THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

The obstacle problem for the porous medium equation

V. Bögelein, T. Lukkari and C. Scheven

REPORT No. 33, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -33-13/14- -SE+fall

THE OBSTACLE PROBLEM FOR THE POROUS MEDIUM EQUATION

VERENA BÖGELEIN, TEEMU LUKKARI, AND CHRISTOPH SCHEVEN

ABSTRACT. We prove existence results for the obstacle problem related to the porous medium equation. For sufficiently regular obstacles, we find continuous solutions whose time derivative belongs to the dual of a parabolic Sobolev space. We also employ the notion of weak solutions and show that for more general obstacles, such a weak solution exists. The latter result is a consequence of a stability property of weak solutions with respect to the obstacle.

1. INTRODUCTION

The porous medium equation (PME for short)

$$\partial_t u - \Delta u^m = 0 \tag{1.1}$$

is an important prototype example of a nonlinear parabolic equation. The name stems from modeling the flow of a gas in a porous medium: a combination of the continuity equation, Darcy's law, and an equation of state lead to equation (1.1) for the density of the gas, after scaling out various physical constants. There is an extensive literature concerned with this equation, and we refer to the monographs [5, 7, 21, 22] for the basic theory and numerous further references in a variety of directions.

In this work, we are concerned with the so-called *obstacle problem* related to this equation. Roughly speaking, in the obstacle problem we want to find solutions to the porous medium equation subject to the constraint that they lie above a given function ψ, the obstacle. This leads to a variational inequality; formally, a function u solves the obstacle problem for the porous medium equation if $u \geq \psi$ and

$$\int_{\Omega_T} \partial_t u(v^m - u^m) + \nabla u^m \cdot (\nabla v^m - \nabla u^m)\,\mathrm{d}z \geq 0 \tag{1.2}$$

for all comparison maps v such that $v \geq \psi$, and with the same boundary values as u; see Definitions 2.1 and 2.2 below for the rigorous interpretation of this inequality. The classic references for the obstacle problem to parabolic equations include [1] and the monograph [18], and some of the more recent ones are [2, 3, 20]. Alternatively, the obstacle problem can be thought of as finding the smallest supersolution to an equation staying above the obstacle function. This approach is analogous to the balayage concept in classical potential theory, and it is used in a nonlinear parabolic context in [15, 17]. One of our motivations is the need for a method of constructing supersolutions with favorable properties in nonlinear potential theory, see e.g. [11, 12, 16]. A recent example of this for the PME can be found in [14]. Our main interest is in the degenerate case, meaning that one has $m > 1$ in (1.1). Since the arguments also work – with occasional minor modifications – in the supercritical singular range $(n-2)_+/n < m < 1$, we included this case as well.

Our main results concern the existence of solutions to the obstacle problem for the porous medium equation. The first of them is the existence of strong solutions. Here strong refers to the fact that the time derivative $\partial_t u$ of a solution u belongs to the dual of a parabolic Sobolev space. This makes (1.2) meaningful, since we can then use the usual

---

Date: March 31, 2014.
1991 *Mathematics Subject Classification.* Primary 35K65, Secondary 35K86, 35D05, 47J20.
Key words and phrases. obstacle problem, porous medium equation, existence.

duality pairing. Apart from the time regularity, the strong solutions we find are locally Hölder continous. The continuity comes out as a byproduct of the existence proof. As far as we know, no previous existence result for the obstacle problem related to (1.1) provides solutions with these regularity properties. To make the inequality (1.2) meaningful in more general situations, we use a similar notion of a weak solution to the obstacle problem as in Alt & Luckhaus [1]. Our second main result is the existence of weak solutions. Here the regularity assumptions on the obstacles can be considerably relaxed, but the control on the time derivative and the continuity of the solution are lost. In both cases, only mild regularity assumptions are made on the obstacles; in particular, contrary to [1, 18], no assumption about boundedness and monotonicity in time is needed.

We use a penalization method to prove the existence of strong solutions. The heuristic idea of our choice of penalization is as in [2]: roughly speaking, we solve the equation

$$\partial_t u - \Delta u^m = (\partial_t \psi - \Delta \psi^m)_+ \tag{1.3}$$

for a given obstacle function ψ, where the positive part $(f)_+ := \max\{f, 0\}$ is taken on the right-hand side. In order to properly define the right-hand side, we certainly need some regularity assumptions on the obstacle ψ. We merge the penalization method with the usual way of constructing solutions to the porous medium equation via approximation by uniformly parabolic equations. The latter is described for instance in Chapter 5 of [21]. Comparison results using ideas from [4] play a key role in proving that the solutions indeed stay above the obstacle function. Note that there are several ways to choose a penalization; for instance, one could use projection operators as in [1, 18]. However, our choice has the advantage that we can obtain uniform estimates for the penalized equations.

The generalization of the existence result to weak solutions and irregular obstacles is a consequence of two facts. The first of them is the natural observation that strong solutions are also weak solutions. The second step is the more difficult one: weak solutions to the obstacle problem turn out to be stable with respect to convergence of the obstacles in a parabolic Sobolev space. In other words, given a sequence of obstacles ψ_i converging to a limit obstacle ψ, we show that one has convergence of the corresponding solutions u_i, up to a subsequence, to a limit function. Further, the limit function turns out to be a solution to the limiting obstacle problem. Thereby, the difficulty relies in identifying the weak limits of u_i and u_i^m. Now the existence of weak solutions follows by choosing a sufficiently smooth approximation of the given obstacle, ensuring that strong solutions exist, and then applying the stability property. Finally, we note that our proof produces local weak solutions, i.e. the variational inequality can be localized on smaller domains. Such a property can be useful when proving regularity of weak solutions, which we postpone to a subsequent work.

The paper is organized as follows. In § 2, we give the rigorous definitions of strong and weak solutions to the obstacle problem and the statements of our existence results. § 3 contains various auxiliary results needed for our existence proofs. In § 4, we start working on the obstacle problem, and establish some basic estimates for later use. § 5 deals with continuity in time, and we prove that the weak solutions to the obstacle problem are continuous in time with values in L^{m+1}. We establish suitable comparison principles for the uniformly parabolic approximating equations in § 6, and these are then employed in § 7 to show the existence of solutions to a penalized porous medium equation. We finally prove the existence of strong and weak solutions in § 8 and § 9, respectively.

Acknowledgements. We acknowledge the warm hospitality of the Institut Mittag-Leffler in the Fall 2013 during the program "Evolutionary problems", during which we initiated our collaboration on this paper.

2. WEAK AND STRONG SOLUTIONS TO THE OBSTACLE PROBLEM

In this section, we define our notions of solution to the obstacle problem and state the main results. We begin by introducing some notation, and recalling several other definitions.

Throughout the paper we always assume that $m > m_c := \frac{(n-2)_+}{n}$ and that Ω is a bounded open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly 2-thick; see Definition 2.4 below. We use the notation $\Omega_T = \Omega \times (0,T)$ and $U_{t_1,t_2} = U \times (t_1,t_2)$, where $U \subset \Omega$ is open and $0 < t_1 < t_2 < T$. The parabolic boundary $\partial_p U_{t_1,t_2}$ of a space-time cylinder U_{t_1,t_2} consists of the initial and lateral boundaries, i.e.

$$\partial_p U_{t_1,t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times [t_1,t_2]).$$

The notation $U_{t_1,t_2} \Subset \Omega_T$ means that the closure $\overline{U_{t_1,t_2}}$ is compact and $\overline{U_{t_1,t_2}} \subset \Omega_T$.

We use $H^1(\Omega)$ to denote the usual Sobolev space, i.e. the space of functions u in $L^2(\Omega)$ such that the weak gradient exists and also belongs to $L^2(\Omega)$. The norm of $H^1(\Omega)$ is

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}.$$

The Sobolev space with zero boundary values, denoted by $H^1_0(\Omega)$, is the completion of $C^\infty_0(\Omega)$ with respect to the norm of $H^1(\Omega)$. The dual of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$ and $\langle\cdot,\cdot\rangle$ will indicate the related duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. We note that $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$, since $m > m_c$.

The parabolic Sobolev space $L^2(0,T;H^1(\Omega))$ consists of all measurable functions $u\colon [0,T] \to H^1(\Omega)$ such that

$$\int_{\Omega_T} |u|^2 + |\nabla u|^2 \,\mathrm{d}z < \infty.$$

The definition of $L^2(0,T;H^1_0(\Omega))$ is identical, apart from the requirement that $u\colon [0,T] \to H^1_0(\Omega)$. We say that u belongs to $L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$ if $u \in L^2(t_1,t_2;H^1(U))$ for all $U_{t_1,t_2} \Subset \Omega_T$. Let $u \in L^\infty(0,T;L^{m+1}(\Omega))$. By $u \in H^1(0,T;H^{-1}(\Omega))$ we mean that there exists $u_t \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T u(\cdot,t)\varphi_t(t)\,\mathrm{d}t = -\int_0^T u_t(\cdot,t)\varphi(t)\,\mathrm{d}t, \qquad \text{for all } \varphi \in C^\infty_0(0,T).$$

The previous equality makes sense due to the inclusion $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ which allows us to identify u as an element of $L^2(0,T;H^{-1}(\Omega))$. Since $u_t \in L^2(0,T;H^{-1}(\Omega))$ implies that $u \in C^0([0,T];H^{-1}(\Omega))$, it is clear what we mean by saying $u(\cdot,0) = u_o \in H^{-1}(\Omega)$ in the H^{-1}-sense.

We consider obstacle functions $\psi\colon \Omega_T \to \mathbb{R}_{\geq 0}$ with

$$\psi^m \in L^2\big(0,T;H^1(\Omega)\big), \quad \partial_t(\psi^m) \in L^{\frac{m+1}{m}}(\Omega_T) \quad \text{and} \quad \psi^m(\cdot,0) \in H^1(\Omega). \tag{2.1}$$

Note that in particular, this implies $\psi \in C^0([0,T];L^{m+1}(\Omega))$. The class of admissible functions is defined by

$$K_\psi(\Omega_T) := \big\{v \in C^0\big([0,T];L^{m+1}(\Omega)\big) : v^m \in L^2\big(0,T;H^1(\Omega)\big), v \geq \psi \text{ a.e. on } \Omega_T\big\}.$$

Furthermore, the class of admissible comparison functions will be denoted by

$$K'_\psi(\Omega_T) := \big\{v \in K_\psi(\Omega_T) : \partial_t(v^m) \in L^{\frac{m+1}{m}}(\Omega_T)\big\}.$$

Note that $\psi \in K_\psi$ and $\psi \in K'_\psi$ and therefore $K_\psi, K'_\psi \neq \emptyset$. As the initial data, we take a function u_o so that

$$u_o^m \in H^1(\Omega) \quad \text{with } u_o \geq \psi(\cdot,0) \text{ a.e. on } \Omega. \tag{2.2}$$

In order to introduce the notion of a weak solution to the obstacle problem, we have to attribute a meaning to the term containing the time derivative even if we do not know that $\partial_t u$ exists in some sense. Therefore, following Alt & Luckhaus [1], for

$u \in L^{2m}(\Omega_T) \cap C^0([0,T]; L^{m+1}(\Omega))$ and $v \in L^{2m}(\Omega_T)$ with $\partial_t(v^m) \in L^{\frac{m+1}{m}}(\Omega_T)$ and $\alpha \in W^{1,\infty}([0,T])$ with $\alpha(T) = 0$ and $\eta \in L^\infty(\Omega)$ we define

$$\langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o} := \int_{\Omega_T} \eta\Big[\alpha'\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] - \alpha u \partial_t v^m\Big]\,\mathrm{d}z + \alpha(0)\int_\Omega \eta\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x,$$

where u_o is the given initial datum. Here, we note that the assumptions on v imply that $v \in C^0([0,T]; L^{m+1}(\Omega))$, so that all integrals are well defined. Moreover, since $m > m_c$, assumption (2.2) implies that $u_o \in L^{m+1}(\Omega)$. Now, we can define what we mean by local weak and strong solution to the obstacle problem.

Definition 2.1. A nonnegative function $u \in K_\psi(\Omega_T)$ is a *local strong solution to the obstacle problem for the porous medium equation* if and only if $\partial_t u \in L^2(0,T; H^{-1}(\Omega))$ and

$$\int_0^T \langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla\big(\eta(v^m - u^m)\big)\,\mathrm{d}z \geq 0 \tag{2.3}$$

holds for all comparison maps $v \in K_\psi(\Omega_T)$, every cut-off function in time $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$ and and every cut-off function in space $\eta \in C_0^1(\Omega, \mathbb{R}_{\geq 0})$.

A nonnegative function $u \in K_\psi(\Omega_T)$ is a *local weak solution to the obstacle problem for the porous medium equation* if and only if

$$\langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o} + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla\big(\eta(v^m - u^m)\big)\,\mathrm{d}z \geq 0 \tag{2.4}$$

holds true for all comparison maps $v \in K'_\psi(\Omega_T)$, every cut-off function in time $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$ and every cut-off function in space $\eta \in C_0^1(\Omega, \mathbb{R}_{\geq 0})$.

We note that if u is a local strong solution with $u(\cdot,0) = u_o$ in the $H^{-1}(\Omega)$-sense, then it is also a local weak solution; see Lemma 3.2. Further, the variational inequality (2.4) implies that $u(\cdot,0) = u_o$; see Lemma 5.2.

The preceding definition is local in the sense that there are no prescribed Dirichlet boundary data on the lateral boundary $\partial\Omega \times (0,T)$. However, our aim in this paper is to prove existence of weak solutions to the obstacle problem with prescribed initial and Dirichlet boundary data. Therefore, we consider Dirichlet boundary data

$$g \in K'_\psi(\Omega_T) \quad \text{and} \quad g(\cdot,0) = u_o. \tag{2.5}$$

Then, the class of admissible functions subject to the Dirichlet boundary data is given by

$$K_{\psi,g}(\Omega_T) := \big\{v \in K_\psi(\Omega_T) : v^m - g^m \in L^2\big(0,T; H_0^1(\Omega)\big)\big\}$$

and the class of admissible comparison functions subject to the Dirichlet boundary data will be denoted by

$$K'_{\psi,g}(\Omega_T) := \big\{v \in K'_\psi(\Omega_T) : v^m - g^m \in L^2\big(0,T; H_0^1(\Omega)\big)\big\}.$$

Note that $g \in K_{\psi,g}$ and $g \in K'_{\psi,g}(\Omega_T)$ and therefore $K_{\psi,g}(\Omega_T), K'_{\psi,g}(\Omega_T) \neq \emptyset$. Now, we can define what we mean by a weak solution to the obstacle problem for the porous medium equation with prescribed boundary data.

Definition 2.2. A nonnegative function $u \in K_{\psi,g}(\Omega_T)$ is a *strong solution to the obstacle problem for the porous medium equation* if and only if $\partial_t u \in L^2(0,T; H^{-1}(\Omega))$ and

$$\int_0^T \langle \partial_t u, \alpha(v^m - u^m)\rangle\,\mathrm{d}t + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla(v^m - u^m)\,\mathrm{d}z \geq 0 \tag{2.6}$$

holds for all comparison maps $v \in K_{\psi,g}(\Omega_T)$ and every cut-off function $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$.

A nonnegative function $u \in K_{\psi,g}(\Omega_T)$ is a *weak solution to the obstacle problem for the porous medium equation* if and only if

$$\langle\!\langle \partial_t u, \alpha(v^m - u^m)\rangle\!\rangle_{u_o} + \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla(v^m - u^m)\,\mathrm{d}z \geq 0 \tag{2.7}$$

holds for all comparison maps $v \in K'_{\psi,g}(\Omega_T)$ and every cut-off function $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$.

Remark 2.3. In the case $m > 1$, the assumptions $\partial_t\psi^m, \partial_t g^m \in L^{\frac{m+1}{m}}(\Omega_T)$ could be replaced by the weaker assumptions $\partial_t\psi^m, \partial_t g^m \in L^{\frac{2m}{2m-1}}(\Omega_T)$. This can be achieved by slight modifications in the proof, in particular by different applications of Hölder's inequality. However, in order to keep the exposition as simple as possible and to have a unified proof for both cases $m > 1$ and $m < 1$, we will work with the assumptions $\partial_t\psi^m, \partial_t g^m \in L^{\frac{m+1}{m}}(\Omega_T)$ in both cases.

We note that under a mild regularity assumption on the domain Ω, any local weak solution $u \in K_\psi(\Omega_T)$ in the sense of Definition 2.1 which additionally satisfies $u^m - g^m \in L^2(0,T; H^1_0(\Omega))$ is a weak solution in the sense of Definition 2.2; see Lemma 3.5. The regularity assumption on the domain is made precise in the following

Definition 2.4. A set $E \subset \mathbb{R}^n$ is *uniformly p-thick* if there exist constants $\gamma, \varrho_o > 0$ such that

$$\mathrm{cap}_p\big(E \cap \overline{B}_\varrho(x), B_{2\varrho}(x)\big) \geq \gamma\, \mathrm{cap}_p\big(\overline{B}_\varrho(x), B_{2\varrho}(x)\big),$$

for all $x \in E$ and for all $0 < \varrho < \varrho_o$. □

Here, cap_p denotes the usual variational p-capacity, cf. [11]. To require that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick is not very restrictive, since all Lipschitz domains or domains satisfying an exterior cone condition are uniformly p-thick for every $p \in (1,\infty)$. If $p > n$, then the complement of every open set $\Omega \subsetneq \mathbb{R}^n$ is uniformly p-thick.

To show the existence of strong solutions, we need the following stronger assumptions on the data:

$$\begin{cases} \psi, g \in L^\infty(\Omega_T, \mathbb{R}_{\geq 0}) \text{ satisfy (2.1), (2.5)}, \quad u_o \in L^\infty(\Omega, \mathbb{R}_{\geq 0}) \text{ satisfies (2.2)}, \\ \Psi := \partial_t\psi - \Delta\psi^m \in L^\infty(\Omega_T). \end{cases} \tag{2.8}$$

Note that these assumptions imply that the obstacle ψ is locally Hölder continuous; indeed, the assumption $\Psi \in L^\infty(\Omega_T)$ contains the fact that ψ is a solution to the porous medium equation with a right-hand side given by a bounded function. Now, the Hölder continuity of ψ follows from [8].

Solutions to the obstacle problem have a relationship to weak solutions and supersolutions. Therefore, we recall what we mean by weak solutions and weak supersolutions to the porous medium equation.

Definition 2.5. *A nonnegative function $u \in C^0([0,T]; L^{m+1}(\Omega))$ is a local weak solution of the porous medium equation*

$$\partial_t u - \Delta u^m = 0 \qquad \text{in } \Omega_T, \tag{2.9}$$

if $u^m \in L^2(0,T; H^1(\Omega))$ and

$$\int_{\Omega_T} \big[-u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi\big]\,\mathrm{d}z = -\int_{\Omega\times\{t\}} u\varphi\,\mathrm{d}x\Big|_{t=0}^{T} \tag{2.10}$$

for all test functions $\varphi \in C^\infty(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times [0,T]$. For weak supersolutions, the requirement is that

$$\int_{\Omega_T} \big[-u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi\big]\,\mathrm{d}z \geq 0 \tag{2.11}$$

for all positive test functions $\varphi \in C_0^\infty(\Omega_T)$. *For weak subsolutions, the inequality in* (2.11) *is reversed.*

A nonnegative function u *is a weak solution in an open set* $V \subset \mathbb{R}^{n+1}$ *if it is a weak solution in the sense defined above in all space-time cylinders* U_{t_1,t_2} *such that* $\overline{U_{t_1,t_2}} \subset V$.

Now we are in a position to state the main results of the present article. We start with the existence result for strong solutions.

Theorem 2.6. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathbb{R}^n \setminus \Omega$ *is uniformly* 2-*thick and* $m > m_c := \frac{(n-2)_+}{n}$. *Assume that the data* ψ, u_o, g *satisfy the regularity and compatibility conditions* (2.8). *Then there exists a local strong solution* u *to the obstacle problem for the porous medium equation in the sense of Definition 2.1 satisfying* $u^m - g^m \in L^2(0,T;H_0^1(\Omega))$ *and* $u(\cdot,0) = u_o$.

The function u *is also locally Hölder continuous, and satisfies* $u \geq \psi$ everywhere *in* Ω_T. *Further,* u *is a weak supersolution to the porous medium equation in* Ω_T*, and a weak solution in the open set* $\{z \in \Omega_T : u(z) > \psi(z)\}$.

Our second result ensures the existence of weak solutions under the presence of an irregular obstacle.

Theorem 2.7. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathbb{R}^n \setminus \Omega$ *is uniformly* 2-*thick and* $m > m_c := \frac{(n-2)_+}{n}$. *Assume that the data* ψ, u_o, g *satisfy the regularity and compatibility conditions* (2.1)*,* (2.2) *and* (2.5)*. Then there exists a local weak solution to the obstacle problem for the porous medium equation in the sense of Definition 2.1 satisfying* $u^m - g^m \in L^2(0,T;H_0^1(\Omega))$. *Again,* u *is also a weak supersolution to the porous medium equation in* Ω_T.

Thanks to Lemma 3.5 this immediately implies the existence of strong and weak solutions to the obstacle problem.

Corollary 2.8. *Under the assumptions of Theorems 2.6 and 2.7 there exists a strong and a weak solution, respectively, to the obstacle problem for the porous medium equation in the sense of Definition 2.2.*

Remark 2.9. We conjecture that (at least) the strong solutions obtained in Theorem 2.6 are unique in a certain sense. To simplify matters, assume for a moment that the obstacle and the boundary values are given by the same function, i.e. $g = \psi$. Define

$$\mathcal{U} := \{v : \Omega_T \to \mathbb{R}_{\geq 0} : v \text{ is a lower semicontinuous weak supersolution to the PME}\},$$

$$\mathcal{U}_\psi := \{v \in \mathcal{U} : v \geq \psi\}.$$

A function $w \in \mathcal{U}_\psi$ is the smallest supersolution above the obstacle ψ, if

$$v \in \mathcal{U}_\psi \quad \text{implies that} \quad w \leq v \text{ in } \Omega_T. \tag{2.12}$$

The smallest supersolution is unique, if it exists: if $u_1, u_2 \in \mathcal{U}_\psi$ both have the property (2.12), then two applications of (2.12) yield that $u_1 \leq u_2$ and $u_2 \leq u_1$, implying that $u_1 = u_2$.

A strong solution u to the obstacle problem with the properties given in Theorem 2.6 clearly belongs to the class $\mathcal{U}_\psi$. Showing that (2.12) also holds would proceed as follows: in the contact set $\{u = \psi\}$ one clearly has $u \leq v$ for any $v \in \mathcal{U}_\psi$. To conclude that $u \leq v$ also outside the contact set, i.e. in $V := \{u > \psi\}$, one would like to apply the comparison principle, since u is a weak solution in V and $u \leq v$ on the boundary ∂V. However, V is a general open set in $\mathbb{R}^{n+1}$, not a space-time cylinder. As far as we know, the comparison principle for general open sets in $\mathbb{R}^{n+1}$ remains open for the porous medium equation.

3. Preliminaries

3.1. **Mollification in time.** The above definitions of weak solution to the obstacle problem do not include a time derivative for u. Nevertheless, we would like to use test functions involving the solution u itself, and the quantity $\frac{\partial u}{\partial t}$ inevitably appears. This situation is usually resolved by employing a mollification procedure in the time direction. The mollification

$$[\![v]\!]_h(x,t) := e^{-\frac{t}{h}}v_o + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}}v(x,s)\,\mathrm{d}s \tag{3.1}$$

with some $v_o \in L^1(\Omega)$ has turned out to be convenient for dealing with the porous medium equation. The aim is to obtain estimates independent of the time derivative of $[\![u]\!]_h$, and then pass to the limit $h \downarrow 0$. The basic properties of the mollification (3.1) are given in the following lemma, see [19].

Lemma 3.1. *Let $p \geq 1$.*

(i) *If $v \in L^p(\Omega_T)$ and $v_o \in L^p(\Omega)$, then*
$$\|[\![v]\!]_h\|_{L^p(\Omega_T)} \leq \|v\|_{L^p(\Omega_T)} + c(p)h^{\frac{1}{p}}\|v_o\|_{L^p(\Omega)},$$
$[\![v]\!]_h \to v$ in $L^p(\Omega_T)$ as $h \downarrow 0$ and
$$\partial_t[\![v]\!]_h = \tfrac{1}{h}(v - [\![v]\!]_h).$$

(ii) *If $\nabla v \in L^p(\Omega_T)$ and $\nabla v_o \in L^p(\Omega)$, then $\nabla[\![v]\!]_h = [\![\nabla v]\!]_h$,*
$$\|\nabla[\![v]\!]_h\|_{L^p(\Omega_T)} \leq \|\nabla v\|_{L^p(\Omega_T)} + c(p)h^{\frac{1}{p}}\|\nabla v_o\|_{L^p(\Omega)},$$
and $\nabla[\![v]\!]_h \to \nabla v$ in $L^p(\Omega_T)$ as $h \downarrow 0$.

(iii) *If $v \in L^\infty(0,T;L^p(\Omega))$ and $v_o \in L^p(\Omega)$, then $[\![v]\!]_h \in C^0([0,T];L^p(\Omega))$ with*
$$\|[\![v]\!]_h\|_{L^\infty(0,T;L^p(\Omega))} \leq \|v\|_{L^\infty(0,T;L^p(\Omega))} + \|v_o\|_{L^p(\Omega)}.$$
Moreover, there holds $[\![v]\!]_h \to v$ in $L^p(\Omega_T)$ as $h \downarrow 0$ and $[\![v]\!]_h(\cdot,0) = v_o$.

(iv) *If $v_k \to v$ in $L^p(\Omega_T)$, then also*
$$[\![v_k]\!]_h \to [\![v]\!]_h \quad \text{and} \quad \partial_t[\![v_k]\!]_h \to \partial_t[\![v]\!]_h \qquad \text{in } L^p(\Omega_T).$$

(v) *If $\nabla v_k \to \nabla v$ in $L^p(\Omega_T)$, then $\nabla[\![v_k]\!]_h \to \nabla[\![v]\!]_h$ in $L^p(\Omega_T)$.*

(vi) *If $v_k \rightharpoonup v$, (or $\nabla v_k \rightharpoonup \nabla v$) weakly in $L^p(\Omega_T)$, then $[\![v_k]\!]_h \rightharpoonup [\![v]\!]_h$, (or $\nabla[\![v_k]\!]_h \rightharpoonup \nabla[\![v]\!]_h$) weakly in $L^p(\Omega_T)$.*

(vii) *If $v, \partial_t v \in L^2(0,T;H^{-1}(\Omega))$ and $v(\cdot,0) = v_o$ in the $H^{-1}(\Omega)$-sense, then $\partial_t[\![v]\!]_h \rightharpoonup \partial_t v$ weakly in $L^2(0,T;H^{-1}(\Omega))$.*

(viii) *If $\varphi \in C(\overline{\Omega_T})$, then*
$$[\![\varphi]\!]_h(x,t) \to \varphi(x,t)$$
uniformly in Ω_T as $h \downarrow 0$.

The next lemma ensures that the weak formulation of the variational inequality in (2.4), respectively (2.7) coincides with the strong form (see Theorem 2.6), if the time derivative $\partial_t u$ of the solution belongs to $L^2(0,T;H^{-1}(\Omega))$.

Lemma 3.2. *Let $u \in L^\infty(0,T;L^{m+1}(\Omega))$ be a non-negative function with $u^m \in L^2(0,T;H^1(\Omega))$, $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ and $u(\cdot,0) = u_o$ in the $H^{-1}(\Omega)$-sense and $u_o \in L^{m+1}(\Omega) \subset H^{-1}(\Omega)$. Then, there holds*

$$\int_0^T \langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t = \langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o} \tag{3.2}$$

for any $v\colon \Omega_T \to \mathbb{R}_{\geq 0}$ satisfying $v^m \in L^2(0,T;H^1(\Omega))$ and $\partial_t(v^m) \in L^{\frac{m+1}{m}}(\Omega_T)$ and $\alpha \in W^{1,\infty}([0,T],\mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$ and $\eta \in C_0^1(\Omega,\mathbb{R}_{\geq 0})$.

Proof. We define $[\![u]\!]_h$ according to (3.1) with the choice $v_o = u_o$ and similarly, for the definition of $[\![u^m]\!]_h$, we choose $v_o = u_o^m$. We first re-write

$$\begin{aligned} &\int_0^T \langle \partial_t [\![u]\!]_h, \alpha\eta(v^m - [\![u^m]\!]_h)\rangle \,\mathrm{d}t \\ &\qquad = \int_{\Omega_T} \alpha\eta\partial_t [\![u]\!]_h (v^m - [\![u]\!]_h^m)\,\mathrm{d}z + \int_{\Omega_T} \alpha\eta\partial_t [\![u]\!]_h ([\![u]\!]_h^m - [\![u^m]\!]_h)\,\mathrm{d}z. \end{aligned} \tag{3.3}$$

For the first integral on the right-hand side we compute

$$\begin{aligned} &\int_{\Omega_T} \alpha\eta\partial_t [\![u]\!]_h (v^m - [\![u]\!]_h^m)\,\mathrm{d}z \\ &\quad = -\int_{\Omega_T} \Big[\alpha'\eta [\![u]\!]_h (v^m - [\![u]\!]_h^m) + \alpha\eta [\![u]\!]_h \partial_t (v^m - [\![u]\!]_h^m)\Big]\,\mathrm{d}z \\ &\qquad - \alpha(0)\int_\Omega \eta u_o (v^m(\cdot,0) - u_o^m)\,\mathrm{d}x \\ &\quad = \int_{\Omega_T} \Big[\alpha'\eta\big[[\![u]\!]_h^{m+1} - [\![u]\!]_h v^m\big] + \tfrac{m}{m+1}\alpha\eta\partial_t [\![u]\!]_h^{m+1} - \alpha\eta [\![u]\!]_h \partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad + \alpha(0)\int_\Omega \eta\big[u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x \\ &\quad = \int_{\Omega_T} \Big[\alpha'\eta\big[\tfrac{1}{m+1}[\![u]\!]_h^{m+1} - [\![u]\!]_h v^m\big] - \alpha\eta [\![u]\!]_h \partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad + \alpha(0)\int_\Omega \eta\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x. \end{aligned}$$

For the integrand of the second integral in (3.3) we have due to Lemma 3.1 (i) that

$$\begin{aligned} \partial_t [\![u]\!]_h ([\![u]\!]_h^m - [\![u^m]\!]_h) &= \partial_t [\![u]\!]_h ([\![u]\!]_h^m - u^m) + \partial_t [\![u]\!]_h (u^m - [\![u^m]\!]_h) \\ &= -\tfrac{1}{h}([\![u]\!]_h - u)([\![u]\!]_h^m - u^m) + \partial_t [\![u]\!]_h (u^m - [\![u^m]\!]_h) \\ &\le \partial_t [\![u]\!]_h (u^m - [\![u^m]\!]_h). \end{aligned}$$

Altogether, we have shown that

$$\begin{aligned} &\int_0^T \langle \partial_t [\![u]\!]_h, \alpha\eta(v^m - [\![u^m]\!]_h)\rangle\,\mathrm{d}t \\ &\quad \le \int_{\Omega_T} \Big[\alpha'\eta\big[\tfrac{1}{m+1}[\![u]\!]_h^{m+1} - [\![u]\!]_h v^m\big] - \alpha\eta [\![u]\!]_h \partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad + \alpha(0)\int_\Omega \eta\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x \\ &\qquad + \int_0^T \langle \partial_t [\![u]\!]_h, \alpha\eta(u^m - [\![u^m]\!]_h)\rangle\,\mathrm{d}t. \end{aligned}$$

Now, Lemma 3.1 ensures that all appearing terms converge in the limit $h \downarrow 0$. In particular, the last term on the right-hand side disappears in the limit $h \downarrow 0$. This proves that

$$\int_0^T \langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t \le \langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o}.$$

In order to prove the reversed inequality, we assume first that $u \ge \varepsilon$ for $\varepsilon > 0$ if $m > 1$, and $u \le k < \infty$ if $m < 1$. We note from (3.1) that $[\![u^m]\!]_h \ge \varepsilon^m$ and $[\![u^m]\!]_h \le k^m$ in the cases $m > 1$ and $m < 1$, respectively. Then we write

$$\begin{aligned} &\int_0^T \langle \partial_t [\![u]\!]_h, \alpha\eta(v^m - [\![u^m]\!]_h)\rangle\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} \Big[\alpha\eta\partial_t [\![u]\!]_h v^m - \alpha\eta\partial_t [\![u^m]\!]_h^{\frac{1}{m}} [\![u^m]\!]_h\Big]\,\mathrm{d}z \end{aligned}$$

$$+\int_{\Omega_T} \alpha\eta\partial_t\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h \,\mathrm{d}z. \tag{3.4}$$

Note that the quantity $\partial_t\llbracket u^m\rrbracket_h^{\frac{1}{m}}$ does not cause any problems, since

$$|\partial_t\llbracket u^m\rrbracket_h^{\frac{1}{m}}| = \tfrac{1}{m}\llbracket u^m\rrbracket_h^{\frac{1}{m}-1}|\partial_t\llbracket u^m\rrbracket_h| \le \begin{cases} \tfrac{1}{m}\varepsilon^{1-m}|\partial_t\llbracket u^m\rrbracket_h|, & \text{if } m>1,\\ \tfrac{1}{m}k^{1-m}|\partial_t\llbracket u^m\rrbracket_h|, & \text{if } m<1.\end{cases}$$

For the first integral on the right-hand side we compute

$$\begin{aligned}
&\int_{\Omega_T}\Big[\alpha\eta\partial_t\llbracket u\rrbracket_h v^m - \alpha\eta\partial_t\llbracket u^m\rrbracket_h^{\frac{1}{m}}\llbracket u^m\rrbracket_h\Big]\,\mathrm{d}z\\
&\quad= -\int_{\Omega_T}\Big[\alpha'\eta\llbracket u\rrbracket_h v^m + \alpha\eta\llbracket u\rrbracket_h\partial_t v^m + \tfrac{1}{m+1}\alpha\eta\partial_t\llbracket u^m\rrbracket_h^{\frac{m+1}{m}}\Big]\,\mathrm{d}z\\
&\qquad - \alpha(0)\int_\Omega \eta u_o v^m(\cdot,0)\,\mathrm{d}x\\
&\quad= \int_{\Omega_T}\Big[\alpha'\eta\big[\tfrac{1}{m+1}\llbracket u^m\rrbracket_h^{\frac{m+1}{m}} - \llbracket u\rrbracket_h v^m\big] - \alpha\eta\llbracket u\rrbracket_h\partial_t v^m\Big]\,\mathrm{d}z\\
&\qquad + \alpha(0)\int_\Omega \eta\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x.
\end{aligned}$$

Moreover, by Lemma 3.1 (i) we have

$$\begin{aligned}
&\int_{\Omega_T}\alpha\eta\partial_t\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\,\mathrm{d}z\\
&\quad= -\int_{\Omega_T}\Big[\alpha\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\partial_t\llbracket u^m\rrbracket_h + \alpha'\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\Big]\,\mathrm{d}z\\
&\quad= \frac{1}{h}\int_{\Omega_T}\alpha\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - u\big)\big(\llbracket u^m\rrbracket_h - u^m\big) + \alpha\eta\big(u-\llbracket u\rrbracket_h\big)\big(\llbracket u^m\rrbracket_h - u^m\big)\,\mathrm{d}z\\
&\qquad + \int_{\Omega_T}\alpha'\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\,\mathrm{d}z\\
&\quad\ge \frac{1}{h}\int_{\Omega_T}\alpha\eta\big(u-\llbracket u\rrbracket_h\big)\big(\llbracket u^m\rrbracket_h - u^m\big)\,\mathrm{d}z + \int_{\Omega_T}\alpha'\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\,\mathrm{d}z\\
&\quad= \int_{\Omega_T}\alpha\eta\partial_t\llbracket u\rrbracket_h\big(\llbracket u^m\rrbracket_h - u^m\big)\,\mathrm{d}z + \int_{\Omega_T}\alpha'\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\,\mathrm{d}z.
\end{aligned}$$

Inserting this above, we conclude that

$$\begin{aligned}
&\int_0^T\langle\partial_t\llbracket u\rrbracket_h, \alpha\eta(v^m - \llbracket u^m\rrbracket_h)\rangle\,\mathrm{d}t\\
&\quad\ge \int_{\Omega_T}\Big[\alpha'\eta\big[\tfrac{1}{m+1}\llbracket u^m\rrbracket_h^{\frac{m+1}{m}} - \llbracket u\rrbracket_h v^m\big] - \alpha\eta\llbracket u\rrbracket_h\partial_t v^m\Big]\,\mathrm{d}z\\
&\qquad + \alpha(0)\int_\Omega \eta\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v^m(\cdot,0)\big]\,\mathrm{d}x\\
&\qquad + \int_0^T\big\langle\partial_t\llbracket u\rrbracket_h, \alpha\eta\big(\llbracket u^m\rrbracket_h - u^m\big)\big\rangle\,\mathrm{d}t + \int_{\Omega_T}\alpha'\eta\big(\llbracket u^m\rrbracket_h^{\frac{1}{m}} - \llbracket u\rrbracket_h\big)\llbracket u^m\rrbracket_h\,\mathrm{d}z.
\end{aligned}$$

Recall that this holds with the assumptions $u \ge \varepsilon > 0$ if $m>1$, and $u \le k$ if $m<1$. To establish the inequality also for a general function u, we apply the estimate proved so far to the functions $u_\varepsilon = (u^m + \varepsilon^m)^{\frac{1}{m}}$ and $u_k = \min(u,k)$, and then appeal to Lemma 3.1 for letting $\varepsilon \downarrow 0$ and $k\to\infty$, respectively. Here, we note that $\partial_t\llbracket u_\varepsilon\rrbracket_h = \frac{1}{h}(u_\varepsilon - \llbracket u_\varepsilon\rrbracket_h) \to \frac{1}{h}(u - \llbracket u\rrbracket_h) = \partial_t\llbracket u\rrbracket_h$ in $L^{m+1}(\Omega_T)$ and $\llbracket u^m\rrbracket_h \to u^m$ in $L^{\frac{m+1}{m}}(\Omega_T)$ in the limit $\varepsilon\downarrow 0$ holds, respectively the analogous convergences for $k\to\infty$ if $m<1$. Finally, another application of Lemma 3.1 shows that all integrals converge as $h \downarrow 0$ and in particular the last two integrals disappear. This proves the asserted identity. □

3.2. **Hardy's inequality.** Another tool we need is the Hardy inequality; see [9, 10, 13] for the proof.

Theorem 3.3. *Let Ω be an open bounded subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly 2-thick, and $u \in H_0^1(\Omega)$. Then there is a constant c depending only on n, γ, where γ is the constant from Definition 2.4, such that*

$$\int_\Omega \frac{|u|^2}{d_\Omega(x)^2}\,\mathrm{d}x \le c\int_\Omega |\nabla u|^2\,\mathrm{d}x,$$

where

$$d_\Omega(x) := \operatorname{dist}(x, \partial\Omega).$$

The next lemma will be a helpful tool to localize certain arguments. The main tool in the proof will be Hardy's inequality.

Lemma 3.4. *Let Ω be an open bounded subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly 2-thick, and let $\eta_\lambda \in C_0^\infty(\Omega, [0,1])$ be a cutoff function such that $\eta_\lambda = 1$ in $\{x \in \Omega : d_\Omega(x) \ge \lambda\}$, and $|\nabla \eta_\lambda| \le \frac{c}{\lambda}$. Then for any $\varphi \in L^2(0,T; H_0^1(\Omega))$ we have*

$$\varphi\eta_\lambda \rightharpoonup \varphi \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)) \quad \text{as } \lambda \downarrow 0.$$

Proof. Pick any functional $f \in L^2(0,T;H^{-1}(\Omega))$. Then $f = g - \operatorname{div} h$ for some $g \in L^2(\Omega_T)$ and $h \in L^2(\Omega_T, \mathbb{R}^n)$. This means that

$$\int_0^T \langle f, \varphi(1-\eta_\lambda)\rangle\,\mathrm{d}t = \int_{\Omega_T} (1-\eta_\lambda)\varphi g\,\mathrm{d}z + \int_{\Omega_T} \nabla(\varphi(1-\eta_\lambda))\cdot h\,\mathrm{d}z,$$

and the claim follows by showing that the right-hand side tends to zero as $\lambda \to 0$.

The first term tends to zero by the dominated convergence theorem. For the second term, we have

$$\int_{\Omega_T} \nabla(\varphi(1-\eta_\lambda))\cdot h\,\mathrm{d}z = \int_{\Omega_T}(1-\eta_\lambda)\nabla\varphi\cdot h\,\mathrm{d}z - \int_{\{d_\Omega(x)<\lambda\}} \varphi\nabla\eta_\lambda\cdot h\,\mathrm{d}z,$$

where we used the fact that $\nabla\eta_\lambda = 0$ when $d_\Omega(x) \ge \lambda$. The first term on the right-hand side again tends to zero by the dominated convergence theorem. For the second one, we use Hölder's inequality and the Hardy inequality from Theorem 3.3 to get

$$\begin{aligned}\left|\int_{\{d_\Omega(x)<\lambda\}} \varphi\nabla\eta_\lambda\cdot h\,\mathrm{d}z\right| &\le c\bigg(\int_{\{d_\Omega(x)<\lambda\}} \frac{|\varphi|^2}{\lambda^2}\,\mathrm{d}z\bigg)^{\frac12}\bigg(\int_{\{d_\Omega(x)<\lambda\}} |h|^2\,\mathrm{d}z\bigg)^{\frac12}\\ &\le c\bigg(\int_{\Omega_T} \frac{|\varphi|^2}{d_\Omega(x)^2}\,\mathrm{d}z\bigg)^{\frac12}\bigg(\int_{\{d_\Omega(x)<\lambda\}} |h|^2\,\mathrm{d}z\bigg)^{\frac12}\\ &\le c\bigg(\int_{\Omega_T} |\nabla\varphi|^2\,\mathrm{d}z\bigg)^{\frac12}\bigg(\int_{\{d_\Omega(x)<\lambda\}} |h|^2\,\mathrm{d}z\bigg)^{\frac12} \to 0\end{aligned}$$

since

$$|\{(x,t)\in\Omega_T : d_\Omega(x) < \lambda\}| \to 0$$

as $\lambda \to 0$. □

As an immediate consequence of Lemma 3.4 we can show that local weak or strong solutions possessing the right lateral boundary data are indeed weak or strong solutions.

Lemma 3.5. *Any local weak solution $u \in K_\psi(\Omega_T)$ in the sense of Definition 2.1 which additionally satisfies $u^m - g^m \in L^2(0,T;H_0^1(\Omega))$ is a weak solution in the sense of Definition 2.2. The same holds for strong solutions.*

Proof. In the variational inequalities (2.3), respectively (2.4) we simply choose $\eta = \eta_\lambda$ with η_λ as in Lemma 3.4 and then pass to the limit $\lambda \downarrow 0$. □

3.3. **Some elementary inequalities.** For $u, v \geq 0$ we define

$$I(u,v) := \tfrac{1}{m+1}\big(u^{m+1} - v^{m+1}\big) - v^m(u-v).$$

Lemma 3.6. *For any $m \geq 1$ and $u, v \geq 0$ we have*

$$\tfrac{1}{m+1}|u-v|^{m+1} \leq I(u,v).$$

Proof. The estimate is trivial for $v = 0$, and in the other case we can divide by v^{m+1} and let $x = \frac{u}{v}$ to see that it suffices to show that the inequality

$$h(x) := \tfrac{1}{m+1}\big(x^{m+1} - 1\big) - (x-1) - \tfrac{1}{m+1}|x-1|^{m+1} \geq 0$$

holds for every $x \geq 0$. But one-dimensional calculus shows that h has critical points in 0 and 1, is strictly concave on $[0, \frac{1}{2}]$ and strictly convex on $[\frac{1}{2}, \infty)$. This implies that $h(1) = 0$ is the minimum value of h on the interval $[0, \infty)$. □

Lemma 3.7. *For any $0 < m < 1$ and any $u, v \geq 0$ we have*

$$I(u,v) \geq m2^{m-2}\begin{cases} v^{m-1}|u-v|^2 & \text{if } |u-v| < v, \\ |u-v|^{m+1} & \text{if } |u-v| \geq v. \end{cases}$$

Proof. In the special case $v = 0$, the estimate holds true since $m2^{m-2} \leq \frac{m}{2} \leq \frac{1}{m+1}$. If $v \neq 0$, we divide by v^{m+1} and let $x = \frac{u}{v}$. It thus remains to prove

$$\tfrac{1}{m+1}\big(x^{m+1} - 1\big) - (x-1) \geq m2^{m-2}\begin{cases} (x-1)^2 & \text{if } 0 \leq x < 2, \\ (x-1)^{m+1} & \text{if } x \geq 2. \end{cases}$$

For the case $0 \leq x < 2$ this is satisfied because the function

$$g(x) := \tfrac{1}{m+1}\big(x^{m+1} - 1\big) - (x-1) - m2^{m-2}(x-1)^2$$

is strictly convex on $[0, 2]$ and possesses its only critical point in $x = 1$ with minimum value $g(1) = 0$. For the case $x > 2$, we introduce the auxiliary function

$$h(x) := \tfrac{1}{m+1}\big(x^{m+1} - 1\big) - (x-1) - m2^{m-2}(x-1)^{m+1}.$$

The concavity of $x \mapsto x^m$ and the inequality $x < 2(x-1)$ for $x > 2$ imply $x^m - 1 > mx^{m-1}(x-1) > m2^{m-1}(x-1)^m$ and therefore

$$h'(x) > \big[m2^{m-1} - m(m+1)2^{m-2}\big](x-1)^m > 0$$

for $x > 2$. We infer that h is strictly increasing on $[2, \infty)$ and consequently, $h(x) > h(2) = g(2) \geq 0$ for any $x > 2$. This completes the proof. □

Corollary 3.8. *For any $0 < m < 1$ and any two functions $u, v \in L^{m+1}(\Omega)$ there holds*

$$\int_\Omega |u-v|^{m+1}\,\mathrm{d}x \leq c\int_\Omega I(u,v)\,\mathrm{d}x + c\bigg(\int_\Omega I(u,v)\,\mathrm{d}x\bigg)^{\frac{m+1}{2}}\bigg(\int_\Omega v^{m+1}\,\mathrm{d}x\bigg)^{\frac{1-m}{2}}$$

for a constant $c = c(m)$.

Proof. We begin by using Hölder's inequality with exponents $\frac{2}{m+1}$ and $\frac{2}{1-m}$ in order to estimate

$$\begin{aligned}\int_{\{|u-v|<v\}} |u-v|^{m+1}\,\mathrm{d}x &= \int_{\{|u-v|<v\}} v^{\frac{(m-1)(m+1)}{2}}|u-v|^{m+1}\,v^{\frac{(1-m)(m+1)}{2}}\,\mathrm{d}x \\ &\leq \bigg(\int_{\{|u-v|<v\}} v^{m-1}|u-v|^2\,\mathrm{d}x\bigg)^{\frac{m+1}{2}}\bigg(\int_\Omega v^{m+1}\,\mathrm{d}x\bigg)^{\frac{1-m}{2}}.\end{aligned}$$

We thereby get

$$\int_\Omega |u-v|^{m+1}\,\mathrm{d}x \le \int_{\{|u-v|\ge v\}} |u-v|^{m+1}\,\mathrm{d}x + \bigg(\int_{\{|u-v|<v\}} v^{m-1}|u-v|^2\,\mathrm{d}x\bigg)^{\frac{m+1}{2}} \bigg(\int_\Omega v^{m+1}\,\mathrm{d}x\bigg)^{\frac{1-m}{2}}.$$

Now the claim follows by estimating the integrands with the help of Lemma 3.7. □

Next, we prove yet another elementary inequality:

Lemma 3.9. *For any* $u, v \ge 0$ *and* $m > 1$ *we have* $|v-u|^m \le |v^m - u^m|$.

Proof. Without loss of generality, we may assume $0 \le u \le v$. Then we can estimate

$$\begin{aligned} v^m - u^m &= \int_0^1 m(u+t(v-u))^{m-1}\,\mathrm{d}t\,(v-u) \\ &\ge \int_0^1 mt^{m-1}(v-u)^{m-1}\,\mathrm{d}t\,(v-u) = (v-u)^m, \end{aligned}$$

where we used the assumption $m > 1$. This implies the asserted inequality. □

As a first consequence, we get the following Lebesgue space estimate.

Corollary 3.10. *For any two maps* $u, v \in L^{2m}(\Omega_T)$ *we have*

$$\|v-u\|^m_{L^{2m}(\Omega_T)} \le \|v^m - u^m\|_{L^2(\Omega_T)} \qquad \textit{in the case } m > 1$$

and

$$\|v^m - u^m\|_{L^2(\Omega_T)} \le \|v-u\|^m_{L^{2m}(\Omega_T)} \qquad \textit{in the case } 0 < m < 1.$$

As a second consequence of Lemma 3.9 we have

Corollary 3.11. *For any* $u, v \ge 0$ *there holds*

$$|v-u|^{m+1} \le (v^m - u^m)(v-u) \qquad \textit{in the case } m > 1$$

and

$$|v^m - u^m|^{\frac{m+1}{m}} \le (v^m - u^m)(v-u) \qquad \textit{in the case } 0 < m < 1.$$

Proof. For the proof of the first inequality, we assume without loss of generality that $0 \le u \le v$ and apply Lemma 3.9 to bound the first factor. The second inequality follows from the first by replacing m by $\frac{1}{m}$ and v, u by v^m, u^m in the first inequality. □

4. Comparison estimates

In this chapter we will prove certain comparison estimates for functions u satisfying the variational inequality (2.4). Throughout this Section we assume that ψ, u_o, g satisfy the regularity and compatibility conditions (2.1), (2.2), and (2.5). Note that we do not assume that u belongs to $C^0([0,T]; L^{m+1}(\Omega))$ here.

Lemma 4.1. *Assume that* $u\colon \Omega_T \to \mathbb{R}_{\ge 0}$ *with* $u^m - g^m \in L^2(0,T; H^1_0(\Omega))$ *and* $u \ge \psi$ *a.e. in* Ω_T *satisfies the variational inequality* (2.4)*. Then, for all comparison maps* $v \in K'_{\psi,g}(\Omega_T)$ *and every cut-off function* $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\ge 0})$ *with* $\alpha(T) = 0$ *and* $\alpha' \le 0$*, we have the estimate*

$$\int_{\Omega_T} |\alpha'| I(u,v)\,\mathrm{d}z \le \int_{\Omega_T} \alpha\big[\partial_t v^m (v-u) + \nabla u^m \cdot \nabla(v^m - u^m)\big]\,\mathrm{d}z + \alpha(0)\int_\Omega I(u_o, v_o)\,\mathrm{d}x,$$

where we abbreviated $v_o := v(\cdot, 0)$ *and*

$$I(u,v) := \tfrac{1}{m+1}(u^{m+1} - v^{m+1}) - v^m(u-v).$$

Proof. For $\lambda > 0$ we let $\eta_\lambda \in C_0^\infty(\Omega, [0,1])$ be a cutoff function in space satisfying $\eta_\lambda = 1$ in $\{x \in \Omega : d_\Omega(x) \geq \lambda\}$, and $|\nabla \eta_\lambda| \leq \frac{c}{\lambda}$. The variational inequality (2.4) implies

$$\begin{aligned}\int_{\Omega_T} &\alpha \nabla u^m \cdot \nabla[\eta_\lambda(v^m - u^m)]\,\mathrm{d}z \\ &\geq \int_{\Omega_T} \Big[(-\alpha')\eta_\lambda\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] + \alpha\eta_\lambda u \partial_t v^m\Big]\,\mathrm{d}z \\ &\quad - \alpha(0)\int_\Omega \eta_\lambda\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x. \end{aligned} \tag{4.1}$$

For the the right-hand side of (4.1) we have

$$\begin{aligned}\lim_{\lambda\downarrow 0}&\int_{\Omega_T} \Big[(-\alpha')\eta_\lambda\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] + \alpha\eta_\lambda u\partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad - \alpha(0)\int_\Omega \eta_\lambda\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x \\ &= \int_{\Omega_T} \Big[(-\alpha')\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] + \alpha u \partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad - \alpha(0)\int_\Omega \big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x \\ &= \int_{\Omega_T} \Big[|\alpha'|\big[\tfrac{1}{m+1}(u^{m+1} - v^{m+1}) - v^m(u-v)\big] + \alpha u\partial_t v^m\Big]\,\mathrm{d}z \\ &\qquad + \int_{\Omega_T} \alpha' \tfrac{m}{m+1} v^{m+1}\,\mathrm{d}z - \alpha(0)\int_\Omega \big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x \\ &= \int_{\Omega_T} \Big[|\alpha'|\big[\tfrac{1}{m+1}(u^{m+1} - v^{m+1}) - v^m(u-v)\big] - \alpha\partial_t v^m(v-u)\Big]\,\mathrm{d}z \\ &\qquad - \alpha(0)\int_\Omega \big[\tfrac{1}{m+1}(u_o^{m+1} - v_o^{m+1}) - v_o^m(u_o - v_o)\big]\,\mathrm{d}x, \end{aligned}$$

where we also used the assumption $\alpha' \leq 0$ and an integration by parts. For the left-hand side of (4.1) we find by Lemma 3.4 that

$$\lim_{\lambda\downarrow 0}\int_{\Omega_T} \alpha\nabla u^m \cdot \nabla[\eta_\lambda(v^m - u^m)]\,\mathrm{d}z = \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla(v^m - u^m)\,\mathrm{d}z.$$

Therefore, using the preceding computations in (4.1), passing to the limit $\lambda \downarrow 0$ and re-arranging terms we arrive at

$$\begin{aligned}\int_{\Omega_T} &|\alpha'|\big[\tfrac{1}{m+1}(u^{m+1} - v^{m+1}) - v^m(u-v)\big]\,\mathrm{d}z \\ &\leq \int_{\Omega_T} \alpha\partial_t v^m(v-u)\,\mathrm{d}z + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla(v^m - u^m)\,\mathrm{d}z \\ &\quad + \alpha(0)\int_\Omega \big[\tfrac{1}{m+1}(u_o^{m+1} - v_o^{m+1}) - v_o^m(u_o - v_o)\big]\,\mathrm{d}x. \end{aligned}$$

This finishes the proof of the Lemma. □

Later on, we need a local version of Lemma 4.1 for comparison maps v, which do not admit the right boundary values on the lateral boundary $\partial\Omega \times (0,T)$.

Lemma 4.2. *Assume that $u\colon \Omega_T \to \mathbb{R}_{\geq 0}$ with $u^m - g^m \in L^2(0,T; H_0^1(\Omega))$ and $u \geq \psi$ a.e. in Ω_T satisfies the variational inequality* (2.4)*. Then, for all comparison maps $v \in K'_\psi(\Omega_T)$, and every cut-off function in time $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$ and $\alpha' \leq 0$ we have the estimate*

$$\int_{B_r(x_o)\times(0,T)} |\alpha'| I(u,v)\,\mathrm{d}z \leq \int_{B_r(x_o)\times(0,T)} \alpha\big[\partial_t v^m(v-u) + \nabla u^m \cdot \nabla(v^m - u^m)\big]\,\mathrm{d}z$$

$$+ 2\int_{\partial B_r(x_o)\times(0,T)} \alpha|\nabla u^m||v^m - u^m|\,\mathrm{d}z$$
$$+ \alpha(0)\int_{B_r(x_o)} I(u_o, v_o)\,\mathrm{d}x,$$

for a.e. $r \in (0, \mathrm{dist}(x_o, \partial\Omega))$, where we abbreviated $v_o := v(\cdot, 0)$ and

$$I(u,v) := \tfrac{1}{m+1}(u^{m+1} - v^{m+1}) - v^m(u - v).$$

Proof. By $\eta_\lambda \in C^1_0(B_r, [0,1])$ we denote a cut-off function in space satisfying $\eta_\lambda = 1$ on $B_{r-\lambda}$ and $|D\eta_\lambda| \le 2/\lambda$. Using this in the variational inequality (2.4), we infer

$$0 \le \langle\!\langle \partial_t u, \alpha\eta_\lambda(v^m - u^m)\rangle\!\rangle_{u_o} + \int_{B_r\times(0,T)} \alpha\nabla u^m\cdot\nabla[\eta_\lambda(v^m - u^m)]\,\mathrm{d}z. \tag{4.2}$$

For the first term, we have

$$\begin{aligned}
\langle\!\langle \partial_t u, \alpha\eta_\lambda(v^m - u^m)\rangle\!\rangle_{u_o} &= \int_{\Omega_T} \Big[\alpha'\eta_\lambda\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] - \alpha\eta_\lambda u\partial_t v^m\Big]\,\mathrm{d}z \\
&\quad + \alpha(0)\int_\Omega \eta_\lambda\big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x \\
&\xrightarrow[\lambda\downarrow 0]{} \int_{B_r\times(0,T)} \Big[\alpha'\big[\tfrac{1}{m+1}u^{m+1} - uv^m\big] - \alpha u\partial_t v^m\Big]\,\mathrm{d}z \\
&\quad + \alpha(0)\int_{B_r} \big[\tfrac{1}{m+1}u_o^{m+1} - u_o v_o^m\big]\,\mathrm{d}x \\
&= \langle\!\langle \partial_t u, \alpha\chi_{B_r}(v^m - u^m)\rangle\!\rangle_{u_o}.
\end{aligned}$$

For the second term in (4.2), we have

$$\begin{aligned}
&\int_{B_r\times(0,T)} \alpha\nabla u^m\cdot\nabla[\eta_\lambda(v^m - u^m)]\,\mathrm{d}z \\
&\le \int_{B_r\times(0,T)} \alpha\eta_\lambda\nabla u^m\cdot\nabla(v^m - u^m)\,\mathrm{d}z + \tfrac{2}{\lambda}\int_{(B_r\setminus B_{r-\lambda})\times(0,T)} \alpha|\nabla u^m||v^m - u^m|\,\mathrm{d}z \\
&\xrightarrow[\lambda\downarrow 0]{} \int_{B_r\times(0,T)} \alpha\nabla u^m\cdot\nabla(v^m - u^m)\,\mathrm{d}z + 2\int_{\partial B_r\times(0,T)} \alpha|\nabla u^m||v^m - u^m|\,\mathrm{d}z
\end{aligned}$$

for a.e. $r > 0$. Joining the two preceeding formulae with (4.2), we arrive at

$$\begin{aligned}
0 \le \langle\!\langle \partial_t u, \alpha\chi_{B_r}(v^m - u^m)\rangle\!\rangle_{u_o} &+ \int_{B_r\times(0,T)} \alpha\nabla u^m\cdot\nabla(v^m - u^m)\,\mathrm{d}z \\
+ 2\int_{\partial B_r\times(0,T)} \alpha|\nabla u^m||v^m - u^m|\,\mathrm{d}z.
\end{aligned}$$

At this stage, we can proceed in the same way as in Lemma 4.1 on B_r instead of Ω, with the last integral as additional term. This yields the claim. □

5. Continuity in time

Here, we prove that the variational inequality (2.4) already implies $u \in C^0([0,T]; L^{m+1}(\Omega))$ and $u(\cdot, 0) = u_o$. The following lemma is a version of [2, Lemma 2.5]; see also [3, Lemma 2.4].

Lemma 5.1. *Assume that $u, \psi, g \in L^\infty(0,T; L^{m+1}(\Omega))$ satisfy $u \ge \psi$ a.e. on Ω_T and $\partial_t\psi^m, \partial_t g^m \in L^{\frac{m+1}{m}}(\Omega_T)$. Then the maps w_h defined by*

$$(w_h)^m := \max\big\{\psi^m, [\![u^m - g^m]\!]_h + g^m\big\},$$

where

$$[\![u^m - g^m]\!]_h(x,t) := e^{-\frac{t}{h}}\big(u_o^m(x) - g^m(x,0)\big) + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}}\big(u^m - g^m\big)(x,s)\,\mathrm{d}s$$

satisfy

$$\limsup_{h\downarrow 0}\sup_{t_o\in(0,T)}\int_{\Omega_{t_o}} \partial_t(w_h^m)(w_h - u)\,\mathrm{d}z \le 0.$$

Proof. We define mollifications of $\psi^m - g^m$ by

$$[\![\psi^m - g^m]\!]_h(x,t) := e^{-\frac{t}{h}}\big(\psi^m - g^m\big)(x,0) + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}}\big(\psi^m - g^m\big)(x,s)\,\mathrm{d}s.$$

Due to the assumption $u \ge \psi$, we have

$$w_h^m \ge [\![\psi^m - g^m]\!]_h + g^m \tag{5.1}$$

and moreover, the assumption $\partial_t\psi^m \in L^{\frac{m+1}{m}}(\Omega_T)$ implies by Lemma 3.1 that

$$\partial_t[\![\psi^m - g^m]\!]_h \to \partial_t\psi^m + \partial_t g^m \qquad \text{strongly in } L^{\frac{m+1}{m}}(\Omega_T) \text{ as } h\downarrow 0. \tag{5.2}$$

For a fixed $h > 0$ we split the term under consideration into

$$\begin{aligned}
&\sup_{t_o\in(0,T)}\int_{\Omega_{t_o}} \partial_t(w_h^m)(w_h - u)\,\mathrm{d}z\\
&\quad\le \sup_{t_o\in(0,T)}\int_{\Omega_{t_o}\cap\{w_h>\psi\}} \partial_t[\![u^m - g^m]\!]_h(w_h - u)\,\mathrm{d}z\\
&\qquad + \sup_{t_o\in(0,T)}\int_{\Omega_{t_o}} \partial_t g^m(w_h - u)\,\mathrm{d}z\\
&\qquad + \sup_{t_o\in(0,T)}\int_{\Omega_{t_o}\cap\{w_h=\psi\}} (\partial_t\psi^m - \partial_t g^m)(\psi - u)\,\mathrm{d}z\\
&\quad =: \mathrm{I}_h + \mathrm{II}_h + \mathrm{III}_h.
\end{aligned}$$

The integrand in the first integral is nonpositive since

$$\begin{aligned}
\partial_t[\![u^m - g^m]\!]_h(w_h - u) &= -\tfrac{1}{h}\big([\![u^m - g^m]\!]_h - (u^m - g^m)\big)(w_h - u)\\
&= -\tfrac{1}{h}(w_h^m - u^m)(w_h - u) \le 0
\end{aligned}$$

by Lemma 3.1 (i) and the monotonicity of the function $r \mapsto r^m$. This implies $\mathrm{I}_h \le 0$. Next, we consider the second term II_h. Here, we note that $w_h \to u$ in $L^{m+1}(\Omega_T)$ by Lemma 3.1 and therefore $\lim_{h\downarrow 0}\mathrm{II}_h = 0$. Finally, we turn our attention to the third term III_h, for which we only have to consider points $z \in \Omega_{t_o}$ with $\psi(z) = w_h(z)$. In such points we have

$$\partial_t[\![\psi^m - g^m]\!]_h = \tfrac{1}{h}\big(\psi^m - g^m - [\![\psi^m - g^m]\!]_h\big) = \tfrac{1}{h}\big(w_h^m - g^m - [\![\psi^m - g^m]\!]_h\big) \ge 0,$$

where we used (5.1) for the last estimate. Since $u \ge \psi$ a.e. by assumption, this implies $\partial_t[\![\psi^m - g^m]\!]_h(\psi - u) \le 0$ on the domain of integration of III_h. We thereby obtain the bound

$$\begin{aligned}
\mathrm{III}_h &\le \sup_{t_o\in(0,T)}\int_{\Omega_{t_o}\cap\{w_h=\psi\}} \partial_t[\![\psi^m - g^m]\!]_h(\psi - u)\,\mathrm{d}z\\
&\qquad + \sup_{t_o\in(0,T)}\int_{\Omega_{t_o}\cap\{w_h=\psi\}} \big(\partial_t(\psi^m - g^m) - \partial_t[\![\psi^m - g^m]\!]_h\big)(\psi - u)\,\mathrm{d}z\\
&\le \big\|\partial_t(\psi^m - g^m) - \partial_t[\![\psi^m - g^m]\!]_h\big\|_{L^{\frac{m+1}{m}}(\Omega_T)}\|\psi - u\|_{L^{m+1}(\Omega_T)} \xrightarrow[h\downarrow 0]{} 0
\end{aligned}$$

by the convergence (5.2). This proves that $\limsup_{h\downarrow 0}\mathrm{III}_h \le 0$. Since we have shown already that $\mathrm{I}_h \le 0$ for every $h > 0$ and $\mathrm{II}_h \to 0$ as $h \downarrow 0$, the proof is complete. □

Lemma 5.2. *Let $m > 0$ and assume that ψ, u_o, g satisfy* (2.1), (2.2), (2.5) *and that $u \in L^\infty(0,T;L^{m+1}(\Omega))$ with $u^m - g^m \in L^2(0,T;H_0^1(\Omega))$ and $u \ge \psi$ a.e. in Ω_T satisfies the variational inequality* (2.4). *Then, we have $u \in C^0([0,T];L^{m+1}(\Omega))$ and $u(\cdot,0) = u_o$.*

Proof. For $h > 0$ we define the maps w_h by

$$(w_h)^m := \max\big\{\psi^m, [\![u^m - g^m]\!]_h + g^m\big\},$$

where $[\![u^m - g^m]\!]_h$ is defined as in Lemma 5.1. In Lemma 4.1 we use $v = w_h$ as comparison map. Since $w_h(\cdot,0) = u_o$, this implies the estimate

$$\int_{\Omega_T} |\alpha'| I(u,w_h)\,\mathrm{d}z \le \int_{\Omega_T} \alpha\partial_t(w_h^m)(w_h - u)\,\mathrm{d}z + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla(w_h^m - u^m)\,\mathrm{d}z,$$

for any cut-off function $\alpha \in W^{1,\infty}(0,T,\mathbb{R}_{\ge 0})$ with $\alpha(T) = 0$ and $\alpha' \le 0$. For a time $t_o \in (0,T)$ and $0 < \varepsilon < t_o$ we define the cut-off function by $\alpha \equiv 1$ on $[0, t_o - \varepsilon]$, $\alpha(t) := \frac{1}{\varepsilon}(t_o - t)$ on $[t_o - \varepsilon, t_o]$ and $\alpha \equiv 0$ on $[t_o, T]$. Plugging this function into the preceding estimate and letting $\varepsilon \downarrow 0$, we infer

$$\begin{aligned} (5.3)\qquad &\operatorname*{ess\,sup}_{t_o\in(0,T)} \int_{\Omega\times\{t_o\}} I(u,w_h)\,\mathrm{d}x \\ &\quad\le \sup_{t_o\in(0,T)} \int_{\Omega_{t_o}} \partial_t(w_h^m)(w_h - u)\,\mathrm{d}z + \int_{\Omega_T} |\nabla u^m|\,|\nabla w_h^m - \nabla u^m|\,\mathrm{d}z. \end{aligned}$$

The last integral vanishes in the limit $h \downarrow 0$ by Lemma 3.1 (ii), while the first integral on the right-hand side can be estimated in the limit $h \downarrow 0$ using Lemma 5.1. We thereby arrive at

$$\lim_{h\downarrow 0} \operatorname*{ess\,sup}_{t_o\in(0,T)} \int_{\Omega\times\{t_o\}} I(u,w_h)\,dx = 0.$$

By Lemma 3.6 if $m \ge 1$, respectively Corollary 3.8 and the fact that w_h is bounded in $L^\infty(0,T;L^{m+1}(\Omega))$ independently of h if $0 < m < 1$, this implies that $w_h \to u$ strongly in $L^\infty(0,T;L^{m+1}(\Omega))$ as $h \downarrow 0$. Moreover, we have $w_h \in C^0([0,T];L^{m+1}(\Omega))$ by Lemma 3.1 (iii). We have thus shown $C^0([0,T];L^{m+1}(\Omega)) \ni w_h \to u$ in $L^\infty(0,T;L^{m+1}(\Omega))$ as $h \downarrow 0$. This implies the claim $u \in C^0([0,T];L^{m+1}(\Omega))$. Since $w_h(\cdot,0) = u_o$, we also have $u(\cdot,0) = u_o$. □

6. COMPARISON LEMMAS FOR PSEUDOMONOTONE OPERATORS

Our aim in this section is to derive certain comparison principles for porous medium type equations. We will also use some other, truncated porous medium equations as part of our proofs. Let $a : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function satisfying

$$0 < \alpha \le a(s) \le \tfrac{1}{\alpha},$$

for some $0 < \alpha < 1$. More precisely, we assume that the mapping $\mathbb{R}_{\ge 0} \times \mathbb{R}^n \ni (s,\xi) \mapsto a(s)\xi$ has the following properties:

$$(6.1)\qquad \begin{cases} |a(s)\xi| \le \frac{1}{\alpha}|\xi| \\ \big(a(s)\xi_1 - a(s)\xi_2\big)\cdot(\xi_1 - \xi_2) \ge \alpha|\xi_1 - \xi_2|^2 \\ |a(s_1)\xi - a(s_2)\xi| \le L|\xi||s_1 - s_2|, \end{cases}$$

for any $s, s_1, s_2 \in \mathbb{R}_{\ge 0}$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$. The above properties of the map $(s,\xi) \mapsto a(s)\xi$ imply that we can define weak solutions as follows.

Definition 6.1. *A nonnegative function u is a local weak solution to the equation*

$$(6.2)\qquad \partial_t u - \operatorname{div}(a(u)\nabla u) = 0 \qquad \textit{in } \Omega_T$$

if $u \in C^0([0,T];L^2(\Omega)) \cap L^2_{\rm loc}(0,T;H^1_{\rm loc}(\Omega))$*, and*

$$\int_{\Omega_T} \big[-u\partial_t\varphi + a(u)\nabla u\cdot\nabla\varphi\big]\,\mathrm{d}z = -\int_{\Omega\times\{t\}} u\varphi\,\mathrm{d}x\bigg|_{t=0}^{T} \tag{6.3}$$

for all test functions $\varphi \in C^\infty(\Omega_T)$ *with* $\varphi = 0$ *on* $\partial\Omega\times[0,T]$*. The requirement for a weak supersolution is that*

$$\int_{\Omega_T} \big[-u\partial_t\varphi + a(u)\nabla u\cdot\nabla\varphi\big]\,\mathrm{d}z \geq 0, \tag{6.4}$$

for all positive test functions $\varphi \in C_0^\infty(\Omega_T)$*. For weak subsolutions, the inequality in* (6.4) *is reversed.*

For the existence of solutions in the above sense we refer to [18]. In order to make the following computations rigorous concerning the use of the time derivative, we need to reformulate the equation in terms of the mollified solution $[\![u]\!]_h$. Here, we choose $v_o = 0$ in (3.1). If u is a weak solution to the porous medium equation (2.9), then $[\![u]\!]_h$ satisfies

$$\int_{\Omega_T} \big[\partial_t[\![u]\!]_h\varphi + \nabla[\![u^m]\!]_h\cdot\nabla\varphi\big]\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{h}\int_\Omega u(\cdot,0)\int_0^T \varphi(\cdot,s)e^{-\frac{s}{h}}\,\mathrm{d}s\,\mathrm{d}x, \tag{6.5}$$

for all test functions $\varphi \in L^2(0,T;H^1_0(\Omega))$. The equation (6.5) follows from (2.10) by straightforward manipulations involving a change of variables and Fubini's theorem. Similarly, the mollification $[\![u]\!]_h$ of a solution to (6.2) satisfies

$$\int_{\Omega_T} \big[\partial_t[\![u]\!]_h\varphi + [\![a(u)\nabla u]\!]_h\cdot\nabla\varphi\big]\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{h}\int_\Omega u(\cdot,0)\int_0^T \varphi(\cdot,s)e^{-\frac{s}{h}}\,\mathrm{d}s\,\mathrm{d}x \tag{6.6}$$

for all $\varphi \in L^2(0,T;H^1_0(\Omega))$. For inhomogeneous equations involving a zero order term $f(x,t,u)$, one has to add the integral $\int_{\Omega_T}[\![f]\!]_h(x,t,u)\varphi\,\mathrm{d}x\,\mathrm{d}t$ on the right-hand side.

Our first aim is to establish some comparison principles for weak solutions, respectively sub- and supersolutions to (6.2). We adapt the arguments from [4] for the proof.

Lemma 6.2. *Let* u_1 *and* u_2 *be weak solutions to* (6.2) *such that* $(u_1-u_2)_+ \in L^2(0,T;H^1_0(\Omega))$ *and* $u_1(\cdot,0)\le u_2(\cdot,0)$ *a.e. on* Ω*. Then*

$$u_1 \le u_2 \quad \textit{almost everywhere in } \Omega_T.$$

The proof is based on the following estimate. In the proof, we make use of the usual two-sided truncation operator T_ε with $\varepsilon > 0$, given by

$$T_\varepsilon(s) := \begin{cases} s, & \text{if } |s|\le\varepsilon \\ \varepsilon\,\mathrm{sgn}(s), & \text{if } |s|>\varepsilon. \end{cases}$$

Lemma 6.3. *Let* u_1 *and* u_2 *be as in Lemma 6.2. Then*

$$\begin{aligned}\sup_{t\in(0,T)}\int_{\{0<u_1-u_2<\varepsilon\}}(u_1-u_2)^2(\cdot,t)\,\mathrm{d}x + \varepsilon\sup_{t\in(0,T)}\int_{\{u_1-u_2\ge\varepsilon\}}(u_1-u_2)(\cdot,t)\,\mathrm{d}x&\\ + \alpha\int_{\{0<u_1-u_2<\varepsilon\}}|\nabla(u_1-u_2)|^2\,\mathrm{d}z&\\ \le \frac{L^2\varepsilon^2}{\alpha}\int_{\{0<u_1-u_2<\varepsilon\}}|\nabla u_1|^2\,\mathrm{d}z.&\end{aligned}$$

Proof. For $\tau\in(0,T]$ we let $\chi_{[0,\tau]}\colon[0,T]\to[0,1]$ be the characteristic function of the interval $[0,\tau]$. We take $\varphi = \chi_{[0,\tau]}T_\varepsilon(u_1-u_2)_+$ with $\varepsilon>0$ as test-function in (6.6) written for u_1 and u_2, subtract the results and use the assumption $u_1(\cdot,0)\le u_2(\cdot,0)$. This leads to

$$\int_{\Omega_\tau}\Big[\partial_t[\![u_1-u_2]\!]_hT_\varepsilon(u_1-u_2)_+ + [\![a(u_1)\nabla u_1-a(u_2)\nabla u_2]\!]_h\nabla T_\varepsilon(u_1-u_2)_+\Big]\,\mathrm{d}z\le 0.$$

We want to get rid of the time derivative in the first term. To achieve this, we write

$$\begin{aligned}\partial_t[\![u_1-u_2]\!]_h T_\varepsilon(u_1-u_2)_+ &= \partial_t[\![u_1-u_2]\!]_h T_\varepsilon\big([\![u_1-u_2]\!]_h\big)_+\\ &\quad+\partial_t[\![u_1-u_2]\!]_h\big[T_\varepsilon(u_1-u_2)_+-T_\varepsilon\big([\![u_1-u_2]\!]_h\big)_+\big].\end{aligned}$$

With the abbreviation $v=u_1-u_2$, we can write the second term as

$$\partial_t[\![v]\!]_h\big(T_\varepsilon(v)_+-T_\varepsilon([\![v]\!]_h)_+\big)=\frac{v-[\![v]\!]_h}{h}\big[T_\varepsilon(v)_+-T_\varepsilon([\![v]\!]_h)_+\big]\geq 0,$$

where the positivity follows from the fact that $v\mapsto T_\varepsilon(v)_+$ is increasing. Thus, we have

$$\begin{aligned}\int_{\Omega_\tau}\partial_t[\![u_1-u_2]\!]_h T_\varepsilon(u_1-u_2)_+\,\mathrm{d}z &\geq \int_{\Omega_\tau}\partial_t[\![u_1-u_2]\!]_h T_\varepsilon\big([\![u_1-u_2]\!]_h\big)_+\,\mathrm{d}z\\ &=\frac12\int_{\{0<[\![u_1-u_2]\!]_h<\varepsilon\}}[\![u_1-u_2]\!]_h^2(\cdot,\tau)\,\mathrm{d}x\\ &\quad+\varepsilon\int_{\{[\![u_1-u_2]\!]_h\geq\varepsilon\}}[\![u_1-u_2]\!]_h(\cdot,\tau)\,\mathrm{d}x.\end{aligned}$$

The time derivative has been eliminated, and we may let $h\downarrow 0$ to get

$$\begin{aligned}&\frac12\int_{\{0<u_1-u_2<\varepsilon\}}(u_1-u_2)^2(\cdot,\tau)\,\mathrm{d}x+\varepsilon\int_{\{u_1-u_2\geq\varepsilon\}}(u_1-u_2)(\cdot,\tau)\,\mathrm{d}x\\ &\qquad+\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}\big(a(u_1)\nabla u_1-a(u_2)\nabla u_2\big)\cdot(\nabla u_1-\nabla u_2)\,\mathrm{d}z\leq 0.\end{aligned}$$

We note that

$$\begin{aligned}&\big(a(u_1)\nabla u_1-a(u_2)\nabla u_2\big)\cdot(\nabla u_1-\nabla u_2)\\ &\quad=\big(a(u_1)-a(u_2)\big)\nabla u_1\cdot(\nabla u_1-\nabla u_2)+a(u_2)|\nabla u_1-\nabla u_2|^2\\ &\quad\geq -L|u_1-u_2||\nabla u_1||\nabla u_1-\nabla u_2|+\alpha|\nabla u_1-\nabla u_2|^2,\end{aligned}$$

where in the last line we used the structure conditions (6.1). Therefore, the preceding inequality yields that

$$\begin{aligned}&\frac12\int_{\{0<u_1-u_2<\varepsilon\}}(u_1-u_2)^2(\cdot,\tau)\,\mathrm{d}x+\varepsilon\int_{\{u_1-u_2\geq\varepsilon\}}(u_1-u_2)(\cdot,\tau)\,\mathrm{d}x\\ &\qquad\qquad+\alpha\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|\nabla u_1-\nabla u_2|^2\,\mathrm{d}z\\ &\quad\leq L\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|u_1-u_2||\nabla u_1||\nabla u_1-\nabla u_2|\,\mathrm{d}z\\ &\quad\leq L\varepsilon\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|\nabla u_1||\nabla(u_1-u_2)|\,\mathrm{d}z\\ &\quad\leq\frac{\alpha}{2}\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|\nabla(u_1-u_2)|^2\,\mathrm{d}z+\frac{L^2\varepsilon^2}{2\alpha}\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|\nabla u_1|^2\,\mathrm{d}z.\end{aligned}$$

This leads to the estimate

$$\begin{aligned}&\int_{\{0<u_1-u_2<\varepsilon\}}(u_1-u_2)^2(\cdot,\tau)\,\mathrm{d}x+\varepsilon\int_{\{u_1-u_2\geq\varepsilon\}}(u_1-u_2)(\cdot,\tau)\,\mathrm{d}x\\ &\qquad+\alpha\int_{\Omega_\tau\cap\{0<u_1-u_2<\varepsilon\}}|\nabla(u_1-u_2)|^2\,\mathrm{d}z\\ &\qquad\qquad\leq\frac{L^2\varepsilon^2}{\alpha}\int_{\{0<u_1-u_2<\varepsilon\}}|\nabla u_1|^2\,\mathrm{d}z.\end{aligned}$$

The desired estimate follows from this inequality by taking the supremum over $\tau\in(0,T]$ in the first two terms and $\tau=T$ in the third one. □

Proof of Lemma 6.2. From Lemma 6.3 we have

$$\sup_{t\in(0,T)} \int_{\{u_1-u_2\geq\varepsilon\}} (u_1-u_2)(\cdot,t)\,\mathrm{d}x \leq \frac{L^2\varepsilon}{\alpha} \int_{\{0<u_1-u_2<\varepsilon\}} |\nabla u_1|^2\,\mathrm{d}z \to 0$$

in the limit $\varepsilon \downarrow 0$. Thus, we have proved

$$\sup_{t\in(0,T)} \int_{\{u_1-u_2\geq 0\}} (u_1-u_2)(\cdot,t)\,\mathrm{d}x \leq 0$$

which implies that $u_1 \leq u_2$ almost everywhere in Ω_T. □

The following variants of Lemma 6.2 follow by essentially the same arguments as above.

Lemma 6.4. *Let u_1 be a weak subsolution, and u_2 a weak supersolution to* (6.2) *such that $(u_1-u_2)_+ \in L^2(0,T;H_0^1(\Omega))$ and $u_1(\cdot,0) \leq u_2(\cdot,0)$ on Ω. Then*

$$u_1 \leq u_2 \quad \textit{almost everywhere in } \Omega_T.$$

We will use the following version when proving that solutions to penalized equations stay above the obstacle function.

Lemma 6.5. *Let $u,\psi \in L^2(0,T;H^1(\Omega))$ be such that $(\psi-u)_+ \in L^2(0,T;H_0^1(\Omega))$ and*

$$\frac{1}{2}\int_{\{0<\psi-u<\varepsilon\}} (\psi-u)^2(\cdot,\tau)\,\mathrm{d}x + \varepsilon \int_{\{\psi-u\geq\varepsilon\}} (\psi-u)(\cdot,\tau)\,\mathrm{d}x + \int_{\Omega_\tau\cap\{0<\psi-u<\varepsilon\}} \big(a(\psi)\nabla\psi - a(u)\nabla u\big)\cdot\nabla(\psi-u)\,\mathrm{d}z \leq 0$$

for all $\tau\in[0,T]$ and all $\varepsilon>0$. Then

$$\psi \leq u \quad \textit{almost everywhere in } \Omega_T.$$

7. Solutions of the penalized PME

In this section, we construct solutions to a penalized porous medium equation by approximating the PME with pseudomonotone equations. In the next two sections, we impose the stronger regularity assumptions (2.8) on the functions ψ, g and u_o.

We fix a (small) number $\delta > 0$. For the penalty term, we pick an increasing function $\zeta_\delta \in C^\infty(\mathbb{R},[0,1])$ such that $\zeta_\delta \equiv 1$ on $[0,\infty)$, $\zeta_\delta \equiv 0$ on $(-\infty,-\delta]$ and $|\nabla\zeta_\delta| \leq c/\delta$. The aim in this section is to construct a weak solution to the Cauchy-Dirichlet problem for the penalized PME

$$\begin{cases} \partial_t u - \Delta u^m = \Psi_+\zeta_\delta(\psi^m-u^m) & \text{in } \Omega_T, \\ u = g & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_o & \text{in } \Omega, \end{cases} \tag{7.1}$$

by an approximation process. Therefore, we will first consider the following equation

$$\partial_t u - \operatorname{div}(a(u)\nabla u) = \Psi_+\zeta_\delta(\psi^m-u^m) \qquad \text{in } \Omega_T, \tag{7.2}$$

where a satisfies the assumptions (6.1). The definition of weak solutions to (7.1) and (7.2) are analogous to Definitions 2.5 and 6.1. The first result concerning equation (7.2) is a comparison lemma.

Lemma 7.1. *Let u_1 and u_2 be weak solutions of* (7.2) *under the assumptions* (6.1) *and* (2.8) *and such that $(u_1-u_2)_+ \in L^2(0,T;H_0^1(\Omega))$ and $u_1(x,0) \leq u_2(x,0)$. Then*

$$u_1 \leq u_2 \quad \textit{almost everywhere in } \Omega_T.$$

Proof. The right-hand side in (7.2) is decreasing in u, so it is straightforward to incorporate this term into the proof of Lemma 6.2. More specifically, testing with $T_\varepsilon(u_1-u_2)_+$ and letting $h\downarrow 0$ yields

$$\int_{\Omega_T}\Psi_+\big[\zeta_\delta(\psi^m-u_1^m)-\zeta_\delta(\psi^m-u_2^m)\big]T_\varepsilon(u_1-u_2)_+\,\mathrm{d}z\le 0.$$

To see that the sign is indeed negative, we note that $T_\varepsilon(u_1-u_2)_+$ is nonzero only when $u_2<u_1$, and the function $t\mapsto\zeta_\delta(\psi^m-t)$ is decreasing. □

To pass to the limit in the approximation scheme, we need an energy estimate for equation (7.1).

Lemma 7.2. *Let u be a weak solution to the Cauchy-Dirichlet problem* (7.1) *under the assumptions* (6.1) *and* (2.8). *Then, we have*

$$\sup_{t\in[0,T]}\int_{\Omega\times\{t\}}u^{m+1}\,\mathrm{d}x+\int_{\Omega_T}|\nabla u^m|^2\,\mathrm{d}z\le c\,M$$

and

$$\|\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))}\le c\,M,$$

where $c=c(n,m,\mathrm{diam}(\Omega),T)$ and

$$M:=\sup_{t\in[0,T]}\int_{\Omega\times\{t\}}g^{m+1}\,\mathrm{d}x+\int_\Omega u_o^{m+1}\,\mathrm{d}x+\int_{\Omega_T}\Big[|\Psi_+|^2+|\nabla g^m|^2+|\partial_t g^m|^{\frac{m+1}{m}}\Big]\,\mathrm{d}z.$$

Proof. For $\tau\in(0,T]$ we let $\chi_{[0,\tau]}\colon[0,T]\to[0,1]$ be the characteristic function of the interval $[0,\tau]$. We test the regularized equation (6.5) with $\varphi=\chi_{[0,\tau]}(u^m-g^m)$ to get

$$\begin{aligned}\int_{\Omega_\tau}\Big[\partial_t[\![u]\!]_h(u^m-g^m)+\nabla[\![u^m]\!]_h\cdot\nabla(u^m-g^m)\Big]\,\mathrm{d}z\\ =\int_{\Omega_\tau}[\![\zeta_\delta(\psi^m-u^m)\Psi_+]\!]_h(u^m-g^m)\,\mathrm{d}z\\ +\frac1h\int_\Omega u_o\int_0^\tau(u^m-g^m)e^{-\frac sh}\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

For the first term on the left-hand side, we find, using Lemma 3.1 (i) that

$$\begin{aligned}\int_{\Omega_\tau}\partial_t[\![u]\!]_h u^m\,\mathrm{d}z&=\int_{\Omega_\tau}\partial_t[\![u]\!]_h[\![u]\!]_h^m\,\mathrm{d}z+\int_{\Omega_\tau}\partial_t[\![u]\!]_h\big(u^m-[\![u]\!]_h^m\big)\,\mathrm{d}z\\&\ge\int_{\Omega_\tau}\partial_t[\![u]\!]_h[\![u]\!]_h^m\,\mathrm{d}z=\frac1{m+1}\int_{\Omega_\tau}\partial_t[\![u]\!]_h^{m+1}\,\mathrm{d}z\\&=\frac1{m+1}\int_{\Omega\times\{\tau\}}[\![u]\!]_h^{m+1}\,\mathrm{d}x.\end{aligned}$$

Moreover, integrating by parts, we find that

$$\int_{\Omega_\tau}\partial_t[\![u]\!]_h g^m\,\mathrm{d}z=-\int_{\Omega_\tau}[\![u]\!]_h\partial_t g^m\,\mathrm{d}z+\int_{\Omega\times\{\tau\}}[\![u]\!]_h g^m\,\mathrm{d}x.$$

Inserting this above, letting $h\downarrow 0$ with the help of Lemma 3.1 and rearranging terms we find that

$$\begin{aligned}&\frac1{m+1}\int_{\Omega\times\{\tau\}}u^{m+1}\,\mathrm{d}x+\int_{\Omega_\tau}|\nabla u^m|^2\,\mathrm{d}z\\&\quad\le\int_{\Omega\times\{\tau\}}ug^m\,\mathrm{d}x-\int_{\Omega_\tau}u\partial_t g^m\,\mathrm{d}z+\int_{\Omega_\tau}\nabla u^m\cdot\nabla g^m\,\mathrm{d}z\\&\qquad+\int_{\Omega_\tau}\zeta_\delta(\psi^m-u^m)\Psi_+(u^m-g^m)\,\mathrm{d}z+\int_{\Omega\times\{0\}}\big(u_o^{m+1}-u_og^m\big)\,\mathrm{d}x.\end{aligned}$$

We proceed by taking absolute values, applying Young's inequality, and taking the supremum over $t \in (0,T]$ to get

$$\begin{aligned}
&\frac{1}{m+1}\int_{\Omega\times\{\tau\}} u^{m+1}\,\mathrm{d}x + \int_{\Omega_\tau} |\nabla u^m|^2\,\mathrm{d}z \\
&\quad\le \frac{1}{2(m+1)}\sup_{t\in(0,T]}\int_{\Omega\times\{t\}} u^{m+1}\,\mathrm{d}x + \frac14\int_{\Omega_\tau}|\nabla u^m|^2\,\mathrm{d}z \\
&\qquad + c\sup_{t\in(0,T]}\int_{\Omega\times\{t\}} g^{m+1}\,\mathrm{d}x + \int_{\Omega_\tau}|\nabla g^m|^2\,\mathrm{d}z + c(T)\int_{\Omega_\tau}|\partial_t g^m|^{\frac{m+1}{m}}\,\mathrm{d}z \\
&\qquad + \int_{\Omega_\tau}\zeta_\delta(\psi^m-u^m)\Psi_+(u^m-g^m)\,\mathrm{d}z + \int_\Omega u_o^{m+1}\,\mathrm{d}x,
\end{aligned}$$

where we have also taken into account that $u_o g^m(\cdot,0)\ge 0$. For the second last term on the right, we have for $\varepsilon>0$ that

$$\begin{aligned}
&\left|\int_{\Omega_\tau}\zeta_\delta(\psi^m-u^m)\Psi_+(u^m-g^m)\,\mathrm{d}z\right| \\
&\quad\le \varepsilon\int_{\Omega_\tau}|u^m-g^m|^2\,\mathrm{d}z + c_\varepsilon\int_{\Omega_\tau}\Psi_+^2\,\mathrm{d}z \\
&\quad\le c\varepsilon\int_{\Omega_\tau}|\nabla(u^m-g^m)|^2\,\mathrm{d}z + c_\varepsilon\int_{\Omega_\tau}\Psi_+^2\,\mathrm{d}z \\
&\quad\le c\varepsilon\int_{\Omega_\tau}|\nabla u^m|^2\,\mathrm{d}z + c\int_{\Omega_\tau}|\nabla g^m|^2\,\mathrm{d}z + c_\varepsilon\int_{\Omega_\tau}\Psi_+^2\,\mathrm{d}z,
\end{aligned}$$

where we used Young's and Poincaré's inequalities and the fact that $|\zeta_\delta|\le 1$. Choosing $\varepsilon>0$ small enough, we can absorb the terms containing ∇u^m from the right-hand side into the left. We thus have arrived at

$$\begin{aligned}
&\frac{1}{m+1}\int_{\Omega\times\{\tau\}} u^{m+1}\,\mathrm{d}x + \frac12\int_{\Omega_\tau}|\nabla u^m|^2\,\mathrm{d}z \\
&\quad\le \frac{1}{2(m+1)}\sup_{t\in(0,T]}\int_{\Omega\times\{t\}} u^{m+1}\,\mathrm{d}x + c\,M,
\end{aligned}$$

with a constant c depending on $n, m, \mathrm{diam}(\Omega), T$ and where M is defined in the statement of the lemma. We now take the supremum over $\tau\in(0,T]$ in the first term on the left-hand side and $\tau=T$ in the second one. Absorbing the matching terms from the right-hand side to the left, we deduce the first estimate claimed in the lemma.

The estimate for the time derivative follows from the first estimate. For any $\varphi\in C_0^\infty(\Omega_T)$ we have by $(7.1)_1$ and Hölder's and Poincaré's inequality that

$$\begin{aligned}
\left|\int_{\Omega_T} u\partial_t\varphi\,\mathrm{d}z\right| &\le \left|\int_{\Omega_T}\nabla u^m\cdot\nabla\varphi\,\mathrm{d}z\right| + \left|\int_{\Omega_T}\zeta_\delta(\psi^m-u^m)\Psi_+\varphi\,\mathrm{d}z\right| \\
&\le \|\nabla u^m\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)} + \|\Psi_+\|_{L^2(\Omega_T)}\|\varphi\|_{L^2(\Omega_T)} \\
&\le c\left[\|\nabla u^m\|_{L^2(\Omega_T)} + \|\Psi_+\|_{L^2(\Omega_T)}\right]\|\varphi\|_{L^2(0,T;H^1(\Omega))} \\
&\le c\,M^{\frac12}\|\varphi\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Since $C_0^\infty(\Omega_T)$ is dense in $L^2(0,T;H_0^1(\Omega))$ this proves the claim and therefore finishes the proof of the lemma. □

Proposition 7.3. *Let g, ψ, and u_o satisfy* (2.8). *Then there exists a function u, which is a weak solution of* (7.1) *satisfying*

$$u\ge\psi \quad \textit{almost everywhere in } \Omega_T.$$

Proof. Fix $\varepsilon, \gamma, \delta \in (0,1]$, and define

$$\psi_\varepsilon := \psi + \varepsilon, \quad g_{\varepsilon,\gamma} := (g^m + \gamma^m)^{\frac{1}{m}} + \varepsilon, \quad u_{o,\varepsilon,\gamma} := u_o + \varepsilon + \gamma.$$

Then $g_{\varepsilon,\gamma} \geq g + \varepsilon \geq \psi_\varepsilon$, and $u_{o,\varepsilon,\gamma} \geq \psi_\varepsilon(\cdot, 0)$. Let

$$M := \max\left\{\sup_{\Omega_T}(\psi_\varepsilon^m + \delta)^{\frac{1}{m}},\ \sup_{\Omega_T} g_{\varepsilon,\gamma},\ \sup_{\Omega} u_{o,\varepsilon,\gamma}\right\}.$$

Then $\psi_\varepsilon \leq M$, $g_{\varepsilon,\gamma} \leq M$, and $u_{o,\varepsilon,\gamma} \leq M$. Choose ε and γ smaller if necessary, so that

$$M \leq \frac{1}{\gamma + \varepsilon}. \tag{7.3}$$

The adjustment can be made independently of δ. Now, we define $a_\varepsilon \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$a_\varepsilon(s) := \begin{cases} m\varepsilon^{m-1} & \text{if } 0 \leq s \leq \varepsilon, \\ ms^{m-1} & \text{if } \varepsilon < s < \frac{1}{\varepsilon}, \\ m\varepsilon^{1-m} & \text{if } s \geq \frac{1}{\varepsilon}. \end{cases} \tag{7.4}$$

Note that a_ε satisfies the structure assumptions (6.1) with $\alpha = m \min\{\varepsilon^{m-1}, \varepsilon^{1-m}\}$ and $L = m|m-1| \max\{\varepsilon^{m-2}, \varepsilon^{2-m}\}$ and let $u_{\varepsilon,\gamma} \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ be the weak solution of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_{\varepsilon,\gamma} - \operatorname{div}\big(a_\varepsilon(u_{\varepsilon,\gamma})\nabla u_{\varepsilon,\gamma}\big) = (\Psi_\varepsilon)_+ \zeta_\delta(\psi_\varepsilon^m - u_{\varepsilon,\gamma}^m) & \text{in } \Omega_T \\ u_{\varepsilon,\gamma} = g_{\varepsilon,\gamma} & \text{on } \partial\Omega \times (0,T) \\ u_{\varepsilon,\gamma}(\cdot, 0) = u_{o,\varepsilon} & \text{in } \Omega. \end{cases} \tag{7.5}$$

We claim that

$$\gamma + \varepsilon \leq u_{\varepsilon,\gamma} \leq \frac{1}{\gamma + \varepsilon}. \tag{7.6}$$

The first inequality follows from Lemma 6.4, since $u_{\gamma,\varepsilon}$ is a weak supersolution and the constant function $\varepsilon + \gamma$ is a weak solution of the equation

$$\partial_t u - \operatorname{div}\big(a_\varepsilon(u)\nabla u\big) = 0$$

and moreover, $(\gamma + \varepsilon - u_{\gamma,\varepsilon})_+ \in L^2(0,T; H^1_0(\Omega))$ and $\gamma + \varepsilon \leq u_{\varepsilon,\gamma}(\cdot, 0)$. For the second inequality, we note that $\psi_\varepsilon^m - M^m \leq -\delta$. Thus the constant function M is a solution of $(7.5)_1$, and since $(u_{\gamma,\varepsilon} - M)_+ \in L^2(0,T; H^1_0(\Omega))$ and $u_{\varepsilon,\gamma}(\cdot, 0) \leq M$, the claim follows by combining Lemma 7.1 and (7.3).

The final property of the function $u_{\varepsilon,\gamma}$ we need is that $u_{\varepsilon,\gamma} \geq \psi_\varepsilon$. We aim at applying Lemma 6.5. We let $\tau \in (0,T]$ and start with the mollified form of (7.5) (as before, we may replace Ω_T by Ω_τ in (7.5) after multiplying the test-function by the characteristic function of $[0,\tau]$). This gives

$$\begin{aligned} &\int_{\Omega_\tau} \Big[-\partial_t [\![u_{\varepsilon,\gamma}]\!]_h \varphi - [\![a_\varepsilon(u_{\varepsilon,\gamma})\nabla u_{\varepsilon,\gamma}]\!]_h \cdot \nabla\varphi \Big] \,\mathrm{d}z \\ &\quad = -\int_{\Omega_\tau} [\![(\Psi_\varepsilon)_+ \zeta_\delta(\psi_\varepsilon^m - u_{\varepsilon,\gamma}^m)]\!]_h \varphi \,\mathrm{d}z - \frac{1}{h}\int_\Omega u_{o,\varepsilon,\gamma} \int_0^\tau \varphi(\cdot, s) e^{-\frac{s}{h}} \,\mathrm{d}s \,\mathrm{d}x. \end{aligned}$$

Next, we observe that

$$\partial_t [\![\psi_\varepsilon]\!]_h = [\![\partial_t \psi_\varepsilon]\!]_h + \tfrac{1}{h}\psi_\varepsilon(\cdot, 0) e^{-\frac{t}{h}}$$

which can be checked by an integration by parts, and

$$[\![a_\varepsilon(\psi_\varepsilon)\nabla\psi_\varepsilon]\!]_h = [\![\nabla\psi_\varepsilon^m]\!]_h$$

for $\varepsilon > 0$ small enough so that $\psi_\varepsilon \le \frac{1}{\varepsilon}$. Taking into account these identities, we add $\partial_t [\![\psi_\varepsilon]\!]_h \varphi + [\![a_\varepsilon(\psi_\varepsilon)\nabla\psi_\varepsilon]\!]_h \cdot \nabla\varphi$ to both sides and get

$$\begin{aligned}\int_{\Omega_\tau} &\partial_t [\![\psi_\varepsilon - u_{\varepsilon,\gamma}]\!]_h \varphi + [\![a_\varepsilon(\psi_\varepsilon)\nabla\psi_\varepsilon - a_\varepsilon(u_{\varepsilon,\gamma})\nabla u_{\varepsilon,\gamma}]\!]_h \cdot \nabla\varphi \,\mathrm{d}z \\ &= \int_{\Omega_\tau} [\![\Psi_\varepsilon - (\Psi_\varepsilon)_+ \zeta_\delta(\psi_\varepsilon^m - u_{\varepsilon,\gamma}^m)]\!]_h \varphi \,\mathrm{d}z \\ &\quad + \frac{1}{h}\int_\Omega (\psi_\varepsilon(\cdot,0) - u_{o,\gamma,\varepsilon}) \int_0^\tau \varphi(\cdot,s) e^{-\frac{s}{h}} \,\mathrm{d}s \,\mathrm{d}x.\end{aligned}$$

Now, we choose $\varphi = T_\lambda(\psi_\varepsilon - u_{\varepsilon,\gamma})_+$ as test-function, discard the initial term which has a negative sign, integrate in the time term and let $h \downarrow 0$ as in the proof of Lemma 6.3. This yields

$$\begin{aligned}\frac{1}{2}\int_{\{0<\psi_\varepsilon - u_{\varepsilon,\gamma}<\lambda\}} &(\psi_\varepsilon - u_{\varepsilon,\gamma})^2(\cdot,\tau)\,\mathrm{d}x \\ &+ \lambda \int_{\{\psi_\varepsilon - u_{\varepsilon,\gamma}\ge\lambda\}} (\psi_\varepsilon - u_{\varepsilon,\gamma})(\cdot,\tau)\,\mathrm{d}x \\ &+ \int_{\Omega_\tau \cap \{0<\psi_\varepsilon - u_{\varepsilon,\gamma}<\lambda\}} \big(a(\psi_\varepsilon)\nabla\psi_\varepsilon - a(u_{\varepsilon,\gamma})\nabla u_{\varepsilon,\gamma}\big)\cdot\nabla(\psi_\varepsilon - u_{\varepsilon,\gamma})\,\mathrm{d}z \\ &\le \int_{\Omega_\tau \cap \{\psi_\varepsilon - u_{\varepsilon,\gamma}>0\}} \big(\Psi_\varepsilon - (\Psi_\varepsilon)_+\zeta_\delta(\psi_\varepsilon^m - u_{\varepsilon,\gamma}^m)\big) T_\lambda(\psi_\varepsilon - u_{\varepsilon,\gamma})\,\mathrm{d}z \\ &= -\int_{\Omega_\tau \cap \{\psi_\varepsilon - u_{\varepsilon,\gamma}>0\}} (\Psi_\varepsilon)_- T_\lambda(\psi_\varepsilon - u_{\varepsilon,\gamma})\,\mathrm{d}z \le 0,\end{aligned}$$

ensuring that the assumption of Lemma 6.5 is satisfied. Therefore, we conclude that $u_{\varepsilon,\gamma} \ge \psi_\varepsilon$, as desired.

Next, we pass to the limit $\varepsilon \downarrow 0$ using the energy estimate of Lemma 7.2. We have $a_\varepsilon(u_{\varepsilon,\gamma})\nabla u_{\varepsilon,\gamma} = \nabla u_{\varepsilon,\gamma}^m$ by (7.4) and (7.6). Thus $u_{\varepsilon,\gamma}$ is also a weak solution of the penalized PME. Since the right-hand side $(\Psi_\varepsilon)_+\zeta_\delta(\psi_\varepsilon^m - u_{\varepsilon,\gamma}^m)$ is bounded independently from ε, we know from [8, Theorem 1.2] that the functions $u_{\varepsilon,\gamma}$ are locally Hölder continuous with a quantitative estimate which is uniform in ε. Hence there is a function u_γ such that $u_{\varepsilon,\gamma} \to u_\gamma$ as $\varepsilon \downarrow 0$ locally uniformly in Ω_T. Since $u_{\varepsilon,\gamma} \ge \psi_\varepsilon$, we also have that $u_\gamma \ge \psi$ almost everywhere. The lateral and initial boundary values of $u_{\varepsilon,\gamma}$ are

$$g_{\varepsilon,\gamma} = (g^m + \gamma^m)^{1/m} + \varepsilon \quad \text{and} \quad u_{o,\varepsilon,\gamma} = u_o + \varepsilon + \gamma$$

respectively. We have

$$\nabla g_{\varepsilon,\gamma}^m = \big((g^m+\gamma^m)^{\frac{1}{m}} + \varepsilon\big)^{m-1} (g^m+\gamma^m)^{\frac{1}{m}-1}\nabla g^m,$$

and similarly for the time derivative. Thus for a fixed γ, the energy estimate of Lemma 7.2 is independent of ε for small values of ε. We recall that Lemma 7.2 is applicable since $u_{\varepsilon,\gamma}$ is also a weak solution of the penalized PME. It follows that $u_\gamma \in L^\infty(0,T;L^{m+1}(\Omega))$, $u_\gamma^m \in L^2(0,T;H^1(\Omega))$, $\partial_t u_\gamma \in L^2(0,T;H^{-1}(\Omega))$, $\nabla u_{\varepsilon,\gamma}^m \rightharpoonup \nabla u_\gamma^m$ weakly in $L^2(\Omega_T,\mathbb{R}^n)$, $u_{\varepsilon,\gamma} \overset{*}{\rightharpoonup} u_\gamma$ weakly star in $L^\infty(0,T;L^{m+1}(\Omega))$, $\partial_t u_{\varepsilon,\gamma} \rightharpoonup \partial_t u_\gamma$ weakly in $L^2(0,T;H^{-1}(\Omega))$ and $u_\gamma^m - g^m - \gamma^m \in L^2(0,T;H_0^1(\Omega))$. The last of these holds since $g_{\varepsilon,\gamma}^m \rightharpoonup g^m + \gamma^m$ weakly in $L^2(0,T;H^1(\Omega))$ as $\varepsilon \downarrow 0$. Moreover, $u_\gamma(\cdot,0) = u_{o,o,\gamma}$ in the $H^{-1}(\Omega)$-sense. We let $\varepsilon \downarrow 0$ in the penalized PME by using the above facts. For $\varphi \in C^\infty(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times [0,T]$, we have

$$\lim_{\varepsilon\downarrow 0}\int_0^T \langle \partial_t u_{\varepsilon,\gamma}, \varphi\rangle\,\mathrm{d}t = \int_0^T \langle \partial_t u_\gamma, \varphi\rangle\,\mathrm{d}t,$$
$$\lim_{\varepsilon\downarrow 0}\int_{\Omega_T} \nabla u_{\varepsilon,\gamma}^m \cdot \nabla\varphi\,\mathrm{d}z = \int_{\Omega_T} \nabla u_\gamma^m \cdot \nabla\varphi\,\mathrm{d}z$$

by the weak convergences, and the fact that

$$\lim_{\varepsilon\downarrow 0}\int_{\Omega_T}\zeta_\delta(\psi_\varepsilon^m-u_{\varepsilon,\gamma}^m)(\Psi_\varepsilon)_+\varphi\,\mathrm{d}z=\int_{\Omega_T}\Psi_+\zeta_\delta(\psi^m-u_\gamma^m)\varphi\,\mathrm{d}z$$

follows from the dominated convergence theorem. Thus we have

$$\int_0^T\langle\partial_t u_\gamma,\varphi\rangle\,\mathrm{d}t+\int_{\Omega_T}\nabla u_\gamma^m\cdot\nabla\varphi\,\mathrm{d}z=\int_{\Omega_T}\Psi_+\zeta_\delta(\psi^m-u_\gamma^m)\varphi\,\mathrm{d}z$$

for all test-functions $\varphi\in C^\infty(\Omega_T)$ with $\varphi=0$ on $\partial\Omega\times[0,T]$. In particular, this implies that u_γ is a local strong solution to the obstacle problem for the PME with obstacle $\psi\equiv 0$ and additional inhomogeneity $\Psi_+\zeta_\delta(\psi^m-u_\gamma)\in L^\infty(\Omega_T)$; see (2.3). From Lemma 3.2 we then infer that u_γ is also a local weak solution to this obstacle problem in the sense of Definition 2.1. Therefore, we can apply Lemma 5.2 (which continues to hold for obstacle problems with an additional right-hand side in L^∞) to conclude that $u\in C^0([0,T];L^{m+1}(\Omega))$ and $u(\cdot,0)=u_o$.

We want to finish the proof by letting $\gamma\downarrow 0$. To this end, we note that by (7.6) and the pointwise convergence $u_{\varepsilon,\gamma}\to u_\gamma$, we have

$$\gamma\le u_\gamma\le\frac{1}{\gamma}.$$

This means that u_γ is also a weak solution of the equation

$$\partial_t u_\gamma-\operatorname{div}\big(a_\gamma(u_\gamma)\nabla u_\gamma\big)=\Psi_+\zeta_\delta(\psi^m-u_\gamma^m).$$

Thus Lemma 7.1 shows that $u_{\gamma_1}\le u_{\gamma_2}$ if $\gamma_1<\gamma_2$, since both functions are solutions of the equation involving a_{γ_1}. We therefore may define the function u by setting

$$u(x,t):=\lim_{\gamma\downarrow 0}u_\gamma(x,t).$$

Since $u_\gamma\ge\psi$ almost everywhere, we also have $u\ge\psi$ almost everywhere. Showing that this function u is the weak solution with the right boundary values proceeds similarly to the limit process $\varepsilon\downarrow 0$ above. However, for the sake of completeness we briefly explain the argument. The lateral and initial boundary values of u_γ are

$$g_\gamma=(g^m+\gamma^m)^{1/m}\quad\text{and}\quad u_{o,o,\gamma}=u_o+\gamma$$

respectively. We have $\nabla g_\gamma^m=\nabla g^m$ and $\partial_t g_\gamma^m=\partial_t g^m$. Moreover, since $\gamma\le u_\gamma\le\frac{1}{\gamma}$, u_γ is also a weak solution of the penalized PME. Therefore, we can apply Lemma 7.2 to u_γ and the resulting energy estimate is independent of γ for small values of γ. It follows that $u^m\in L^2(0,T;H^1(\Omega))$, $\nabla u_\gamma^m\rightharpoondown\nabla u^m$ weakly in $L^2(\Omega_T,\mathbb{R}^n)$, and $u^m-g^m\in L^2(0,T;H_0^1(\Omega))$ and $\partial_t u\in L^2(0,T;H^{-1}(\Omega))$, $\partial_t u_\gamma\rightharpoondown\partial_t u$ in $L^2(0,T;H^{-1}(\Omega))$ and $u(\cdot,0)=u_o$ in the H^{-1}-sense. We now let $\gamma\downarrow 0$ in the penalized PME by using the above facts similarly as before when passing to the limit $\varepsilon\downarrow 0$. In conclusion, we find that

$$\int_0^T\langle\partial_t u,\varphi\rangle\,\mathrm{d}t+\int_{\Omega_T}\nabla u^m\cdot\nabla\varphi\,\mathrm{d}z=\int_{\Omega_T}\Psi_+\zeta_\delta(\psi^m-u^m)\varphi\,\mathrm{d}z$$

for all test-functions $\varphi\in C^\infty(\Omega_T)$ with $\varphi=0$ on $\partial\Omega\times[0,T]$. Arguing as above, we find that $u\in C^0([0,T];L^{m+1}(\Omega))$ and $u(\cdot,0)=u_o$. We also have $u\ge\psi$ almost everywhere and therefore, u is the weak solution of (7.1) we were looking for. □

Remark 7.4. The solutions constructed in the previous proposition are the unique *limit solutions* to the problem: no other solution may arise as a pointwise limit of strictly positive solutions. To see this, note that the approximating equations in the proof hold for strictly positive solutions, so the comparison principle continues to hold for limit solutions.

8. Existence of strong solutions to the obstacle problem

In this section, we prove Theorem 2.6 by passing to the limit $\delta \downarrow 0$ in the penalized equations of the previous section.

Proof of Theorem 2.6. For $\delta > 0$, let u_δ be the weak solution to the Cauchy-Dirichlet problem (7.1) constructed in Proposition 7.3. Since the right-hand side $\Psi_+ \zeta_\delta(\psi^m - u_\delta^m)$ is bounded independently from δ, we know from [8, Theorem 1.2] that the functions u_δ are locally Hölder continuous with a quantitative estimate which is uniform in δ. Hence there is a function u such that $u_\delta \to u$ as $\delta \downarrow 0$ locally uniformly in Ω_T. We will show that this function u is the desired solution of the obstacle problem. First, we note that clearly $u \geq \psi$, since $u_\delta \geq \psi$ for all $\delta > 0$. From the a priori estimate of Lemma 7.2, we deduce that for a subsequence still indexed by δ that

$$\begin{cases} \partial_t u_\delta \rightharpoonup \partial_t u & \text{weakly in } L^2(0,T;H^{-1}(\Omega)), \\ \nabla u_\delta^m \rightharpoonup \nabla u^m & \text{weakly in } L^2(\Omega_T, \mathbb{R}^n) \end{cases} \tag{8.1}$$

as $\delta \downarrow 0$. Moreover, we have that $u^m - g^m \in L^2(0,T;H_0^1(\Omega))$ and $u(\cdot,0) = u_o$ in the $H^{-1}(\Omega)$-sense. By the lower semicontinuity of the L^2 norm, we also have

$$\|\nabla u^m\|_{L^2(\Omega_T)} \leq \liminf_{\delta\downarrow 0} \|\nabla u_\delta^m\|_{L^2(\Omega_T)} \tag{8.2}$$

for a subsequence.

We let $\eta_\delta \in C_0^\infty(\Omega,[0,1])$ be a cutoff function such that $\eta_\delta = 1$ in $\{x \in \Omega : d_\Omega(x) \geq \delta\}$, and $|\nabla \eta_\delta| \leq c/\delta$. We take $\varphi = \alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)$, with α, η, v as in the statement of the theorem, as a test-function in the weak formulation of $(7.1)_1$ to get

$$\begin{aligned} \int_0^T \langle \partial_t u_\delta, \alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)\rangle \,\mathrm{d}t + \int_{\Omega_T} \nabla u_\delta^m \cdot \nabla\big(\alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)\big)\,\mathrm{d}z \\ = \int_{\Omega_T} \alpha\eta\Psi_+\zeta_\delta(\psi^m - u_\delta^m)(v^m - u_\delta^m + \delta\eta_\delta)\,\mathrm{d}z. \end{aligned} \tag{8.3}$$

We write the first term in (8.3) as

$$\int_0^T \langle \partial_t u_\delta, \alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)\rangle \,\mathrm{d}t = \int_0^T \langle \partial_t u_\delta, \alpha\eta v^m - \alpha\eta u_\delta^m + \delta\alpha\eta\eta_\delta\rangle\,\mathrm{d}t.$$

We have

$$\lim_{\delta\downarrow 0} \int_0^T \langle \partial_t u_\delta, \alpha\eta v^m + \delta\alpha\eta\eta_\delta\rangle\,\mathrm{d}t = \int_0^T \langle \partial_t u, \alpha\eta v^m\rangle\,\mathrm{d}t,$$

by $(8.1)_1$ and the fact that $\delta\alpha\eta\eta_\delta \to 0$ strongly in $H^1(\Omega)$ as $\delta \downarrow 0$. Two applications of Lemma 3.2 with $v \equiv 0$ yield in view of the local uniform convergence $u_\delta \to u$ that

$$\begin{aligned} \lim_{\delta\downarrow 0} \int_0^T \langle \partial_t u_\delta, \alpha\eta u_\delta^m\rangle\,\mathrm{d}t &= -\tfrac{1}{m+1} \lim_{\delta\downarrow 0} \left[\int_{\Omega_T} \eta\alpha' u_\delta^{m+1}\,\mathrm{d}z + \alpha(0)\int_\Omega \eta u_o^{m+1}\,\mathrm{d}x \right] \\ &= -\tfrac{1}{m+1} \left[\int_{\Omega_T} \eta\alpha' u^{m+1}\,\mathrm{d}z + \alpha(0)\int_\Omega \eta u_o^{m+1}\,\mathrm{d}x \right] \\ &= \int_0^T \langle \partial_t u, \alpha\eta u^m\rangle\,\mathrm{d}t. \end{aligned}$$

Putting all of the above together, we see that the limit as $\delta \downarrow 0$ of the first term is

$$\lim_{\delta\downarrow 0} \int_0^T \langle \partial_t u_\delta, \alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)\rangle\,\mathrm{d}t = \int_0^T \langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t.$$

Let us now turn our attention to the second term on the left-hand side of (8.3). By $(8.1)_2$ and the fact that $u_\delta \to u$ locally uniformly as $\delta \downarrow 0$, we find that

$$\lim_{\delta\downarrow 0} \int_{\Omega_T} \alpha \nabla u_\delta^m \cdot \nabla\eta (v^m - u_\delta^m + \delta\eta_\delta)\,\mathrm{d}z = \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla\eta(v^m - u^m)\,\mathrm{d}z.$$

Again by $(8.1)_2$, we have

$$\lim_{\delta\downarrow 0} \int_{\Omega_T} \alpha\eta\nabla u_\delta^m \cdot \nabla v^m\,\mathrm{d}z = \int_{\Omega_T} \alpha\eta\nabla u^m \cdot \nabla v^m\,\mathrm{d}z.$$

Moreover, from (8.2) and the assumption $\alpha, \eta \geq 0$, we conclude that

$$-\liminf_{\delta\downarrow 0} \int_{\Omega_T} \alpha\eta|\nabla u_\delta^m|^2\,\mathrm{d}z \leq -\int_{\Omega_T} \alpha\eta|\nabla u^m|^2\,\mathrm{d}z.$$

Finally, by $(8.1)_2$ and the fact that $\nabla\delta\eta_\delta \to 0$ in $L^2(\Omega_T)$, we find that

$$\lim_{\delta\downarrow 0} \int_{\Omega_T} \alpha\eta\nabla u_\delta^m \cdot \nabla(\delta\eta_\delta)\,\mathrm{d}z = 0.$$

Combining the above convergences, we conclude for the limit of the second term on the left-hand side of (8.3) that

$$\limsup_{\delta\downarrow 0} \int_{\Omega_T} \nabla u_\delta^m \cdot \nabla\big(\alpha\eta(v^m - u_\delta^m + \delta\eta_\delta)\big)\,\mathrm{d}z \leq \int_{\Omega_T} \nabla u^m \cdot \nabla\big(\alpha\eta(v^m - u^m)\big)\,\mathrm{d}z.$$

The next step is to show that the limit as $\delta \downarrow 0$ of the right-hand side of (8.3) is nonnegative. Denoting $g(s) := \zeta_\delta(\psi^m - s)$, $a := v^m + \delta\zeta_\delta$, and $b := u_\delta^m$, and taking into account that g is decreasing, we have

$$\begin{aligned}\zeta_\delta(\psi^m - u_\delta^m)(v^m - u_\delta^m + \delta\eta_\delta) - \zeta_\delta(\psi^m - v^m - \delta\eta_\delta)(v^m - u_\delta^m + \delta\eta_\delta)&\\ = (g(b) - g(a))(a-b) \geq 0.&\end{aligned}$$

Since $\alpha\eta\Psi_+$ is non-negative, this shows for the integrand on the right-hand side of (8.3) that

$$\alpha\eta\Psi_+\zeta_\delta(\psi^m - u_\delta^m)(v^m - u_\delta^m + \delta\eta_\delta) \geq \alpha\eta\Psi_+\zeta_\delta(\psi^m - v^m - \delta\eta_\delta)(v^m - u_\delta^m + \delta\eta_\delta).$$

On the set where $\eta_\delta = 1$ we have $\psi^m - v^m - \delta\eta_\delta \leq -\delta$, since $v \geq \psi$, and hence $\zeta_\delta(\psi^m - v^m - \delta\eta_\delta) = 0$. This is the only point where we use the fact that v lies above the obstacle ψ. We combine this with the above inequality to see that for the right-hand side of (8.3) we have

$$\begin{aligned}&\int_{\Omega_T} \alpha\eta\Psi_+\zeta_\delta(\psi^m - u_\delta^m)(v^m - u_\delta^m + \delta\eta_\delta)\,\mathrm{d}z\\ &\quad\geq \int_{\{(x,t)\in\Omega_T : d_\Omega(x)<\delta\}} \alpha\eta\Psi_+\zeta_\delta(\psi^m - v^m - \delta\eta_\delta)(v^m - u_\delta^m + \delta\eta_\delta)\,\mathrm{d}z\\ &\quad\geq -\bigg(\int_{\{(x,t)\in\Omega_T : d_\Omega(x)<\delta\}} |\alpha\eta\Psi_+|^2\,\mathrm{d}z\bigg)^{\frac12}\bigg(\int_{\Omega_T} |v^m - u_\delta^m + \delta\eta_\delta|^2\,\mathrm{d}z\bigg)^{\frac12},\end{aligned}$$

where in the last line we used Hölder's inequality and the fact that $\zeta_\delta \leq 1$. The second factor is bounded for small values of δ, and the first tends to zero as $\delta \downarrow 0$, so that

$$\liminf_{\delta\downarrow 0} \int_{\Omega_T} \alpha\eta\Psi_+\zeta_\delta(\psi^m - u_\delta^m)(v^m - u_\delta^m + \delta\eta_\delta)\,\mathrm{d}z \geq 0.$$

Combining the preceding argumentation, we have so far shown that

$$\int_0^T \langle\partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t + \int_{\Omega_T} \nabla u^m \cdot \nabla\big(\alpha\eta(v^m - u^m)\big)\,\mathrm{d}z \geq 0.$$

This shows that u solves the variational inequality (2.3). Finally, from Lemma 3.2 and Lemma 5.2, we conclude that $u \in C^0([0,T]; L^{m+1}(\Omega))$ and $u(\cdot, 0) = u_o$. Therefore, u is the strong solution to the obstacle problem we were looking for.

To show that u is a weak supersolution, we pick an arbitrary nonnegative $\varphi \in C_0^\infty(\Omega_T)$ and choose

$$v^m = u^m + \varphi \geq u^m \geq \psi^m$$

in (2.3), and cut-off functions α, η with $\alpha\eta \equiv 1$ on $\operatorname{supp}\varphi$. This gives

$$\int_{\Omega_T} \big[-u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi \big] \,\mathrm{d}z \geq 0, \tag{8.4}$$

proving that u is a weak supersolution of the PME.

Next, we show that u is a weak solution of the PME in $\{z \in \Omega_T : u(z) > \psi(z)\}$. To this aim we pick any function $\varphi \in C_0^\infty(\Omega_T)$ which is compactly supported in the set $\{z \in \Omega_T : u(z) > \psi(z)\}$ and claim that (8.4) is valid also for this testing function. Without loss of generality, we may assume that $\inf\varphi < 0$, for otherwise there is nothing to prove. Now, we choose

$$v^m = u^m + \varepsilon\varphi$$

in (2.3), where

$$0 < \varepsilon < \frac{\inf_{\operatorname{supp}\varphi}(u^m - \psi^m)}{-\inf\varphi}.$$

Note that $v \geq \psi$. As a result, we conclude that (8.4) holds also for sign changing test functions with compact support in $\{z \in \Omega_T : u(z) > \psi(z)\}$. It follows that u is a weak solution of the PME in $\{z \in \Omega_T : u(z) > \psi(z)\}$. □

9. Existence of weak solutions to the obstacle problem

Before we can prove the existence of weak solutions to the obstacle problem for the porous medium equation, we still need the following energy estimate.

Lemma 9.1. *Assume that $u \in K_{\psi,g}$ is a local weak solution of the obstacle problem for the porous medium equation in the sense of Definition 2.1. Then, there holds the following energy estimate*

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} u^{m+1} \,\mathrm{d}x + \int_{\Omega_T} \big(u^{2m} + |\nabla u^m|^2\big) \,\mathrm{d}z \\ &\leq c\bigg[\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} g^{m+1} \,\mathrm{d}x + \int_\Omega u_o^{m+1} \,\mathrm{d}x + \int_{\Omega_T} \big(g^{2m} + |\nabla g^m|^2 + |\partial_t g^m|^{\frac{m+1}{m}}\big) \,\mathrm{d}z \bigg] \end{aligned}$$

with a constant c depending on $n, m, \operatorname{diam}(\Omega)$, and T.

Proof. In Lemma 4.1 we choose the cut-off function $\alpha \in W^{1,\infty}([0,T])$ by $\alpha \equiv 1$ on $[0, \tau - \varepsilon]$ and $\alpha(t) := \frac{1}{\varepsilon}(\tau - t)$ on $(\tau - \varepsilon, \tau)$ and $\alpha \equiv 0$ on $[\tau, T]$, where $\tau \in (0,T]$ and $\varepsilon \in (0, \tau)$. Letting $\varepsilon \downarrow 0$ we obtain

$$\begin{aligned} &\int_{\Omega\times\{\tau\}} I(u,v) \,\mathrm{d}x + \int_{\Omega_\tau} \nabla u^m \cdot (\nabla u^m - \nabla v^m) \,\mathrm{d}z \\ &\leq \int_{\Omega_\tau} \partial_t v^m (v - u) \,\mathrm{d}z + \int_\Omega I(u_o, v(\cdot, 0)) \,\mathrm{d}x, \end{aligned} \tag{9.1}$$

for all comparison maps $v \in K'_{\psi,g}(\Omega_T)$. Here, we choose $v = g$, which is admissible, since $g \in K'_{\psi,g}(\Omega_T)$. We recall from Lemma 4.1 that $I(u,g) \equiv \frac{1}{m+1}(u^{m+1} - g^{m+1}) - g^m(u - g)$. By elementary computations, we infer that

$$\tfrac{1}{2(m+1)} u^{m+1} - c(m) g^{m+1} \leq I(u,g) \leq u^{m+1} + g^{m+1}.$$

This yields for any $\delta \in (0,1)$ that

$$\begin{aligned}
&\int_{\Omega\times\{\tau\}} u^{m+1}\,\mathrm{d}x + \int_{\Omega_\tau} |\nabla u^m|^2\,dz \\
&\quad\le \int_{\Omega_\tau} \nabla u^m\cdot\nabla g^m + \partial_t g^m (g-u)\,\mathrm{d}z + c\int_\Omega \big(u_o^{m+1} + g^{m+1}(\cdot,0) + g^{m+1}(\cdot,\tau)\big)\,\mathrm{d}x \\
&\quad\le \frac12 \int_{\Omega_\tau} \big(|\nabla u^m|^2 + |\nabla g^m|^2\big)\,\mathrm{d}z + \frac{1}{2T}\int_{\Omega_\tau}\big(u^{m+1}+g^{m+1}\big)\,\mathrm{d}z \\
&\qquad + c(m,T)\bigg[\int_{\Omega_\tau} |\partial_t g^m|^{\frac{m+1}{m}}\,\mathrm{d}z + \sup_{t\in[0,T]}\int_{\Omega\times\{t\}} g^{m+1}\,\mathrm{d}x + \int_\Omega u_o^{m+1}\,\mathrm{d}x\bigg].
\end{aligned}$$

Now, we absorb the term containing $|\nabla u^m|^2$ from the right-hand side into the left. We use the resulting inequality in two directions. In the first term on the left-hand side we take the supremum over $\tau\in(0,T]$, while in the second term we choose $\tau = T$. Finally, we absorb the term containing u^{m+1} from the right-hand side into the left, taking into account that $\int_{\Omega_T} u^{m+1}\,\mathrm{d}x \le T\sup_{\tau\in[0,T]}\int_{\Omega\times\{\tau\}} u^{m+1}\,\mathrm{d}x$. In this way, we find that

$$\begin{aligned}
&\sup_{\tau\in[0,T]}\int_{\Omega\times\{\tau\}} u^{m+1}\,\mathrm{d}x + \int_{\Omega_T}|\nabla u^m|^2\,\mathrm{d}z \\
&\quad\le c\bigg[\int_{\Omega_T}\big(|\nabla g^m|^2 + |\partial_t g^m|^{\frac{m+1}{m}}\big)\,\mathrm{d}z + \sup_{t\in[0,T]}\int_{\Omega\times\{t\}} g^{m+1}\,\mathrm{d}x + \int_\Omega u_o^{m+1}\,\mathrm{d}x\bigg],
\end{aligned}$$

for a constant $c = c(m,T)$. Moreover, applying Poincaré's inequality slice-wise to $u^m - g^m$, we find that

$$\begin{aligned}
\int_{\Omega_T} u^{2m}\,\mathrm{d}z &\le 2\int_{\Omega_T}\big(|u^m - g^m|^2 + g^{2m}\big)\,\mathrm{d}z \\
&\le c(n,\mathrm{diam}(\Omega))\int_{\Omega_T}|\nabla u^m - \nabla g^m|^2\,\mathrm{d}z + 2\int_{\Omega_T} g^{2m}\,\mathrm{d}z.
\end{aligned}$$

Combining the last two inequalities proves the energy estimate. □

Now, we have all the prerequisites to prove the main result of the paper.

Proof of Theorem 2.7. The proof will be divided into several steps.

Step 1: Regularization. We begin by approximating the obstacle ψ by functions $\psi_i \in L^\infty(\Omega_T,\mathbb{R}_{\ge 0})$, $i\in\mathbb{N}$, satisfying $(2.8)_2$ and

$$\psi_i^m \in L^2\big(0,T;H^1(\Omega)\big),\quad \partial_t(\psi_i^m)\in L^{\frac{m+1}{m}}(\Omega_T)\quad\text{and}\quad \psi_i^m(\cdot,0)\in H^1(\Omega)$$

such that

$$\psi_i^m \to \psi^m \quad\text{in } L^2(0,T;H^1(\Omega))\quad\text{and}\quad \partial_t\psi_i^m \to \partial_t\psi^m \quad\text{in } L^{\frac{m+1}{m}}(\Omega_T) \tag{9.2}$$

and

$$\psi_i^m(\cdot,0) \rightharpoonup \psi^m(\cdot,0) \qquad\text{weakly in } H^1(\Omega), \tag{9.3}$$

as $i\to\infty$. Next, we approximate the lateral boundary data g by functions $g_i \in L^\infty(\Omega_T,\mathbb{R}_{\ge 0})$, $i\in\mathbb{N}$, satisfying $g_i \in K'_{\psi_i}(\Omega_T)$ and

$$g_i^m \to g^m \quad\text{in } L^2(0,T;H^1(\Omega))\quad\text{and}\quad \partial_t g_i^m \to \partial_t g^m \quad\text{in } L^{\frac{m+1}{m}}(\Omega_T) \tag{9.4}$$

and

$$g_i^m(\cdot,0) \rightharpoonup g^m(\cdot,0) \qquad\text{weakly in } H^1(\Omega), \tag{9.5}$$

as $i\to\infty$. Finally, we approximate the initial values u_o by $u_{o,i}\in L^\infty(\Omega,\mathbb{R}_{\ge 0})$ with $u_{o,i}^m \in H^1(\Omega)$, $u_{o,i}\ge \psi_i(\cdot,0)$ and

$$u_{o,i}^m \to u_o^m \qquad\text{strongly in } H^1(\Omega). \tag{9.6}$$

By $u_i \in K_{\psi_i,g_i}$ we denote the local strong solution to the obstacle problem for the PME with obstacle ψ_i, lateral boundary data g_i and initial boundary data $u_{o,i}$ constructed in Theorem 2.6. From Lemma 3.2 we know that u_i is also a local weak solution in the sense of Definition 2.1, i.e. u_i solves the variational inequality

$$\langle\!\langle \partial_t u_i, \alpha\eta(v^m - u_i^m)\rangle\!\rangle_{u_{o,i}} + \int_{\Omega_T} \alpha\nabla u_i^m \cdot \nabla[\eta(v^m - u_i^m)]\,\mathrm{d}z \geq 0 \tag{9.7}$$

for every cut-off function in time $\alpha \in W^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ with $\alpha(T) = 0$ and every cut-off function in space $\eta \in C_0^\infty(\Omega, \mathbb{R}_{\geq 0})$ and for every comparison map $v \in K'_{\psi_i,g_i}(\Omega_T)$.

Step 2: Energy bounds and weak convergence. In view of the energy bound from Lemma 9.1 and our assumptions on g_i, we have

$$\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} u_i^{m+1}\,\mathrm{d}x + \int_{\Omega_T} \big(u_i^{2m} + |Du_i^m|^2\big)\,\mathrm{d}z \leq c, \tag{9.8}$$

where $c = c(n, m, \mathrm{diam}(\Omega), T, g)$ is independent of $i \in \mathbb{N}$. Therefore, passing to a (not relabeled) subsequence, we can assume weak convergence

$$u_i^m \rightharpoonup u^m \qquad \text{weakly in } L^2(0,T; H^1(\Omega)) \tag{9.9}$$

as $i \to \infty$ for a limit map u^m satisfying $u^m - g^m \in L^2(0,T; H_0^1(\Omega))$. Applying Gagliardo-Nirenberg's inequality to u_i^m, cf. [6, Proposition I.3.1] and (9.8), we infer that

$$\int_{\Omega_T} u_i^q\,\mathrm{d}z \leq \int_{\Omega_T} \big(u_i^{2m} + |Du_i^m|^2\big)\,\mathrm{d}z \bigg(\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} u_i^{m+1}\,\mathrm{d}x\bigg)^{\frac{2}{n}} \leq c, \tag{9.10}$$

where $q = 2\big(m + \frac{m+1}{n}\big) > 2m$.

Step 3: Strong convergence of the time mollifications as $i \to \infty$. We define mollifications of u_i^m in time by

$$[\![u_i^m]\!]_h(\cdot,t) := e^{-\frac{t}{h}} u_i^m(\cdot,0) + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} u_i^m(\cdot,s)\,\mathrm{d}s$$

for $h \in (0,T]$, and similarly, we mollify the obstacles by

$$[\![\psi_i^m]\!]_h := e^{-\frac{t}{h}} \psi_i^m(\cdot,0) + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} \psi_i^m(\cdot,s)\,\mathrm{d}s.$$

Moreover, in order to restore the obstacle condition, we define the maps $w_{i,h}$ by

$$w_{i,h}^m := [\![u_i^m]\!]_h - [\![\psi_i^m]\!]_h + \psi_i^m \qquad \text{and} \qquad w_h^m := [\![u^m]\!]_h - [\![\psi^m]\!]_h + \psi^m.$$

The definition ensures the obstacle condition $w_{i,h} \geq \psi_i$ a.e. in Ω_T, and the initial values are $w_{i,h}(\cdot,0) = u_{o,i}$. The convergences (9.2), (9.3), (9.6), and (9.9), together with Lemma 3.1 (vi) yield $[\![u_i^m]\!]_h \rightharpoonup [\![u^m]\!]_h$ and $[\![\psi_i^m]\!]_h \rightharpoonup [\![\psi^m]\!]_h$ weakly in $L^2(0,T; H^1(\Omega))$ as $i \to \infty$, and therefore

$$w_{i,h}^m \rightharpoonup w_h^m \qquad \text{weakly in } L^2(0,T; H^1(\Omega)) \text{ as } i \to \infty. \tag{9.11}$$

The maps $v_{i,h}^m := w_{i,h}^m - \psi_i^m = [\![u_i^m]\!]_h - [\![\psi_i^m]\!]_h$ satisfy

$$\|\nabla v_{i,h}^m\|_{L^2(\Omega_T)} \leq \|\nabla u_i^m\|_{L^2(\Omega_T)} + \|\nabla \psi_i^m\|_{L^2(\Omega_T)}$$

and

$$\partial_t v_{i,h}^m = \partial_t [\![u_i^m]\!]_h - \partial_t [\![\psi_i^m]\!]_h = -\tfrac{1}{h}(w_{i,h}^m - u_i^m) \in L^2(\Omega_T).$$

From the energy bounds in (9.8), the strong convergence (9.2) of the obstacles and the convergence (9.11), we thus get L^2-bounds for both $\nabla v_{i,h}^m$ and $\partial_t v_{i,h}^m$, uniformly in $i \in \mathbb{N}$ for a fixed $h > 0$. Rellich's theorem implies strong subconvergence of $v_{i,h}^m$ in $L^2(\Omega_T)$. This implies

$$w_{i,h}^m = v_{i,h}^m + \psi_i^m \to w_h^m \qquad \text{in } L^2(\Omega_T), \text{ as } i \to \infty, \tag{9.12}$$

for any $h>0$, and this convergence even holds without passing to a subsequence because we already have identified the limit by (9.11). In view of Corollary 3.10 we get in the case $m\ge 1$ that

$$w_{i,h}\to w_h \qquad \text{in } L^{2m}(\Omega_T),\text{ as } i\to\infty, \tag{9.13}$$

for every $h>0$.

Step 4: Locally uniform convergence of the time mollifications as $h\downarrow 0$. Since $v=w_{i,h}$ does not admit the right boundary values on the lateral boundary, it is not admissible as comparison function in Lemma 4.1, so that we have to apply Lemma 4.2 instead. For a suitable cut-off construction, we consider a fixed ball $B_R=B_R(x_o)$ with $B_{2R}(x_o)\subset\Omega$. By the mean value theorem, we may choose a radius $r\in(R,2R)$, possibly depending on i, with

$$\int_0^T\int_{\partial B_r}|\nabla u_i^m|^2\,\mathrm{d}x\,\mathrm{d}t\le\frac{1}{R}\int_0^T\int_{B_{2R}}|\nabla u_i^m|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{9.14}$$

We can also choose r in such a way that the application of Lemma 4.2 is possible with the comparison function $v=w_{i,h}$. Keeping in mind that $I(u_i,w_{i,h})$ is non-negative and that $w_{i,h}(0)=u_{o,i}$, this yields the bound

$$\begin{aligned}-\int_{B_r\times(0,T)}\alpha\partial_t w_{i,h}^m(w_{i,h}-u_i)\,\mathrm{d}z\le&\int_{B_r\times(0,T)}\alpha\nabla u_i^m\cdot(\nabla w_{i,h}^m-\nabla u_i^m)\,\mathrm{d}z\\&+2\int_{\partial B_r\times(0,T)}\alpha|\nabla u_i^m|\,|w_{i,h}^m-u_i^m|\,\mathrm{d}z.\end{aligned}$$

Here, we define the cut-off function $\alpha\in W^{1,\infty}([0,T])$ by $\alpha\equiv 1$ on $[0,T-\varepsilon]$ and $\alpha(t):=\frac{1}{\varepsilon}(T-t)$ on $(T-\varepsilon,T]$. Letting $\varepsilon\downarrow 0$, we infer

$$\begin{aligned}-\int_{B_r\times(0,T)}\partial_t w_{i,h}^m(w_{i,h}-u_i)\,\mathrm{d}z\le&\int_{B_r\times(0,T)}\nabla u_i^m\cdot(\nabla w_{i,h}^m-\nabla u_i^m)\,\mathrm{d}z\\&+2\int_{\partial B_r\times(0,T)}|\nabla u_i^m|\,|w_{i,h}^m-u_i^m|\,\mathrm{d}z.\end{aligned} \tag{9.15}$$

Next, we compute

$$-\partial_t w_{i,h}^m=\tfrac{1}{h}(w_{i,h}^m-u_i^m)-\partial_t\psi_i^m. \tag{9.16}$$

We now plug (9.16) into (9.15) to deduce that

$$\begin{aligned}&\int_{B_r\times(0,T)}(w_{i,h}^m-u_i^m)(w_{i,h}-u_i)\,\mathrm{d}z\\&\quad\le h\bigg[\int_{B_r\times(0,T)}\partial_t\psi_i^m(w_{i,h}-u_i)\,\mathrm{d}z+\int_{B_r\times(0,T)}\nabla u_i^m\cdot(\nabla w_{i,h}^m-\nabla u_i^m)\,\mathrm{d}z\\&\qquad\quad+2\int_{\partial B_r\times(0,T)}|\nabla u_i^m|\,|w_{i,h}^m-u_i^m|\,\mathrm{d}z\bigg]\\&\quad=:h[\mathrm{I}+\mathrm{II}+\mathrm{III}]\end{aligned}$$

with the obvious labeling of the terms I, II, III. Here, we can estimate

$$\begin{aligned}\mathrm{I}&\le\|\partial_t\psi_i^m\|_{L^{\frac{m+1}{m}}}\big(\|w_{i,h}\|_{L^{m+1}}+\|u_i\|_{L^{m+1}}\big)\\&\le c\|\partial_t\psi_i^m\|_{L^{\frac{m+1}{m}}}\big(\|u_i\|_{L^{m+1}}+\|\psi_i\|_{L^{m+1}}+\|u_{o,i}\|_{L^{m+1}}+\|\psi_i(\cdot,0)\|_{L^{m+1}}\big),\end{aligned}$$

and

$$\begin{aligned}\mathrm{II}&\le\|\nabla u_i^m\|_{L^2}\big(\|\nabla w_{i,h}^m\|_{L^2}+\|\nabla u_i^m\|_{L^2}\big)\\&\le c\big(\|\nabla u_i^m\|_{L^2}^2+\|\nabla\psi_i^m\|_{L^2}^2+\|\nabla u_{o,i}\|_{L^{m+1}}+\|\nabla\psi_i(\cdot,0)\|_{L^{m+1}}\big).\end{aligned}$$

Finally, by (9.14) and the embedding $H^1(B_r) \hookrightarrow L^2(\partial B_r)$ we have

$$\begin{aligned} \mathrm{III} &\le \int_{\partial B_r(x_o)\times(0,T)} |\nabla u_i^m|^2 + |w_{i,h}^m - u_i^m|^2 \,\mathrm{d}z \\ &\le \frac{1}{R}\int_{B_{2R}\times(0,T)} |\nabla u_i^m|^2 + u_i^{2m} + |\nabla w_{i,h}^m|^2 + w_{i,h}^{2m}\,\mathrm{d}z. \end{aligned}$$

From the energy bounds (9.8) and the assumptions (9.2) and (9.3) on ψ_i we thereby conclude that $\mathrm{I}+\mathrm{II}+\mathrm{III}$ is bounded independently from $i \in \mathbb{N}$ and $h > 0$ and hence

$$\int_{B_r\times(0,T)} (w_{i,h}^m - u_i^m)(w_{i,h} - u_i)\,\mathrm{d}z \le C\,h$$

with a constant C independent from $i \in \mathbb{N}$ and $h > 0$. We now employ Corollary 3.11 to deduce that in the case $m \ge 1$ there holds

$$\|w_{i,h} - u_i\|_{L^{m+1}(B_R\times(0,T))} \le Ch^{\frac{1}{m+1}} \qquad \text{for all } i \in \mathbb{N}, \tag{9.17}$$

while in the case $m_c < m < 1$, we have

$$\|w_{i,h}^m - u_i^m\|_{L^{\frac{m+1}{m}}(B_R\times(0,T))} \le Ch^{\frac{m}{m+1}} \qquad \text{for all } i \in \mathbb{N}, \tag{9.18}$$

with a constant C independent from $i \in \mathbb{N}$ and $h > 0$.

Step 5: Pointwise convergence of the solutions. As in the last step, we consider a fixed ball $B_R = B_R(x_o)$ with $B_{2R}(x_o) \subset \Omega$. In the following, we abbreviate $B_R^T := B_R \times (0,T)$. We start by treating the case $m \ge 1$. For any $i \in \mathbb{N}$ and $h > 0$, we can estimate

$$\begin{aligned} &\|u_i - u\|_{L^{m+1}(B_R^T)} \\ &\le \|u_i - w_{i,h}\|_{L^{m+1}(B_R^T)} + \|w_{i,h} - w_h\|_{L^{m+1}(B_R^T)} + \|w_h - u\|_{L^{m+1}(B_R^T)}. \end{aligned} \tag{9.19}$$

For an arbitrary $\varepsilon > 0$, we can exploit the uniform convergence (9.17) in order to choose $h > 0$ so small that

$$\|u_i - w_{i,h}\|_{L^{m+1}(B_R^T)} \le Ch^{\frac{1}{m+1}} \le \tfrac{1}{2}\varepsilon$$

holds for all $i \in \mathbb{N}$. By Lemma 3.1, we moreover can achieve

$$\|w_h - u\|_{L^{m+1}(B_R^T)} \le \|w_h^m - u^m\|_{L^{\frac{m+1}{m}}(B_R^T)}^{\frac{1}{m}} \le \tfrac{1}{2}\varepsilon$$

by choosing $h > 0$ smaller if necessary. Plugging the two preceding estimates into (9.19) and letting $i \to \infty$, we arrive at

$$\limsup_{i\to\infty} \|u_i - u\|_{L^{m+1}(B_R^T)} \le \varepsilon + \lim_{i\to\infty} \|w_{i,h} - w_h\|_{L^{m+1}(B_R^T)} = \varepsilon,$$

where the last step follows from (9.13). Since $\varepsilon > 0$ and the ball B_R were arbitrary, this implies $u_i \to u$ strongly in $L^{m+1}_{\mathrm{loc}}(\Omega_T)$, as $i \to \infty$. From this, we can conclude that $u_i^m \to u^m$ strongly in $L^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega_T)$, as $i \to \infty$. At this point, we recall from (9.10) that u_i is uniformly bounded in $L^q(\Omega_T)$ with $q > 2m$. Therefore, the interpolation property of L^p-spaces ensures that

$$u_i^m \to u^m \qquad \text{strongly in } L^2_{\mathrm{loc}}(\Omega_T), \text{ as } i \to \infty. \tag{9.20}$$

Now, we turn our attention to the case $m < 1$. Here, we estimate for any $i \in \mathbb{N}$ and $h > 0$:

$$\begin{aligned} &\|u_i^m - u^m\|_{L^2(B_R^T)} \\ &\le \|u_i^m - w_{i,h}^m\|_{L^2(B_R^T)} + \|w_{i,h}^m - w_h^m\|_{L^2(B_R^T)} + \|w_h^m - u^m\|_{L^2(B_R^T)}. \end{aligned} \tag{9.21}$$

For an arbitrary $\varepsilon > 0$, since $\frac{m+1}{m} > 2$ we can exploit the uniform convergence (9.18) in order to choose $h > 0$ so small that

$$\|u_i^m - w_{i,h}^m\|_{L^2(B_R^T)} \le C\|u_i^m - w_{i,h}^m\|_{L^{\frac{m+1}{m}}(B_R^T)} \le Ch^{\frac{m}{m+1}} \le \tfrac{1}{2}\varepsilon$$

holds for all $i \in \mathbb{N}$. By Lemma 3.1, we moreover can achieve

$$\|w_h^m - u^m\|_{L^2(B_R^T)} \le \tfrac{1}{2}\varepsilon$$

by choosing $h > 0$ smaller if necessary. Plugging the two preceding estimates into (9.21) and letting $i \to \infty$, we arrive at

$$\limsup_{i\to\infty} \|u_i^m - u^m\|_{L^2(B_R^T)} \le \varepsilon + \lim_{i\to\infty} \|w_{i,h}^m - w_h^m\|_{L^2(B_R^T)} = \varepsilon,$$

where the last step follows from (9.12). Since $\varepsilon > 0$ was arbitrary, this implies

$$u_i^m \to u^m \qquad \text{strongly in } L^2_{\mathrm{loc}}(\Omega_T), \text{ as } i \to \infty. \tag{9.22}$$

As a consequence of (9.20), respectively (9.22) we conclude that $u \ge \psi$ a.e. on Ω_T. By passing to another subsequence, we can also assume almost everywhere convergence. We note that this is the crucial step to identify the limit map. If we would know only weak convergence, the weak limit of u_i and the weak limit of u_i^m might not be related. But knowing (9.20), respectively (9.22), we can now conclude

$$\begin{cases} u_i^m \rightharpoonup u^m & \text{weakly in } L^2(0,T;H^1(\Omega)), \text{ as } i \to \infty, \\ u_i \rightharpoonup u & \text{weakly in } L^{m+1}(\Omega_T), \text{ as } i \to \infty. \end{cases} \tag{9.23}$$

Step 6: Passage to the limit. Here, we will to pass to the limit $i \to \infty$ in the variational inequality (9.7). Therefore, we consider $\alpha \in W^{1,\infty}([0,T],\mathbb{R}_{\ge 0})$ with $\alpha(T) = 0$, $\eta \in C_0^\infty(\Omega,\mathbb{R}_{\ge 0})$ and $v \in K'_{\psi_i,g_i}(\Omega_T)$. By (9.23) and (9.6) we have

$$\lim_{i\to\infty} \langle\!\langle \partial_t u_i, \alpha\eta(v^m - u_i^m)\rangle\!\rangle_{u_{o,i}} = \langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o}.$$

Again by (9.23) we find that

$$\lim_{i\to\infty} \int_{\Omega_T} \alpha \nabla u_i^m \cdot \nabla(\eta v^m)\,\mathrm{d}z = \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla(\eta v^m)\,\mathrm{d}z.$$

and

$$-\liminf_{i\to\infty} \int_{\Omega_T} \alpha\eta \nabla u_i^m \cdot \nabla u_i^m\,\mathrm{d}z \le -\int_{\Omega_T} \alpha\eta \nabla u^m \cdot \nabla u^m\,\mathrm{d}z.$$

holds true. Finally, the strong convergence (9.20), respectively (9.22) and the weak convergence (9.23) imply

$$\lim_{i\to\infty} \int_{\Omega_T} \alpha u_i^m \nabla u_i^m \cdot \nabla\eta\,\mathrm{d}z = \int_{\Omega_T} \alpha u^m \nabla u^m \cdot \nabla\eta\,\mathrm{d}z.$$

In conclusion, the above considerations allow us to pass to the limit $i \to \infty$ in the variational inequality (9.7) to deduce that

$$\langle\!\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\!\rangle_{u_o} + \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla[\eta(v^m - u^m)]\,\mathrm{d}z \ge 0$$

holds for any v, α and η as in Definition 2.1. At this point, we can apply Lemma 5.2 to deduce that $u \in C^0([0,T];L^{m+1}(\Omega))$ and $u(\cdot,0) = u_o$. This shows that u is the local weak solution to the obstacle problem for the PME we were looking for. Finally, to check that u is also a weak supersolution, we note that

$$\int_{\Omega_T} -u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi\,\mathrm{d}z = \lim_{i\to\infty} \int_{\Omega_T} -u_i\partial_t\varphi + \nabla u_i^m \cdot \nabla\varphi\,\mathrm{d}z \ge 0$$

for all nonnegative $\varphi \in C_0^\infty(\Omega_T)$ by (9.23) and the fact that the functions u_i are weak supersolutions. □

REFERENCES

[1] H. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[2] V. Bögelein, F. Duzaar, and G. Mingione. Degenerate problems with irregular obstacles. J. *Reine Angew. Math.*, 650:107–160, 2011.

[3] V. Bögelein and C. Scheven. Higher integrability in parabolic obstacle problems. *Forum Math.*, 24(5):931–972, 2012.

[4] J. Casado-Díaz, F. Murat, and A. Porretta. Uniqueness results for pseudomonotone problems with $p > 2$. *C. R. Math. Acad. Sci. Paris*, 344(8):487-492, 2007.

[5] P. Daskalopoulos and C. E. Kenig. *Degenerate diffusions – Initial value problems and local regularity theory*. European Mathematical Society (EMS), Zürich, 2007.

[6] E. DiBenedetto. *Degenerate Parabolic Equations.* Springer Universitext, Springer, New York, 1993.

[7] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack's inequality for degenerate and singular parabolic equations.* Springer Monographs in Mathematics. Springer, New York, 2012.

[8] E. DiBenedetto and A. Friedman. Hölder estimates for nonlinear degenerate parabolic systems. *J. Reine Angew. Math.*, 357:1–22, 1985.

[9] P. Hajłasz. Pointwise Hardy inequalities. *Proc. Amer. Math. Soc.*, 127(2):417–423, 1999.

[10] L. I. Hedberg. On certain convolution inequalities. *Proc. Amer. Math. Soc.*, 36:505–510, 1972.

[11] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Oxford University Press, 1993.

[12] J. Kinnunen and P. Lindqvist. Definition and properties of supersolutions to the porous medium equation. *J. Reine Angew. Math.*, 618:135–168, 2008.

[13] J. Kinnunen and O. Martio. Hardy's inequalities for Sobolev functions. *Math. Res. Lett.*, 4(4):489–500, 1997.

[14] J. Kinnunen, P. Lindqvist, and T. Lukkari. Perron's method for the porous medium equation. Submitted, available at `http://arxiv.org/abs/1401.4277`.

[15] R. Korte, T. Kuusi, and J. Siljander. Obstacle problem for nonlinear parabolic equations. *J. Differential Equations*, 246(9):3668–3680, 2009.

[16] P. Lindqvist. On the definition and properties of p-superharmonic functions. *J. Reine Angew. Math.*, 365:67–79, 1986.

[17] P. Lindqvist and M. Parviainen. Irregular time dependent obstacles. *J. Funct. Anal.*, 263(8):2458–2482, 2012.

[18] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.

[19] J. Naumann. *Einführung in die Theorie parabolischer Variationsungleichungen*, volume 64 of *Teubner-Texte zur Mathematik*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984.

[20] C. Scheven. Existence of localizable solutions to nonlinear parabolic problems with irregular obstacles. Submitted.

[21] J. L. Vázquez. *The porous medium equation– Mathematical theory.* Oxford University Press, Oxford, 2007.

[22] Z. Wu, J. Zhao, J. Yin, and H. Li. *Nonlinear diffusion equations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

VERENA BÖGELEIN, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, CAUERSTR. 11, 91056 ERLANGEN, GERMANY

E-mail address: `boegelein@math.fau.de`

TEEMU LUKKARI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 35 (MAD), 40014 UNIVERSITY OF JYVÄSKYLÄ, JYVÄSKYLÄ, FINLAND

E-mail address: `teemu.j.lukkari@jyu.fi`

CHRISTOPH SCHEVEN, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STR. 9, 45127 ESSEN, GERMANY

E-mail address: `christoph.scheven@uni-due.de`