\ &\ge-\frac{4mC_1^2}{C_o}\iint_{Q^\tau}\psi(u)u^{m-1}|D\zeta|^2\,dxd\tau\end{aligned}$$

Here, we discarded in the last line the positive term. As for the remaining term, i.e. the one with the measure, we observe that

$$\psi(u)\le\ln\Big(\frac{H}{\mathbf{c}}\Big)\quad\text{and}\quad\psi'(u)\le\frac{1}{\mathbf{c}}$$

holds true, and therefore we conclude

$$2\int_{Q^t}\big[\psi\psi'\big](u)\zeta^2\,d\mu\le\frac{2}{\mathbf{c}}\ln\Big(\frac{H}{\mathbf{c}}\Big)\int_{Q_{\varrho,\theta}}\chi_{\{u>k\}}\,d\mu.$$

Here, we also used the fact that $\zeta^2\in[0,1]$. Collecting these estimates and taking the supremum over $\tau\in(-\theta,0)$, establishes the claim of the proposition. □

6. Proof of Theorem 1.2

For a cylinder $Q\Subset E_T$ we define

$$\mathbb{I}^\mu_{2,Q}(r,\theta):=\sup_{z\in Q}\mathbf{I}^\mu_2(z,r,\theta).$$

Theorem 1.2 will be a consequence of the following result.

Proposition 6.1. *There exist $C>1$, $\nu_*\in(0,1)$, and $\delta\in[\frac56,1)$ depending only on the structural constants m, C_o, C_1, and the dimension N, such that with $\eta:=\sqrt{\frac18\nu_*\delta^{1-m}}$ the following assertion holds true: Let u be a weak energy solution of (1.2) in E_T in the sense of Definition 1.1 and $z_o\in E_T$. There exist $\varrho>0$, and $\omega>0$ such that with $\varrho_o:=\varrho$, $\omega_o:=\omega$, and*

$$c_n:=\omega_n^{\frac{m-1}{2}},\quad\varrho_n:=\eta^n\varrho_o,\quad Q_n:=Q_{c_n\varrho_n,\varrho_n^2}(z_o)\tag{6.1}$$

and

$$\omega_{n+1}:=\max\big\{\delta\omega_n,C\,\mathbb{I}^\mu_{2,Q_n}(4c_n\varrho_n,16\varrho_n^2)\big\}$$

for $n\in\mathbb{N}_0$, the assertions

$$Q_n\subset Q_{n-1}\subset\cdots\subset Q_o\subset E_T,\tag{6.2}$$

and

$$\operatorname*{osc}_{Q_n}u\le\omega_n,\quad\textit{for all } n\in\mathbb{N}_0\tag{6.3}$$

hold true. □

Before entering the proof of Proposition 6.1, we assume for the moment that its assertion holds true, and proceed with the proof of Theorem 1.2. The idea of the proof is to convert the discrete decay of the oscillation on the cylinders Q_n into a quantitative decay in terms of the radius.

Proof of Theorem 1.2. By the definition of ω_n, we have

$$\begin{aligned}\omega_n^{\frac{m-1}{2}} &= \min\Big\{(\delta\omega_{n-1})^{\frac{m-1}{2}}, \big(C\,\mathbb{I}^{\mu}_{2,Q_{n-1}}\big(4c_{n-1}\varrho_{n-1}, 16\varrho_{n-1}^2\big)\big)^{\frac{m-1}{2}}\Big\}\\ &\le (\delta\omega_{n-1})^{\frac{m-1}{2}} \le \cdots \le \big(\delta^{n-k}\omega_k\big)^{\frac{m-1}{2}}, \quad \text{for all } k\in\{0,1,\dots,n-1\}.\end{aligned}$$

The definition of c_n therefore yields

$$c_n \le \big(\delta^{n-k}\omega_k\big)^{\frac{m-1}{2}}, \quad \text{for all } k\in\{0,1,\dots,n-1\}.$$

Using this and $\varrho_n = \eta^n\varrho$ in the expression for ω_{n+1} we get

$$\begin{aligned}\omega_{n+1} &\le \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4\omega_o^{\frac{m-1}{2}}(\delta^{\frac{m-1}{2}}\eta)^n\varrho, 16\eta^{2n}\varrho^2\big)\\ &\le \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_o}\big(4\omega_o^{\frac{m-1}{2}}\varrho, 16\varrho^2\big).\end{aligned}\tag{6.4}$$

In the last line we used the inclusion $Q_n \subset Q_o$ from (6.2) and $\delta^{\frac{m-1}{2}}\eta < 1$ which follows from the definition of η and the fact that $\nu_* < 1$. We iterate the preceding inequality starting on the left-hand side with ω_n. This will imply a first rough bound for ω_n. Abbreviating

$$\mathbb{I}_o := \mathbb{I}^{\mu}_{2,Q_o}\big(4\omega_o^{\frac{m-1}{2}}\varrho, 16\varrho^2\big),$$

the iteration gives

$$\omega_n \le \delta^n\omega_o + C\,\mathbb{I}_o\sum_{j=0}^{n-1}\delta^{n-1-j} \le \omega_o + \frac{C\,\mathbb{I}_o}{1-\delta} \quad \text{for all } n\in\mathbb{N}.\tag{6.5}$$

Now, we utilize the inequalities $\eta \le (\tfrac18\nu_*)^{\frac12}$ and $\delta^{\frac{m-1}{2}}\eta \le \tfrac18\nu_*$, which also follows from the definition of η. Instead of (6.4) we now get

$$\omega_{n+1} \le \delta\omega_n + C\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\nu_*}{8}\Big)^{\frac{n}{2}}\varrho, 16\Big(\frac{\nu_*}{8}\Big)^{n}\varrho^2\Big),\tag{6.6}$$

for any $n\in\mathbb{N}_0$. Now, for $\bar\varrho\in(0,\varrho)$ there exists $k\in\mathbb{N}$, such that

$$\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho \le \bar\varrho < \Big(\frac{\nu_*}{8}\Big)^{\frac{k-1}{2}}\varrho.$$

Note that k is uniquely determined by

$$k-1 < \frac{\ln\frac{\bar\varrho}{\varrho}}{\ln\sqrt{\frac18\nu_*}} \le k.\tag{6.7}$$

We now iterate (6.6) starting with ω_n on the left-hand side k-times. This yields

$$\begin{aligned}\omega_n &\le \delta^{n-k}\omega_k + C\sum_{j=k}^{n-1}\mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\nu_*}{8}\Big)^{\frac{j}{2}}\varrho, 16\Big(\frac{\nu_*}{8}\Big)^{j}\varrho^2\Big)\delta^{n-1-j}\\ &\le \delta^{n-k}\omega_k + C\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho, 16\Big(\frac{\nu_*}{8}\Big)^{k}\varrho^2\Big)\sum_{j=k}^{n-1}\delta^{n-1-j}\\ &\le \delta^{n-k}\omega_k + \tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\nu_*}{8}\Big)^{\frac{k}{2}}\varrho, 16\Big(\frac{\nu_*}{8}\Big)^{k}\varrho^2\Big)\end{aligned}$$

$$\leq \delta^{n-k}\omega_k + \tfrac{C}{1-\delta}\, \mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\,\bar\varrho, 16\bar\varrho^2\Big)\,.$$

Here we used the choice of k from before. Now, we consider a fixed $\bar r \in (0,\bar\varrho)$ and argue similarly as in [11, Chapter III, § 3]. First, we fix a number $0<b<\delta$ and choose $\ell\in\mathbb{N}$ such that $b^\ell\bar\varrho \leq \bar r < b^{\ell-1}\bar\varrho$. The number ℓ is uniquely determined by

$$\ell - 1 < \frac{\ln\frac{\bar r}{\bar\varrho}}{\ln b} \leq \ell. \tag{6.8}$$

Then, with $n = k+\ell$ we estimate

$$\delta^{n-k} = \delta^\ell = \exp(\ell\ln\delta) = b^{\alpha\ell} \leq \Big(\frac{\bar r}{\bar\varrho}\Big)^\alpha,$$

where α is defined by

$$\alpha := \frac{\ln\delta}{\ln b} \in (0,1)\,.$$

Up to this point we have shown that there exists $\alpha=\alpha(\delta,b)\in(0,1)$ (note that δ depends only on the data), such that

$$\omega_n \leq \Big(\frac{\bar r}{\bar\varrho}\Big)^\alpha\omega_k + \tfrac{C}{1-\delta}\, \mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\,\bar\varrho, 16\bar\varrho^2\Big)$$

holds true for any $0<\bar r<\bar\varrho<\varrho$, where $n=k+\ell$ and the integers k,ℓ are uniquely defined by (6.7) and (6.8). The strategy now is as follows: We fix $\beta\in(0,1)$ and consider radii $0<\bar r<\varrho$. We choose an intermediate radius $\bar\varrho\in(\bar r,\varrho)$ according to $\bar\varrho=\varrho^{1-\beta}\bar r^\beta$ and determine the integers k,ℓ according to (6.7) and (6.8) which by the particular choice of $\bar\varrho$ is equivalent to

$$k-1<\frac{\beta\ln\frac{\bar r}{\varrho}}{\ln\sqrt{\frac18\nu_*}}\leq k, \quad\text{and}\quad \ell-1<\frac{(1-\beta)\ln\frac{\bar r}{\varrho}}{\ln b}\leq\ell\,. \tag{6.9}$$

Finally, we let $n=k+\ell$. With these choices the last inequality can be re-written as follows:

$$\operatorname*{osc}_{Q_n} u \leq \omega_n \leq \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\omega_k + \tfrac{C}{1-\delta}\, \mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\bar r}{\varrho}\Big)^\beta\varrho, 16\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big).$$

Using the boundedness of ω_k from (6.5) the preceding inequality implies that

$$\begin{aligned}\operatorname*{osc}_{Q_n} u \leq \omega_n &\leq \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\Big[\omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o\Big] + \tfrac{C}{1-\delta}\, \mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\bar r}{\varrho}\Big)^\beta\varrho, 16\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big)\\ &=: \mathrm{I}(\bar r)+\mathrm{II}(\bar r)\end{aligned}$$

holds true. Observe, that both terms of the right-hand side vanish as $\bar r\downarrow 0$. Hence, we can choose $\varrho_o\in(0,\varrho)$ such that $\mathrm{I}(\bar r)+\mathrm{II}(\bar r)\leq\omega_o$ for any $0<\bar r\leq\varrho_o$. Via (6.9) we determine $n_o\in\mathbb{N}$ such that $\omega_n\leq\omega_o$ holds true for $n\geq n_o$. Actually, we can take

$$n_o = \left\lceil \frac{\beta\ln\frac{\varrho_o}{\varrho}}{\ln\sqrt{\frac18\nu_*}}\right\rceil + \left\lceil\frac{(1-\beta)\ln\frac{\varrho_o}{\varrho}}{\ln b}\right\rceil.$$

Then, for $n\geq n_o$, we have, note that $c_o=\omega_o^{\frac{m-1}{2}}$,

$$\widetilde Q_n := B_{c_o\varrho_n}\times(-\varrho_n^2,0]\subset B_{c_n\varrho_n}\times(-\varrho_n^2,0]=Q_n,$$

and therefore

$$\operatorname*{osc}_{\widetilde Q_n} u \leq \Big(\frac{\bar r}{\varrho}\Big)^{\alpha(1-\beta)}\Big[\omega_o + \tfrac{C}{1-\delta}\,\mathbb{I}_o\Big] + \tfrac{C}{1-\delta}\, \mathbb{I}^{\mu}_{2,Q_o}\Big(4\omega_o^{\frac{m-1}{2}}\Big(\frac{\bar r}{\varrho}\Big)^\beta\varrho, 16\Big(\frac{\bar r}{\varrho}\Big)^{2\beta}\varrho^2\Big).$$

From (6.9) we obtain that

$$n-2<\left(\frac{\beta}{\ln\sqrt{\frac{1}{8}\nu_*}}+\frac{(1-\beta)}{\ln b}\right)\ln\frac{\bar r}{\varrho}\le n.$$

The previous inequality motivates to define

$$\sigma:=\left(\frac{\beta}{\ln\sqrt{\frac{1}{8}\nu_*}}+\frac{(1-\beta)}{\ln b}\right)\ln\eta.$$

This choice of σ allows us to estimate the radii ϱ_n from below by

$$\varrho_n=\eta^n\varrho=\varrho e^{n\ln\eta}>\varrho\exp\left(\sigma\ln\frac{\bar r}{\varrho}+2\ln\eta\right)=\eta^2\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\sigma}.$$

Finally we set

$$r:=\eta^2\varrho\Big(\frac{\bar r}{\varrho}\Big)^{\sigma}.$$

Since $Q_{c_o r,r^2}\subset\tilde Q_n$, we conclude that

$$\begin{aligned}\operatorname*{osc}_{Q_{c_o r,r^2}}u\le{}&\frac{1}{\eta^{\frac{2\alpha(1-\beta)}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\alpha(1-\beta)}{\sigma}}\Big[\omega_o+\tfrac{C}{1-\delta}\,\mathbb{I}_o\Big]\\&+\tfrac{C}{1-\delta}\,\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\omega_o^{\frac{m-1}{2}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}},\frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right)\end{aligned}$$

holds true. We now distinguish two cases. First, if $\omega_o\ge1$, we estimate

$$\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\omega_o^{\frac{m-1}{2}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}},\frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right)\le\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}},\frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right).$$

On the other hand, if $\omega_o<1$, we conclude that

$$\begin{aligned}&\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\omega_o^{\frac{m-1}{2}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}},\frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right)\\&\qquad\le\mathbb{I}^{\mu}_{2,Q_o}\left(\frac{4\varrho}{\eta^{\frac{2\beta}{\sigma}}}\omega_o^{\frac{m-1}{2}}\Big(\frac{r}{\varrho}\Big)^{\frac{\beta}{\sigma}},\frac{16\varrho^2}{\eta^{\frac{4\beta}{\sigma}}}\omega_o^{m-1}\Big(\frac{r}{\varrho}\Big)^{\frac{2\beta}{\sigma}}\right).\end{aligned}$$

Joining both cases, a further re-definition of r, leads to consider $\sup_{z\in Q_o}\mathbf{I}^{\mu}_2(z,r,r^2)$, that is, the classical parabolic, truncated Riesz potential, with no intrinsic scaling with respect to time. This proves the continuity of u on $Q_{c_o r,r^2}$. Statements concerning the continuity over a compact set, follow easily by a standard covering argument. □

We now come to the proof of the main Proposition 6.1. We consider a fixed generic point $z_o\in E_T$, and $\varepsilon,R\in(0,1)$ such that $Q_{2R^{1-\varepsilon},4R^2}(z_o)\subset E_T$. Without loss of generality, we may assume that $z_o\equiv(0,0)$. Henceforth we omit the center in our notation and write $Q_{R^{1-\varepsilon},R^2}$ instead of $Q_{R^{1-\varepsilon},R^2}(z_o)$. Now, if

$$\operatorname*{osc}_{Q_{\varrho^{1-\varepsilon},\varrho^2}}u\le\varrho^{\frac{2\varepsilon}{1-m}}\qquad\text{for any }\varrho\in(0,R],$$

there is nothing to prove, since the essential oscillation of u has a power-like decay. Otherwise, there exists a radius $\varrho \in (0, R]$ such that

$$\operatorname*{osc}_{Q_{\varrho^{1-\varepsilon},\varrho^2}} u > \varrho^{\frac{2\varepsilon}{1-m}}.$$

Then, we define

$$\mu_o^+ := \sup_{Q_{\varrho^{1-\varepsilon},\varrho^2}} u, \qquad \mu_o^- := \inf_{Q_{\varrho^{1-\varepsilon},\varrho^2}} u, \qquad \omega_o := \mu_o^+ - \mu_o^- > 0. \tag{6.10}$$

By the preceding inequality and the definition of ω_o, we have that

$$\omega_o^{\frac{1-m}{2}} = \Big(\operatorname*{osc}_{Q_{\varrho^{1-\varepsilon},\varrho^2}} u \Big)^{\frac{1-m}{2}} > \varrho^{\varepsilon},$$

which, letting $c_o := \omega_o^{\frac{m-1}{2}}$, guarantees that

$$Q_o := Q_{c_o\varrho,\varrho^2} \subset Q_{\varrho^{1-\varepsilon},\varrho^2}, \qquad \operatorname*{osc}_{Q_{c_o\varrho,\varrho^2}} u \le \omega_o. \tag{6.11}$$

Thus (6.11) ensures that (6.2)–(6.3) hold for $n = 0$. We remark that the role of introducing the cylinder $Q_{\varrho^{1-\varepsilon},\varrho^2}$, is to ensure that the upper bound for the oscillation in (6.11) holds true for the constructed cylinder $Q_o = Q_{c_o\varrho,\varrho^2}$. The main part of the proof of Proposition 6.1 consists in establishing that at each step of the induction argument, the cylinders Q_n and the essential oscillation of u within them, satisfy the right geometry. Apart from this, ε plays no other role in this context.

The remaining part of the section is devoted to the proof of Proposition 6.1: we will determine constants $\delta, \nu_* \in (0,1)$ and $C > 1$, depending only on the data m, N, C_o, C_1, but independent of u and z_o for which (6.2)–(6.3) hold inductively for all n.

6.1. **The Induction Argument.** We assume that (6.2)–(6.3) hold for some $n \in \mathbb{N}_0$. Without loss of generality we can assume that there holds

$$\mu_n^+ - \mu_n^- \ge \tfrac{3}{4}\omega_n. \tag{6.12}$$

Otherwise, $(6.3)_{n+1}$ trivially holds true. In the following we remove the index n in our notation by writing

$$\varrho = \varrho_n, \qquad \omega = \omega_n, \qquad c = c_n = \omega^{\frac{m-1}{2}} \qquad Q_{c\varrho,\varrho^2} = Q_n.$$

Moreover, we set

$$\mu^+ = \sup_{Q_{c\varrho,\varrho^2}} u = \mu_n^+, \qquad \mu^- = \inf_{Q_{c\varrho,\varrho^2}} u = \mu_n^-.$$

By $(6.3)_n$ and (6.12) we have

$$\omega \ge \mu^+ - \mu^- \ge \tfrac{3}{4}\omega. \tag{6.13}$$

Denote by a, ξ and A fixed numbers in $(0,1]$, and let

$$\tilde{c} := A^{\frac{1-m}{2}} c = \Big(\frac{\omega}{A}\Big)^{\frac{m-1}{2}}. \tag{6.14}$$

Then, for radii $r \in (0, \frac{\varrho}{2}]$ we have the inclusions

$$Q_{2\tilde{c}r,4r^2} \subset Q_{c\varrho,\varrho^2}, \qquad Q_{4\tilde{c}r,16r^2} \subset E_T.$$

From [10] we recall the following DeGiorgi-type result.

Lemma 6.1. *Let u be a weak energy solution of* (1.2)*, in E_T in the sense of Definition* 1.1*. There exists a positive number ν_-, depending on the parameters a, ξ, A and the data m, N, C_o, C_1, such that if*

$$\big|\{u \le \xi\omega\} \cap Q_{2\tilde{c}r,4r^2}\big| \le \nu_-\big|Q_{2\tilde{c}r,4r^2}\big|,$$

then

$$u \ge a\xi\omega \qquad \textit{a.e. in } Q_{\tilde{c}r,r^2}.$$

Proof. The proof follows from the energy estimate (5.1) in Proposition 5.1; see [10, Lemma 10.1, Chapter 3]. Indeed, in [10] the proof is done considering the intrinsic scaling with respect to the t variable: it is a matter of straightforward calculations to adapt that result to our context, where the intrinsic scaling is performed with respect to the x variable. Moreover, note that in the case considered here, the constant C from the structural conditions [10, Chapter 3, (5.2)] is zero, and therefore, the first alternative $C\varrho > 1$ from Lemma 10.1 will never occur. □

Remark 6.1. The functional dependence of ν_- on the indicated parameters can be retrieved form [10, Chapter 3, (10.5)]. We have

$$\nu_- = \boldsymbol{\gamma}^{-1}(1-a)^{N+2}\frac{(\xi A)^{1-m}}{[1+(\xi A)^{1-m}]^{\frac{N+2}{2}}} \tag{6.15}$$

for a quantitative constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, N, C_o, C_1) > 1$, independent of a, ξ and A. □

Lemma 6.2. *Let u be a weak energy solution of* (1.2)*, in E_T in the sense of Definition* 1.1*. Suppose that $\xi \in (0, \frac{1}{2}]$, and*

$$\tfrac{1}{2}\omega \le \mu^+ - \tfrac{1}{4}\omega \le \tfrac{5}{6}\omega. \tag{6.16}$$

There exist $\nu_+ \in (0,1)$ and $B > 1$ both depending on a, A and the data m, N, C_o, C_1, such that if

$$\big|\{u \ge \mu^+ - \xi\omega\} \cap Q_{2\tilde{c}r,4r^2}\big| \le \nu_+\big|Q_{2\tilde{c}r,4r^2}\big|, \tag{6.17}$$

then either

$$\xi\omega < B\,\mathbb{I}^{\mu}_{2,Q_{2\tilde{c}r,4r^2}}\big(4\tilde{c}r, 16r^2\big) \tag{6.18}$$

or

$$u \le \mu^+ - a\xi\omega \qquad \textit{a.e. in } Q_{\tilde{c}r,r^2}. \tag{6.19}$$

Remark 6.2. The energy estimates have been proved working in cylinders with the scaling

$$Q_\varrho^{(a)} = B_{a^{\frac{m-1}{2}}\varrho}(x_o) \times (t_o - \varrho^2, t_o).$$

In the following we work in cylinders

$$Q_\varrho^{(\frac{\omega}{A})} = B_{(\frac{\omega}{A})^{\frac{m-1}{2}}\varrho}(x_o) \times (t_o - \varrho^2, t_o).$$

The presence of the measure μ prevents the use of the classical DeGiorgi iteration scheme, as adapted to singular parabolic equations by DiBenedetto (see [11, Chapter IV]). Therefore, we use the Kilpeläinen-Malý approach, which has been successfully used in the singular case of parabolic p-Laplacian measure data problems in [23]. Throughout the proof we assume without loss of generality that

$$\xi\omega \ge B\,\mathbb{I}^{\mu}_{2,Q_{2\tilde{c}r,4r^2}}(4\tilde{c}r, 16r^2), \tag{6.20}$$

where $B > 1$ will be determined later in a universal way. Otherwise, the claim of the Lemma holds trivially. Let $z_1 = (x_1, t_1) \in Q_{\tilde{c}r,r^2}$ be an arbitrary point. In the sequel we establish that

$$u(z_1) \le \mu^+ - a\xi\omega, \tag{6.21}$$

which of course implies the claim of the Lemma. The proof of (6.21) is divided into several steps.

Step 1: The iteration scheme. For $j \in \{-1, 0, 1, \dots\}$ we define (recall the definition of $\tilde{c}$ from (6.14))

$$r_j := \frac{r}{2^j}, \quad B_j := B_{\tilde{c}r_j}(x_1), \quad Q_j := B_j \times \big(t_1 - r_j^2, t_1\big],$$

and

$$\alpha_j = \int_0^{\tilde{c}r_j} \frac{\mu\big(Q_{\varrho,\varrho^2/\tilde{c}^2}(z_1)\big)}{\varrho^N} \frac{d\varrho}{\varrho}.$$

Moreover, we set

$$a_o = \mu^+ - \xi\omega.$$

We now assume that for $j \ge 0$ numbers $a_o, a_1, \dots, a_j$ have already been selected. Then we choose a_{j+1} in the following way. We let $\lambda \in (0, \frac{\beta}{N(1+m)}]$, where β is defined in (4.1) and define for values $a \in (a_j, \infty)$ the function

$$\mathbf{K}_j(a) = \frac{1}{\big|\frac12 Q_j\big|} \iint_{\frac12 Q_j \cap \{u > a_j\}} \left(\frac{u - a_j}{a - a_j}\right)^{1+\lambda} dxdt. \tag{6.22}$$

We note that the function $(a_j, \infty) \ni a \mapsto \mathbf{K}_j(a)$ is a monotone decreasing, continuous function, with $\lim_{a \downarrow a_j} \mathbf{K}_j(a) = +\infty$ and $\lim_{a \to \infty} \mathbf{K}_j(a) = 0$. Now, let $\kappa \in (0,1)$ be a fixed parameter. Later on κ will be fixed in dependence on N, m, C_0, C_1, λ. Now, if

$$\mathbf{K}_j\big(a_j + \tfrac14(\alpha_{j-1} - \alpha_j)\big) < \kappa$$

we choose a_{j+1} as

$$a_{j+1} := a_j + \tfrac14(\alpha_{j-1} - \alpha_j).$$

Otherwise, i.e. if

$$\mathbf{K}_j\big(a_j + \tfrac14(\alpha_{j-1} - \alpha_j)\big) \ge \kappa,$$

we choose $a_{j+1} > a_j + \frac14(\alpha_{j-1} - \alpha_j)$ according to

$$\mathbf{K}_j\big(a_{j+1}\big) = \kappa.$$

The choice of a_{j+1} is possible due to the above mentioned properties of $\mathbf{K}_j$. In any case we have

$$\mathbf{K}_j\big(a_{j+1}\big) \le \kappa \quad \text{and} \quad a_{j+1} \ge a_j + \tfrac14(\alpha_{j-1} - \alpha_j). \tag{6.23}$$

Step 2: Preliminary bounds for a_j. Here, we establish that

$$a_j < \tfrac12(\mu^+ + a_{j-1}) - \tfrac14 B\alpha_{j-1}, \quad \text{and} \quad a_j < \mu^+ - \tfrac12 B\alpha_{j-1}, \tag{6.24}$$

hold true for any $j \in \mathbb{N}$. We begin with the proof of (6.24) for $j = 1$. By the definition of a_o and (6.20) we have

$$a_o = \mu^+ - \xi\omega \le \mu^+ - B\alpha_{-1}.$$

This implies that

$$\bar{a} := \tfrac12(\mu^+ + a_o) - \tfrac14 B\alpha_o > a_o + \tfrac14(\alpha_{-1} - \alpha_o).$$

Moreover, using again (6.20) and the definition of a_o we conclude that

$$\bar a - a_o = \tfrac12\xi\omega - \tfrac14 B\alpha_o \ge \tfrac12\xi\omega - \tfrac14\xi\omega = \tfrac14\xi\omega.$$

Using in turn (6.16), (6.17) and $\lambda \le \frac1N \le 1$ we end up with the estimate

$$\begin{aligned}\mathbf{K}_o(\bar a) &= \frac{1}{|\frac12 Q_o|}\iint_{\frac12 Q_o\cap\{u>a_o\}}\Big(\frac{u-a_o}{\bar a - a_o}\Big)^{1+\lambda}dxdt\\ &\le \frac{2^{N+2}}{\boldsymbol{\alpha}_N\tilde c^N r^{N+2}}\Big(\frac{\mu^+-(\mu^+-\xi\omega)}{\xi\omega/4}\Big)^{1+\lambda}\big|\{u>a_o\}\cap Q_o\big|\\ &\le \frac{4^{1+\lambda}2^{N+2}}{\boldsymbol{\alpha}_N\tilde c^N r^{N+2}}\big|\{u>a_o\}\cap Q_{2\tilde c r,4r^2}\big|\\ &\le \frac{4^{1+\lambda}2^{N+2}\nu_+}{\boldsymbol{\alpha}_N\tilde c^N r^{N+2}}\big|Q_{2\tilde c r,4r^2}\big| \le 2^{2(N+4)}\,\nu_+.\end{aligned}$$

Now, we choose ν_+ small enough to satisfy

$$2^{2(N+4)}\nu_+ \le \tfrac12\kappa \quad\iff\quad \nu_+ \le 2^{-(2N+9)}\kappa. \tag{6.25}$$

This fixes ν_+ in dependence of N, and κ. Note that κ will be chosen later in the course of the proof in a universal way. With this choice of ν_+, we have

$$\mathbf{K}_o(\bar a) \le \tfrac12\kappa.$$

The choice of a_1 and the monotonicity of $a\mapsto \mathbf{K}_o(a)$ imply that $a_o+\frac14(\alpha_{-1}-\alpha_o)\le a_1 < \bar a$. By the definition of $\bar a$ this proves $(6.24)_1$. The second inequality in (6.24) follows from the first one, using the preliminary bound for a_o from above. This finally proves $(6.24)_{j=1}$.

Now, we assume that (6.24) holds for $1,2,\dots,j$ with $j\in\mathbb{N}$. From the definition of α_j one can easily retrieve by simple computations that there holds

$$\begin{cases} \alpha_{j-1}-\alpha_j \ge \dfrac{\boldsymbol{\alpha}_N}{2N}\dfrac{\mu\big(Q_{\tilde c r_j, r_j^2}(z_1)\big)}{|B_j|} = \dfrac{\boldsymbol{\alpha}_N}{2N}\dfrac{\mu(Q_j)}{|B_j|},\\[2ex] \alpha_{j-1}-\alpha_j \le \dfrac{2^N\boldsymbol{\alpha}_N}{N}\dfrac{\mu\big(Q_{\tilde c r_{j-1}, r_{j-1}^2}(z_1)\big)}{|B_{j-1}|} = \dfrac{2^N\boldsymbol{\alpha}_N}{N}\dfrac{\mu(Q_{j-1})}{|B_{j-1}|}.\end{cases} \tag{6.26}$$

We now define

$$\bar a := \tfrac12(\mu^+ + a_j) - \tfrac14 B\alpha_j.$$

Moreover, again by the definition of $\bar a$ and $(6.24)_2$, we obtain

$$\bar a - a_j = \tfrac12(\mu^+ - a_j) - \tfrac14 B\alpha_j \ge \tfrac14 B(\alpha_{j-1}-\alpha_j), \tag{6.27}$$

while $(6.24)_1$ implies

$$\begin{aligned}\bar a - a_j &= \tfrac12(\mu^+ + a_j - 2a_j) - \tfrac14 B\alpha_j\\ &\ge \tfrac12(a_j - a_{j-1}) + \tfrac14 B(\alpha_{j-1}-\alpha_j) \ge \tfrac12(a_j-a_{j-1}).\end{aligned} \tag{6.28}$$

Finally, using $(6.24)_2$, the assumption (6.16), the definition of a_o, and the fact that the sequence $a_o, a_1,\dots$ in increasing we obtain that

$$\bar a \le \mu^+ \le \tfrac{13}{12}\omega = \tfrac{13}{3}\,\tfrac14\omega < \tfrac{13}{3}(\mu^+ - \tfrac12\omega) \le \tfrac{13}{3}a_o < \tfrac{13}{3}a_i \quad \forall i\in\{0,1,\dots,j\}. \tag{6.29}$$

From now on we proceed much as we did in the proof of Theorem 1.1. We consider $\mathbf{K}_j(\bar a)$. Using Lemmas 2.5 and 2.2 we obtain for any given $\tilde\varepsilon\in(0,1]$ that

$$\mathbf{K}_j(\bar a) = \frac{1}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{\bar a - a_j}\Big)^{1+\lambda}dxdt$$

$$
\begin{aligned}
&\le \frac{\boldsymbol\gamma}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u^m-a_j^m}{(\bar a-a_j)^m}\Big)^{\frac{1+\lambda}{m}}dxdt\\
&\quad+\frac{\boldsymbol\gamma}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{u^m-a_j^m}{(\bar a-a_j)^m}\bigg)^{1+\lambda}dxdt\\
&\le \boldsymbol\gamma\Big[\tilde\varepsilon^{\frac{1+\lambda}{m}}+\tilde\varepsilon^{1+\lambda}\Big]\frac{\big|\frac12 Q_j\cap\{u>a_j\}\big|}{|\frac12 Q_j|}\\
&\quad+\frac{\boldsymbol\gamma_{\tilde\varepsilon}}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\bigg|V_\lambda\Big(\frac{u^m-a_j^m}{(\bar a-a_j)^m}\Big)\bigg|^{\frac{2(1+\lambda)}{m(1-\lambda)}}dxdt\\
&\quad+\frac{\boldsymbol\gamma_{\tilde\varepsilon}}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{u^m-a_j^m}{(\bar a-a_j)^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{1-\lambda}}dxdt\\
&\le \boldsymbol\gamma\tilde\varepsilon^{1+\lambda}\frac{\big|\frac12 Q_j\cap\{u>a_j\}\big|}{|\frac12 Q_j|}+\frac{\boldsymbol\gamma_{\tilde\varepsilon}}{|\frac12 Q_j|}(\mathrm{I}+\mathrm{II}),
\end{aligned}
$$

holds true with constants $\boldsymbol\gamma=\boldsymbol\gamma(m)$ and $\boldsymbol\gamma_{\tilde\varepsilon}=\boldsymbol\gamma(m,\lambda,\tilde\varepsilon)$ and where we have abbreviated

$$
\begin{aligned}
\mathrm{I}&:=\iint_{\frac12 Q_j\cap\{u>a_j\}}\bigg|V_\lambda\Big(\frac{u^m-a_j^m}{(\bar a-a_j)^m}\Big)\bigg|^{\frac{2(1+\lambda)}{m(1-\lambda)}}dxdt,\\
\mathrm{II}&:=\iint_{\frac12 Q_j\cap\{u>a_j\}}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{u^m-a_j^m}{(\bar a-a_j)^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{1-\lambda}}dxdt.
\end{aligned}
$$

In the last line we have also taken into account that $\tilde\varepsilon^{\frac{1+\lambda}{m}}\le\tilde\varepsilon^{1+\lambda}$, since $m<1$. The first term on the right-hand side of the preceding inequality can be estimated as follows

$$
\begin{aligned}
\big|\tfrac12 Q_j\cap\{u>a_j\}\big|&\le\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}dxdt\\
&\le\iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}dxdt\\
&=\big|\tfrac12 Q_{j-1}\big|\mathbf K_j(a_{j-1})\le\big|\tfrac12 Q_{j-1}\big|\kappa=|Q_j|\kappa.
\end{aligned}
$$

Here, we used in the last line that by construction $\mathbf K_j(a_{j-1})\le\kappa$, and the definition of the cylinders Q_j. Inserting this above we obtain

$$
\boldsymbol K_j(\bar a)\le\boldsymbol\gamma\tilde\varepsilon^{1+\lambda}\kappa+\frac{\boldsymbol\gamma_{\tilde\varepsilon}}{|\frac12 Q_j|}(\mathrm{I}+\mathrm{II}). \tag{6.30}
$$

The appearing constants have the following dependencies: $\boldsymbol\gamma=\boldsymbol\gamma(N,m)$ and $\boldsymbol\gamma_{\tilde\varepsilon}=\boldsymbol\gamma_{\tilde\varepsilon}(m,\lambda,\tilde\varepsilon)$. To estimate the terms I and II, we use Gagliardo-Nirenberg's inequality and the energy estimates (3.2) and (3.4) from § 3. We start with the estimation of I. Using Gagliardo-Nirenberg's inequality from Lemma 2.1 with exponents $p=2$, $q=\frac{2(1+\lambda)}{m(1-\lambda)}$ and $r=N\big[\frac{1+\lambda}{m(1-\lambda)}-1\big]$, we find that

$$
\begin{aligned}
\mathrm{I}&\le\boldsymbol\gamma\bigg[\sup_{t\in\Lambda_j}\int_{\frac12 B_j\times\{t\}}\bigg|V_\lambda\Big(\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\Big)\bigg|^{N[\frac{1+\lambda}{m(1-\lambda)}-1]}dx\bigg]^{\frac2N}\\
&\quad\cdot\iint_{\frac12 Q_j}\frac{1}{(\tilde c r_j)^2}\bigg|V_\lambda\Big(\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\Big)\bigg|^2+\bigg|DV_\lambda\Big(\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\Big)\bigg|^2dxdt
\end{aligned}
$$

$$=: \boldsymbol{\gamma}\, \mathrm{I}_1(\mathrm{I}_2 + \mathrm{I}_3),$$

with the obvious meaning of I_1, I_2 and I_3. For the estimate of I_1 we abbreviate $q := 1+\lambda-m(1-\lambda)$ and apply in turn Lemma 2.2, Hölder's inequality (note that $\frac{Nq}{2}\le 1$ by the choice of λ in (4.1)), Lemma 2.4 and the energy estimate (4.19). In this way, we get for any $\varepsilon\in(0,1)$ that

$$\begin{aligned}
\mathrm{I}_1 &\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\sup_{t\in\Lambda_j} \frac{1}{|\frac12 B_j|}\int_{\frac12 B_j(t)} \bigg(\frac{(u^m-a_j^m)_+}{\mathbf{d}_{j-1}^m}\bigg)^{\frac{Nq}{2m}} dx\Bigg]^{\frac{2}{N}} \\
&\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\sup_{t\in\Lambda_j} \frac{1}{|\frac12 B_j|}\int_{\frac12 B_j(t)} \bigg(\frac{(u^m-a_j^m)_+}{\mathbf{d}_{j-1}^m}\bigg)^{\frac{1}{m}} dx\Bigg]^{q} \\
&\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\varepsilon + \boldsymbol{\gamma}_\varepsilon \sup_{t\in\Lambda_j} \frac{1}{|\frac12 B_j|}\int_{\frac12 B_j^+(t)} \frac{1}{\mathbf{d}_{j-1}}\int_{a_j}^u 1-\bigg(1+\frac{s^m-a_j^m}{\mathbf{d}_{j-1}^m}\bigg)^{-\lambda} ds dx\Bigg]^{q}.
\end{aligned}$$

Here we used the abbreviations $B_j(t) := B_j\times\{t\}$ and $B_j^+(t) := B_j\times\{t\}\cap\{u>a_j\}$ and $\mathbf{d}_{j-1} := \bar a - a_j$. In order to estimate the integral on the right-hand side, we rely on the energy estimates from § 3. We apply the energy estimate (3.4) on Q_j with $a\equiv a_j$, $d\equiv \mathbf{d}_{j-1}$ and $\tilde a\equiv \frac{\omega}{A}$ and obtain (abbreviating $Q_j^+ := Q_j\cap\{u>a_j\}$ and $d_{j-1} := a_j - a_{j-1}$)

$$\begin{aligned}
\mathrm{I}_1 &\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\varepsilon + \frac{\boldsymbol{\gamma}_\varepsilon}{|Q_j|}\iint_{Q_j^+} \bigg[\Big(\frac{\omega}{A\mathbf{d}_{j-1}}\Big)^{1-m}\bigg(1+\frac{u^m-a_j^m}{\mathbf{d}_{j-1}^m}\bigg)^{1+\lambda} + \frac{u-a_j}{\mathbf{d}_{j-1}}\bigg] dxdt \\
&\qquad\qquad + \frac{\boldsymbol{\gamma}_\varepsilon\, \mu(Q_j)}{|\frac12 B_j|\mathbf{d}_{j-1}}\Bigg]^q \\
&\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\varepsilon + \frac{\boldsymbol{\gamma}_\varepsilon}{|Q_j|}\iint_{Q_j^+} \bigg[\Big(\frac{a_{j-1}}{d_{j-1}}\Big)^{1-m}\bigg(1+\frac{u^m-a_j^m}{d_{j-1}^m}\bigg)^{1+\lambda} + \frac{u-a_j}{d_{j-1}}\bigg] dxdt \\
&\qquad\qquad + \frac{\boldsymbol{\gamma}_\varepsilon\, \mu(Q_j)}{|\frac12 B_j| B(\alpha_{j-1}-\alpha_j)}\Bigg]^q,
\end{aligned}$$

where in the last line we have used (6.27) and

$$\mathbf{d}_{j-1} = \bar a - a_j \ge \tfrac12 (a_j - a_{j-1}) = \tfrac12 d_{j-1} \quad\text{and}\quad \omega \le 2(\mu^+ - \tfrac14\omega) \le 2a_o \le 2a_{j-1}.$$

which is a consequence of (6.28), the hypothesis (6.16) and the definition of a_o. By the arguments leading us to (4.20) and (6.26)we further estimate

$$\begin{aligned}
\mathrm{I}_1 &\le |\tfrac12 B_j|^{\frac{2}{N}} \Bigg[\varepsilon + \frac{\boldsymbol{\gamma}_\varepsilon}{|\frac12 Q_{j-1}|}\iint_{\frac12 Q_{j-1}\cap\{u>a_{j-1}\}} \bigg(\frac{u-a_{j-1}}{a_j-a_{j-1}}\bigg)^{1+\lambda} dxdt + \frac{\boldsymbol{\gamma}_\varepsilon}{B}\Bigg]^q \\
&\le \big|\tfrac12 B_j\big|^{\frac{2}{N}} \Big[\varepsilon + \boldsymbol{\gamma}_\varepsilon\big[\kappa + \tfrac{1}{B}\big]\Big]^q,
\end{aligned}$$

for a constant $\boldsymbol{\gamma}_\varepsilon$ depending on $m, N, C_o, C_1, \lambda, A$ and ε and where where $q = 1+\lambda-m(1-\lambda)$. To estimate I_2, we use in turn $\omega < 2a_j$, Lemma 2.2, (6.28) and the choice of a_j. This leads us to

$$\begin{aligned}
\mathrm{I}_2 &\le \frac{1}{r_j^2}\Big(\frac{\omega}{A}\Big)^{1-m}\iint_{\frac12 Q_j\cap\{u>a_j\}} \bigg(\frac{u^m-a_j^m}{(\bar a - a_j)^m}\bigg)^{1-\lambda} dxdt \\
&\le \frac{2a_j^{1-m}}{r_j^2}\iint_{\frac12 Q_j\cap\{u>a_j\}} \Big(\frac{u-a_j}{\bar a - a_j}\Big)^{m(1-\lambda)} dxdt
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\boldsymbol\gamma a_j^{1-m}}{r_j^2}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{m(1+\lambda)}dxdt\\
&\le \frac{\boldsymbol\gamma a_j^{1-m}|\frac12 Q_j|\kappa}{r_j^2}\le \boldsymbol\gamma a_j^{1-m}|B_j|\kappa,
\end{aligned}
$$

for a constant $\boldsymbol\gamma$ depending only on A. Hence, it remains to estimate I_3. By the energy estimate (3.4), the computation we did in the estimate of I_1 and the fact that $\mathbf{d}_{j-1}^{1-m}=\bar a-a_j\le\frac76 a_j$, which follows from (6.29) we find that

$$
\begin{aligned}
\mathrm{I}_3 &\le \frac{\boldsymbol\gamma \mathbf{d}_{j-1}^{1-m}}{r_j^2}\iint_{Q_j^+}\Big[\Big(\frac{\omega}{A\mathbf{d}_{j-1}}\Big)^{1-m}\Big(1+\frac{u^m-a_j^m}{\mathbf{d}_{j-1}^m}\Big)^{1+\lambda}+\frac{u-a_j}{\mathbf{d}_{j-1}}\Big]dxdt+\frac{\boldsymbol\gamma\mu(Q_j)}{\mathbf{d}_{j-1}^m}\\
&\le \boldsymbol\gamma a_j^{1-m}|B_j|\big[\kappa+\tfrac1B\big],
\end{aligned}
$$

where $\boldsymbol\gamma=\boldsymbol\gamma(N,m,C_o,C_1,\lambda,A)$. Inserting the estimates obtained for I_1 – I_3 above, leads us to the final estimate of I:

$$
\begin{aligned}
\mathrm{I}&\le \boldsymbol\gamma\,\big|\tfrac12 B_j\big|^{\frac2N}a_j^{1-m}|B_j|\Big[\varepsilon+\boldsymbol\gamma_\varepsilon\big[\kappa+\tfrac1B\big]\Big]^q\big[\kappa+\tfrac1B\big]\\
&\le \boldsymbol\gamma\,|Q_j|\Big[\varepsilon+\boldsymbol\gamma_\varepsilon\big[\kappa+\tfrac1B\big]\Big]^q\big[\kappa+\tfrac1B\big]\\
&\le \boldsymbol\gamma\varepsilon|Q_j|\big[\kappa+\tfrac1B\big]+\boldsymbol\gamma_\varepsilon|Q_j|\big[\kappa+\tfrac1B\big]^{q+1},
\end{aligned}\tag{6.31}
$$

for constants $\boldsymbol\gamma=\boldsymbol\gamma(N,m,C_o,C_1,\lambda,A)$ and $\boldsymbol\gamma_\varepsilon=\boldsymbol\gamma_\varepsilon(N,m,C_o,C_1,\lambda,A,\varepsilon)$. In the second last line we used the fact that $a_j\le\frac{13}{12}\omega$.

Now, we turn our attention to the estimate of II in (6.30). Using Gagliardo-Nirenberg's inequality from Lemma 2.1 with $p=2$ and $q=\frac{2(1+\lambda)}{1-\lambda}$ and $r=\frac{2\lambda N}{1-\lambda}$ we find that

$$
\mathrm{II}=\iint_{\frac12 Q_j}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\bigg)\bigg|^{\frac{2(1+\lambda)}{1-\lambda}}dxdt\le\boldsymbol\gamma\,\mathrm{II}_1\big(\mathrm{II}_2+\mathrm{II}_3\big),
$$

where we have set

$$
\begin{aligned}
\mathrm{II}_1&:=\bigg[\sup_{t\in\Lambda_j}\int_{\frac12 B_j\times\{t\}}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\bigg)\bigg|^{\frac{2\lambda N}{1-\lambda}}dx\bigg]^{\frac2N},\\
\mathrm{II}_2&:=\frac{4}{r_j^2}\Big(\frac{\omega}{A}\Big)^{1-m}\iint_{\frac12 Q_j}\bigg|V_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\bigg)\bigg|^2dxdt,\\
\mathrm{II}_3&:=\iint_{\frac12 Q_j}\bigg|DV_\lambda\bigg(\Big(\frac{a_j}{\bar a-a_j}\Big)^{1-m}\frac{(u^m-a_j^m)_+}{(\bar a-a_j)^m}\bigg)\bigg|^2dxdt.
\end{aligned}
$$

For the estimate of II_1 we apply in turn Lemma 2.2, Hölder's inequality (remember that $\lambda\le\frac{\beta}{N(1+m)}\le\frac1N$), Lemma 2.3, the energy estimate (3.2), the fact that $\omega<2a_j$, (6.28), (6.26), and the choice of a_{j-1}. To simplify and shorten the notation, we again define $\mathbf{d}_j:=\bar a-a_j$ and $\frac12B_j^+(t):=\frac12B_j\times\{t\}\cap\{u>a_j\}$. Proceeding in this way, we obtain for any $\varepsilon\in(0,1]$ that

$$
\begin{aligned}
\frac{\mathrm{II}_1}{|\frac12B_j|^{\frac2N}}&\le\bigg[\sup_{t\in\Lambda_j}\frac{1}{|\frac12B_j|}\int_{\frac12B_j^+(t)}\bigg(\Big(\frac{a_j}{\mathbf{d}_j}\Big)^{1-m}\frac{u^m-a_j^m}{\mathbf{d}_j^m}\bigg)^{\lambda N}dx\bigg]^{\frac2N}\\
&\le\bigg[\sup_{t\in\Lambda_j}\frac{1}{|\frac12B_j|}\int_{\frac12B_j^+(t)}\Big(\frac{a_j}{\mathbf{d}_j}\Big)^{1-m}\frac{u^m-a_j^m}{\mathbf{d}_j^m}\,dx\bigg]^{2\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\le \Bigg[\varepsilon + \sup_{t\in\Lambda_j} \frac{\boldsymbol\gamma}{\varepsilon \mathbf{d}_j|\frac12 B_j|}\int_{\frac12 B_j^+(t)}\int_{a_j}^u 1-\bigg(1+\Big(\frac{a_j}{\mathbf{d}_j}\Big)^{1-m}\frac{s^m-a_j^m}{\mathbf{d}_j^m}\bigg)^{-\lambda} ds\,dx\Bigg]^{2\lambda}\\
&\le \Bigg[\varepsilon + \frac{\boldsymbol\gamma}{\varepsilon}\Bigg[\frac{1+\big(\frac{\omega}{a_j}\big)^{1-m}}{r_j^2|B_j|}\iint_{Q_j\cap\{u>a_j\}}\bigg(1+\frac{u-a_j}{\mathbf{d}_j}\bigg)^{1+\lambda}dxdt + \frac{\mu(Q_j)}{\mathbf{d}_j|B_j|}\Bigg]\Bigg]^{2\lambda}\\
&\le \Bigg[\varepsilon + \frac{\boldsymbol\gamma}{\varepsilon}\Bigg[\frac{1}{|Q_j|}\iint_{Q_j\cap\{u>a_j\}}\bigg(1+\frac{u-a_{j-1}}{a_j-a_{j-1}}\bigg)^{1+\lambda}dxdt + B^{-1}\Bigg]\Bigg]^{2\lambda}\\
&\le \Big[\varepsilon + \frac{\boldsymbol\gamma}{\varepsilon}\big[\kappa+\tfrac{1}{B}\big]\Big]^{2\lambda}
\end{aligned}
$$

for a constant $\boldsymbol\gamma=\boldsymbol\gamma(m,N,C_o,C_1,\lambda,A)$. Next, we estimate the term II_2. This is done via Lemma 2.2, the elementary inequality $a_j^{1-m}(u^m-a_j^m)\le u-a_j$, which holds true on the set $\{u>a_j\}$, the fact that $\omega<2a_j$, (6.28) and the choice of a_{j-1}, i.e. that $\mathbf{K}_j(a_{j-1})\le\kappa$. This leads us to

$$
\begin{aligned}
\mathrm{II}_2 &\le \frac{\boldsymbol\gamma a_j^{1-m}}{r_j^2}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{\bar a-a_j}\Big)^{1-\lambda}dxdt\\
&\le \frac{\boldsymbol\gamma a_j^{1-m}|B_j|}{|\frac12 Q_j|}\iint_{\frac12 Q_j\cap\{u>a_j\}}\Big(\frac{u-a_{j-1}}{a_j-a_{j-1}}\Big)^{1+\lambda}dxdt\\
&\le \boldsymbol\gamma a_j^{1-m}|B_j|\kappa,
\end{aligned}
$$

where the constant $\boldsymbol\gamma$ depends only on N. Finally, by the energy estimate (3.2), applied with $a=a_j$, $d=\bar a-a_j$, $\tilde a=\omega/A$ and $\varrho=r_j$, and the same arguments as above, we immediately have

$$
\begin{aligned}
\mathrm{II}_3 &\le \frac{\boldsymbol\gamma a_j^{1-m}}{r_j^2}\Big[1+\Big(\frac{\omega}{Aa_j}\Big)^{1-m}\Big]\iint_{Q_j\cap\{u>a_j\}}\bigg(1+\frac{u-a_j}{\bar a-a_j}\bigg)^{1+\lambda}dxdt + \frac{\boldsymbol\gamma a_j^{1-m}\mu(Q_j)}{\bar a-a_j}\\
&\le \boldsymbol\gamma\, a_j^{1-m}|B_j|\big[\kappa+\tfrac1B\big],
\end{aligned}
$$

for a constant $\boldsymbol\gamma=\boldsymbol\gamma(N,m,C_o,C_1,\lambda,A)$. Inserting the estimates for II_1 – II_3 into the inequality for II we find

$$
\begin{aligned}
\mathrm{II} &\le \boldsymbol\gamma\,\big|\tfrac12 B_j\big|^{\frac2N}a_j^{1-m}|B_j|\Big[\varepsilon+\frac{\boldsymbol\gamma}{\varepsilon}\big[\kappa+\tfrac1B\big]\Big]^{2\lambda}\big[\kappa+\tfrac1B\big]\\
&\le \boldsymbol\gamma\varepsilon^{2\lambda}|Q_j|\big[\kappa+\tfrac1B\big]+\frac{\boldsymbol\gamma|Q_j|}{\varepsilon^{2\lambda}}\big[\kappa+\tfrac1B\big]^{1+2\lambda},
\end{aligned}
\tag{6.32}
$$

for a constant $\boldsymbol\gamma=\boldsymbol\gamma(N,m,C_o,C_1,\lambda,A)$. Combining (6.31) and (6.32) with (6.30), we obtain that

$$
\begin{aligned}
\mathbf{K}_j(\bar a) &\le \boldsymbol\gamma\,\tilde\varepsilon^{1+\lambda}\kappa+\boldsymbol\gamma_{\tilde\varepsilon}(\varepsilon+\varepsilon^{2\lambda})\big[\kappa+\tfrac1B\big]+\boldsymbol\gamma_{\tilde\varepsilon}\boldsymbol\gamma_\varepsilon\big[\kappa+\tfrac1B\big]^{1+q}+\frac{\boldsymbol\gamma_{\tilde\varepsilon}}{\varepsilon^{2\lambda}}\big[\kappa+\tfrac1B\big]^{1+2\lambda}\\
&\le \boldsymbol\gamma\,\tilde\varepsilon^{1+\lambda}\kappa+\boldsymbol\gamma_{\tilde\varepsilon}(\varepsilon+\varepsilon^{2\lambda})\big[\kappa+\tfrac1B\big]+\boldsymbol\gamma_{\tilde\varepsilon}\boldsymbol\gamma_\varepsilon\big[\kappa+\tfrac1B\big]^{1+q}
\end{aligned}
$$

holds true with constants $\boldsymbol\gamma,\boldsymbol\gamma_\varepsilon$ and $\boldsymbol\gamma_{\tilde\varepsilon}$ all depending on N,m,C_o,C_1,λ and A. The constant $\boldsymbol\gamma_\varepsilon$ additionally depends as indicated on the parameter ε, while $\boldsymbol\gamma_{\tilde\varepsilon}$ additionally depends on $\tilde\varepsilon$. In the last line we used that $1<1+2\lambda\le 2+\lambda-m(1-\lambda)=1+q$. Note also that $\varepsilon,\tilde\varepsilon,\kappa\in(0,1]$ are still at our disposal. We now perform the choices of these parameters. First, we choose

$$
\tfrac1B\le\kappa,
\tag{6.33}
$$

which yields

$$\mathbf{K}_j(\bar a) \le \Big[\boldsymbol\gamma\,\tilde\varepsilon^{1+\lambda} + \boldsymbol\gamma_{\tilde\varepsilon}\boldsymbol\gamma(\varepsilon+\varepsilon^{2\lambda}) + \boldsymbol\gamma_{\tilde\varepsilon}\boldsymbol\gamma_\varepsilon\kappa^q\Big]\kappa.$$

Then, we choose $\tilde\varepsilon$ to satisfy $\boldsymbol\gamma\tilde\varepsilon^{1+\lambda} = \frac16$. This fixes $\tilde\varepsilon$ (and also $\boldsymbol\gamma_{\tilde\varepsilon}$) in dependence on N, m, C_o, C_1, λ, and A. Next, we choose ε such that $\boldsymbol\gamma_{\tilde\varepsilon}(\varepsilon+\varepsilon^{2\lambda}) = \frac16$, which fixes ε (and therefore also $\boldsymbol\gamma_\varepsilon$) in dependence on the same parameters. Having fixed $\tilde\varepsilon$ and ε, we choose κ such that $\boldsymbol\gamma_\varepsilon\boldsymbol\gamma_{\tilde\varepsilon}\kappa^q \le \frac16$. Note that up to now ε and κ depend only on N, m, C_o, C_1, λ and A. With these choices, we conclude that

$$\mathbf{K}_j(\bar a) \le \tfrac12\kappa.$$

Since, by (6.27) we have $\bar a > a_j + \frac14 B(\alpha_{j-1}-\alpha_j)$, the construction procedure leading to a_{j+1} yields that $a_{j+1} < \bar a$ holds true. By the definition of $\bar a$ this proves that the claim $(6.24)_1$ holds also true for $j+1$. For the bound $(6.24)_2$ we recall that $a_j < \mu^+ - \frac12 B\alpha_{j-1}$ (an immediate consequence of (6.24) at step j). Therefore

$$a_{j+1} < \bar a = \tfrac12(\mu^+ + a_j) - \tfrac14 B\alpha_j < \mu^+ - \tfrac14 B(\alpha_{j-1}+\alpha_j) \le \mu^+ - \tfrac12 B\alpha_j,$$

proving $(6.24)_2$ for $j+1$. Altogether we have established that (6.24) also holds for $j+1$. This proves the claim for any $j\in\mathbb{N}$.

Step 3: Improved bounds for a_j. For the quantity

$$d_j := a_{j+1} - a_j$$

we establish, with a constant $\boldsymbol\gamma$ depending on m, N, C_o, C_1, λ and A, that there holds

$$d_j \le \tfrac12 d_{j-1} + \boldsymbol\gamma\,\frac{\mu(2Q_j)}{|B_j|} \qquad \forall\, j\in\mathbb{N}. \tag{6.34}$$

For the proof we may assume without loss of generality, that

$$d_j \ge \tfrac12 d_{j-1}, \qquad d_j > \tfrac14(\alpha_{j-1}-\alpha_j),$$

because otherwise, there is nothing to prove; cf. (6.26) for the case that $d_j \le \frac14(\alpha_{j-1} - \alpha_j)$. The second inequality implies (see the construction of a_{j+1}), that $\mathbf{K}_j(a_{j+1}) = \kappa$ holds true. We now argue as in the Step 2, but instead of estimating the term involving the measure as we systematically did, we keep the measure and replace $\bar a - a_j$ in the denominator by $d_j = a_{j+1} - a_j$. In this way we obtain

$$\kappa \le \bigg[\boldsymbol\gamma\tilde\varepsilon^{1+\lambda} + \boldsymbol\gamma_{\tilde\varepsilon}(\varepsilon+\varepsilon^{2\lambda}) + \frac{\boldsymbol\gamma_{\tilde\varepsilon}\kappa^{2\lambda}}{\varepsilon^{2\lambda}} + \boldsymbol\gamma_\varepsilon\boldsymbol\gamma_{\tilde\varepsilon}\kappa^q\bigg]\kappa + \boldsymbol\gamma_\varepsilon\boldsymbol\gamma_{\tilde\varepsilon}\bigg[\frac{\mu(2Q_j)}{d_j|B_j|} + \bigg(\frac{\mu(2Q_j)}{d_j|B_j|}\bigg)^{1+q}\bigg].$$

With the same choices for κ, ε, $\tilde\varepsilon$ as in the proof of Step 2, and the argument from the end of [2, § 4.3], we conclude that

$$d_j \le \boldsymbol\gamma\,\frac{\mu(2Q_j)}{|B_j|}$$

with a constant $\boldsymbol\gamma = \boldsymbol\gamma(N, m, C_o, C_1, \lambda, A)$. This proves the claim (6.34).

Step 4: Quantitative bound for u. For $J > 1$ we sum up the inequalities (6.34) for $j = 1, \dots, J-1$. Taking into account the definition of d_j, and the fact that $\{a_j\}$ is a monotone increasing sequence, we deduce that

$$\begin{aligned} a_J - a_1 &= \sum_{j=1}^{J-1}(a_{j+1}-a_j) \\ &\le \tfrac12\sum_{j=1}^{J-1}(a_j - a_{j-1}) + \boldsymbol\gamma\sum_{j=1}^{J-1}\frac{\mu(2Q_j)}{|B_j|} \end{aligned}$$

$$\leq \tfrac{1}{2}(a_J - a_o) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{|B_j|}.$$

From this, we easily obtain that

$$a_J \leq 2a_1 - a_o + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{|B_j|}. \tag{6.35}$$

From the construction, we have two alternatives for the value of a_1, i.e. either

$$a_1 = a_o + \tfrac{1}{4}(\alpha_{-1} - \alpha_o),$$

or

$$a_1 > a_o + \tfrac{1}{4}(\alpha_{-1} - \alpha_o)$$

holds true. In the first case, recalling the definition $a_o = \mu^+ - \xi\omega$ and using $(6.26)_2$ for $j = 0$, we find that

$$a_J \leq \mu^+ - \xi\omega + \tfrac{1}{2}(\alpha_{-1} - \alpha_o) + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{|B_j|} \leq \mu^+ - \xi\omega + \boldsymbol{\gamma} \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{|B_j|},$$

By the definition of Q_j, we have

$$\begin{aligned}
\sum_{j=0}^{J-1} \frac{\mu(2Q_j)}{|B_j|} &\leq \sum_{j=0}^{\infty} \frac{\mu(2Q_j)}{|B_j|} = \sum_{j=0}^{\infty} \frac{\mu(Q_{\tilde{c}r_{j-1}, r_{j-1}^2})}{(\tilde{c}r_j)^N} \\
&\leq \sum_{j=0}^{\infty} \int_{\tilde{c}r_{j-1}}^{\tilde{c}r_{j-2}} \frac{\mu(Q_{\varrho, \varrho^2/\tilde{c}^2})}{(\tilde{c}r_j)^N} \frac{d\varrho}{\tilde{c}r_{j-2} - \tilde{c}r_{j-1}} \\
&\leq \boldsymbol{\gamma} \sum_{j=0}^{\infty} \int_{\tilde{c}r_{j-1}}^{\tilde{c}r_{j-2}} \frac{\mu(Q_{\varrho, \varrho^2/\tilde{c}^2})}{\varrho^N} \frac{d\varrho}{\varrho} = \boldsymbol{\gamma}\, \mathbf{I}_2^{\mu}(4\tilde{c}r, 16r^2),
\end{aligned}$$

from which we conclude

$$a_J \leq \mu - \xi\omega + \boldsymbol{\gamma}\, \mathbf{I}_2^{\mu}(4\tilde{c}r, 16r^2).$$

Here the constant $\boldsymbol{\gamma}$ depends on N, m, C_o, C_1, λ and A. In the second case, by the construction of a_1, we have that $\mathbf{K}_o(a_1) = \kappa$. Therefore, taking (6.17) and the definition of a_o into account, we obtain

$$\kappa = \frac{1}{|\frac{1}{2}Q_o|} \iint_{\frac{1}{2}Q_o \cap \{u > a_o\}} \left(\frac{u - a_o}{a_1 - a_o}\right)^{1+\lambda} dxdt \leq \frac{|Q_{-1}|}{|\frac{1}{2}Q_o|} \left(\frac{\mu^+ - (\mu^+ - \xi\omega)}{a_1 - a_o}\right)^{1+\lambda} \nu_+,$$

implying the inequality

$$a_1 \leq a_o + \boldsymbol{\gamma}\, \xi\omega\nu_+^{\frac{1}{1+\lambda}}$$

for the explicit constant $\boldsymbol{\gamma} = 2^{2(N+2)}$. We substitute this back into (6.35), and obtain

$$a_J \leq \mu^+ - \xi\omega + \boldsymbol{\gamma}\, \xi\omega\nu_+^{\frac{1}{1+\lambda}} + \boldsymbol{\gamma}\, \mathbf{I}_2^{\mu}(4\tilde{c}r, 16r^2),$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N, m, C_o, C_1, \lambda, A)$. Combining the two alternatives, we obtain that the preceding inequality holds true in any case for any $J \in \mathbb{N}$. Now, either the inequality (6.18) holds true, or

$$a_J \leq \mu^+ - \xi\omega + \boldsymbol{\gamma}\, \xi\omega\nu_+^{\frac{1}{1+\lambda}} + \tfrac{\boldsymbol{\gamma}}{B}\, \xi\omega.$$

Since $\{a_j\}$ is monotone increasing, the previous bound shows that $\lim_{j\to\infty} a_j = a_\infty$ exists and is finite, so that that $\lim_{j\to\infty}(a_{j+1}-a_j)=0$, and

$$a_\infty \le \mu^+ - \xi\omega\Big[1-\boldsymbol{\gamma}\,\nu_+^{\frac{1}{1+\lambda}} - \tfrac{\boldsymbol{\gamma}}{B}\Big]$$

holds true. Since $a_\infty \ge a_o > 0$, and $\bigcap_{j\in\mathbb{N}} Q_j = \{z_1\}$ we conclude that

$$\begin{aligned}(u(z_1)-a_\infty)_+^{1+\lambda} &= \lim_{j\to\infty}\frac{1}{|\frac12 Q_j|}\iint_{\frac12 Q_j}(u-a_j)_+^{1+\lambda}dxdt\\ &= \lim_{j\to\infty}\frac{(a_{j+1}-a_j)^{1+\lambda}}{|\frac12 Q_j|}\iint_{\frac12 Q_j}\Big(\frac{u-a_j}{a_{j+1}-a_j}\Big)_+^{1+\lambda}dxdt\\ &\le \lim_{j\to\infty}(a_{j+1}-a_j)^{1+\lambda}\kappa = 0.\end{aligned}$$

Here, we have also taken the construction of a_j into account, which guarantees that the integral in the previous chain of identities is bounded by κ. Therefore, we conclude that $u(z_1)\le a_\infty$, and by the bound on a_∞ this amounts to the estimate

$$u(z_1) \le \mu^+ - \xi\omega\Big[1-\boldsymbol{\gamma}\,\nu_+^{\frac{1}{1+\lambda}} - \tfrac{\boldsymbol{\gamma}}{B}\Big].$$

Now, we fix $\lambda = \frac{\beta}{N(1+m)}$ in dependence on N, m and choose B large enough such that

$$\tfrac{1}{B} \le \nu_+^{\frac{1}{1+\lambda}}. \tag{6.36}$$

This fixes B in dependence on ν_+. With this choice of B we have

$$u(z_1) \le \mu^+ - \xi\omega\Big[1-2\boldsymbol{\gamma}\,\nu_+^{\frac{1}{1+\lambda}}\Big].$$

Finally, we choose ν_+ such that

$$1-2\boldsymbol{\gamma}\,\nu_+^{\frac{1}{1+\lambda}} \ge a \iff \nu_+ \le \Big(\frac{1-a}{2\boldsymbol{\gamma}}\Big)^{1+\lambda}. \tag{6.37}$$

This fixes ν_+ in dependence on $\boldsymbol{\gamma}, \kappa$. With this choice of ν_+ we arrive at

$$u(z_1) \le \mu^+ - a\xi\omega,$$

and since z_1 is an arbitrary point in $Q_{r,\tilde\theta r^2}$, the claim follows. Finally, a few remarks concerning the dependencies of the constants are in order. Firstly, for the constant ν_+ we imposed the smallness conditions (6.25) and (6.37), i.e.

$$\nu_+ \le 2^{-(2N+9)}\kappa \quad\text{and}\quad \nu_+ \le \Big(\frac{1-a}{2\boldsymbol{\gamma}}\Big)^{1+\lambda}.$$

Since κ and $\boldsymbol{\gamma}$ both depend on m, N, C_o, C_1 and A, we can fulfill both requirements by a ν_+ of the form $\big(\frac{1-a}{\boldsymbol{\gamma}}\big)^{1+\lambda}$, with a constant $\boldsymbol{\gamma}$ depending on m, N, C_o, C_1 and A. Secondly, for the constant B we required in (6.33) and (6.36) that

$$B \ge \kappa^{-1} \quad\text{and}\quad B \ge \nu_+^{-\frac{1}{1+\lambda}}.$$

Taking here the functional dependence of ν_+ into account, we obtain a functional dependence of B in the form $\frac{\boldsymbol{\gamma}}{1-a}$, with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, N, C_o, C_1, A) \ge 1$. This finishes the proof of Lemma 6.2. □

Now, we assume that the hypotheses of Lemma 6.1 are fulfilled with the choices

$$r = \tfrac12\varrho, \quad a = \tfrac12, \quad \xi = \tfrac12, \quad A = 1, \quad \tilde c = c = \omega^{\frac{m-1}{2}}.$$

We denote by ν_* the corresponding value of ν_- given by (6.15), i.e. we set

$$\nu_* := \nu_-\big(\xi = \tfrac{1}{2}, a = \tfrac{1}{2}, A = 1\big),$$

and record that ν_* is now a quantity that depends only on m, N, C_o, and C_1. Furthermore, taking into account the definition of μ^+ and ω, from here on we assume that (6.16) holds true, so that this hypothesis of Lemma 6.2 is always fulfilled. We note that the left-hand inequality of (6.16) is always satisfied by (6.13). Note that (6.13) implies that $\mu^+ \geq \frac{3}{4}\omega$, which is equivalent to the left-hand inequality in (6.16). The case in which the right-hand inequality in (6.16) fails to hold, will be examined later. We have two alternatives, which we now discuss separately.

6.2. **The first alternative.** Suppose that

$$\big|\big\{u \leq \tfrac{1}{2}\omega\big\} \cap Q_{c\varrho,\varrho^2}\big| \leq \nu_*\big|Q_{c\varrho,\varrho^2}\big| \tag{6.38}$$

holds true. Then, by Lemma 6.1 we conclude that

$$u \geq \tfrac{1}{4}\omega \qquad \text{a.e. in } Q_{\frac{1}{2}c\varrho,\frac{1}{4}\varrho^2},$$

yielding

$$-\inf_{Q_{\frac{1}{2}c\varrho,\frac{1}{4}\varrho^2}} u \leq -\tfrac{1}{4}\omega.$$

To the preceding inequality we add the supremum of u over $Q_{\frac{1}{2}c\varrho,\frac{1}{4}\varrho^2}$ on the left-hand side and μ^+ on the right-hand side. By (6.16) the result leads to

$$\operatorname*{osc}_{Q_{\frac{1}{2}c\varrho,\frac{1}{4}\varrho^2}} u \leq \mu^+ - \tfrac{1}{4}\omega \leq \tfrac{5}{6}\omega.$$

Now, we recall the meaning of $\varrho = \varrho_n$, $c = c_n$ and $\omega = \omega_n$ from the beginning of § 6.1. In the original terminology the preceding estimate reads as

$$\operatorname*{osc}_{Q_{\frac{1}{2}c_n\varrho_n,(\frac{1}{2}\varrho_n)^2}} u \leq \tfrac{5}{6}\omega_n, \tag{6.39}$$

and this finishes the induction step in the case of the first alternative.

6.3. **The second alternative.** Here, we assume that (6.38) does not hold true, which means that

$$\big|\big\{u > \tfrac{1}{2}\omega\big\} \cap Q_{c\varrho,\varrho^2}\big| \leq (1-\nu_*)\big|Q_{c\varrho,\varrho^2}\big|. \tag{6.40}$$

Since we are assuming that (6.16) holds, (6.40) easily yields

$$\big|\big\{u > \mu^+ - \tfrac{1}{4}\omega\big\} \cap Q_{c\varrho,\varrho^2}\big| \leq (1-\nu_*)\big|Q_{c\varrho,\varrho^2}\big|. \tag{6.41}$$

We have the following

Lemma 6.3. *There exists a time level* $\bar{s} \in [-\varrho^2, -\frac{1}{2}\nu_*\varrho^2]$*, such that*

$$\big|\big\{u(\cdot,\bar{s}) > \mu^+ - \tfrac{1}{4}\omega\big\} \cap B_{c\varrho}\big| \leq \frac{1-\nu_*}{1-\frac{1}{2}\nu_*}|B_{c\varrho}|. \tag{6.42}$$

Proof. The proof goes by contradiction. If (6.42) does not hold for some s in the indicated range, then

$$\big|\big\{u(\cdot,s) > \mu^+ - \tfrac{1}{4}\omega\big\} \cap B_{c\varrho}\big| > \frac{1-\nu_*}{1-\frac{1}{2}\nu_*}|B_{c\varrho}| \qquad \forall s \in [-\varrho^2, -\tfrac{1}{2}\nu_*\varrho^2].$$

Integrating the preceding inequality over the interval $[-\varrho^2, -\frac12\nu_*\varrho^2]$ yields

$$\begin{aligned}\big|\big\{u>\mu^+-\tfrac14\omega\big\}\cap Q_{c\varrho,\varrho^2}\big| &\ge \int_{-\varrho^2}^{-\frac12\nu_*\varrho^2}\big|\big\{u(\cdot,s)>\mu^+-\tfrac14\omega\big\}\cap B_{c\varrho}\big|\,ds\\ &> \varrho^2(1-\nu_*)\big|B_{c\varrho}\big| = (1-\nu_*)\big|Q_{c\varrho,\varrho^2}\big|,\end{aligned}$$

contradicting (6.41). □

In the following, we will omit the reference point of the potential, i.e. we shall write $\mathbf{I}_2^\mu(r,\theta)$, instead of $\mathbf{I}_2^\mu(z_o,r,\theta)$. From (6.26) we obtain (take $j=0$ and $\tilde c = c$ there)

$$\mathbf{I}_2^\mu(2cr,4r^2)\ge \frac{\boldsymbol{\alpha}_N}{2N}\frac{\mu(Q_{cr,r^2})}{|B_{cr}|}. \tag{6.43}$$

Lemma 6.4. *There exists a positive integer $s_1\ge 4$ depending only on m,N,C_o,C_1 such that either*

$$\omega < 2^{s_1}\mathbf{I}_2^\mu(2c\varrho,4\varrho^2), \tag{6.44}$$

or

$$\Big|\Big\{u(\cdot,t)>\mu^+-\frac{\omega}{2^{s_1}}\Big\}\cap B_{c\varrho}\Big|\le\big(1-\tfrac14\nu_*^2\big)\,|B_{c\varrho}| \tag{6.45}$$

holds true for all $t\in[\bar s,0]$.

Proof. We let $k=\mu^+-\frac14\omega$ and define

$$H_k^+ := \sup_{B_{c\varrho}\times[\bar s,0]}(u-k)_+\le\tfrac14\omega.$$

Furthermore, we let $\mathbf{c}=\frac{\omega}{2^{\ell+2}}$ for some integer $\ell\in\mathbb{N}$, with $\ell\ge2$. To apply Proposition 5.2, we need $0<\mathbf{c}<H_k^+$. This can be achieved by choosing ℓ later on large enough in a universal way. The integer s_1 will then be $\ell+2$. We apply Proposition 5.2 on the cylinder $B_{c\varrho}\times[\bar s,0]$ with the logarithmic function $\psi\circ u=\psi_{(\mathbf{a},\mathbf{b},\mathbf{c})}\circ u$ for the choice $(\mathbf{a},\mathbf{b},\mathbf{c})=\big(H_k,k,\frac{\omega}{2^{\ell+2}}\big)$. More in detail, we consider

$$\psi(u)(x,t):=\big(\psi_{(H_k,k,\frac{\omega}{2^{\ell+2}})}\circ u\big)(x,t)=\ln_+\Bigg[\frac{H_k^+}{H_k^+-(u(x,t)-k)_+ +\frac{\omega}{2^{\ell+2}}}\Bigg].$$

The cutoff function $\zeta\in W_0^{1,\infty}\big(B_{c\varrho},[0,1]\big)$ is taken to be 1 in $B_{(1-\sigma)c\varrho}$, where $\sigma\in(0,1)$ has to be chosen, and such that $|D\zeta|\le\frac{1}{\sigma c\varrho}$. With these choices the Logarithmic estimate from Proposition 5.2 takes the form

$$\begin{aligned}\int_{B_{(1-\sigma)c\varrho}\times\{t\}}\psi^2(u)\,dx &\le \int_{B_{c\varrho}\times\{\bar s\}}\psi^2(u)\,dx+\frac{\gamma}{\sigma^2c^2\varrho^2}\iint_{B_{c\varrho}\times[\bar s,0]}u^{m-1}\psi(u)\,dxdt\\ &\quad+\frac{2^{\ell+3}}{\omega}\ln\Big(\frac{2^{\ell+2}H_k^+}{\omega}\Big)\int_{B_{c\varrho}\times[\bar s,0]}\chi_{\{u>k\}}\,d\mu,\\ &=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3\end{aligned} \tag{6.46}$$

for a.e. $t\in[\bar s,0]$. Before starting to estimate the right-hand side integrals, we observe that

$$\psi(u)\le\ln\Big(\frac{H_k^+2^{\ell+2}}{\omega}\Big)\le\ln2^\ell=\ell\ln2. \tag{6.47}$$

This follows immediately from the very definition of $\psi(u)$ and the fact that $H_k^+\le\frac14\omega$. Now, we start with the **estimate of** I_1. Using the preceding inequality and the fact that the logarithmic function $\psi(u)$ vanishes whenever $u \le \mu^+ - \frac14\omega + 2^{-(\ell+2)}\omega$, and taking into account also Lemma 6.3, we find that

$$\begin{aligned} \mathrm{I}_1 &\le \ell^2 \ln^2 2 \Big| B_{c\varrho} \cap \Big\{ u(\cdot,\bar s) > \mu^+ - \tfrac14\omega + \frac{\omega}{2^{\ell+2}} \Big\} \Big| \\ &\le \ell^2 \ln^2 2 \big| B_{c\varrho} \cap \big\{ u(\cdot,\bar s) > \mu^+ - \tfrac14\omega \big\} \big| \\ &\le \ell^2 \ln^2 2 \, \frac{1-\nu_*}{1-\frac12\nu_*} |B_{c\varrho}|. \end{aligned}$$

Next, we **estimate** I_2. Using again the bound for $\psi(u)$, the first inequality in (6.16) (more precisely that $u \ge \mu^+ - \frac14\omega \ge \frac12\omega$), and $c^2 = \omega^{m-1}$ we obtain the estimate

$$\mathrm{I}_2 \le \frac{\boldsymbol\gamma\,\ell\ln 2}{\sigma^2 c^2 \varrho^2} \left(\tfrac12\omega\right)^{m-1} \varrho^2 |B_{c\varrho}| = \frac{\boldsymbol\gamma\,\ell\ln 2}{\sigma^2}|B_{c\varrho}|,$$

for a constant $\boldsymbol\gamma = \boldsymbol\gamma(m,N,C_o,C_1)$. Finally, we **estimate** I_3. Using the second inequality of (6.47) and (6.43) we obtain

$$\mathrm{I}_3 \le \frac{2^{\ell+3}\,\ell\ln 2}{\omega}\,\mu(Q_{c\varrho,\varrho^2}) \le \frac{2^{\ell+4}\ell N\ln 2}{\boldsymbol\alpha_N\omega}\,\mathbf{I}_2^\mu(2c\varrho,4\varrho^2)|B_{c\varrho}|.$$

Assume for the moment that the integer ℓ has been chosen, and that

$$\tfrac{2^\ell}{\omega}\,\mathbf{I}_2^\mu(2c\varrho,4\varrho^2) < 1\,.$$

Otherwise, (6.44) holds trivially for $s_1 = \ell+2$. Under this assumption, we have the upper bound $\mathrm{I}_3 \le \boldsymbol\gamma(N)\ell|B_{c\varrho}|$. Inserting the inequalities for I_1 - I_3 into (6.46) we obtain

$$\int_{B_{(1-\sigma)c\varrho}\times\{t\}} \psi^2(u)\,dx \le \ell^2\ln^2 2\,\frac{1-\nu_*}{1-\frac12\nu_*}|B_{c\varrho}| + \frac{\boldsymbol\gamma\,\ell}{\sigma^2}|B_{c\varrho}|, \tag{6.48}$$

for a constant $\boldsymbol\gamma = \boldsymbol\gamma(N,m,C_o,C_1)$. Now we estimate the left-hand side in (6.48) from below, integrating over the smaller set $B_{(1-\sigma)c\varrho}\cap\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\}$. Since $\psi(u)(x,t)$ is for any fixed x a decreasing function in the parameter H_k^+, we can first replace H_k^+ by $\frac14\omega$ in the expression for $\psi(u)$, and then estimate the denominator by $\frac{\omega}{2^{\ell+2}}$. This leads us to the following bound from below:

$$\psi(u)^2 \ge \ln^2 \frac{\frac{\omega}{4}}{\frac{\omega}{2^{\ell+1}}} = (\ell-1)^2\ln^2 2. \tag{6.49}$$

Inserting (6.49) into (6.48), and dividing through by $(\ell-1)^2\ln^2 2$, we obtain the measure bound

$$\Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\Big\}\cap B_{(1-\sigma)c\varrho}\Big| \le \Big(\frac{\ell}{\ell-1}\Big)^2\,\frac{1-\nu_*}{1-\frac12\nu_*}|B_{c\varrho}| + \frac{\boldsymbol\gamma\ell}{\sigma^2(\ell-1)^2}|B_{c\varrho}|.$$

On the other hand, we have

$$\begin{aligned} &\Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\Big\}\cap B_{c\varrho}\Big| \\ &\qquad \le \Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\Big\}\cap B_{(1-\sigma)c\varrho}\Big| + \big|B_{c\varrho}\backslash B_{(1-\sigma)c\varrho}\big| \\ &\qquad \le \Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\Big\}\cap B_{(1-\sigma)c\varrho}\Big| + N\sigma|B_{c\varrho}|. \end{aligned}$$

Therefore, we conclude that

$$\Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{\ell+2}}\Big\}\cap B_{c\varrho}\Big| \le \bigg[\Big(\frac{\ell}{\ell-1}\Big)^2\frac{1-\nu_*}{1-\frac12\nu_*} + \frac{\boldsymbol\gamma\ell}{\sigma^2(\ell-1)^2} + N\sigma\bigg]|B_{c\varrho}|,$$

holds true for all $t \in [\bar{s}, 0]$. Now choose σ small enough, such that $N\sigma \le \frac{3}{8}\nu_*^2$. This fixes σ in dependence of m, N, C_o, C_1. Note that ν_* depends only on these quantities. Then we choose ℓ so large that

$$\Big(\frac{\ell}{\ell-1}\Big)^2 < \big(1-\tfrac{1}{2}\nu_*\big)\,(1+\nu_*) \quad \text{and} \quad \frac{\gamma\ell}{\sigma^2(\ell-1)^2} \le \tfrac{3}{8}\nu_*^2.$$

Of course this fixes $\ell = \ell(m, N, C_o, C_1)$. Now, the claim follows with $s_1 := \ell + 2$. We note, that s_1 is in particular independent of ω, ϱ, $\bar{s}$. □

Corollary 6.1. *Under the assumptions of Lemma* 6.4*, either*

$$\omega < 2^{s_1}\mathbf{I}_2^{\mu}(2c\varrho, 4\varrho^2),$$

or

$$\Big|\Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^j}\Big\} \cap B_{c\varrho}\Big| \le \big(1-\tfrac{1}{4}\nu_*^2\big)\,|B_{c\varrho}|$$

holds true for all $j \ge s_1$, and for all $t \in [-\frac{1}{2}\nu_*\varrho^2, 0]$.

We now work in the cylinder $Q_{c\varrho,\frac{1}{2}\nu_*\varrho^2} = B_{c\varrho} \times \big(-\frac{1}{2}\nu_*\varrho^2, 0\big]$. As a consequence of Corollary 6.1 and the energy estimate (5.2), we have the following lemma. The proof goes exactly as in [3, Lemma 4.5] and therefore we omit it.

Lemma 6.5. *For every $\bar{\nu} \in (0,1)$, there exists a positive integer q_*, depending only on m, N, C_o, C_1, and $\bar{\nu}$, such that either*

$$\omega < 2^{s_1+q_*}\,\mathbf{I}_2^{\mu}(4c\varrho, 16\varrho^2),$$

or

$$\Big|\Big\{u > \mu^+ - \frac{\omega}{2^{s_1+q_*}}\Big\} \cap Q_{c\varrho,\frac{1}{2}\nu_*\varrho^2}\Big| < \bar{\nu}\big|Q_{c\varrho,\frac{1}{2}\nu_*\varrho^2}\big|.$$

Now, referring to the notation of Lemma 6.2, we let

$$r = \sqrt{\tfrac{1}{8}\nu_*}\varrho, \quad a = \tfrac{1}{2}, \quad \xi = 2^{-(s_1+q_*)}, \quad A = 1, \quad \tilde{c} = c.$$

We apply Lemma 6.2, and take into account the first alternative of Lemma 6.5. This allows us to conclude that either

$$\omega < C\,\mathbb{I}^{\mu}_{2,Q_{c\sqrt{\frac{1}{2}\nu_*}\varrho,\frac{1}{2}\nu_*\varrho^2}}(4c\varrho, 16\varrho^2), \qquad C := \max\{B, 2^{s_1+q_*}\} \tag{6.50}$$

where B is the constant in (6.18), or

$$u \le \mu^+ - \frac{\omega}{2^{s_1+q_*+1}} \qquad \text{a.e. in } B_{c\sqrt{\frac{1}{8}\nu_*}\varrho} \times \big(-\tfrac{1}{8}\nu_*\varrho^2, 0\big], \tag{6.51}$$

provided $\bar{\nu}$ is chosen equal to ν_+ as defined in (6.37), of course with the choices $a = \frac{1}{2}$ and $A = 1$, and then in turn the integer q_* is chosen according to Lemma 6.5.We note that the constant C depends only on m, N, C_o, C_1. The estimate (6.51) can be rewritten in the form

$$\sup_{Q_{c\sqrt{\frac{1}{8}\nu_*}\varrho,\frac{1}{8}\nu_*\varrho^2}} u \le \mu^+ - \frac{\omega}{2^{\ell_*+q_*+1}}.$$

But this implies that either (6.50) holds true, or that

$$\begin{aligned}\operatorname*{osc}_{Q_{c\sqrt{\frac{1}{8}\nu_*}\varrho,\frac{1}{8}\nu_*\varrho^2}} u &= \sup_{Q_{c\sqrt{\frac{1}{8}\nu_*}\varrho,\frac{1}{8}\nu_*\varrho^2}} u - \inf_{Q_{c\sqrt{\frac{1}{8}\nu_*}\varrho,\frac{1}{8}\nu_*\varrho^2}} u \\ &\le \mu^+ - \frac{\omega}{2^{\ell_*+q_*+1}} - \mu^- \le \Big(1 - \frac{1}{2^{\ell_*+q_*+1}}\Big)\omega.\end{aligned}$$

Now, we recall the set up introduced at the beginning of § 6.1, in particular the abbreviations $\varrho = \varrho_n$, $\omega = \omega_n$ and $c = c_n = \omega_n^{\frac{m-1}{2}}$. We re-write the preceding inequalities using the original quantities and infer that either

$$\omega_n < C\, \mathbb{I}^{\mu}_{2,Q_{c_n\sqrt{\frac12\nu_*}\varrho_n,\frac12\nu_*\varrho_n^2}}\big(4c_n\varrho_n, 16\varrho_n^2\big),$$

or

$$\operatorname*{osc}_{Q_{c_n\sqrt{\frac18\nu_*}\varrho_n,\frac18\nu_*\varrho_n^2}} u \le \Big(1-\frac{1}{2^{\ell_*+q_*+1}}\Big)\omega_n$$

holds true. Since $Q_{c_n\sqrt{\frac18\nu_*}\varrho_n,\frac18\nu_*\varrho_n^2} \subset Q_n$ and $\sup_{Q_n} u \le \omega_n$ by our induction assumption, we conclude that in any case there holds

$$\operatorname*{osc}_{Q_{c_n\sqrt{\frac18\nu_*}\varrho_n,\frac18\nu_*\varrho_n^2}} u \le \max\Big\{\Big(1-\frac{1}{2^{\ell_*+q_*+1}}\Big)\omega_n,\ C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4c_n\varrho_n, 16\varrho_n^2\big)\Big\}. \tag{6.52}$$

6.4. **Pasting the two alternatives together.** Recalling the set up from (6.1), we define

$$\delta := 1-\frac{1}{2^{\ell_*+q_*+1}},\quad \omega_{n+1} := \max\Big\{\delta\omega_n, C\,\mathbb{I}^{\mu}_{2,Q_n}\big(4c_n\varrho_n, 16\varrho_n^2\big)\Big\}.$$

In view of the oscillation estimates of u from the first alternative in (6.39) and the second alternative in (6.52), we must define $\eta \in (0,1)$, in such a way that

$$Q_{n+1} = B_{c_{n+1}\eta\varrho_n} \times \big(-(\eta\varrho_n)^2, 0\big] \subset B_{c_n\sqrt{\frac18\nu_*}\varrho_n} \times \big(-\tfrac18\nu_*\varrho_n^2, 0\big].$$

As we have already pointed out, the definition of ω_{n+1} implies that $\omega_{n+1}^{\frac{m-1}{2}} \le (\delta\omega_n)^{\frac{m-1}{2}}$ (note that $m \le 1$). Therefore, the inclusion holds true, if we define

$$\eta := \sqrt{\tfrac18\nu_*\delta^{1-m}} < \tfrac12.$$

With these choices for δ and η, we can paste together the first alternative (6.39) (note that $\delta \ge \frac56$) and the second alternative (6.52) to conclude that

$$\operatorname*{osc}_{Q_{n+1}} u \le \omega_{n+1}$$

holds true. The induction argument is now completed, provided the assumption (6.16) is satisfied.

6.5. **The Proof of Proposition 6.1 concluded.** In this final section we have to deal with the case that assumption (6.16) does not hold for some index n. In such a situation we have

$$\mu_n^+ > \tfrac{13}{12}\omega_n,$$

which by (6.13) implies that $\mu_n^- \ge \frac{1}{12}\omega_n$. By the definition of μ_n^-, this means that u is uniformly bounded away from zero in Q_n, and therefore, by the structure conditions (1.3), the porous medium type equation (1.2) is non-singular in Q_n, and behaves like a quasilinear parabolic equation with growth of order 2 and measurable vector-field, with a measure data right-hand side, as considered, for example, in [12].

The argument is exactly the same as considered in [3, § 4.5], with the only difference that here we deal with the change of variables

$$\Phi(x,s) := (c^*x, s),\quad \text{where } c^* := \big(\tfrac{12}{13}\mu_o^+\big)^{\frac{m-1}{2}},$$

and the new function $v(x,s) := (\mu_o^+)^{-1}u(c^*x,s)$, whereas in [3] we had

$$\Phi(x,s) := (x, \theta^* s),\quad \text{where } \theta^* := \big(\tfrac{12}{13}\mu_o^+\big)^{1-m},$$

and $v(x,s) := (\mu_o^+)^{-1} u(x, \theta^* s)$.

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