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VERY WEAK SOLUTIONS OF SINGULAR POROUS MEDIUM EQUATIONS WITH MEASURE DATA

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ABSTRACT. We consider non-homogeneous, singular ($0 < m < 1$) porous medium type equations with a non-negative Radon-measure μ having finite total mass $\mu(E_T)$ on the right-hand side. We deal with a Cauchy-Dirichlet problem for these type of equations, with homogeneous boundary conditions on the parabolic boundary of the domain E_T, and we establish the existence of a solution in the sense of distributions. Finally, we show that the constructed solution satisfies linear pointwise estimates via linear Riesz potentials.

1. INTRODUCTION

In this paper we consider non-homogeneous porous medium type equations and the associated Cauchy-Dirichlet problems in a space-time cylinder $E_T := E \times (0, T)$, where $E \subset \mathbb{R}^n$ is a bounded open domain, $n \geq 2$, and $T > 0$. The equations are of the type

$$u_t - D_j\big[\mathbf{a}_{ij}(x,t,u)D_i u^m\big] = \mu \qquad \text{in } E_T, \tag{1.1}$$

where μ is a Radon-measure in E_T with finite total mass $\mu(E_T) < \infty$. Without loss of generality, we assume that the measure is defined on $\mathbb{R}^{n+1}$ by letting $\mu\lfloor(\mathbb{R}^{n+1} \setminus E_T) = 0$. For the coefficients $\mathbf{a}_{ij}\colon E_T \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $1 \leq i,j \leq n$, we assume measurability with respect to $(x,t) \in E_T$ for all $u \in \mathbb{R}$, and continuity with respect to u for a.e. $(x,t) \in E_T$. Moreover, we assume the following growth and ellipticity conditions

$$\mathbf{a}_{ij}(x,t,u)\xi_i\xi_j \geq C_o|\xi|^2 \qquad \text{and} \qquad \mathbf{a}_{ij}(x,t,u)\xi_i\eta_j \leq C_1|\xi||\eta|, \tag{1.2}$$

whenever $(x,t) \in E_T$, $u \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$, for some $0 < C_o \leq C_1 < \infty$. The most prominent example for equations treated in the sequel is given by the classical *singular* (i.e. with $0 < m < 1$) porous medium equation (also called *fast diffusion equation*), in which case $\mathbf{a}_{ij}(x,t,u) \equiv \mathbf{a}(x,t,u)\delta_{ij}$ and therefore

$$u_t - \operatorname{div}\big(\mathbf{a}(x,t,u)Du^m\big) = \mu \qquad \text{in } E_T. \tag{1.3}$$

Throughout the paper we deal only with the *singular case* $0 < m < 1$. The main aim of this paper is to establish the existence of a solution of the Cauchy-Dirichlet problem

$$\begin{cases} u_t - D_j\big[\mathbf{a}_{ij}(x,t,u)D_i u^m\big] = \mu, & \text{in } E_T, \\ u = 0 & \text{on } \partial_{\mathrm{par}} E_T, \end{cases} \tag{1.4}$$

in the sense of distributions, when μ is a bounded non-negative Radon-measure. In this context we use the notion of very weak solutions, which is defined as follows:

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Definition 1.1 (very weak solution)**.** A non-negative function $u \in L^1(E_T)$ satisfying

$$u^m \in L^1\big(0,T; W_0^{1,1}(E)\big)$$

is termed *very weak solution* of the Cauchy-Dirichlet problem (1.4) for the singular porous medium type equation, if and only if the following identity

$$\iint_{E_T} \big[-u\varphi_t + \mathbf{a}_{ij}\big(x,t,u\big)D_i u^m D_j\varphi\big]\,dxdt = \int_{E_T} \varphi\,d\mu \tag{1.5}$$

holds true for any testing function $\varphi \in C^\infty(\overline{E_T})$ vanishing in a neighborhood of $[\partial E \times (0,T)] \cup [\overline{E} \times \{T\}]$. □

In terms of this notion we have the following existence result for very weak solutions.

Theorem 1.1 (Existence of very weak solutions)**.** *Assume that hypotheses* (1.2) *hold with*

$$\frac{(n-2)_+}{n} < m < 1, \tag{1.6}$$

and μ is a non-negative Radon-measure in E_T. Then, there exists at least one non-negative very weak solution u of (1.4) *satisfying*

$$u^m \in L^p(E_T) \cap L^q\big(0,T; W_0^{1,q}(E)\big) \tag{1.7}$$

for any p and q such that

$$1 \le p < 1 + \frac{2}{nm} \quad \textit{and} \quad 1 \le q < 1 + \frac{1}{1+nm}\,. \tag{1.8}$$

Note that in the case $m = 1$, we recover the existence result for measure data problems for non-linear parabolic equations with linear growth of the coefficients in the gradient variable from [3, Theorem 1.2]. Indeed, with the choices $r = q$ and $p = 2$ the conditions [3, (1.7), (1.8)] reduce to the hypothesis $1 \le q < 1 + \frac{1}{n+1}$, which is exactly condition (1.8) from above. The very weak solution constructed in Theorem 1.1 is a *Solution Obtained by Limit of Approximations* (SOLA) to (1.4). The approximating sequence is given by weak energy solutions

$$u_\ell \in C^0\big([0,T]; L^{1+m}(E)\big), \quad u_\ell^m \in L^2\big(0,T; W_0^{1,2}(E)\big) \tag{1.9}$$

of Cauchy-Dirichlet problems

$$\begin{cases} \partial_t u_\ell - D_j\big[\mathbf{a}_{ij}(x,t,u_\ell)D_i u_\ell^m\big] = \mu_\ell, & \text{in } E_T, \\ u_\ell = 0 & \text{on } \partial_{\mathrm{par}} E_T, \end{cases} \tag{1.10}$$

where $\mu_\ell \in L^\infty(E_T)$ is a regularization of μ such that $\mu_\ell\,dxdt \to \mu$ in the limit $\ell \to \infty$ in the sense of Radon-measures, and $\mu_\ell(E_T) \le \mu(E_T)$ for $\ell \ge 1$. The precise definition of a local weak energy solution is as follows:

Definition 1.2 (Weak energy solution)**.** A non-negative function $u\colon E_T \to \mathbb{R}$ satisfying

$$u \in C^0_{\mathrm{loc}}\big(0,T; L^{1+m}_{\mathrm{loc}}(E)\big), \quad u^m \in L^2_{\mathrm{loc}}\big(0,T; W^{1,2}_{\mathrm{loc}}(E)\big)$$

is termed *local weak energy solution of the porous medium type equation* (1.1) if and only if for every subset $U \Subset E$ and every subinterval $[t_1,t_2] \subset (0,T)$ the following equation

$$\int_U u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_U \big[-u\varphi_t + \mathbf{a}_{ij}(x,t,u)D_i u^m D_j\varphi\big]\,dxdt = \int_{U\times(t_1,t_2)} \varphi\,d\mu \tag{1.11}$$

holds true for any testing function

$$\varphi \in L^2_{\mathrm{loc}}\big(0,T; W_0^{1,2}(U)\big) \quad \text{with} \quad \varphi, \varphi_t \in L^\infty(E_T).$$

Moreover, we say that u satisfying (1.9) is a *global weak energy solution of the Cauchy-Dirichlet problem* (1.10) if and only if

$$\iint_{E_T} \big[-u\varphi_t + \mathbf{a}_{ij}(x,t,u)D_i u^m D_j\varphi \big]\,dxdt = \int_{E_T} \varphi\,d\mu \tag{1.12}$$

holds true for any testing function

$$\varphi \in L^2\big(0,T;W^{1,2}(E)\big) \quad \text{with} \quad \varphi, \varphi_t \in L^\infty(E_T),$$

vanishing in a neighbourhood of $\overline{E} \times \{T\} \cup \partial\Omega \times (0,T)$. □

Although in § 4 the approximating sequence will be built assuming $\mu_\ell \in L^\infty(E_T)$, without loss of generality the definition of weak energy solutions can be given assuming $\mu \in \mathcal{M}(E_T)$ as in (1.11). The assumption that the testing function φ and its time derivative φ_t must be bounded has to be imposed to guarantee that the terms involving the time derivative and the right-hand side of (1.11) are well defined. All other integrals appearing there are finite, due to the other assumptions on u and φ. The existence of weak energy solutions in the sense of Definition 1.2 with the right-hand side $\mu \in L^\infty(E_T)$ follows for example from [1]; see Proposition 4.1 below.

The construction of very weak solutions to measure data problems is a well established procedure; we refer to [2, 3, 4, 5, 12, 14] for the existence of very weak solutions of measure data problems related to elliptic and parabolic p-Laplacian equations. Measure data problems for the porous medium equation with $\mathbf{a}(x,t) \equiv 1$ in (1.3) have been studied in [21, 22] in the degenerate and singular case, respectively. General coefficients have been considered in [6] only for the degenerate case $m > 1$. Finally, in [13] a somewhat related existence problem for inhomogeneous porous medium equations of the type

$$u_t - \operatorname{div}(\mathbf{a}(x,t,u)Du) = \mathbf{b}(x,t,u,Du) \quad \text{in } E_T$$

has been investigated. To our knowledge, up to now, the general case of multiplicative coefficients $\mathbf{a}(x,t,u)Du^m$ with a behavior as described in (1.2) has not been studied in the singular case $m < 1$.

To formulate the next main result, we need to define the localized (or truncated) parabolic Riesz-potential by

$$\mathbf{I}^\mu_\beta(z_o,r,\theta) := \int_0^r \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2}(z_o))}{\varrho^{n+2-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad \beta \in (0, n+2]\,,$$

for $z_o \in E_T$ and $r,\theta > 0$ such that $Q_{r,\theta}(z_o) \in E_T$. Here, $Q_{\varrho,\varrho^2\theta/r^2}(z_o)$ stands for a general parabolic cylinder in E_T, see § 2.1. In the standard case when $\theta = r^2$, the potential $\mathbf{I}^\mu_\beta$ reduces to the standard localized parabolic Riesz potential. By a further truncation procedure in the approximation procedure described above we obtain the following pointwise bound for very weak solutions.

Theorem 1.2 (Potential estimates for general measure data problems)**.** *Suppose that u is the very weak solution built in Theorem 1.1, with m satisfying (1.6). Then, for any given $\lambda \in \big(0, \frac{\kappa}{n(1+m)}\big]$, where $\kappa := 2 - n(1-m)$, almost every $z_o \in E_T$, and every parabolic cylinder $Q_{4r,\theta}(z_o) \subset E_T$, the following potential estimate*

$$u(z_o) \le \boldsymbol{\gamma}\bigg(\frac{1}{\theta^{1+\frac{N}{2}}}\iint_{Q_{r,\theta}} u^{1+\lambda}\,dxdt\bigg)^{\frac{2}{\kappa+2\lambda}} + \boldsymbol{\gamma}\Big(\frac{\theta}{r^2}\Big)^{\frac{1}{1-m}} + \boldsymbol{\gamma}\left(\frac{r^2}{\theta}\right)^{\frac{n}{\kappa}}\mathbf{I}_2(z_o,4r,\theta)^{\frac{2}{\kappa}}$$

holds true with a universal constant $\boldsymbol{\gamma}$ depending only on n, m, C_o, C_1 and λ.

Indeed, Theorem 1.2 continues to hold true for any very weak solution u satisfying the additional regularity requirement that $\min\{u,k\}^m \in L^2(0,T;W^{1,2}(E))$ for any $k>0$.

It would be nice, if we could test our results against the *Barenblatt fundamental solution*

$$\mathcal{B}_m(x,t) := \begin{cases} \frac{1}{t^{\frac{n}{\kappa}}} \frac{1}{\left[1+b\left(\frac{|x|}{t^{1/\kappa}}\right)^2\right]^{\frac{1}{1-m}}}, & t>0 \\ 0 & t \le 0 \end{cases}$$

where $\frac{n-2}{n} < m < 1$ and

$$\kappa = n(m-1)+2, \quad \text{and} \quad b = b(n,m) = \frac{n(1-m)}{2nm\kappa}\,.$$

Unfortunately, as it is well known, the Barenblatt fundamental solution is the very weak solution in $\mathbb{R}^n \times [-\varepsilon, +\infty)$ of the porous medium equation

$$\partial_t u - \Delta u^m = \delta_{(0,0)},$$

where $\varepsilon > 0$ and $\delta_{(0,0)}$ is the delta-function at the origin (see [10], and also [11, Chapter 3]). Therefore, it cannot be a solution to (1.4), which is stated in E_T, E being a bounded domain, with homogeneous conditions on the parabolic boundary.

Finally, for more regular data, we can improve in Theorem 1.1 the integrability properties of the very weak solution.

Theorem 1.3. *Assume that hypothesis* (1.2) *holds and that the Radon-measure is given by a non-negative function $\mu \in L^s(E_T)$ for some s in the range*

$$1 < s < 1 + \frac{nm}{nm+2+2m}.$$

Then, there exists at least one non-negative very weak solution u of the measure data problem (1.4) *satisfying*

$$u^m \in L^q\big(0,T;W_0^{1,q}(E)\big),$$

where q is given by

$$q = \frac{s(nm+2)}{nm+m+1-ms}\,.$$

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2. Preliminaries

2.1. **Notations.** For a point $z \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ we shall always write $z = (x,t)$. By $B_r(x_o) \equiv \{x \in \mathbb{R}^n : |x - x_o| < r\}$ we denote the open ball in $\mathbb{R}^n$ with center $x_o \in \mathbb{R}^n$ and radius $r > 0$. Moreover, we write

$$Q_{r,\theta}(z_o) := B_r(x_o) \times (t_o - \theta, t_o),$$

where $z_o = (x_o, t_o) \in \mathbb{R}^{n+1}$ and $r, \theta > 0$. Whenever writing $2Q$ for a cylinder $Q \equiv Q_{r,\theta}(z_o)$, we mean $2Q = Q_{2r,4\theta}(z_o)$. Finally, by $\mathcal{M}(E_T)$ we denote the set of all non-negative Radon-measures.

2.2. **Auxiliary lemmas.** The first result which will frequently be used is the Gagliardo-Nirenberg estimate, which for instance can be retrieved from [15, 16].

Lemma 2.1. *Let $Q_{\varrho,\theta}(z_o)$ be a parabolic cylinder with $0 < \varrho, \theta \le 1$ and $1 < p < \infty$, $0 < r < \infty$. Then, there exists a constant $\boldsymbol{\gamma}$ depending only on n, p, r such that for every*

$$u \in L^\infty\big(t_o-\theta, t_o; L^r(B_\varrho(x_o))\big) \cap L^p\big(t_o-\theta, t_o; W^{1,p}(B_\varrho(x_o))\big)$$

there holds

$$\iint_{Q_{\varrho,\theta}(z_o)} |u|^q \,dxdt \le \boldsymbol{\gamma}\bigg(\sup_{t\in(t_o-\theta,t_o)} \int_{B_\varrho(x_o)\times\{t\}} |u|^r \,dx\bigg)^{\frac{p}{n}} \iint_{Q_{\varrho,\theta}(z_o)} \Big|\frac{u}{\varrho}\Big|^p + |Du|^p \,dxdt\,,$$

where

$$q = \frac{p(n+r)}{n}\,.$$

The following Lemma, which can be found in [9, Corollary 3.8], may be interpreted as a technical tool yielding a certain weak coercivity for the naturally appearing quantity u^{m+1}. It will later be used to ensure that the weak energy solutions constructed in [1], which are of class $L^\infty(0,T;L^{m+1}(E))$ are actually in $C^0([0,T];L^{m+1}(E))$. The Lemma reads as follows:

Lemma 2.2. *For $m \in (0,1)$ let*

$$I(u,v) := \tfrac{1}{m+1}\big(u^{m+1} - v^{m+1}\big) - v^m(u-v). \tag{2.1}$$

For any two functions $u, v \in L^{m+1}(E)$ there holds

$$\int_E |u-v|^{m+1}\,dx \le c\int_E I(u,v)\,dx + c\bigg(\int_E I(u,v)\,dx\bigg)^{\frac{m+1}{2}}\bigg(\int_E v^{m+1}\,dx\bigg)^{\frac{1-m}{2}}$$

with a constant $c = c(m)$.

2.3. **Mollification in time.** For $v \in L^1(E_T)$ we define the following mollification in time

$$[\![v]\!]_h(\cdot,t) := \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} v(\cdot,s)\,ds, \qquad \text{for } h \in (0,T] \text{ and } t \in [0,T], \tag{2.2}$$

For the main properties of this mollification we refer to [19, Lemma 2.2], or [8, Appendix B]. If u is a weak energy solution of (1.10), then its time regularization $[\![u]\!]_h$ satisfies

$$\iint_{E_T} \partial_t [\![u]\!]_h \varphi + [\![\mathbf{a}_{ij}(x,\tau,u) D_i u^m]\!]_h D_j\varphi \,dxdt = \int_{E_T} [\![\varphi]\!]_{\bar h}\,d\mu, \tag{2.3}$$

for any $\varphi \in L^2(0,T;W^{1,2}(E)) \cap L^\infty(E_T)$ with compact support in E_T; cf. [6, §2.4].

2.4. **Auxiliary functions.** For $\sigma, k > 0$ we define the auxiliary function $F_{\sigma,k}\colon [0,k] \to \mathbb{R}_+$ by

$$F_{\sigma,k}(\tau) := \sigma \int_0^\tau \frac{ds}{\sigma + k^m - s^m}\,. \tag{2.4}$$

For the function $F_{\sigma,k}$ we have the following

Lemma 2.3. *The function $F_{\sigma,k}\colon [0,k] \to \mathbb{R}_+$ converges uniformly to 0 as $\sigma \downarrow 0$.*

Proof. We define $\beta := (\sigma + k^m)^{\frac{1}{m}}$. Then $\beta \ge \max\{\sigma^{\frac{1}{m}}, k\}$. Now, we first transform the integral as follows:

$$F_{\sigma,k}(\tau) = \frac{\sigma}{\beta^m} \int_0^\tau \frac{ds}{1 - \big(\frac{s}{\beta}\big)^m} = \sigma\beta^{1-m} \int_0^{\frac{\tau}{\beta}} \frac{ds}{1 - s^m} .$$

Using the fact that $\frac{1-s}{1-s^m} \le \frac{1}{m}$ for $s \in [0, 1)$, we can further estimate

$$F_{\sigma,k}(\tau) \le \frac{\sigma\beta^{1-m}}{m} \int_0^{\frac{\tau}{\beta}} \frac{ds}{1-s} = \frac{\sigma\beta^{1-m}}{m} \log\Big(\frac{\beta}{\beta-\tau}\Big) \le \frac{\sigma\beta^{1-m}}{m} \log\Big(\frac{\beta}{\beta-k}\Big) .$$

Now, we use the elementary estimate

$$\frac{\beta}{\beta-k} = \frac{(\sigma+k^m)^{\frac{1}{m}}}{(\sigma+k^m)^{\frac{1}{m}} - k} \le \frac{2^{\frac{1}{m}}k}{(\sigma+k^m)^{\frac{1}{m}} - k} \le \frac{2^{\frac{1}{m}}mk}{k^{1-m}\sigma} = \frac{2^{\frac{1}{m}}mk^m}{\sigma} ,$$

which holds true for $0 \le \sigma \le k^m$. Therefore, we can conclude that for $\sigma \in [0, k^m]$

$$F_{\sigma,k}(\tau) \le \frac{\sigma\beta^{1-m}}{m} \log\bigg(\frac{2^{\frac{1}{m}}mk^m}{\sigma}\bigg) ,$$

proving the claim. □

For $\lambda \in (0, 1)$ and $s \ge 0$ we define the following function, which appears in a natural way in § 4.2:

$$G_\lambda(s) := \int_0^s 1 - (1+\sigma)^{-\lambda}\, d\sigma \equiv s - \tfrac{1}{1-\lambda}\big((1+s)^{1-\lambda} - 1\big).$$

We state an auxiliary estimates for the function G_λ from [6, §2.2] which will be useful later on.

Lemma 2.4. *For any $\varepsilon \in (0, 1]$ and $s \ge 0$ there holds*

$$s \le \varepsilon + \boldsymbol{\gamma}_\varepsilon G_\lambda(s)$$

for a constant $\boldsymbol{\gamma}_\varepsilon \equiv \boldsymbol{\gamma}(\lambda)\varepsilon^{-1}$.

3. WEAK SUPER-SOLUTIONS

In this section we develop the theory of weak super-solutions of the singular porous medium equation, assuming the possibly minimal regularity. The first Lemma ensures that given a non-negative weak energy solution u, for a given $k > 0$, the truncated function $T_k(u) := \min\{u, k\}$ is a weak super-solution; in principle this can be retrieved from [15, Chapter 3, Lemma 5.1]. For sake of completeness we give the proof.

Lemma 3.1. *Let $u\colon E_T \to \mathbb{R}_+$ be a non-negative weak energy solution of* (1.10) *under the structure conditions* (1.2) *and with a non-negative right-hand side $\mu \in \mathcal{M}(E_T)$. Moreover, for $k > 0$ assume that the truncation $u_k := \min\{u, k\}$ satisfies $u_k^m \in L^2(0, T; W_0^{1,2}(E))$. Then, u_k is a weak energy super-solution, that is*

$$\iint_{E_T} \big[- u_k\partial_t\varphi + \mathbf{a}_{ij}(x, t, u_k)D_i u_k^m D_j\varphi\big]\, dxdt \ge 0$$

holds for any non-negative $\varphi \in C_0^1(E_T)$.

Proof. In order to overcome the missing regularity of u with respect to time, we use the time-regularization $[\![u]\!]_h$, introduced in § 2.3. In (2.3) we choose the testing function

$$\widetilde{\varphi} := \varphi\, f_{\sigma,k}(u) := \varphi\, \frac{k^m - T_k^m(u)}{\sigma + k^m - T_k^m(u)}\,,$$

where $\sigma > 0$, k is as above, and $\varphi \geq 0$ is a non-negative function in $C_0^1(E_T)$. Here, for values $s \geq 0$ we abbreviated

$$f_{\sigma,k}(s) := \frac{k^m - T_k^m(s)}{\sigma + k^m - T_k^m(s)}\,, \qquad \text{with } T_k(s) := \min\{u, k\}.$$

We note that $s \mapsto f_{\sigma,k}(s)$ is non-increasing and $f_{\sigma,k}(s) = 0$ for $s \geq k$. To proceed further, we consider the terms appearing in (2.3) one by one. We start with the term involving the time derivative. Taking the definition of the testing function into account, this term can be re-written as follows:

$$\begin{aligned}\iint_{E_T} &\partial_t [\![u]\!]_h \widetilde{\varphi}\, dxdt \\ &= \iint_{E_T} \varphi \partial_t [\![u]\!]_h f_{\sigma,k}([\![u]\!]_h)\, dxdt + \iint_{E_T} \varphi \partial_t [\![u]\!]_h \big(f_{\sigma,k}(u) - f_{\sigma,k}([\![u]\!]_h)\big)\, dxdt \\ &=: \mathrm{I}_h + \mathrm{II}_h\,.\end{aligned}$$

The term II_h is negative. This can be easily inferred, since $\partial_t [\![u]\!]_h = \frac{1}{h}(u - [\![u]\!]_h)$, and $f_{\sigma,k}$ is non-increasing. For the corresponding integrand this implies that

$$\partial_t [\![u]\!]_h \big(f_{\sigma,k}(u) - f_{\sigma,k}([\![u]\!]_h)\big) = \tfrac{1}{h}\big(u - [\![u]\!]_h\big)\big(f_{\sigma,k}(u) - f_{\sigma,k}([\![u]\!]_h)\big) \leq 0,$$

so that $\mathrm{II}_h \leq 0$. Next, we consider the term I_h, which we re-write as follows:

$$\begin{aligned}\iint_{E_T} &\varphi \partial_t [\![u]\!]_h f_{\sigma,k}([\![u]\!]_h)\, dxdt \\ &= \iint_{E_T} \mathbf{1}_{\{[\![u]\!]_h < k\}} \varphi \partial_t [\![u]\!]_h\, dxdt - \sigma \iint_{E_T} \frac{\mathbf{1}_{\{[\![u]\!]_h < k\}} \varphi \partial_t [\![u]\!]_h}{\sigma + k^m - [\![u]\!]_h^m}\, dxdt \\ &=: \mathrm{I}_h^{(1)} + \mathrm{I}_{h,\sigma}^{(2)}\,.\end{aligned}$$

Observing that $\mathbf{1}_{\{[\![u]\!]_h < k\}} \partial_t [\![u]\!]_h = \partial_t \min\{[\![u]\!]_h, k\} \equiv \partial_t T_k([\![u]\!]_h)$, after an integration by parts we obtain

$$\mathrm{I}_h^{(1)} = -\iint_{E_T} \varphi \partial_t T_k([\![u]\!]_h)\, dxdt = \iint_{E_T} \partial_t \varphi T_k([\![u]\!]_h)\, dxdt\,.$$

Since $u \in C^0\big([0,T]; L^{1+m}(E)\big)$, then $u \in L^\infty\big(0,T; L^1(E)\big)$, and we obtain that

$$\lim_{h \downarrow 0} \mathrm{I}_h^{(1)} = \iint_{E_T} \partial_t \varphi T_k(u)\, dxdt = -\iint_{E_T} u_k \partial_t \varphi\, dxdt$$

holds true. The treatment of the integral $\mathrm{I}_{h,\sigma}^{(2)}$ is more involved. The preceding definition allows us to re-write the integral $\mathrm{I}_{h,\sigma}^{(2)}$ in terms of the function $F_{\sigma,k}$ defined in (2.4). We obtain

$$\begin{aligned}\mathrm{I}_{h,\sigma}^{(2)} &= \sigma \iint_{E_T} \frac{\varphi\, \partial_t T_k([\![u]\!]_h)}{\sigma + k^m - T_k^m([\![u]\!]_h)}\, dxdt \\ &= \sigma \iint_{E_T} \varphi\, \partial_t \bigg[\int_0^{T_k([\![u]\!]_h)} \frac{ds}{\sigma + k^m - s^m}\bigg]\, dxdt\end{aligned}$$

$$= -\iint_{E_T} \partial_t\varphi F_{\sigma,k}\big(T_k([\![u]\!]_h)\big)\,dxdt\,.$$

In the preceding identity we pass to the limit $h \downarrow 0$, thereby obtaining

$$\lim_{h\downarrow 0} \mathrm{I}^{(2)}_{h,\sigma} = \iint_{E_T} \partial_t\varphi F_{\sigma,k}\big(T_k(u)\big)\,dxdt\,.$$

Using Lemma 2.3 we infer that

$$\lim_{\sigma\downarrow 0}\lim_{h\downarrow 0} \mathrm{I}^{(2)}_{h,\sigma} = 0\,.$$

Next, we turn our attention to the integral containing the diffusion term. For the sake of simplicity, we write u_k instead of $T_k(u)$. Here we first pass to the limit $h \downarrow 0$. We find that

$$\begin{aligned}
\lim_{h\downarrow 0} &\iint_{E_T} [\![\mathbf{a}_{ij}(x,t,u)D_iu^m]\!]_h D_j\widetilde{\varphi}\,dx \\
&= \iint_{E_T} \mathbf{a}_{ij}(x,t,u)D_iu^m\, D_j\bigg[\frac{\varphi(k^m-u_k^m)}{\sigma+k^m-u_k^m}\bigg]\,dxdt \\
&= \iint_{E_T} \mathbf{a}_{ij}(x,t,u)D_iu^m\, D_j\varphi\,\frac{k^m-u_k^m}{\sigma+k^m-u_k^m}\,dxdt \\
&\quad -\sigma\iint_{E_T} \varphi\,\mathbf{a}_{ij}(x,t,u)D_iu^m\,\frac{D_ju_k^m}{[\sigma+k^m-u_k^m]^2}\,dxdt \\
&=: \mathrm{II}_\sigma + \mathrm{III}_\sigma\,.
\end{aligned}$$

Taking into account the fact that $u_k = \min\{u,k\} = u$ on the set $\{u<k\}$, the integral II_σ can be re-written in the form

$$\begin{aligned}
\mathrm{II}_\sigma &= \iint_{E_T} \mathbf{1}_{\{u<k\}}\frac{k^m-u^m}{\sigma+k^m-u^m}\mathbf{a}_{ij}(x,t,u)D_iu^mD_j\varphi\,dxdt \\
&= \iint_{E_T} \frac{k^m-u^m}{\sigma+k^m-u^m}\mathbf{a}_{ij}(x,t,u_k)D_iu_k^mD_j\varphi\,dxdt\,.
\end{aligned}$$

Now, by the dominated convergence theorem we can pass to the limit $\sigma \downarrow 0$, to infer that

$$\lim_{\sigma\downarrow 0}\mathrm{II}_\sigma = \iint_{E_T} \mathbf{a}_{ij}(x,t,u_k)D_iu_k^mD_i\varphi\,dxdt.$$

Next, we treat the integral III_σ. Since

$$D_ju_k^m = \mathbf{1}_{\{u<k\}}D_ju^m\,,$$

the term can be re-written in the form

$$\mathrm{III}_\sigma = -\sigma\iint_{E_T} \mathbf{1}_{\{u<k\}}\varphi\,\frac{\mathbf{a}_{ij}(x,t,u)D_iu^mD_ju^m}{[\sigma+k^m-u^m]^2}\,dxdt \le 0\,,$$

where the last inequality follows by the ellipticity condition $(1.2)_1$. At this point, it remains to handle the right-hand side integral containing the non-negative $\mu \in \mathcal{M}(E_T)$. Here, we observe that

$$\lim_{h\downarrow 0}\int_{E_T} [\![\widetilde{\varphi}]\!]_h\,d\mu = \int_{E_T} \varphi\frac{k^m-u_k^m}{\sigma+k^m-u_k^m}\,d\mu = \int_{E_T} \mathbf{1}_{\{u<k\}}\varphi\frac{k^m-u^m}{\sigma+k^m-u^m}\,d\mu$$

holds true. In the preceding identity we pass to the limit $\sigma \downarrow 0$ and obtain that

$$\lim_{\sigma\downarrow 0}\lim_{h\downarrow 0}\int_{E_T} [\![\widetilde{\varphi}]\!]_{\bar h}\,d\mu = \int_{E_T} \mathbf{1}_{\{u<k\}}\varphi\,d\mu \ge 0\,.$$

Using the results obtained so far in (2.3) (this means that we insert the deduced identities in (2.3) and pass first to the limit $h \downarrow 0$ and then to the limit $\sigma \downarrow 0$), we arrive at

$$\iint_{E_T} \big[-u_k \partial_t \varphi + \mathbf{a}_{ij}(x,t,u_k) D_i u_k^m D_j \varphi \big] \, dxdt \geq \int_{E_T} \mathbf{1}_{\{u<k\}} \varphi \, d\mu \geq 0 \, .$$

This proves the claim, i.e. that $u_k = \min\{u,k\}$ is a weak super-solution, and finishes the proof of the lemma. □

In the sequel we consider non-negative weak super-solutions $w \colon E_T \to \mathbb{R}_+$ satisfying $w^m \in L^2(0,T; W_0^{1,2}(E))$, that is for any $0 \leq \varphi \in C_0^1(E_T)$ there holds

$$\iint_{E_T} \big[-w \partial_t \varphi + \mathbf{a}_{ij}(x,t,w) D_i w^m D_j \varphi \big] \, dxdt \geq 0 \, . \tag{3.1}$$

We start our considerations with the following Lemma (see [20, Lemma 2.15] for a related result concerning the model case (1.3) without coefficients in the degenerate (i.e. $m > 1$) framework).

Lemma 3.2. *Let $w \colon E_T \to \mathbb{R}_+$ with $w^m \in L^2(0,T; W_0^{1,2}(E))$ be a bounded, non-negative weak super-solution of the singular porous medium type equation in the sense of* (3.1)*. Moreover, assume that $0 \leq w \leq M$. Then, for any $0 \leq t_1 < t_2 \leq T$ and $\eta \in C_0^1(E)$ the following Caccioppoli type estimate*

$$\iint_{E_{t_1,t_2}} \eta^2 |Dw^m|^2 \, dxdt \leq \boldsymbol{\gamma} M^m \bigg[M^m (t_2 - t_1) \int_E |D\eta|^2 \, dx + M \int_E \eta^2 \, dx \bigg] \tag{3.2}$$

holds true with a constant $\boldsymbol{\gamma}$ depending only on C_o and C_1.

Proof. From the regularized version of (3.1) (see the derivation of [20, (2.13)]) we infer that

$$\iint_{E_T} \big[\partial_t [\![w]\!]_h \varphi + [\![\mathbf{a}_{ij}(x,t,w) D_i w^m]\!]_h D_j \varphi \big] \, dxdt \geq 0.$$

Here, we choose the non-negative testing function

$$\varphi(x,t) = \eta^2(x) \xi(t) \big(M^m - w^m(x,t) \big) \, ,$$

where $\eta \in C_0^1(E)$, and $\xi \in W_0^{1,\infty}(\mathbb{R}, [0,1])$ satisfies $\xi \equiv 1$ on $[t_1, t_2] \subset (0,T)$, $\xi \equiv 0$ on $\mathbb{R} \setminus (t_1 - \tau, t_2 + \tau)$ where $0 < \tau < \min\{t_1, T - t_2\}$, and $\xi(t) = \frac{1}{\tau}(t - t_1 + \tau)$ on $(t_1 - \tau, t_1)$ respectively $\xi(t) = -\frac{1}{\tau}(t - t_2 - \tau)$ on $(t_2, t_2 + \tau)$. Note that $\varphi \in L^\infty(E_T)$ and $D\varphi \in L^2(E_T, \mathbb{R}^n)$. Therefore, we find that

$$\begin{aligned} \mathrm{I}_\tau &:= \iint_{E_T} \eta^2 \xi [\![\mathbf{a}_{ij}(x,t,w) D_i w^m]\!]_h D_j w^m \, dxdt \\ &\leq 2 \iint_{E_T} \eta \xi \big(M^m - w^m \big) [\![\mathbf{a}_{ij}(x,t,w) D_i w^m]\!]_h D_j \eta \, dxdt \\ &\qquad\qquad + \iint_{E_T} \eta^2 \xi \big(M^m - w^m \big) \partial_t [\![w]\!]_h \, dxdt \\ &= \mathrm{II}_\tau + \mathrm{III}_\tau + \mathrm{IV}_\tau + \mathrm{V}_\tau, \end{aligned} \tag{3.3}$$

where we have set

$$\mathrm{II}_\tau := 2 \iint_{E_T} \eta \xi \big(M^m - w^m \big) [\![\mathbf{a}_{ij}(x,t,w) D w^m]\!]_h D_j \eta \, dxdt$$

$$\mathrm{III}_\tau := M^m \iint_{E_T} \eta^2 \xi \partial_t [\![w]\!]_h \, dxdt$$

$$\mathrm{IV}_\tau := -\iint_{E_T} \eta^2\xi [\![w]\!]_h^m \partial_t [\![w]\!]_h \,dxdt$$
$$\mathrm{V}_\tau := \iint_{E_T} \eta^2\xi \big([\![w]\!]_h^m - w^m\big) \partial_t [\![w]\!]_h \,dxdt.$$

First, we treat the integrals containing the time derivative $\partial_t [\![w]\!]_h$. We start by considering the term III_τ. We first perform an integration by parts with respect to time, and then use the assumption $w \le M$, which implies that $[\![w]\!]_h \le M$. In this way we obtain

$$\begin{aligned}\lim_{\tau\downarrow 0} \mathrm{III}_\tau &= -M^m \lim_{\tau\downarrow 0} \iint_{E_T} \eta^2 \partial_t\xi [\![w]\!]_h \,dxdt \\ &\le M^{m+1} \lim_{\tau\downarrow 0} \iint_{E_T} \eta^2 |\partial_t \xi| \,dxdt \\ &= 2M^{m+1} \int_E \eta^2 \,dx\,.\end{aligned}$$

Next, we consider the term IV_τ. Using the fact $[\![w]\!]_h^m \partial_t [\![w]\!]_h = \frac{1}{m+1}\partial_t [\![w]\!]_h^{m+1}$, after an integration by parts and after using the fact that $[\![w]\!]_h \le M$, we obtain the following estimate

$$\begin{aligned}\lim_{\tau\downarrow 0} \mathrm{IV}_\tau &= -\tfrac{1}{m+1} \lim_{\tau\downarrow 0} \iint_{E_T} \eta^2\xi \partial_t [\![w]\!]_h^{m+1} \,dxdt \\ &= \tfrac{1}{m+1} \lim_{\tau\downarrow 0} \iint_{E_T} \eta^2 \partial_t\xi [\![w]\!]_h^{m+1} \,dxdt \\ &\le \tfrac{1}{m+1} M^{m+1} \lim_{\tau\downarrow 0} \iint_{E_T} \eta^2 |\partial_t\xi| \,dxdt \\ &= \tfrac{2}{m+1} M^{m+1} \int_E \eta^2 \,dx\,.\end{aligned}$$

Finally, we need to control V_τ. But this can easily be achieved by [8, Lemma B.1] in the following way:

$$\mathrm{V}_\tau = -\tfrac{1}{h} \iint_{E_T} \eta^2\xi \big(w^m - [\![w]\!]_h^m\big)\big(w - [\![w]\!]_h\big) \,dxdt \le 0\,.$$

For the estimate of II_τ we pass to the limit $h \downarrow 0$ and use the assumptions $w \le M$ and $(1.2)_2$. Using also Young's inequality we arrive at

$$\begin{aligned}\lim_{h\downarrow 0} \mathrm{II}_\tau &\le 2C_1 M^m \iint_{E_T} \eta\xi |Dw^m||D\eta| \,dxdt \\ &\le \delta \iint_{E_T} \eta^2\xi |Dw^m|^2 \,dxdt + \frac{C_1^2 M^{2m}}{\delta} \iint_{E_T} \xi |D\eta|^2 \,dxdt\,,\end{aligned}$$

for any $\delta \in (0,1)$. Finally, we estimate the integral I_τ from below. First, we pass to the limit $h \downarrow 0$ and then use the ellipticity condition $(1.2)_1$. This implies

$$\lim_{h\downarrow 0} \mathrm{I}_\tau = \iint_{E_T} \eta^2\xi \mathbf{a}_{ij}(x,t,w) D_i w^m D_j w^m \,dxdt \ge C_o \iint_{E_T} \eta^2\xi |Dw^m|^2 \,dxdt.$$

Joining the estimates for I_τ – V_τ with (3.3) and choosing $\delta = \frac{1}{2}C_o$, we can re-absorb the energy term appearing on the right-hand side into the left. At this point, we pass to the limit $\tau \downarrow 0$ and conclude that the claim (3.2) holds true for any choice of $0 < t_1 < t_2 < T$ for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(C_o, C_1)$. This implies the claim for any $0 \le t_1 < t_2 \le T$. □

Let w be a non-negative weak super-solution as in (3.1). Then by the Riesz representation theorem there exists a non-negative Radon-measure λ_w such that

$$\iint_{E_T} \big[- w\partial_t\varphi + \mathbf{a}_{ij}(x,t,w)D_i w^m D_j\varphi\big]\,dxdt = \int_{E_T} \varphi\,d\lambda_w \tag{3.4}$$

holds true for any $\varphi \in C_0^1(E_T)$. In the following lemma we shall derive a local bound for the associated measure λ_w to a bounded weak super-solution.

Lemma 3.3. *Let $w\colon E_T \to \mathbb{R}_+$ with $w^m \in L^2(0,T;W_0^{1,2}(E))$ be a bounded, non-negative, weak super-solution of the singular porous medium type equation with $w \le M$, and let λ_w be the associated Radon-measure according to* (3.4). *Then, for any $U \Subset E$, any $0 < t_1 < t_2 < T$, and any $\eta \in C_0^1(E)$ with $\eta \equiv 1$ on U and $0 \le \eta \le 1$, there holds*

$$\lambda_w\big(U_{t_1,t_2}\big) \le \boldsymbol{\gamma}\bigg[M^m(t_2-t_1)\int_E |D\eta|^2\,dxdt + M\int_E \eta^2\,dx\bigg], \tag{3.5}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(C_o, C_1)$.

Proof. Let $\eta \in C_0^1(E)$ be as in the statement of the lemma. Furthermore, for $\tau > 0$ we choose the cut-off function in time $\xi(t)$ as in the proof of the energy estimate in Lemma 3.2. Testing (3.4) with $\varphi(x,t) = \eta^2(x)\xi(t)$, taking into account that w is bounded above by M, using (1.2) and Cauchy-Schwarz inequality we infer that

$$\begin{aligned}
\lambda_w(U_{t_1,t_2}) &\le \int_{E_T} \eta^2\xi\,d\lambda_w \\
&= \iint_{E_T} \big[-\eta^2\partial_t\xi w + 2\eta\xi\mathbf{a}_{ij}(x,t,w)D_i w^m D_j\eta\big]\,dxdt \\
&\le M\iint_{E_T} \eta^2|\partial_t\xi|\,dxdt + 2C_1\iint_{E_T} \eta\xi|Dw^m||D\eta|\,dxdt \\
&=: M\,\mathrm{I}_\tau + 2C_1\,\mathrm{II}_\tau\,.
\end{aligned}$$

In the right-hand side we shall pass to the limit $\tau \downarrow 0$. Using the definition of $\xi(t)$, for the integral I_τ we find that

$$\lim_{\tau\downarrow 0} \mathrm{I}_\tau = 2\int_E \eta^2\,dx\,.$$

Passing also in the second integral to the limit $\tau \downarrow 0$, and applying the energy estimate from Lemma 3.2 we conclude that there holds

$$\begin{aligned}
\lim_{\tau\downarrow 0} \mathrm{II}_\tau &= \iint_{E_{t_1,t_2}} \eta|Dw^m||D\eta|\,dxdt \\
&\le \bigg[\iint_{E_{t_1,t_2}} \eta^2|Dw^m|^2\,dxdt\bigg]^{\frac12}\bigg[(t_2-t_1)\int_E |D\eta|^2\,dxdt\bigg]^{\frac12} \\
&\le \boldsymbol{\gamma}\bigg[M^m(t_2-t_1)\int_E |D\eta|^2\,dxdt + M\int_E \eta^2\,dx\bigg].
\end{aligned}$$

Inserting the last two inequalities in the estimate for λ_w above, we infer that (3.5) holds true with a constant $\boldsymbol{\gamma}$ depending only on C_o and C_1. This proves the claim of the lemma. □

As an immediate consequence of Lemmas 3.2 and 3.3, we obtain for any pair of sets $U \Subset W \Subset E$, and $0 < t_1 < t_2 < T$ the following local energy and measure bound for a bounded weak super-solution w and its associated Radon-measure λ_w:

$$\iint_{U_{t_1,t_2}} |Dw^m|^2\,dxdt + M^m\lambda_w(U_{t_1,t_2}) \le \boldsymbol{\gamma}|W|M^m\left[\frac{M^m(t_2-t_1)}{\operatorname{dist}(U,\partial W)^2}+M\right]. \tag{3.6}$$

At this point, we can use equation (3.4) together with the energy estimate (3.6) to deduce a local bound for the time derivative of w.

Lemma 3.4. *Let w be a bounded weak super-solution as in Lemmas 3.2 and 3.3. Moreover, let $U \Subset W \Subset E$ and $0 < t_1 < t_2 < T$ as in* (3.6)*. Then, there holds*

$$\|\partial_t w\|_{L^1(t_1,t_2,W^{-1,1}(U))} \le \boldsymbol{\gamma}|W|\left[\frac{M^m(t_2-t_1)}{\operatorname{dist}(U,\partial W)^2}+M\right],$$

where $\boldsymbol{\gamma}$ depends only on C_o and C_1.

Proof. In equation (3.4) we consider testing functions $\varphi \in C^1_0(E_T)$ with compact support in U_{t_1,t_2} and $\|\varphi\|_{L^\infty(t_1,t_2;W^{1,\infty}(U))} \le 1$. Using in turn the growth condition $(1.2)_2$, the fact that w is bounded by M, and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\left|\iint_{E_T} w\partial_t\varphi\,dxdt\right| &\le \left|\int_{E_T}\varphi\,d\lambda_w\right| + \left|\iint_{E_T}\mathbf{a}_{ij}(x,t,w)D_iw^mD_j\varphi\,dxdt\right| \\
&\le \lambda_w(U_{t_1,t_2}) + C_1\iint_{U_{t_1,t_2}}|Dw^m|\,dxdt \\
&\le \lambda_w(U_{t_1,t_2}) + C_1|U_{t_1,t_2}|^{\frac12}\left[\iint_{U_{t_1,t_2}}|Dw^m|^2\,dxdt\right]^{\frac12} \\
&\le \lambda_w(U_{t_1,t_2}) + C_1\left[M^m|U|(t_2-t_1) + M^{-m}\iint_{U_{t_1,t_2}}|Dw^m|^2\,dxdt\right].
\end{aligned}$$

Using (3.6) we finally obtain that

$$\left|\iint_{E_T} w\partial_t\varphi\,dxdt\right| \le \boldsymbol{\gamma}|W|\left[\frac{M^m(t_2-t_1)}{\operatorname{dist}(U,\partial W)^2}+M\right].$$

Taking the supremum over all $\varphi \in C^1_0(U_{t_1,t_2})$ with $\|\varphi\|_{L^\infty(t_1,t_2;W^{1,\infty}(U))} \le 1$ yields the claim. □

4. Existence of Very Weak Solutions and Proof of Theorem 1.1

In this section we deal with the Cauchy-Dirichlet problem for a general porous medium type equation with a right-hand side $\mu \in \mathcal{M}(E_T)$. To be more specific, we want to show that the Cauchy-Dirichlet problem (1.4) admits a very weak solution. These very weak solutions are obtained as limits of solutions to regularized problems. Here *regularized* means that the right-hand side μ is assumed to be bounded, i.e. that $\mu \in L^\infty(E_T)$. For such right-hand sides the existence of a weak energy solution is guaranteed by the following result.

Proposition 4.1. *Let $\mu \in L^\infty(E_T)$ be a non-negative right-hand side and hypotheses* (1.2) *be satisfied. Then, there exists at least one non-negative (bounded) weak energy solution*

$$u \in C^0\big([0,T];L^{1+m}(E)\big), \quad \textit{with}\ \ u^m \in L^2\big(0,T;W^{1,2}_0(E)\big)$$

of the Cauchy-Dirichlet problem (1.10)*.*

Proof. If we let

$$b(v) := v^{\frac{1}{m}}, \qquad 0 < m < 1,$$

then Theorem 1.7, Lemma 1.5, and Remark 1.10 of [1] give the existence of a solution

$$v \in L^\infty\big(0,T; L^{\frac{1+m}{m}}(E)\big) \cap L^2\big(0,T; W_0^{1,2}(E)\big)$$

of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t b(v) - D_j\big[\mathbf{a}_{ij}(x,t,v)D_i v\big] = \mu, & \text{in } E_T, \\ v = 0 \quad \text{on } \partial_{\mathrm{par}} E_T, \end{cases}$$

with $\mu \in L^\infty(E_T)$. If we let $u := v^{\frac{1}{m}}$, this yields the existence of a (weak energy) solution

$$u \in L^\infty\big(0,T; L^{1+m}(E)\big), \quad \text{with } u^m \in L^2\big(0,T; W_0^{1,2}(E)\big)$$

of the Cauchy-Dirichlet problem (1.4). In order to conclude the proof, we have to show that $u \in C^0(0,T; L^{1+m}(E))$. First, we observe that $u_t \in L^2(0,T; W^{-1,2}(E))$. Indeed, letting $\varphi \in C_0^\infty(E_T)$, we conclude from (1.12) that

$$\begin{aligned} \left| \iint_{E_T} u\varphi_t \, dxdt \right| &= \left| \iint_{E_T} \big[\mathbf{a}_{ij}\big(x,t,u\big)D_i u^m D_j\varphi - \varphi\mu\big] \, dxdt \right| \\ &\le \iint_{E_T} \big[C_1 |Du^m||D\varphi| + |\varphi||\mu|\big] \, dxdt \Big| \\ &\le \boldsymbol{\gamma}\Big[\|Du^m\|_{L^2(E_T)} + \|\mu\|_{L^2(E_T)}\Big]\|\varphi\|_{L^2(0,T;W^{1,2}(E))}, \end{aligned}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(C_1)$. By the density of $C_0^\infty(E_T)$ in $L^2(0,T; W_0^{1,2}(E))$, this implies the claim $u_t \in L^2(0,T; W^{-1,2}(E))$. This allows us to re-write the weak form of (1.4) as follows:

$$\int_0^T \langle u_t, \varphi\rangle \, dt + \iint_{E_T} \mathbf{a}_{ij}\big(x,t,u\big)D_i u^m D_j\varphi \, dxdt = \iint_{E_T} \varphi\mu \, dxdt, \tag{4.1}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing of $W^{-1,2}(E)$ and $W_0^{1,2}(E)$. To proceed further, let $\alpha \in W^{1,\infty}([0,T],[0,1])$ with $\alpha(T) = 0$ and $\alpha' \le 0$, and choose in (4.1) as testing function $\varphi = \alpha\varphi_h$ with $\varphi_h := [\![u^m]\!]_h - u^m$, where $[\![u^m]\!]_h$ is defined according to (2.2) and $h \in (0,T]$. Then, by [9, Lemma 3.2] we have that

$$\int_0^T \langle u_t, \alpha\varphi_h\rangle \, dt = \iint_{E_T} \Big[\alpha'\big[\tfrac{1}{m+1}u^{m+1} - u[\![u^m]\!]_h\big] - \alpha u \partial_t [\![u^m]\!]_h\Big] \, dxdt.$$

With the definition of I from (2.1) and by an integration by parts, we find that

$$\begin{aligned} \int_0^T \langle u_t, \alpha\varphi_h\rangle \, dt &= \iint_{E_T} \Big[\alpha' I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big) - \tfrac{m}{m+1}\alpha'[\![u^m]\!]_h^{\frac{m+1}{m}} - \alpha u\partial_t[\![u^m]\!]_h\Big] \, dxdt \\ &= \iint_{E_T} \Big[\alpha' I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big) + \alpha\partial_t[\![u^m]\!]_h\big([\![u^m]\!]_h^{\frac{1}{m}} - u\big)\Big] \, dxdt \\ &\le \iint_{E_T} \alpha' I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big) \, dxdt, \end{aligned}$$

where in the last line we used the identity $\partial_t[\![u^m]\!]_h = \frac{1}{h}(u^m - [\![u^m]\!]_h)$, which implies that in the second last line the last term on the right-hand side is non-positive. Taking into account that $\alpha' \le 0$ we arrive at

$$\begin{aligned}\iint_{E_T} |\alpha'| I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big)\, dxdt &\le -\int_0^T \langle u_t, \alpha\varphi_h\rangle\, dt \\ &= \iint_{E_T} \alpha\, \mathbf{a}_{ij}\big(x,t,u\big) D_i u^m D_j \varphi_h\, dxdt - \iint_{E_T} \alpha\varphi_h \mu\, dxdt.\end{aligned}$$

For a time $t_o \in (0,T)$ and $0 < \varepsilon < t_o$ we choose the cut-off function $\alpha \equiv 1$ on $[0, t_o - \varepsilon]$, $\alpha(t) := \frac{1}{\varepsilon}(t_o - t)$ on $[t_o - \varepsilon, t_o]$ and $\alpha \equiv 0$ on $[t_o, T]$ and then let $\varepsilon \downarrow 0$. This leads to

$$\int_{E\times\{t_o\}} I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big)\, dx \le \iint_{E_{t_o}} \mathbf{a}_{ij}\big(x,t,u\big) D_i u^m D_j \varphi_h\, dxdt - \iint_{E_{t_o}} \varphi_h \mu\, dxdt\,.$$

Taking into account that $t_o \in (0,T)$ is arbitrary, that $\varphi_h \downarrow 0$ in $L^2(0,T; W^{1,2}(E))$ in the limit $h \downarrow 0$, and the fact that $I(u,v) \ge 0$, we conclude that

$$\lim_{h\downarrow 0} \mathbf{I}_h := \lim_{h\downarrow 0} \sup_{t_o\in(0,T)} \int_{E\times\{t_o\}} I\big(u, [\![u^m]\!]_h^{\frac{1}{m}}\big)\, dx = 0.$$

Now, we apply Lemma 2.2 and use a standard estimate for the mollification $[\![\cdot]\!]_h$ to infer that

$$\begin{aligned}&\sup_{t_o\in(0,T)} \int_{E\times\{t_o\}} \big|u - [\![u^m]\!]_h^{\frac{1}{m}}\big|^{m+1}\, dx \\ &\qquad\le \boldsymbol{\gamma}\, \mathbf{I}_h + \boldsymbol{\gamma}\, \mathbf{I}_h^{\frac{m+1}{2}} \bigg(\sup_{t_o\in(0,T)} \int_{E\times\{t_o\}} [\![u^m]\!]_h^{\frac{m+1}{m}}\, dx\bigg)^{\frac{1-m}{2}} \\ &\qquad\le \boldsymbol{\gamma}\, \mathbf{I}_h + \boldsymbol{\gamma}\, \mathbf{I}_h^{\frac{m+1}{2}} \bigg(\sup_{t_o\in(0,T)} \int_{E\times\{t_o\}} u^{m+1}\, dx\bigg)^{\frac{1-m}{2}}\end{aligned}$$

holds true with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m)$. Note that the last integral is finite, since $u \in L^\infty(0,T; L^{1+m}(E))$. Therefore, passing to the limit $h \downarrow 0$, we find that

$$\lim_{h\downarrow 0} \sup_{t_o\in(0,T)} \int_{E\times\{t_o\}} \big|u - [\![u^m]\!]_h^{\frac{1}{m}}\big|^{m+1}\, dx = 0,$$

i.e. $[\![u^m]\!]_h^{\frac{1}{m}} \to u$ strongly in $L^\infty(0,T; L^{1+m}(E))$. Since $[\![u^m]\!]_h^{\frac{1}{m}} \in C^0([0,T]; L^{1+m}(E))$, this proves that $u \in C^0([0,T]; L^{1+m}(E))$. □

4.1. **The approximation scheme.** In this section we set up the approximation scheme which will finally lead us to the existence of a very weak solution of the general porous medium type equation. Given a non-negative measure $\mu \in \mathcal{M}(E_T)$, we denote by $\mu_\ell \in L^\infty(E_T)$ a sequence of non-negative bounded functions such that

$$\iint_{E_T} \mu_\ell\, dxdt \le \mu(E_T) \quad \text{and} \quad \mu_\ell dxdt \to \mu \text{ in the weak } * \text{ sense of measures.} \tag{4.2}$$

From Proposition 4.1 there exists a weak energy solution $u_\ell \in C^0([0,T]; L^{1+m}(E))$ with $u_\ell^m \in L^2(0,T; W_0^{1,2}(E))$ of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_\ell - D_j\big[\mathbf{a}_{ij}(x,t,u_\ell) D_i u_\ell^m\big] = \mu_\ell & \text{in } E_T, \\ u_\ell = 0 & \text{on } \partial_{\mathrm{par}} E_T. \end{cases} \tag{4.3}$$

4.2. **A priori estimates below the natural growth.** In this section we consider a weak energy solution u of the Cauchy-Dirichlet problem (1.10) with a right-hand side $\mu \in L^\infty(E_T)$; for instance u could be one function of our sequence $(u_\ell)_{\ell\in\mathbb{N}}$ from our approximation scheme. Our goal is to derive certain energy bounds below the natural energy space which are uniform in the L^1-norm of the right-hand side. This will be a crucial step in the existence proof, since for our approximating sequence μ_ℓ we will only have the uniform L^1-bound from (4.2) at hand.

We start with the choice of a suitable testing function. Since the a priori estimates we have in mind should only contain the L^1-norm of μ, the testing function has to be bounded in E_T. One possibility to achieve this, is suggested by the Kilpeläinen & Maly testing function [17, 18]. We fix a parameter $\lambda \in (0,1)$, which is at our disposal, and define the testing function

$$\varphi := \xi v := \xi g_\lambda(u) := \xi\bigg(1 - \frac{1}{(1+u^m)^\lambda}\bigg),$$

where $\xi \in W^{1,\infty}(\mathbb{R})$ satisfies $\xi \equiv 1$ for $t \le \tau - \delta$, $\xi \equiv 0$ for $t \ge \tau$, and $\xi(t) = -\frac{1}{\delta}(t-\tau)$ for $\tau - \delta < t < \tau$. Here, $\tau \in (0,T)$, $\delta \in (0,\tau)$, and moreover

$$g_\lambda(s) := 1 - \frac{1}{(1+s^m)^\lambda}\,.$$

We note that $\varphi = 0$ on $\partial_{\rm par} E_T$. In the following, we argue formally concerning the use of the time derivatives. As before, the argument can be made rigorous by use of the time mollification procedure introduced in Section 2.3. Testing the weak formulation (1.11) of $(1.4)_1$ with φ we obtain that

$$\iint_{E_T} \big[u_t\varphi + \mathbf{a}_{ij}(x,t,u) D_i u^m D_j\varphi\big]\,dxdt = \iint_{E_T} \mu\varphi\,dxdt$$

holds true. For the term involving the time derivative we obtain that

$$\begin{aligned}\iint_{E_T} u_t\varphi\,dxdt &= \iint_{E_T} \xi\frac{\partial}{\partial t}\bigg[\int_0^{u(\cdot,t)} g_\lambda(s)\,ds\bigg]\,dxdt\\ &= -\iint_{E_T} \partial_t\xi \int_0^{u(\cdot,t)} g_\lambda(s)\,ds\,dxdt\\ &= \frac{1}{\delta}\iint_{E\times(\tau-\delta,\tau)} \int_0^{u(\cdot,t)} g_\lambda(s)\,ds\,dxdt\,.\end{aligned}$$

In the preceding identity we let $\delta \downarrow 0$ and obtain that for every $\tau \in (0,T)$ (note that $u \in C^0\big([0,T]; L^{1+m}(E)\big)$)

$$\lim_{\delta\downarrow 0}\iint_{E_T} u_t\varphi\,dxdt = \int_E \int_0^{u(\cdot,\tau)} g_\lambda(s)\,ds\,dx\,.$$

Next we compute Dv; the computation yields

$$Dv = \lambda\frac{Du^m}{(1+u^m)^{1+\lambda}}\,.$$

The diffusion term in the weak formulation can now be estimated from below with the help of the ellipticity condition $(1.2)_1$ in the following way:

$$\iint_{E_T} \mathbf{a}_{ij}(x,t,u) D_i u^m D_j\varphi\,dxdt$$

$$= \lambda \iint_{E_T} \xi \frac{\mathbf{a}_{ij}(x,t,u) D_i u^m D_j u^m}{(1+u^m)^{1+\lambda}}\, dxdt \geq C_o \lambda \iint_{E_T} \frac{\xi\, |Du^m|^2}{(1+u^m)^{1+\lambda}}\, dxdt\,.$$

For the right-hand side we obtain

$$\iint_{E_T} \mu\varphi\, dxdt \leq \iint_{E_T} \mu\, dxdt,$$

where we used the fact that $0 \leq \varphi \leq 1$. Joining the preceding inequalities, we finally arrive at the preliminary energy estimate

$$\sup_{t\in(0,T)} \int_E \int_0^{u(\cdot,t)} g_\lambda(s)\, ds\, dx + C_o \lambda \iint_{E_T} \frac{|Du^m|^2}{(1+u^m)^{1+\lambda}}\, dxdt \leq \iint_{E_T} \mu\, dxdt\,.$$

From Lemma 2.4 we know that $\int_0^u g_\lambda(s)\, ds \geq (u-1)/\boldsymbol{\gamma}$. This leads us to the following *energy estimate*:

$$\sup_{t\in(0,T)} \int_E u\, dx + \iint_{E_T} \frac{|Du^m|^2}{(1+u^m)^{1+\lambda}}\, dxdt \leq \boldsymbol{\gamma} \bigg[\iint_{E_T} \mu\, dxdt + |E| \bigg], \tag{4.4}$$

where the constant $\boldsymbol{\gamma}$ depends on C_o and λ. Note that this estimate is exactly of the form we are looking for, that is, in the right-hand side, only the L^1-norm of μ is involved. We use the preceding estimate by an application of Hölder's inequality in order to bound the L^q-norm of Du^m for some $q \in [1,2)$. The range of q which allows this procedure will be specified in the course of the proof. As mentioned before, by Hölder's inequality we obtain

$$\begin{aligned}
\iint_{E_T} |Du^m|^q\, dxdt &= \iint_{E_T} \frac{|Du^m|^q}{(1+u^m)^{\frac{q(1+\lambda)}{2}}} (1+u^m)^{\frac{q(1+\lambda)}{2}}\, dxdt \\
&\leq \bigg(\iint_{E_T} \frac{|Du^m|^2}{(1+u^m)^{(1+\lambda)}}\, dxdt \bigg)^{\frac{q}{2}} \bigg(\iint_{E_T} (1+u^m)^{\frac{q(1+\lambda)}{2-q}}\, dxdt \bigg)^{1-\frac{q}{2}} \\
&\leq \boldsymbol{\gamma} \bigg(\iint_{E_T} \mu\, dxdt + |E| \bigg)^{\frac{q}{2}} \bigg(|E_T| + \iint_{E_T} u^{m\frac{q(1+\lambda)}{2-q}}\, dxdt \bigg)^{1-\frac{q}{2}}. \qquad (4.5)
\end{aligned}$$

For the last term of the right-hand side of the preceding inequality we use the parabolic Sobolev embedding from Lemma 2.1 to conclude that

$$\iint_{E_T} u^{m\frac{q(1+\lambda)}{2-q}}\, dxdt \leq \boldsymbol{\gamma} \sup_{t\in(0,T)} \bigg(\int_{E\times\{t\}} (u^m)^{\frac{1}{m}}\, dx \bigg)^{\frac{q}{n}} \iint_{E_T} |Du^m|^q\, dxdt\,,$$

where q is chosen as follows:

$$\frac{q(1+\lambda)}{2-q} = \frac{q(n+\frac{1}{m})}{n}.$$

The condition on q leads to the identity

$$q = 1 - \frac{1-nm\lambda}{1+nm}\,. \tag{4.6}$$

Now, if we suppose that q fulfills the smallness condition

$$1 \leq q < 1 + \frac{1}{1+nm}\,, \tag{4.7}$$

then we can always find $\lambda > 0$ such that (4.6) is satisfied. Now, using (4.4) in the preceding inequality we see that

$$\iint_{E_T} u^{m\frac{q(1+\lambda)}{2-q}}\, dxdt \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{\frac{q}{n}} \iint_{E_T} |Du^m|^q\, dxdt \tag{4.8}$$

holds true. Inserting this in (4.5) and using in turn Young's inequality we obtain

$$\begin{aligned}
&\iint_{E_T} |Du^m|^q\, dxdt \\
&\qquad \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{\frac{q}{2}} |E_T|^{1-\frac{q}{2}} \\
&\qquad\quad + \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{\frac{q}{2}+\frac{q}{n}(1-\frac{q}{2})} \Big(\iint_{E_T} |Du^m|^q\, dxdt\Big)^{1-\frac{q}{2}} \\
&\qquad \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{\frac{q}{2}} |E_T|^{1-\frac{q}{2}} \\
&\qquad\quad + \tfrac{1}{2}\iint_{E_T} |Du^m|^q\, dxdt + \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{1+\frac{2}{n}(1-\frac{q}{2})} .
\end{aligned}$$

In the preceding inequality we re-absorb the second integral of the right-hand side into the left and compute that $1 + \frac{2}{n}(1 - \frac{q}{2}) = 1 + \frac{2-q}{n}$. This leads to an energy bound below the natural growth. More, precisely we conclude that

$$\begin{aligned}
&\iint_{E_T} |Du^m|^q\, dxdt \\
&\qquad \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{\frac{q}{2}} |E_T|^{1-\frac{q}{2}} + \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{1+\frac{2-q}{n}} \\
&\qquad \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{1+\frac{2-q}{n}} + |E_T|^{1-\frac{q}{n+2}} ,
\end{aligned} \tag{4.9}$$

holds true for any q in the range from (4.7). Note that in the last line we used Young's inequality, and $\boldsymbol{\gamma}$ depends on n, m, C_o and q. Altogether, combining (4.4) and (4.9) we have shown:

Lemma 4.1. *Let $\mu \in L^\infty(E_T)$ be non-negative and $u \in C^0\big([0,T]; L^{1+m}(E)\big)$ with $u^m \in L^2\big(0,T; W_0^{1,2}(E)\big)$ be a non-negative weak energy solution of the Cauchy-Dirichlet problem* (1.10)*, where the coefficients $\mathbf{a}_{ij}$ satisfy* (1.2)*. Then, the following L^∞–L^1 a priori estimate*

$$0 \le \sup_{t\in(0,T)} \int_{E\times\{t\}} u\, dx \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big) .$$

holds true with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, C_o)$. Furthermore, if q satisfies the restriction

$$1 \le q < 1 + \frac{1}{1+nm}\, , \tag{4.10}$$

then the energy bound

$$\iint_{E_T} |Du^m|^q\, dxdt \le \boldsymbol{\gamma}\Big(\iint_{E_T} \mu\, dxdt + |E|\Big)^{1+\frac{2-q}{n}} + |E_T|^{1-\frac{q}{n+2}}$$

holds true. The constant $\boldsymbol{\gamma}$ blows up when q approaches the upper bound in (4.10)*, i.e. $\boldsymbol{\gamma}\to\infty$ when $q\uparrow 1+\frac{1}{1+nm}<2$.*

Remark 4.1. We remark that in the case $m=1$ the upper bound (4.10) for q is in perfect accordance with the upper bound from [3] for $p=2$. Moreover, the L^q-bound for Du^m can also be used to get an estimate for u^m. For this we first observe that

$$\begin{aligned} m\,\frac{q(1+\lambda)}{2-q} &= m\,\frac{\big(1+\frac{1-nm\lambda}{1+nm}\big)(1+\lambda)}{1-\frac{1-nm\lambda}{1+nm}} \\ &= m\,\frac{(2+nm-nm\lambda)(1+\lambda)}{nm+nm\lambda} \\ &= m\bigg(\frac{2}{nm}+1-\lambda\bigg). \end{aligned}$$

Therefore, by (4.8) we have

$$\iint_{E_T} u^{mp}\,dxdt \le \boldsymbol{\gamma}\,,$$

for any p satisfying

$$0<p<1+\frac{2}{nm}\,. \tag{4.11}$$

Finally, to ensure that $mp>1$ holds true, we need that $m+\frac{2}{n}>1$, which is satisfied, provided we assume that $m>m_c:=\frac{(n-2)_+}{n}$. □

Remark 4.2. For later use, we note that testing the weak form (1.12) with $\xi g_\lambda(T_k(u))$ for $k\ge 1$ instead of using $\xi g_\lambda(u)$, we obtain the following energy estimate for $T_k(u)$ (see the derivation of (4.4)):

$$\iint_{E_T}|DT_k(u)^m|^2\,dxdt\le\boldsymbol{\gamma}\,k^{1+\lambda}\bigg[\iint_{E_T}\mu\,dxdt+|E|\bigg].$$

4.3. **Uniform bounds for the approximating sequence.** We start with a fixed sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of weak energy solutions as described in Section 4.1. The right-hand sides $(\mu_\ell)_{\ell\in\mathbb{N}}$ of the corresponding Cauchy-Dirichlet problems (4.3) fulfill the requirements $(4.2)_1$ and $(4.2)_2$. In particular, Lemma 4.1 is applicable to the solutions u_ℓ with

$$u_\ell\in C^0\big([0,T];L^{1+m}(E)\big),\quad\text{with }\; u_\ell^m\in L^2\big(0,T;W^{1,2}_0(E)\big),$$

and we obtain that the uniform $L^\infty\!-\!L^1$ bound

$$0\le\sup_{t\in(0,T)}\int_{E\times\{t\}}u_\ell\,dx\le\boldsymbol{\gamma}\big[\mu(E_T)+|E|\big], \tag{4.12}$$

holds true and moreover that

$$\sup_{\ell\in\mathbb{N}}\iint_{E_T}|Du_\ell^m|^q\,dxdt\le\boldsymbol{\gamma}\big[\mu(E_T)+|E|\big]^{1+\frac{2-q}{n}}+|E_T|^{1-\frac{q}{n+2}} \tag{4.13}$$

for any $\ell\in\mathbb{N}$, whenever q satisfies (4.10). Note that the constants in (4.12) and (4.13) are independent of ℓ. Furthermore, whenever p is in the range of (4.11), we have the uniform bound

$$\sup_{\ell\in\mathbb{N}}\iint_{E_T}u_\ell^{mp}\,dxdt\le\boldsymbol{\gamma}\,.$$

Since we are assuming that $m > m_c$, the range (4.11) of admissible p allows values of p such that $mp > 1$. Therefore, there exist a subsequence (still denoted by $(u_\ell)_{\ell\in\mathbb{N}}$) and functions $u \in L^{mp}(E_T)$ and $v \in L^q(E_T, \mathbb{R}^n)$ such that

$$\begin{cases} u_\ell \rightharpoonup u & \text{weakly in } L^{mp}(E_T), \text{ and} \\ Du_\ell^m \rightharpoonup v & \text{weakly in } L^q(E_T, \mathbb{R}^n). \end{cases}$$

The main aim in the sequel is the identification of the weak limit v as Du^m. This will be achieved in several steps.

4.3.1. *Truncations from above and their a priori estimates.* For fixed $k \geq 1$ we consider the truncations

$$w_\ell^{(k)} := \min\{u_\ell, k\} = \min\{u_\ell^m, k^m\}^{\frac{1}{m}}.$$

Then, we have a pointwise bound for the spatial derivative of $w_\ell^{(k)}$ in the form

$$\big|Dw_\ell^{(k)}\big| \leq \tfrac{1}{m} k^{1-m} \mathbf{1}_{\{u_\ell < k\}} |Du_\ell^m| \leq \boldsymbol{\gamma}_k |Du_\ell^m|.$$

Moreover, from the definition of $w_\ell^{(k)}$, we trivially have the bound $w_\ell^{(k)} \leq u_\ell$. Combining these pointwise bounds with the uniform bounds (4.13), we obtain that

$$\iint_{E_T} \big|w_\ell^{(k)}\big|^{mp} + \big|Dw_\ell^{(k)}\big|^q \,dxdt \leq \boldsymbol{\gamma}_k \left[\iint_{E_T} |u_\ell|^{mp} + |Du_\ell^m|^q \,dxdt\right] \leq \boldsymbol{\gamma}_k \tag{4.14}$$

holds true for a constant $\boldsymbol{\gamma}_k$ independent of $\ell \in \mathbb{N}$. Hence, for fixed $k > 0$ the sequence $(w_\ell^{(k)})_{\ell\in\mathbb{N}}$ is uniformly bounded in $L^r(0,T; W_0^{1,r}(E))$ for some $r > 1$; note that we are considering the case $m > m_c$. In order to apply a compactness theorem of J. Simon guaranteeing the compactness of the sequence of truncations $w_\ell^{(k)}$ in $L^1_{\rm loc}(E_T)$, we need a local uniform estimate for their time derivatives. By Lemma 3.1 we know that the functions $w_\ell^{(k)}$ are bounded weak super-solutions to the singular porous medium equation, and therefore Lemma 3.4 is applicable. The application of the lemma (note that the functions are bounded by k and therefore M has to be replaced by k) yields that the following estimate

$$\big\|\partial_t w_\ell^{(k)}\big\|_{L^1(t_1,t_2,W^{-1,1}(U))} \leq \boldsymbol{\gamma} k |W| \left[\frac{T}{\operatorname{dist}(U, \partial W)^2} + 1\right] \tag{4.15}$$

holds true for any $U \Subset W \Subset E$, any $0 < t_1 < t_2 < T$, and any $\ell \in \mathbb{N}$. The constant $\boldsymbol{\gamma}$ only depends on C_o and C_1.

4.3.2. *Identifying the weak limit v.* In this final step we will identify the limit v of the weakly convergent sequence Du_ℓ^m. For this, we take an exhaustion of E by smooth sets $(U^{(\alpha)})_{\alpha\in\mathbb{N}}$ and nested intervals $(t_1^{(\alpha)}, t_2^{(\alpha)})_{\alpha\in\mathbb{N}}$ with $U^{(\alpha)} \Subset E$, $0 < t_1^{(\alpha)} < t_2^{(\alpha)} < T$, and such that

$$E_T = \bigcup_{\alpha=1}^{\infty} U^{(\alpha)}_{t_1^{(\alpha)}, t_2^{(\alpha)}} =: \bigcup_{\alpha=1}^{\infty} \mathbf{U}^{(\alpha)}.$$

For fixed $k \geq 1$ we apply [23, Corollary 4] with the following choice of spaces $X = W^{1,q}(U^{(1)})$, $B = L^1(U^{(1)})$, $Y = W^{-1,1}(U^{(1)})$ and $q > 1$ to the sequence $(w_\ell^{(k)})_{\ell\in\mathbb{N}}$; note that this is possible due to the uniform bounds (4.14) and (4.15). We conclude the compactness of the sequence in $L^1(\mathbf{U}^{(1)})$ which allows us to extract a subsequence $\mathfrak{K}_1^{(k)} \subset \mathbb{N}$ such that

$$w_\ell^{(k)} \to \widetilde{w}_1 \text{ strongly in } L^1\big(\mathbf{U}^{(1)}\big) \text{ and a.e. in } \mathbf{U}^{(1)} \text{ as } \mathfrak{K}_1^{(k)} \ni \ell \to \infty.$$

On $\mathbf{U}^{(2)}$ we now consider the sequence $(w_\ell^{(k)})_{\ell\in\mathfrak{K}_1^{(k)}}$. The argument from above applies again, and gives a subsequence $\mathfrak{K}_2^{(k)}\subset\mathfrak{K}_1^{(k)}$ such that

$$w_\ell^{(k)}\to\widetilde{w}_2 \text{ strongly in } L^1\big(\mathbf{U}^{(2)}\big) \text{ and a.e. in } \mathbf{U}^{(2)} \text{ as } \mathfrak{K}_2^{(k)}\ni\ell\to\infty.$$

Of course we must have $\widetilde{w}_2\equiv\widetilde{w}_1$ on $\mathbf{U}^{(1)}$, since the pointwise limits are unique. This process can be continued inductively by picking a subsequence $\mathfrak{K}_{\alpha+1}^{(k)}\subset\mathfrak{K}_\alpha^{(k)}$ such that the subsequence $(w_\ell^{(k)})_{\ell\in\mathfrak{K}_{\alpha+1}^{(k)}}$ converges strongly in $L^1(\mathbf{U}^{(\alpha+1)})$ and almost everywhere in $\mathbf{U}^{(\alpha+1)}$ to $\widetilde{w}_{\alpha+1}$. Clearly, we have $\widetilde{w}_{\alpha+1}=\widetilde{w}_\alpha$ on $\mathbf{U}^{(\alpha)}$. Now, we let $\mathfrak{K}_\infty^{(k)}:=\cap_{\alpha\in\mathbb{N}}\mathfrak{K}_\alpha^{(k)}$ be the diagonal sequence. Then, for the corresponding diagonal sequence of functions $(w_\ell^{(k)})_{\ell\in\mathfrak{K}_\infty^{(k)}}$ we have that

$$w_\ell^{(k)}\to w^{(k)} \text{ strongly in } L^1_{\rm loc}(E_T) \text{ and a.e. on } E_T \text{ as } \mathfrak{K}_\infty^{(k)}\ni\ell\to\infty,$$

where $w^{(k)}$ is defined in a natural way by the local limits $\widetilde{w}_\alpha$.

Now, we choose the values of k as $k\in\mathbb{N}$. Then, the argument from above shows that there exists a subsequence $\mathfrak{A}_1:=\mathfrak{K}_\infty^{(1)}\subset\mathbb{N}$ such that $w_\ell^{(1)}\to w^{(1)}$ strongly in $L^1_{\rm loc}(E_T)$ and a.e. on E_T as $\mathfrak{A}_1\ni\ell\to\infty$. Starting with $(w_\ell^{(2)})_{\ell\in\mathfrak{A}_1}$ the argument yields another subsequence $\mathfrak{A}_2\subset\mathfrak{A}_1$ such that $w_\ell^{(2)}\to w^{(2)}$ strongly in $L^1_{\rm loc}(E_T)$ and a.e. on E_T as $\mathfrak{A}_2\ni\ell\to\infty$. Inductively, we find $\mathfrak{A}_{k+1}\subset\mathfrak{A}_k$ such that $w_\ell^{(k+1)}\to w^{(k+1)}$ strongly in $L^1_{\rm loc}(E_T)$ and a.e. on E_T as $\mathfrak{A}_{k+1}\ni\ell\to\infty$. At this stage, we consider again the diagonal sequence $\mathfrak{A}_\infty:=\cap_{k\in\mathbb{N}}\mathfrak{A}_k$ and obtain that for any $k\in\mathbb{N}$ there holds:

$$w_\ell^{(k)}\to w^{(k)} \text{ strongly in } L^1_{\rm loc}(E_T) \text{ and a.e. on } E_T \text{ as } \mathfrak{A}_\infty\ni\ell\to\infty. \tag{4.16}$$

Since

$$w_\ell^{(k)}=\min\{u_\ell,k\}\le\min\{u_\ell,k+1\}=w_\ell^{(k+1)},$$

the almost everywhere convergence of $w_\ell^{(k)}\to w^{(k)}$ and $w_\ell^{(k+1)}\to w^{(k+1)}$ as $\mathfrak{A}_\infty\ni\ell\to\infty$ implies that the sequence $(w^{(k)})_{k\in\mathbb{N}}$ of non-negative functions is non-decreasing and therefore admits a pointwise, non-negative limit $w\in L^1_{\rm loc}(E_T)$, that is

$$w(x,t):=\lim_{k\to\infty}w^{(k)}(x,t)$$

is well defined for almost every $(x,t)\in E_T$. Note that the $L^1_{\rm loc}(E_T)$-convergence (4.16), implies that for any $U\Subset E$ and $0<t_1<t_2<T$ there holds

$$\iint_{U_{t_1,t_2}}\big|w^{(k)}\big|\,dxdt=\lim_{\mathfrak{A}_\infty\ni j\to\infty}\iint_{U_{t_1,t_2}}\big|w_\ell^{(k)}\big|\,dxdt\le\sup_{j\in\mathbb{N}}\iint_{E_T}|u_\ell|\,dxdt.$$

Since u_ℓ is uniformly bounded in $L^1(E_T)$, we have that the sequence of functions $w^{(k)}$ is uniformly bounded in $L^1(U_{t_1,t_2})$. Hence, we can pass to a non-relabeled subsequence such that $w^{(k)}\uparrow w$ weakly in $L^1(U_{t_1,t_2})$. This, however, implies for any $0\le\varphi\in L^\infty(U_{t_1,t_2})$ that

$$0\le\iint_{U_{t_1,t_2}}\big(w-w^{(k)}\big)\varphi\,dxdt\to0\qquad\text{as } k\to\infty,$$

and proves that $w^{(k)}\to w$ in $L^1_{\rm loc}(E_T)$ as $k\to\infty$. Next, we consider the difference between u_ℓ and w in the L^1-norm and obtain for any $U\Subset E$ and $0<t_1<t_2<T$ that

$$\iint_{U_{t_1,t_2}}|u_\ell-w|\,dxdt\le\iint_{U_{t_1,t_2}}\big|u_\ell-w_\ell^{(k)}\big|\,dxdt$$

$$+\iint_{U_{t_1,t_2}} \big|w_\ell^{(k)} - w^{(k)}\big|\,dxdt + \iint_{U_{t_1,t_2}} \big|w^{(k)} - w\big|\,dxdt\,.$$

The first term appearing on the right-hand side of the preceding inequality can be estimated as follows:

$$\begin{aligned}\iint_{U_{t_1,t_2}} \big|u_\ell - w_\ell^{(k)}\big|\,dxdt &\le \iint_{U_{t_1,t_2}\cap\{u_\ell>k\}} u_\ell\,dxdt \\ &\le k^{1-mp}\iint_{E_T} u_\ell^{mp}\,dxdt \le \boldsymbol{\gamma}\,k^{1-mp}\,.\end{aligned}$$

We insert this above and then pass to the limit $\ell\to\infty$, taking into account that the second term converges to zero by (4.16). Hence,

$$\limsup_{j\to\infty}\iint_{U_{t_1,t_2}} \big|u_\ell - w\big|\,dxdt \le \boldsymbol{\gamma}\,k^{1-p} + \iint_{U_{t_1,t_2}} \big|w^{(k)} - w\big|\,dxdt\,.$$

In the previous inequality we can now pass to the limit $k\to\infty$, thereby using the previously established fact that $w^{(k)}\to w$ in $L^1_{\rm loc}(E_T)$ as $k\to\infty$. This implies that $u_\ell\to w$ in $L^1_{\rm loc}(E_T)$ and a.e. on E_T in the limit $\mathfrak{A}_\infty\ni\ell\to\infty$. Hence, $w\equiv u$ on E_T. Since, u_ℓ, u are uniformly bounded in $L^{mp}(E_T)$ for p as in (4.11) and $u_\ell\to u$ a.e. on E_T, we conclude that

$$u_\ell\to u \text{ strongly in } L^{mp}(E_T) \text{ as } \mathfrak{A}_\infty\ni\ell\to\infty \text{ for any } 1\le p<1+\tfrac{2}{nm}.$$

We are now ready to identify the weak limit v by the following computation: for testing functions $\varphi\in C_0^\infty(E_T)$ we have

$$\begin{aligned}\iint_{E_T} v\cdot\varphi\,dxdt &= \lim_{\ell\to\infty}\iint_{E_T} Du_\ell^m\cdot\varphi\,dxdt \\ &= -\lim_{\ell\to\infty}\iint_{E_T} u_\ell^m\cdot D\varphi\,dxdt \\ &= -\iint_{E_T} u^m\cdot D\varphi\,dxdt\,. \end{aligned}\tag{4.17}$$

This implies that the function u^m is weakly differentiable with weak derivative v, i.e. $Du^m=v$. Hence, we can conclude that after passing to a (non re-labelled) subsequence the sequence of functions u_ℓ admits the following convergence properties:

$$\begin{cases} u_\ell\to u & \text{strongly in } L^{mp}(E_T) \text{ for any } 1\le p<1+\frac{2}{nm},\\ u_\ell(x,t)\to u(x,t) & \text{for almost every } (x,t)\in E_T, \text{ and}\\ Du_\ell^m\to Du^m & \text{weakly in } L^q(E_T,\mathbb{R}^n) \text{ for any } 1\le q<1+\frac{1}{1+nm}.\end{cases}\tag{4.18}$$

4.4. **Passing to the limit in the equation.** Due to the convergence properties (4.18) of the approximating sequence u_ℓ, the claim follows easily. Indeed, let φ be a testing function as in Definition 1.1, which means that $\varphi\in C^\infty(\overline{E_T})$ vanishes in a neighborhood of $[\partial E\times(0,T)]\cup[\overline{E}\times\{T\}]$. For such a testing function we have to show that in the weak formulation of (4.3), i.e. in

$$\iint_{E_T}\big[-u_\ell\varphi_t + \mathbf{a}_{ij}(x,t,u_\ell)D_iu_\ell^mD_j\varphi\big]\,dxdt = \iint_{E_T}\mu_\ell\varphi\,dxdt\,,$$

we can pass to the limit $\ell\to\infty$. Recalling that the exponent p in $(4.18)_1$ can be chosen in such a way that $mp\ge1$, we can pass to the limit in the first term of the left-hand side. Moreover, by the weak $*$ convergence of $\mu_\ell dxdt\to\mu$, we can also pass to the limit in the right-hand side. Finally, due to the a.e. convergence $(4.18)_2$, the dominated convergence theorem and (1.2), we conclude that $\mathbf{a}_{ij}(x,t,u_\ell)D_j\varphi$ converges strongly in $L^r(E_T)$ to $\mathbf{a}_{ij}(x,t,u)D_j\varphi$. Therefore, using the weak convergence $(4.18)_3$, we can also pass to the limit in the second term on the left-hand side, and conclude that the limit $u \in L^1(E_T)$ with $u^m \in L^p(E_T) \cap L^q(0,T;W^{1,q}(E))$ is the desired very weak solution. This proves the claim and concludes the proof of Theorem 1.1.

4.5. **Properties of the truncated solution.** For later reference we will show here, that $T_k(u) \in L^2(0,T;W_0^{1,2}(E))$. From Remark 4.2 we infer that

$$\iint_{E_T} |DT_k(u_\ell)^m|^2 \, dxdt \le \boldsymbol{\gamma}\, k^{1+\lambda} \bigg[\iint_{E_T} \mu \, dxdt + |E| \bigg],$$

for any $k \in \mathbb{N}$. Since $T_k(u_\ell) \to T_k(u)$ in $L^1(E_T)$, we also have that $T_k^m(u_\ell) \to T_k^m(u)$ strongly in $L^2(E_T)$ as $\ell \to \infty$. Moreover, by the preceding energy estimate we may extract a non re-labelled subsequence such that $DT_k^m(u_\ell) \rightharpoonup v$ weakly in $L^2(E_T,\mathbb{R}^n)$ for some $v \in L^2(E_T,\mathbb{R}^n)$ as $\ell \to \infty$. Arguing as in (4.17), we identify the weak limit as $v = DT_k^m(u)$. This implies the claim $T_k(u) \in L^2(0,T;W_0^{1,2}(E))$.

5. Proof of Theorem 1.2

We first note that $u \in L^s(E_T)$ for any $1 \le s < m + \frac{2}{n}$ by (1.7). As before, we abbreviate $T_k(u) := \min\{u,k\}$. From § 4.5 we know that $T_k(u) \in L^2(0,T;W_0^{1,2}(E))$. Although we stated and proved Lemma 3.1 for weak energy solutions, it is straightforward to see that it can be extended to very weak solutions. Therefore, such a lemma ensures that $T_k(u)$ is a bounded weak energy super-solution of (1.4). By the Riesz representation theorem, there exists a non-negative Radon-measure ν_k such that

$$\iint_{E_T} \big[-T_k(u)\partial_t\eta + \mathbf{a}_{ij}(x,t,T_k(u))D_i[T_k(u)]^m D_j\eta \big] \, dxdt = \int_{E_T} \eta \, d\nu_k$$

for any $\eta \in C_0^1(E_T)$. Since the left-hand side converges to

$$\iint_{E_T} \big[-u\partial_t\eta + \mathbf{a}_{ij}(x,t,u)D_iu^m D_j\eta \big] \, dxdt,$$

in the limit $k \to \infty$, by the uniqueness of the limit we conclude that $\nu_k \to \mu$ in the sense of Radon-measures. Take the proof of [7, Theorem 1] and build the sequence a_j as done either in (4.5) or (4.7) of that paper, in terms of u. Then, for fixed $j \in \mathbb{N}$, let $k > a_j$ and consider $T_k(u)$. This ensures that $\{u > a_j\} = \{T_k(u) > a_j\}$. Moreover, the energy estimates of [7, § 3] can be applied to $T_k(u)$. Now, in [7, (4.20)], we remark that

$$\kappa = \frac{1}{|Q_j|} \iint_{Q_j \cap \{u > a_j\}} \Big(\frac{u - a_j}{d_j} \Big)^{1+\lambda} dxdt = \lim_{k\to\infty} \kappa_k,$$

where κ_k is defined by

$$\kappa_k := \frac{1}{|Q_j|} \iint_{Q_j \cap \{T_k(u) > a_j\}} \bigg(\frac{T_k(u) - a_j}{d_j} \bigg)^{1+\lambda} dxdt.$$

From here on, we perform all the computations of [7, § 4], with $\kappa_k, T_k(u), \nu_k$ instead of κ, u, μ, to obtain

$$d_j \le \tfrac{1}{2} d_{j-1} + \frac{\gamma\, \nu_k(2Q_{r_j,\theta_j})}{|B_j|}$$

for any k and j, such that $k > a_j$. Using the upper semicontinuity of the convergence of measures on closed sets, we can pass to the limit $k \to \infty$ and recover

$$d_j \le \tfrac{1}{2} d_{j-1} + \frac{\gamma\, \mu(\overline{2Q}_{r_j,\theta_j})}{|B_j|} \le \tfrac{1}{2} d_{j-1} + \frac{\gamma\, \mu(3Q_{r_j,\theta_j})}{|B_j|},$$

where in the last step we used that $\mu(E \times \{t\}) = 0$ for almost every $t \in (0, T)$. This plays the role of estimate [7, (4.13)], with $3Q_{r_j,\theta_j}$ instead of $2Q_{r_j,\theta_j}$. From here on the proof is finished as in [7, § 4].

6. A PRIORI ESTIMATES FOR $\mu \in L^s(E_T)$

In this final section we give the proof of Theorem 1.3. The starting point is again an a priori estimate below the natural growth for weak energy solutions of the Cauchy-Dirichlet problem (1.4) with a right-hand side measure given now by a non-negative function $\mu \in L^s(E_T)$ for some $s > 1$. This assumption allows to test the weak formulation of the singular porous medium equation with the testing function $v := (1+u^m)^{1-\lambda} - 1$ for some $\lambda \in (0, 1)$.

Lemma 6.1. *Let $\lambda \in (0,1)$, and assume that u is a weak energy solution to the Cauchy-Dirichlet problem* (1.10) *with a non-negative right-hand side $\mu \in L^s(E_T)$ for some $s > 1$, where the structure conditions* (1.2) *are in force. Then, the following improved energy estimate*

$$\sup_{0<t<T} \int_{E\times\{t\}} u^{m(1-\lambda)+1}\, dx + \iint_{E_T} \frac{|Du^m|^2}{(1+u^m)^\lambda}\, dxdt \le \boldsymbol{\gamma} \bigg[|E| + \|\mu\|_{L^s(E_T)} \bigg(\iint_{E_T} u^{m\frac{s(1-\lambda)}{s-1}}\, dxdt \bigg)^{1-\frac{1}{s}} \bigg],$$

holds true, for a constant $\boldsymbol{\gamma}$ depending only on m, C_o and λ.

Proof. Let $\lambda \in (0, 1)$. We define the testing function

$$\varphi := \xi v = \xi(t)\big((1+u^m)^{1-\lambda} - 1\big),$$

where $\xi \in W^{1,\infty}(\mathbb{R})$ satisfies $\xi \equiv 1$ for $t \le \tau - \delta$, $\xi \equiv 0$ for $t \ge \tau$, and $\xi(t) = -\frac{1}{\delta}(t - \tau)$ for $\tau - \delta < t < \tau$. Here, $\tau \in (0, T)$ and $\delta \in (0, \tau)$. We note that $\varphi = 0$ on $\partial_{\rm par} E_T$. Testing the weak formulation of $(1.4)_1$ with φ we obtain that

$$\iint_{E_T} \big[u_t \varphi + \mathbf{a}_{ij}(x,t,u) D_i u^m D_j \varphi\big]\, dxdt = \iint_{E_T} \mu\varphi\, dxdt$$

holds true. For the term involving the time derivative we obtain that

$$\begin{aligned}\iint_{E_T} u_t \varphi\, dxdt &= \iint_{E_T} \xi \frac{\partial}{\partial t} \bigg[\int_0^u \big[(1+s^m)^{1-\lambda} - 1\big]\, ds \bigg] dxdt \\ &= -\iint_{E_T} \partial_t \xi \int_0^u \big[(1+s^m)^{1-\lambda} - 1\big]\, ds\, dxdt \\ &= \frac{1}{\delta} \iint_{E\times(\tau-\delta,\tau)} \int_0^u \big[(1+s^m)^{1-\lambda} - 1\big]\, ds\ dxdt\,.\end{aligned}$$

In the preceding identity we let $\delta \downarrow 0$ and obtain for almost every $\tau \in (0, T)$ (note that $u \in C^0([0, T]; L^{1+m}(E))$) that

$$\lim_{\delta\downarrow 0} \iint_{E_T} u_t \varphi\, dxdt = \int_{E\times\{\tau\}} \int_0^u \big[(1+s^m)^{1-\lambda} - 1\big]\, ds\ dx.$$

To proceed further we define

$$\Phi_\lambda(\sigma) := \int_0^\sigma \big[(1+s^m)^{1-\lambda} - 1\big]\, ds.$$

Since $[0,\infty) \ni \sigma \mapsto (1+\sigma^m)^{1-\lambda} - 1$ is strictly increasing, we can bound $\Phi_\lambda(\sigma)$ from below as follows:

$$\begin{aligned}\Phi_\lambda(\sigma) &\ge \int_{\sigma/2}^\sigma \big[(1+s^m)^{1-\lambda} - 1\big]\, ds \ge \Big[\Big(1+\big(\tfrac{\sigma}{2}\big)^m\Big)^{1-\lambda} - 1\Big]\tfrac{\sigma}{2} \ge \big(\tfrac{\sigma}{2}\big)^{m(1-\lambda)+1} - \tfrac{\sigma}{2} \\ &\ge \tfrac12\big(\tfrac{\sigma}{2}\big)^{m(1-\lambda)+1} - 2^{\frac{1}{m(1-\lambda)}} \ge \tfrac18 \sigma^{m(1-\lambda)+1} - 2^{\frac{1}{m(1-\lambda)}}.\end{aligned}$$

This implies that

$$\lim_{\delta\downarrow 0} \iint_{E_T} u_t\varphi\, dxdt \ge \tfrac18 \int_{E\times\{\tau\}} u^{m(1-\lambda)+1}\, dx - \boldsymbol{\gamma}|E|,$$

for a constant $\boldsymbol{\gamma}$ depending only on m and λ. Next, we need to compute Dv. We obtain

$$Dv = \frac{(1-\lambda)Du^m}{(1+u^m)^\lambda}\,.$$

With this expression the diffusion term in the weak formulation can easily be estimated from below by the ellipticity condition $(1.2)_1$ as follows:

$$\begin{aligned}\iint_{E_T} \mathbf{a}_{ij}(x,t,u) D_i u^m D_j\varphi\, dxdt &= (1-\lambda)\iint_{E_T} \xi\, \mathbf{a}_{ij}(x,t,u)\frac{D_i u^m D_j u^m}{(1+u^m)^\lambda}\, dxdt \\ &\ge (1-\lambda) C_o \iint_{E_T} \frac{\xi |Du^m|^2}{(1+u^m)^\lambda}\, dxdt\,.\end{aligned}$$

This, together with the estimate for the term stemming from the time derivative and after passing to the limit $\delta\downarrow 0$, yields

$$\begin{aligned}\tfrac18 \int_{E\times\{\tau\}} u^{m(1-\lambda)+1}\, dx &+ C_o(1-\lambda)\iint_{E_\tau} \frac{|Du^m|^2}{(1+u^m)^\lambda}\, dxdt \\ &\le \boldsymbol{\gamma}|E| + \iint_{E_T} \mu\,\big((1+u^m)^{1-\lambda} - 1\big)\, dxdt \\ &\le \boldsymbol{\gamma}|E| + \iint_{E_T} \mu\, u^{m(1-\lambda)}\, dxdt \\ &\le \boldsymbol{\gamma}|E| + \|\mu\|_{L^s(E_T)}\bigg(\iint_{E_T} u^{m\frac{s(1-\lambda)}{s-1}}\, dxdt\bigg)^{1-\frac1s}\end{aligned}$$

for a.e. $\tau\in(0,T]$. Joining the preceding inequalities, we finally arrive at the *improved energy estimate* for a constant $\boldsymbol{\gamma}$ depending only on m, C_o and λ. We note that the constant $\boldsymbol{\gamma}$ blows up when $\lambda\uparrow 1$. □

Lemma 6.1 can now be used, to improve the a priori estimate for the integrability of Du^m. This is possible, since by assumption the right-hand side μ admits a certain integrability in Lebesgue-spaces. The precise statement is as follows:

Lemma 6.2. *Let the measure μ be given by a function $\mu\in L^s(E_T)$, where*

$$1 < s < 1 + \frac{nm}{nm+2+2m},$$

and let u be a weak energy solution to the Cauchy-Dirichlet problem (1.4)*, where the structure conditions* (1.2) *are in force. Then $u^m \in L^q\big(0,T; W_0^{1,q}(E)\big)$, where*

$$q := \frac{s(nm+2)}{nm+m+1-ms}.$$

Proof. We use the Lemma 6.1 in order to bound the L^q-norm of Du^m for some $q \in [1,2)$. The range of s which allows this procedure and the precise value of q will be specified in the course of the proof. As mentioned before, by Hölder's inequality and Lemma 6.1 we obtain

$$\begin{aligned}
&\iint_{E_T} |Du^m|^q \,dxdt = \iint_{E_T} \frac{|Du^m|^q}{(1+u^m)^{\frac{q\lambda}{2}}} (1+u^m)^{\frac{q\lambda}{2}} \,dxdt \\
&\le \bigg(\iint_{E_T} \frac{|Du^m|^2}{(1+u^m)^{\lambda}} \,dxdt \bigg)^{\frac{q}{2}} \bigg(\iint_{E_T} (1+u^m)^{\frac{q\lambda}{2-q}} \,dxdt \bigg)^{1-\frac{q}{2}} \\
&\le \boldsymbol{\gamma} \bigg[|E| + \|\mu\|_{L^s} \bigg(\iint_{E_T} u^{m\frac{s(1-\lambda)}{s-1}} \,dxdt \bigg)^{1-\frac{1}{s}} \bigg]^{\frac{q}{2}} \bigg(|E_T| + \iint_{E_T} u^{m\frac{q\lambda}{2-q}} \,dxdt \bigg)^{1-\frac{q}{2}}.
\end{aligned}$$

Suppose for the moment that $\lambda \in (0,1)$ and $q < 2$ can be fixed such that

$$\kappa := \frac{s(1-\lambda)}{s-1} = \frac{\lambda q}{2-q}$$

holds true. Then the last inequality turns into

$$\iint_{E_T} |Du^m|^q \,dxdt \le \boldsymbol{\gamma} \,(1 + \|\mu\|_{L^s})^{\frac{q}{2}} \bigg(1 + \iint_{E_T} u^{m\kappa} \,dxdt \bigg)^{1-\frac{q}{2s}}, \tag{6.1}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(m, C_o, \lambda, |E|, T)$. For the integral on the right-hand side of the preceding inequality we use the parabolic Sobolev embedding from Lemma 2.1 to conclude that

$$\iint_{E_T} u^{m\kappa} \,dxdt \le \boldsymbol{\gamma} \sup_{t\in(0,T)} \bigg(\int_{E\times\{t\}} u^{mr} \,dx \bigg)^{\frac{q}{n}} \iint_{E_T} |Du^m|^q \,dxdt\,,$$

where r and q are chosen as follows:

$$r = 1 - \lambda + \frac{1}{m} \quad \text{and} \quad \kappa = \frac{q(n+r)}{n} = q\Big(1 + \frac{1-\lambda}{n} + \frac{1}{nm} \Big).$$

These choices lead after some elementary, but lengthy and exhausting computations to the following values of q, κ and λ:

$$q = \frac{s(nm+2)}{nm+m+1-ms}, \quad \kappa = \frac{s(nm+2)}{m(n+2-2s)} \quad \text{and} \quad \lambda = 1 - \frac{(s-1)(nm+2)}{m(n+2-2s)}.$$

To have $\lambda > 0$, we need to require that

$$1 < s < 1 + \frac{nm}{nm+2+2m}.$$

This hypothesis also implies that $s < 1 + \frac{n}{2}$, which is needed for $\kappa > 0$. Finally, the bound on s also implies that $1 < q < 2$, so that the above application of Lemma 2.1 is justified. With these choices, the last inequality, (6.1), and Lemma 6.1 together yield an estimate for the L^κ-norm of u^m. The precise estimate is as follows:

$$\iint_{E_T} u^{m\kappa} \,dxdt \le \boldsymbol{\gamma} \sup_{t\in(0,T)} \bigg(\int_{E\times\{t\}} u^{m(1-\lambda)+1} \,dx \bigg)^{\frac{q}{n}} \iint_{E_T} |Du^m|^q \,dxdt$$

$$\le \boldsymbol{\gamma}\Big[1+\|\mu\|_{L^s}^{\frac{q}{n}+\frac{q}{2}}\Big]\bigg[1+\iint_{E_T} u^{m\kappa}\,dxdt\bigg]^{1+\frac{q}{n}-\frac{q}{ns}-\frac{q}{2s}}.$$

In order to re-absorb the L^κ-norm of u in the left-hand side we need that

$$1+\frac{q}{n}-\frac{q}{ns}-\frac{q}{2s}<1,$$

which holds true, since $s<1+\frac{n}{2}$. Hence, by Young's inequality we obtain

$$\iint_{E_T} u^{m\kappa}\,dxdt \le \boldsymbol{\gamma}\big[1+\|\mu\|_{L^s}\big]^{\frac{s(n+2)}{n+2-2s}} + \tfrac{1}{2}\iint_{E_T} u^{m\kappa}\,dxdt\,,$$

which gives the desired estimate for the L^κ-norm of u^m in terms of $\|\mu\|_{L^s}$. The precise estimate is as follows:

$$\bigg(\iint_{E_T} u^{m\kappa}\,dxdt\bigg)^{\frac{1}{\kappa}} \le \boldsymbol{\gamma}\big[1+\|\mu\|_{L^s}\big]^{\frac{m(n+2)}{nm+2}}.$$

This finishes the proof of the lemma. □

The proof of Theorem 1.3 is now an easy consequence of the a priori estimate for the $L^q(E_T)$-norm of Du_j^m from Lemma 6.2 for the approximating weak energy solutions u_j. Such an estimate is obviously preserved in the limit $j\to\infty$.

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