THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

First eigenfunctions of the 1-Laplacian are viscosity solutions

B. Kawohl and F. Schuricht

REPORT No. 34, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -34-13/14- -SE+fall

First eigenfunctions of the 1-Laplacian are viscosity solutions

B. Kawohl & F. Schuricht

January 31, 2014

Abstract *We address the question if eigenfunctions of the 1-Laplacian, which are obtained through a variational argument, are also viscosity solutions of the associated strongly degenerate formal Euler equation. The answer is positive, but examples show also that there are many more viscosity solutions than expected.*

Mathematics Subject Classification (2010). 35D40, 35J70, 49G05, 49A52

Keywords. discontinuous viscosity solution, 1-Laplacian, Cheeger set

1 Introduction

In [12] it was shown that a first eigenfunction u_1 of the 1-Laplacian operator on a bounded domain Ω under Dirichlet boundary conditions, scaled in L^∞ to 1, is a discontinuous function $u_1 = \chi_C(x)$ where C is a Cheeger set of Ω. A Cheeger set infimizes the ratio of $(n-1)$–dimensional perimeter over n–dimensional volume among all Cacciopoli subsets of Ω. Formally the Euler–equation of u_1 reads

$$(-\Delta_1 u =) - \operatorname{div}\left(\frac{Du}{|Du|}\right) = \lambda \frac{u}{|u|} \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

and in points in which $u = 0$ or $|Du| = 0$ this is not well–defined. We call any function u that solves the following variational problem *first eigenfunction of the 1-Laplacian under Dirichlet boundary conditions*

$$E(u) = \int_\Omega d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} \to \text{Min!}, \quad u \in BV(\Omega) \tag{1.2}$$

under the side constraint

$$\int_\Omega |u|\, dx = 1. \tag{1.3}$$

Here $\int_\Omega d|Du|$ is the total variation of $u \in BV(\Omega)$ and $E(u)$ is the total variation of the zero extension of u in $BV(I\!R^n)$.

Various attempts have been made to give proper meaning to the formal Euler equation (1.1). In [14] a precise derivation of (1.1) is given where $Du/|Du|$ is replaced by a vector field $z : \Omega \mapsto \overline{B_1(0)} \subset I\!R^n$ and $u/|u|$ is replaced by a suitable signum function $s : \Omega \mapsto [-1, 1]$. However, it turns out that the equation

$$-\operatorname{div} z = \lambda s \tag{1.4}$$

has many solutions, some of which are not minimizers.

It is the purpose of the present note to investigate if the notion of viscosity solutions to (1.1) provides a more restricted class of solutions. In fact, each minimizer u_1 can be shown to be a viscosity solution of (1.1), provided we are willing to accept a suitably adapted notion of a discontinuous viscosity solution u to a Dirichlet problem for a discontinuous elliptic equation $F(u, Du, D^2u) = 0$. On the other hand, we provide a number of examples to illustrate that this notion admits also other unexpected solutions.

In view of this observation, the restrictions by inner variations indicated in [18] seem to select fewer and more reasonable solutions.

2 Viscosity solutions

The notion of viscosity solutions is introduced in [3] for second order equations

$$F(u, Du, D^2u) = 0 \quad \text{on } \Omega$$

where, for degenerate F, its definition uses the lower semicontinuous hull F_* and the upper semicontinuous hull F^*. Notice that our equation (1.1) is not proper (or monotone nondecreasing) in u in the sense of [3]. Therefore we restrict the set of testfunctions φ for sub- and supersolutions as in [11] and [9] to those that touch the graph of u in x. Since we have to handle discontinuous (merely measurable) functions $u \in BV(\Omega)$, it is inappropriate to define sub- and supersolutions by means of the (usual) upper and lower semicontinuous hulls u^* and u_*. Instead we use the approximate lower and upper semicontinuous hull of u given by

$$u_*(x) \quad := \quad \text{ap} \liminf_{y \to x} u(y) := \sup \left\{ t \mid \lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap \{u < t\})}{r^n} = 0 \right\} \tag{2.1}$$

$$u^*(x) \quad := \quad \text{ap} \limsup_{y \to x} u(y) := \inf \left\{ t \mid \lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap \{u > t\})}{r^n} = 0 \right\} \tag{2.2}$$

(cf. [5, Section 5.9]). Notice that u agrees with u^* and u_* a.e. on Ω.

Definition 2.1 *A (measurable) function u, is a* **viscosity solution** *of the degenerate elliptic differential equation $F(u, Du, D^2u) = 0$ in Ω, if and only if u^* is a viscosity subsolution of $F_*(u, Du, D^2u) = 0$ and u_* is a viscosity supersolution of $F^*(u, Du, D^2u) = 0$ (where F_* and F^* denote the lower and upper semicontinuous hulls of F and u_*, u^* are given in (2.1), (2.2)).*

Definition 2.2 *The (approximate) upper semicontinuous hull u^* is a* **viscosity subsolution** *of $F_*(u, Du, D^2u) = 0$ if for every $x \in \Omega$ and $\varphi \in C^2$ s.th. φ touches u^* from above at x the inequality $F_*(\varphi(x), D\varphi(x), D^2\varphi(x)) \leq 0$ holds.*

Definition 2.3 *The (approximate) lower semicontinuous hull u_* is a* **viscosity supersolution** *of $F^*(u, Du, D^2u) = 0$ if for every $x \in \Omega$ and $\varphi \in C^2$ s.th. φ touches u_* from below at x the inequality $F^*(\varphi(x), D\varphi(x), D^2\varphi(x)) \geq 0$ holds.*

Let us apply this concept to equation (1.1). As outlined in [15], if $x \in \Omega$ and $D \subset \Omega$ is the superlevel set $\{y \in \Omega \,:\, \varphi(y) \geq \varphi(x)\}$ of a smooth function φ, and if $D\varphi(x) \neq 0$, then

$$
\begin{aligned}
-\Delta_1\varphi \;=\; -\mathrm{div}\left(\frac{D\varphi}{|D\varphi|}\right) \;&=\; \frac{-\Delta\varphi|D\varphi| + D\varphi\, D^2\varphi\, \frac{D\varphi}{|D\varphi|}}{|D\varphi|^2}\\
&=\; \frac{-\Delta\varphi + \frac{D\varphi}{|D\varphi|}\, D^2\varphi\, \frac{D\varphi}{|D\varphi|}}{|D\varphi|} \;=\; -\frac{\Delta\varphi}{|D\varphi|} + \frac{\varphi_{\nu\nu}}{|D\varphi|}\\
&=\; \frac{-\varphi_{\nu\nu} - (N-1)H_\varphi\varphi_\nu}{|D\varphi|} + \frac{\varphi_{\nu\nu}}{|D\varphi|}\\
&=\; (N-1)H_\varphi \qquad (2.3)
\end{aligned}
$$

where ν is the outer unit normal of D at x and $H_\varphi(x)$ denotes the mean curvature of the level surface of φ through x. Note that the sign of H is positive if the level set D is convex (in $\mathbb{R}^2$) or mean-convex (in $\mathbb{R}^n$). Thus equation (1.1) is related to

$$
F(u,q,X) = \begin{cases} -\frac{1}{|q|}\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij} - \lambda\frac{u}{|u|} & \text{if } q \neq 0 \text{ and } u \neq 0,\\ ? & \text{if } q = 0 \text{ or } u = 0. \end{cases} \qquad (2.4)
$$

Every symmetric matrix X has only real eigenvalues $\mu_1(X) \le \mu_2(X) \le \ldots \le \mu_n(X)$, and in terms of these

$$
F_*(u,q,X) = \begin{cases} -\frac{1}{|q|}\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij} - \lambda\frac{u}{|u|} & \text{if } q \neq 0 \text{ and } u \neq 0,\\ -\,\infty\,\left(\sum_{i=2}^n \mu_i(X)\right) - \lambda\,\frac{u}{|u|} & \text{if } q = 0 \text{ and } u \neq 0,\\ -\,\frac{1}{|q|}\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij} - \lambda & \text{if } q \neq 0 \text{ and } u = 0,\\ -\infty\,\left(\sum_{i=2}^n \mu_i(X)\right) - \lambda & \text{if } q = 0 \text{ and } u = 0, \end{cases} \qquad (2.5)
$$

with the convention that $-\infty \cdot 0 = -\infty$, the lsc-hull. To see the second case in (2.5) one has to realize that we have to maximize $\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij}$. For that we recall that the trace $\delta_{ij}X_{ij}$ represents the sum of the eigenvalues $\sum_{i=1}^n \mu_i(X)$ and we choose q to be the eigenvector associated with $\mu_1(X)$. If $\sum_{i=2}^n \mu_i(X) = 0$, then we can choose $X_k \to X$ such that $\sum_{i=2}^n \mu_i(X_k) > 0$ to explain the convention $-\infty \cdot 0 = -\infty$. In the third case we get $-\lambda$ at $u = 0$ because $\lambda u/|u|$ attains values $\pm\lambda$ near $u = 0$.

Analogously

$$
F^*(u,q,X) = \begin{cases} -\frac{1}{|q|}\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij} - \lambda\frac{u}{|u|} & \text{if } q \neq 0 \text{ and } u \neq 0,\\ -\,\infty\,\left(\sum_{i=1}^{n-1} \mu_i(X)\right) - \lambda\,\frac{u}{|u|} & \text{if } q = 0 \text{ and } u \neq 0,\\ -\,\frac{1}{|q|}\left(\delta_{ij} - \frac{q_iq_j}{|q|^2}\right)X_{ij} + \lambda & \text{if } q \neq 0 \text{ and } u = 0,\\ -\infty\,\left(\sum_{i=1}^{n-1} \mu_i(X)\right) + \lambda & \text{if } q = 0 \text{ and } u = 0, \end{cases} \qquad (2.6)
$$

now with the convention that $-\infty \cdot 0 = +\infty$, the usc-hull.

Since the first eigenfunction u_1 of the 1-Laplacian violates the boundary condition $u = 0$ in the classical sense on parts of the boundary, we also consider the definition of boundary conditions in the viscosity sense along the lines of [3, Def. 7.4].

Definition 2.4 *A (measurable) function u satisfies the* **Dirichlet boundary condition** $u = 0$ *on* $\partial\Omega$ **in the viscosity sense** *if and only if for every $x \in \partial\Omega$ the (approximate) usc hull u^* satisfies* $\min(u, F_*(u, Du, D^2u)) \le 0$ *and the (approximate) lsc hull u_* satisfies* $\max(u, F^*(u, Du, D^2u)) \ge 0$ *in the sense of viscosity solutions.*

Now we are in a situation to state our main result.

Theorem 2.5 *Suppose that $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is a bounded domain with Lipschitz boundary and let $u \in BV(\Omega)$ be a first eigenfunction of the* 1*-Laplacian under Dirichlet boundary conditions. Then u is a viscosity solution of problem* (1.1) *and satisfies the boundary conditions in the viscosity sense.*

Several examples in the next section supplement the assertion of the theorem. They show that there may be other viscosity solutions of the 1-Laplacian and, in particular, that viscosity solutions of (1.1) need not be eigenfunctions of the 1-Laplacian. In Example 6 we see the remarkable fact that only $u \equiv 0$ is viscosity solutions of (1.1) for $n = 1$ and, thus, the theorem is wrong in that case. Before we start the proof of the theorem let us still mention that eigenfunctions of the 1-Laplacian belong to $L^\infty(\Omega)$ (cf. [16] and [4, Proposition 3.3]).

Proof: Let us show by contradiction that u is a viscosity subsolution. The proof for supersolutions is similar and left as an excercise to the reader.

(1) If u^* is not a viscosity subsolution of the differential equation then there exists a point $x_0 \in \Omega$ and a smooth function $\varphi \geq u^*$ touching u^* in x_0 from above such that

$$F_*(\varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) > 0. \tag{2.7}$$

For some small $r > 0$ with $B_r(x_0) \in \Omega$ we can assume that

$$\varphi > u^* \text{ on } B_r(x_0) \setminus \{x_0\}, \quad \gamma := \inf\left\{\varphi(x) - u^*(x) : \, |x - x_0| \geq \frac{r}{2}\right\} > 0 \tag{2.8}$$

(otherwise we replace $\varphi(x)$ by $\tilde{\varphi}(x) = \varphi(x) + |x - x_0|^4$) and

$$u^*(x) \leq \varphi(x) < 0 \text{ on } B_r(x_0) \text{ if } u^*(x_0) < 0\,. \tag{2.9}$$

(1a) Let us first consider the regular case $D\varphi(x_0) \neq 0$. With

$$s := \begin{cases} 1 & \text{if } \varphi(x_0) \geq 0 \\ -1 & \text{if } \varphi(x_0) < 0 \end{cases} \tag{2.10}$$

(2.7) combined with (2.5) says that

$$-\Delta_1\varphi(x_0) > s\lambda \tag{2.11}$$

(cf. also (2.3)). By the C^2-regularity of φ we may assume for sufficiently small $r > 0$ that

$$D\varphi(x) \neq 0 \text{ and } -\Delta_1\varphi(x) - s\lambda > 0 \qquad \text{in } B_r(x_0). \tag{2.12}$$

With γ according to (2.8) we define

$$\phi := \varphi - \tilde{\gamma} \text{ for some } \tilde{\gamma} \in (0, \gamma/2)\,. \tag{2.13}$$

Clearly

$$D\phi(x) \neq 0 \text{ and } -\Delta_1\phi(x) - s\lambda > 0 \quad \text{in } B_r(x_0) \tag{2.14}$$

and

$$\phi(x_0) < u^*(x_0) \text{ but } \phi(x) > u^*(x) \text{ on } B_r(x_0) \setminus B_{r/2}(x_0)\,. \tag{2.15}$$

We obviously have that

$$\eta := u^* - \phi \in BV(\Omega)$$

and, by the coarea formula (cf. [5]), the level sets $\{\eta > t\}$ have finite perimeter in Ω for L^1-a.e. $t \in I\!R$. Thus, possibly after an arbitrarily small change of $\tilde{\gamma}$, we can assume that $\{\eta > 0\}$ has finite perimeter. By $\phi(x_0) < u^*(x_0)$ and the definition of u^*, the set $\{\eta > 0\}$ has positive measure and, since $\phi > u^*$ outside $B_{r/2}(x_0)$,

$$\{\eta > 0\} \subset B_{r/2}(x_0)\,.$$

Let us now consider the positive part η_+ which also belongs to $BV(\Omega)$ by the coarea formula. Moreover $\eta_+ = 0$ on $\partial B_r(x_0)$ in the sense of trace by (2.15). We multiply the inequality in (2.14) with η_+ and integrate over $B_r(x_0)$. Recalling that $\{\eta > 0\}$ has positive measure we get

$$-\int_{B_r(x_0)} \eta_+ \operatorname{div}\left(\frac{D\phi}{|D\phi|}\right) + s\lambda\eta_+ \, dx > 0\,. \tag{2.16}$$

The vector field $D\phi/|D\phi|$ is continuous and, thus, integration by parts gives (cf. [2])

$$0 < \int_{B_r(x_0)} \frac{D\phi}{|D\phi|} \, d(D\eta_+) - \int_{B_r(x_0)} s\lambda\eta_+ \, dx\,. \tag{2.17}$$

Recall that $\{\eta > 0\}$ has finite perimeter. Let $\{\eta > 0\}^*$ denote its measure theoretic interior (i.e. points of density 1) and let $\partial^*\{\eta > 0\}$ be its reduced boundary with (measure theoretic) inner unit normal ν. For $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*\{\eta > 0\}$ there is an inner trace η^+ and an outer trace η^- of η by [1, Theorem 3.77] (notice that $\eta^+ = \eta^*$ and $\eta^- = \eta_*$ by [5, Theorem 5.9.3] which, however, is not relevant for our analysis). By the continuity of ϕ and the sign of η on $\{\eta > 0\}$ and its complement $\{\eta \le 0\}$, there is an inner trace u^+ and an outer trace u^- of u such that

$$0 \le \eta^+ = u^+ - \phi \quad \text{and} \quad 0 \ge \eta^- = u^- - \phi \quad \mathcal{H}^{n-1}\text{-a.e. on } x \in \partial^*\{\eta > 0\}\,.$$

Then we get by [1, Theorem 3.84]

$$\int_{B_r(x_0)} \frac{D\phi}{|D\phi|} \, d(D\eta_+) = \int_{\{\eta>0\}^*} \frac{D\phi}{|D\phi|} \, d(D\eta) + \int_{\partial^*\{\eta>0\}} \eta^+ \frac{D\phi}{|D\phi|} \cdot \nu \, d\mathcal{H}^{n-1}\,. \tag{2.18}$$

With $\eta = u^* - \phi$ and $\eta^+ = u^+ - \phi$ we get

$$\begin{aligned}
\int_{B_r(x_0)} \frac{D\phi}{|D\phi|} \, d(D\eta_+) \;&=\; \int_{\{\eta>0\}^*} \frac{D\phi}{|D\phi|} \, d(Du^*) - \int_{\{\eta>0\}^*} \frac{D\phi}{|D\phi|} \, d(D\phi) \\
&\quad + \int_{\partial^*\{\eta>0\}} (u^+ - \phi) \frac{D\phi}{|D\phi|} \cdot \nu \, d\mathcal{H}^{n-1} \\
&\le\; \int_{\{\eta>0\}^*} d|Du| - \int_{\{\eta>0\}^*} d|D\phi| \\
&\quad + \int_{\partial^*\{\eta>0\}} (u^+ - \phi) \, d\mathcal{H}^{n-1}\,.
\end{aligned}$$

Hence (2.17) implies

$$\begin{aligned}
0 \;&<\; \int_{\{\eta>0\}^*} d|Du| - \int_{\{\eta>0\}^*} d|D\phi| + \int_{\partial^*\{\eta>0\}} (u^+ - \phi) \, d\mathcal{H}^{n-1} \\
&\quad -\lambda \int_{\{\eta>0\}} su \, dx + \lambda \int_{\{\eta>0\}} s\phi \, dx\,.
\end{aligned} \tag{2.19}$$

On the other hand, the minimizing property of u in the Rayleigh quotient implies

$$0 \le \int_\Omega d|D(u-\eta_+)| + \int_{\partial\Omega} |u-\eta_+|\, d\mathcal{H}^{n-1} - \lambda \int_\Omega |u-\eta_+|\, dx\,. \tag{2.20}$$

We have $u - \eta_+ = \phi$ on $\{\eta > 0\}$ and $u - \eta_+ = u$ on $\{\eta \le 0\}$. Thus, analogously as in (2.18), we can decompose the first integral in (2.20) by [1, Theorem 3.84]. We use the notation

$$\overline{\{\eta > 0\}}^* := \{\eta > 0\}^* \cup \partial^*\{\eta > 0\}$$

to get

$$\int_\Omega d|D(u-\eta_+)| = \int_{\{\eta>0\}^*} |D\phi|\, dx + \int_{\partial^*\{\eta>0\}} |\phi - u^-|\, d\mathcal{H}^{n-1} + \int_{\Omega\setminus\overline{\{\eta>0\}}^*} d|Du|$$

Consequently, (2.20) implies (recall $\phi - u^- \ge 0$)

$$\begin{aligned} 0 \quad \le \quad & \int_{\{\eta>0\}^*} |D\phi|\, dx + \int_{\partial^*\{\eta>0\}} \phi - u^-\, d\mathcal{H}^{n-1} + \int_{\Omega\setminus\overline{\{\eta>0\}}^*} d|Du| \\ & + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} - \lambda \int_{\{\eta>0\}} |\phi|\, dx - \lambda \int_{\Omega\setminus\{\eta>0\}} |u|\, dx\,. \end{aligned} \tag{2.21}$$

Adding inequalities (2.19) and (2.21), we obtain

$$\begin{aligned} 0 \quad < \quad & \int_{\{\eta>0\}^*} d|Du| + \int_{\partial^*\{\eta>0\}} u^+ - u^-\, d\mathcal{H}^{n-1} \\ & - \lambda \int_{\{\eta>0\}} su\, dx + \lambda \int_{\{\eta>0\}} s\phi - |\phi|\, dx \\ & + \int_{\Omega\setminus\overline{\{\eta>0\}}^*} d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} - \lambda \int_{\Omega\setminus\{\eta>0\}} |u|\, dx\,. \end{aligned}$$

Recalling that

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} = \lambda \int_\Omega |u|\, dx \tag{2.22}$$

and using that (notice that $u^+ \ge \phi \ge u^-$)

$$\int_{\overline{\{\eta>0\}}^*} d|Du| = \int_{\{\eta>0\}^*} d|Du| + \int_{\partial^*\{\eta>0\}} u^+ - u^-\, d\mathcal{H}^{n-1}\,.$$

we conclude that

$$s \int_{\{\eta>0\}} u - \phi\, dx < \int_{\{\eta>0\}} |u| - |\phi|\, dx\,. \tag{2.23}$$

Now we distinguish two cases. If $\varphi(x_0) < 0$ we have $s = -1$ and, by (2.9), $\phi < u^* < 0$ on $\{\eta > 0\}$. Thus we can put $s\phi = |\phi|$ and $su = |u|$ in the previous inequality and get a contradiction. In the case $\varphi(x_0) \ge 0$ we have $s = 1$. Using the triangle inequality and $\eta = |\eta|$ in $\{\eta > 0\}$, we obtain from the inequality above

$$\begin{aligned} & \int_{\{\eta>0\}} |u-\phi|\, dx + \int_{\{\eta>0\}} |\phi|\, dx \\ & \qquad < \quad \int_{\{\eta>0\}} |u|\, dx \;\le\; \int_{\{\eta>0\}} |u-\phi|\, dx + \int |\phi|\, dx \end{aligned}$$

which is a contradiction. Hence (2.7) cannot be true for some φ with $D\varphi(x_0) \neq 0$.

(1b) Now we study the case $D\varphi(x_0)=0$. From (2.5) and (2.7) we get

$$F_*(\varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) = \infty.$$

Since F_* is lsc,

$$F_*(v,q,X) > 0 \text{ for } (v,q,X) \text{ near } (\varphi(x_0), 0, D^2\varphi(x_0)).$$

Let us consider

$$\psi(x) := \varphi(x_0) + (x-x_0)\cdot D^2\varphi(x_0)\cdot(x-x_0) + \varepsilon|x-x_0|^2, \quad \varepsilon > 0.$$

For $\varepsilon > 0$ and $r > 0$ sufficiently small we have $\psi(x) > \varphi(x)$ on $B_r(x_0)\setminus\{x_0\}$ and, by the C^2-regularity of ψ,

$$F_*(\psi(x), D\psi(x), D^2\psi(x)) > 0 \text{ on } B_r(x_0).$$

Moreover, possibly by an arbitrarily small change of $\varepsilon > 0$, we reach $D\psi(x) \neq 0$ for $x \neq x_0$ and, thus,

$$-\Delta_1\psi(x) - s\lambda > 0 \text{ on } B_r(x_0)\setminus\{x_0\} \tag{2.24}$$

with s as in (2.10).

Now we can argue almost the same way as in part (1a). Instead of (2.16) we have

$$-\int_{B_r(x_0)\setminus\overline{B_\delta(x_0)}} \eta_+ \operatorname{div}\left(\frac{D\phi}{|D\phi|}\right) + s\lambda\eta_+\,dx > 0 \tag{2.25}$$

for some small $\delta > 0$. Partial integration gives

$$\begin{aligned} 0 \;<\; & \int_{B_r(x_0)\setminus\overline{B_\delta(x_0)}} \frac{D\phi}{|D\phi|} d(D\eta_+) - \int_{B_r(x_0)} s\lambda\eta_+\,dx \\ & - \int_{\partial B_\delta(x_0)} \frac{D\phi}{|D\phi|}\cdot\nu\,\eta_+\,d\mathcal{H}^{n-1} \end{aligned} \tag{2.26}$$

with the outer unit normal ν of $B_\delta(x_0)$. Since u^* is bounded and thus also η, the last integral in (2.26) tends to zero for $\delta \to 0$ (here we use that $n \geq 2$). By (2.24), the left hand side in (2.25) increases if $\delta \to 0$. Thus, also the right hand side in (2.26) increases if $\delta \to 0$ and, since this right hand side is bounded, the limit exists. In particular we have strict inequality in the limit in (2.26) and obtain

$$0 < \int_{B_r(x_0)\setminus\{x_0\}} \frac{D\phi}{|D\phi|} d(D\eta_+) - \int_{B_r(x_0)} s\lambda\eta_+\,dx.$$

Since the set $\{x_0\}$ is $\mathcal{H}^{n-1}$-negligible (for $n \geq 2$), the variation measure $|D\eta_+|$ vanishes on $\{x_0\}$. Thus we arrive at (2.17) and can continue exactly as in part (1a). Therefore we get a contradiction also in the case $D\varphi(x_0) = 0$. Consequently (2.7) cannot be true and u^* has to be a viscosity subsolution of the differential equation in (1.1).

(2) Next we take a point $x_0 \in \partial\Omega$. If $u^*(x_0) \leq 0$, then the subsolution condition in Definition 2.4 is satisfied. If $u^*(x_0) > 0$, then we have to show that

$$F_*(\varphi(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

for any $\varphi \in C^2$ touching u^* from above at x_0. For that we identify u^* with its extension by zero on a big open ball $B \subset \mathbb{R}^n$ containing $\bar\Omega$. Then we essentially argue as in part (1) with B instead of Ω. Notice that we work with B instead of $\mathbb{R}^n$ to ensure that ϕ (that we can choose to be in $C^2(\bar{B})$) belongs to $BV(B)$. In the regular case (1a) we choose $\tilde{\gamma} > 0$ so small in (2.13) that $\phi(x) > 0$ on $B_r(x_0)$. This ensures that $\eta_+ = 0$ outside Ω and, thus, the minimizing argument in (2.20) works and now gives

$$0 \le \int_B d|D(u-\eta_+)| - \lambda \int_B |u - \eta_+|\, dx\,.$$

Consequently the integral on $\partial\Omega$ disappears in (2.21). Since it also disappears in (2.22), which becomes $\int_B d|Du| = \lambda \int_B |u|\, dx$, we arrive at (2.23) also in the boundary case. Then we merely have to consider the case $\varphi(x_0) > 0$ and get a contradiction exactly as in (1a). For the singular case we can use (1b) without further adaption. Thus the boundary condition is satisfied in the viscosity sense and the proof is complete. □

Remark: If one considers eigenfunctions of the p-Laplacian for $p \in (1, \infty)$, the corresponding Euler equation $\Delta_p u + \lambda |u|^{p-2} u = 0$ is more benign than for $p = 1$, because the corresponding operator

$$F_p(u, q, X) = -|q|^{p-2} \left(\delta_{ij} + (p-2) \frac{q_i q_j}{|q|^2} \right) X_{ij} - \lambda |u|^{p-2} u$$

is continuous both at $u = 0$ and $q = 0$. In [9], Lemma 1.8, it was shown that the variational first eigenfunction to the p-Laplace operator is always a viscosity solution. Compared to our Theorem 2.5 the proof could start with continuous solutions of a continuous equation. The question if every viscosity solution is also a variational solution was not adressed there.

3 Geometric interpretation of the requirements for viscosity solutions

In this section we present several examples that are viscosity solutions of (1.1). Here we also discuss eigenfunctions that are already covered by Theorem 2.5, but we give more direct arguments that they are viscosity solutions. These arguments then allow us to construct further viscosity solutions of (1.1) that are not eigenfunctions of the 1-Laplacian. In particular we show that there are functions $u \in BV(\Omega)$ that are viscosity solutions for different parameters $\lambda \in \mathbb{R}$. Finally we consider the remarkable case $n = 1$ where Theorem 2.5 fails.

Example 1. We consider the square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. In [10, p. 1935] we find the Cheeger constant of Ω, that equals the first eigenvalue of the 1-Laplacian, to be

$$\lambda = 2 + \sqrt{\pi}\,.$$

The Cheeger set C is the union of all balls in Ω having radius $1/\lambda \approx 0.265$. To verify that $u = \chi_C$ is a viscosity solution of (1.1) we have to check Definitions 2.2 and 2.3.

Subsolution case: Let us first take test functions φ touching u^* in x from above. If $x \in \operatorname{int} C$, then $D\varphi(x) = 0$ and $D^2\varphi(x) \ge 0$ so that the second condition in (2.2) is satisfied. If $x \in \Omega \setminus \bar{C}$, then the fourth condition in (2.2) is satisfied. If $x \in \partial C \cap \Omega$, then it is possible that $D\varphi(x) = 0$ or $D\varphi(x) \ne 0$. In the first case $\mu_2(D^2\varphi(x)) \ge 0$ and the second condition in (2.2) is satisfied. In the second case the first condition in (2.2) is satisfied, because the level curve $\{\varphi = 1\}$ has smaller curvature in x than ∂C. It remains to discuss points $x \in \partial\Omega$. If $x \in \partial C \cap \partial\Omega$, then we argue as in the previous case. If $x \in \partial\Omega \setminus \partial C$, then the Dirichlet condition is satisfied in the classical sense. Thus u^* is a viscosity subsolution of (1.1) under Dirichlet boundary conditions.

Supersolution case: Now φ should be a test function touching u_* from below. We argue analogously up to $x \in \partial C \cap \Omega$. If $D\varphi(x) = 0$, then $\mu_1(D^2\varphi(x)) \leq 0$ and the fourth condition of (2.3) holds. If $D\varphi(x) \neq 0$, then we have that the level curve $\{\varphi = 0\}$ has curvature greater than $-\lambda$ and the third condition in (2.3) is satisfied. Finally u_* is viscosity supersolution of (1.1) under Dirichlet boundary conditions.

Consequently u and, analogously, also $-u$ are viscosity solution of (1.1).

By inspection of all the arguments we realize that u is also viscosity solution of (1.1) for any $\tilde{\lambda} > \lambda$.

Example 2. We again consider a square $\Omega = (-1/2, 1/2) \times (0, 1) \subset I\!R^2$. But now let us discuss the conjectured second eigenfunction u_2. With the Cheeger sets $C^\pm$ of $\Omega^\pm = \{x \in \Omega | \pm x_1 > 0\}$ numerical simulations [8] suggest that

$$u_2 = \chi_{C^+} - \chi_{C^-}$$

should be a second eigenfunction of the 1-Laplacian under Dirichlet boundary conditions. Here the eigenvalue λ_2 equals the curvature of the circular arcs bounding $C^\pm$. To check that these are viscosity solutions we can argue as in Example 1) up to $x \in \partial C^+ \cap \partial C^-$. But for such x we can argue similarly as for $x \in \partial C \cap \Omega$ in Example 1. Therefore u_2 is viscosity solution of (1.1) with $\lambda = \lambda_2$ under Dirichlet boundary conditions.

Example 3. Suppose that $\Omega = (0, 1) \times (0, 1) \subset I\!R^2$ and that $\tilde{C} \subset \Omega$ is a Cheeger type set, i.e. it is the union of discs in Ω with radius $\tilde{r} > 1/\lambda_1$ where λ_1 is the first eigenvalue of the 1-Laplacian. Arguments as in Example 1 show that $\tilde{u} = \chi_{\tilde{C}}$ is viscosity solution of (1.1) under Dirichlet boundary conditions for any $\lambda \geq \tilde{\kappa} := 1/\tilde{r}$.

Moreover it turns out that $\tilde{u}$ is also a weak solution of (1.4) under Dirichlet boundary conditions with $\tilde{\lambda} := |\partial\tilde{C}|/|\tilde{C}| > \lambda_1$. To see that we first determine the Cheeger set of $\tilde{C}$, which is the union C_r of all discs in $\tilde{C}$ with some radius r. The case $r > \tilde{r}$ can be excluded, since a simple calculation shows that $r \to |\partial C_r|/|C_r|$ is decreasing for relevant $r > r_1 := 1/\lambda_1$. Hence $r \leq \tilde{r}$ and $C_r = \tilde{C}$, i.e. $\tilde{C}$ coincides with its Cheeger set. But notice that the Cheeger constant $\tilde{\lambda} = |\partial\tilde{C}|/|\tilde{C}| > \lambda_1$, that $\tilde{\kappa} < \lambda_1$, and thus $\tilde{\kappa} \neq \tilde{\lambda}$ for the curvature $\tilde{\kappa}$ of the arcs of $\partial\tilde{C}$. Using [7, Corollary A.1] we find a solution of the mean curvature equation

$$\operatorname{div} \frac{Dw}{\sqrt{1 + |Dw|^2}} = \tilde{\lambda} \qquad \text{on } \tilde{C} \tag{3.1}$$

such that Dw agrees with the outer unit normal of $\tilde{C}$ on $\partial\tilde{C}$. We set

$$z(x) := \begin{cases} \frac{Dw(x)}{\sqrt{1+|dw|^2}} & \text{on } \tilde{C}\,, \\ -D\,d_{\tilde{C}}(x) & \text{on } \Omega \setminus \tilde{C}\,, \end{cases} \tag{3.2}$$

(where $D\,d_{\tilde{C}}$ denotes the gradient of the distance function $d_{\tilde{C}}$) and obtain (as in [14]) a solution of (1.4) with $\tilde{\lambda}$ for some suitable s.

Example 4. Let $\Omega \subset I\!R^2$ be an L-shape. E. Parini has shown that there exist special L-shapes with a continuum of non-unique Cheeger sets. For the readers's convenience we recall how they were constructed. Suppose that the L-shaped domain has length 2 on the left and bottom side and that the horizontal part has width 1 while the vertical part has width $a < 0.5$ with $a = 0.35599$.

Then $\lambda = 1/a$ happens to be the Cheeger constant of Ω and the boundary of the Cheeger set contains a half circle. Moving that half circle up and down does not change the Cheeger ratio $|\partial C|/|C|$ (cf. Figure 1).

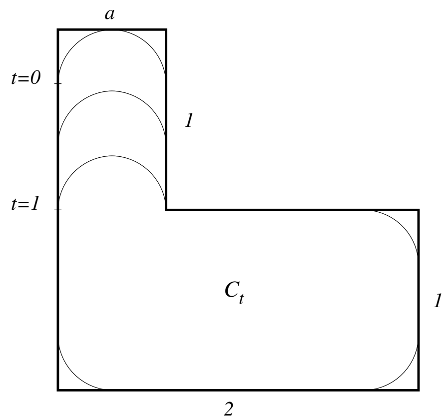

Figure 1: *The figure shows an L-shape Ω with a continuum of Cheeger sets C_t for $t \in [0,1]$. Here C_t is bounded by the quarter circles and by the half-circle touching the left boundary of Ω at level t. Thus C_0 is the largest and C_1 the smallest Cheeger set of Ω.*

We consider the function

$$u(x) = \begin{cases} 1 & \text{for } x \in \bar{C}_1 \\ \alpha(t) & \text{for } x \in \partial C_t \setminus \partial C_0,\ 0 < t < 1 \\ 0 & \text{for } x \in \Omega \setminus C_0 \end{cases}$$

where $\alpha : (0,1) \mapsto (0,1)$ is continuously differentiable and increasing. Notice that $u \in C^1(C_0)$ and that any such u is a first eigenfunction of the 1-Laplacian. By the same arguments as in the previous examples we see that any such u is a viscosity solution of (1.1). Now suppose that $\alpha' > 0$ in $(0,1)$. Then for $x \in \Omega$ with $u(x) \in (0,1)$ the gradient $Du(x) \neq 0$ and the first condition in (2.6) has to be used in the definition of supersolution. It tells us that the curvature of ∂C_t is bounded below by λ. Because u is a subsolution, the curvature of ∂C_t also must be bounded above by λ. Thus, in contrast to our previous examples where the definition of viscosity solution implied only that the curvature is bounded from below by $-\lambda$ and from above by λ, we now get that in fact it has to be equal to λ.

Example 5. Let us consider $\Omega \subset I\!R^n$ with a ball $\bar{B} \subset \Omega$ of radius $r > 0$. In [18] it is shown that $u = \chi_B$ is weak solution of (1.4) with $\lambda = n/r$ and it is also a viscosity solution of (1.1). To see that the interesting points are $x \in \partial B$. Any test function φ touching u^* from above at x with $D\varphi(x) \neq 0$ has a level surface $\{\varphi = 1\}$ with mean curvature less or equal to λ as required by the first condition in (2.2). If $D\varphi(x) = 0$, then $D^2\varphi(x) \geq 0$, because for any $y \in B$

$$0 \leq \varphi(y) - \varphi(x) = (y-x) \cdot D^2\varphi(x) \cdot (y-x) + o(\|y-x\|^2).$$

This in particular implies that $t \cdot D^2\varphi(x) \cdot t \geq 0$ for all tangential directions on ∂B. Thus Definition 2.2 is satisfied with the second line in (2.5). With obvious modifications we get that u_* is a viscosity supersolution. We see that u is a viscosity solution of (1.1) for all $\lambda \geq n/r$.

Example 6. Let us consider $\Omega = (0,1) \subset I\!R^1$. A comprehensive discussion of eigensolutions of the 1-Laplacian in that case can be found in [18] where eigenfunctions are identified as step functions that, except for the first eigenfunction, change sign. Notice that on the set $\{D\varphi \neq 0\}$ always $\Delta_1 \varphi = 0$ by (2.3). Therefore the condition for a viscosity subsolution boils down to

$$-\lambda \frac{\varphi}{|\varphi|} \leq 0 \text{ if } \varphi \neq 0 \text{ and } -\lambda \leq 0 \text{ if } \varphi = 0$$

and for a viscosity supersolution we have

$$-\lambda \frac{\varphi}{|\varphi|} \geq 0 \text{ if } \varphi \neq 0 \text{ and } \lambda \geq 0 \text{ if } \varphi = 0.$$

We readily see that any continuous function $u \geq 0$ is a viscosity subsolution but it is a supersolution only for $u \equiv 0$. Vice versa, any continuous function $u \leq 0$ is a viscosity supersolution but it is a subsolution only for $u \equiv 0$. It is remarkable that only $u \equiv 0$ is a viscosity solution of (1.1), and then for any $\lambda \geq 0$. Notice that in this case no eigenfunction is a viscosity solution and that all higher eigenfunctions are neither viscosity subsolution nor supersolution.

Acknowledgement: This paper was begun at the Oberwolfach workshop "The p-Laplacian Operator and Application" in Feb. 2013 and continued at the special semester on "Nonlinear Evolution Equations" in the fall of 2013 at the Mittag-Leffler-Institute. We thank both institutions for providing excellent research conditions and P. Lindqvist for a helpful discussion. Moreover we are grateful to DFG for supporting the cooperation of the authors within the project "Variational problems related to the 1-Laplace operator".

References

[1] Ambrosio, L., Fusco, N. & Pallara, D., *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, Oxford, 2000

[2] Anzellotti, G. Pairings between measures and bounded functions and compensated compactness. Ann. Mat. Pura Appl. (4) **135** (1983) pp. 193-318.

[3] Crandall, M., Ishii, H. & Lions, P.L. User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. **27** (1992) pp. 1–67.

[4] Degiovanni, M., Margone, P., Linking solutions for quasilinear equations at critical growth involving the 1-Laplace operator. Calc. Var. **36** (2009) pp. 591-609.

[5] Evans, L.C. & Gariepy, R.F., *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992

[6] Gilbarg, D. & Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order.* 2nd ed. Springer-Verlag, Heidelberg, 1983

[7] Giusti, E. On the equation of surfaces of prescribed mean curvature, existence and uniqueness without boundary conditions. Invent. Math. **46** (1978) pp. 111-137.

[8] Horák, J. Numerical investigation of the smallest eigenvalues of the p-Laplace operator on planar domains. Electron. J. Differential Equations (2011), No. 132, 30 pp.

[9] Juutinen, P., Lindqvist, P. & Manfredi, J.J., The ∞-Eigenvalue Problem. Arch Ration Mech. Anal. **148** (1999) pp. 89–105.

[10] Kawohl, B. & Kutev, N., Global behaviour of solutions to a parabolic mean curvature equation. Diff. int. Eqs. **8** (1995) pp. 1923–1946.

[11] Kawohl, B. & Kutev, N., Viscosity solutions for degenerate and nonmonotone elliptic equations, in: *Applied Nonlinear Analysis*, ed.: B. da Vega, A.Sequeira, J.Videman, Plenum Press, New York u. London (1999), pp. 185–210.

[12] Kawohl, B. & Fridman, V., Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant. Comment. Math. Univ. Carolinae **44** (2003) pp. 659-667.

[13] Kawohl, B. & Lachand-Robert, Th., Characterization of Cheeger sets for convex subsets of the plane, Pacific J. Math. **225** (2006) pp. 103–118.

[14] Kawohl, B. & Schuricht, F., Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, Communications in Contemporary Mathematics **9** (2007) pp. 515-543.

[15] Kawohl, B., Variations on the p-Laplacian, in: Nonlinear Elliptic Partial Differential Equations, Eds. D. Bonheure, P. Takač et al., Contemporary Mathematics **540** (2011) pp. 35–46.

[16] Milbers, Z. & Schuricht, F. Existence of a sequence of eigensolutions for the 1-Laplace operator. J. Lond. Math. Soc. **82** (2010) pp. 74–88.

[17] Milbers, Z. & Schuricht, F. Some special aspects related to the 1-Laplace operator. Adv. Calc. Var. **4** (2011) pp. 101–126.

[18] Milbers, Z. & Schuricht, F. Necessary condition for eigensolutions of the 1-Laplace operator by means of inner variations. Math. Ann. **356** (2013) pp. 147–177.

[19] Parini, E., An Introduction to the Cheeger problem. Surveys in Mathematics and its Applications **6** (2011) pp. 9–22.

Author's addresses:

Bernd Kawohl, Mathematisches Institut, Universität zu Köln, 50923 Köln, Germany, kawohl@mi.uni-koeln.de

Friedemann Schuricht, Fachrichtung Mathematik, Technische Universitat Dresden, 01062 Dresden, Germany, friedemann.schuricht@tu-dresden.de