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REPORT No. 37, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -37-13/14- -SE+fall

CHARACTERIZATIONS OF SETS OF FINITE PERIMETER USING HEAT KERNELS IN METRIC SPACES

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Abstract. The overarching goal of this paper is to link the notion of sets of finite perimeter (a concept associated with $N^{1,1}$-spaces) and the theory of heat semigroups (a concept related to $N^{1,2}$-spaces) in the setting of metric measure spaces whose measure is doubling and supports a 1-Poincaré inequality. We prove a characterization of sets of finite perimeter in terms of a short time behavior of the heat semigroup in such metric spaces. We also give a new characterization of BV functions in terms of a near-diagonal energy in this general setting.

Contents

1. Introduction

In this paper we study functions of bounded variation and, in particular, sets of finite perimeter on general metric measure spaces. More precisely, we investigate a relation between the perimeter of a set and the short-time behavior of the action of a

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2010 *Mathematics Subject Classification.* 31C25, 26B30, 46E35, 35K05.

Key words and phrases. Bakry–Émery condition, bounded variation, Dirichlet form, doubling measure, heat kernel, heat semigroup, isoperimetric inequality, metric space, perimeter, Poincaré inequality, sets of finite perimeter, total variation.

heat semigroup on the characteristic function of such a set. First, we provide a new characterization of BV functions in terms of asymptotic behavior of a near-diagonal energy or, in other words, the near-diagonal part of the Korevaar–Schoen type energy [KS]. Second, we shall give a Ledoux-type characterization of sets of finite perimeter in terms of the heat semigroup [L].

The characterizations obtained in this paper connect the theory of functions of bounded variation and sets of finite perimeter to the theory of the heat semigroup and heat flow in metric spaces, thus connecting the nonlinear potential theory associated with the index $p = 1$ to the linear potential theory associated with the index $p = 2$. In the Euclidean setting such a connection is well-established, see for example [DeG] and [L]. In the setting of Riemannian manifolds with lower bounded Ricci curvature, see [CM], [MP], and [MP2]. In the more general metric setting there is precedence; indeed, it was shown in [JK] that when the measure on a metric measure space X is doubling, supports a 2-Poincaré inequality, and satisfies a curvature condition, a global isoperimetric inequality holds in X. In this note we are more interested in characterizing sets of finite perimeter in terms of the asymptotic behavior of the heat extension of the characteristic function of the sets, and to do so we need the stronger assumption of 1-Poincaré inequality. It is not known whether the curvature condition assumed in [JK], together with the doubling property of the measure and the support of a 2-Poincaré inequality, implies the support of a 1-Poincaré inequality.

1.1. **Ledoux-type characterization.** It was observed by Ledoux in [L] that the classical isoperimetric inequality in $\mathbb{R}^n$, and also in more general Gauss spaces, can be characterized by the fact that the L^2-norm of the heat semigroup acting on the characteristic function of sets is increasing under isoperimetric rearrangement. More precisely, condition that a set B is isoperimetric, that is a minimizer of the perimeter measure among all sets E with the same measure as B, is equivalent to the following L^2-inequality

$$\|T_t\chi_E\|_{L^2(\mathbb{R}^n)} \leq \|T_t\chi_B\|_{L^2(\mathbb{R}^n)}$$

for all $t \geq 0$ and sets $E \subset \mathbb{R}^n$ with the same volume as the Euclidean ball B. Here χ_E denotes the characteristic function of the set E, and T_t stands for the heat semigroup defined on $L^2(\mathbb{R}^n)$ by convolution with the classical Gauss–Weierstrass kernel so that T_tf solves the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u \\ u(0,x) = f(x), \end{cases}$$

with $f \in L^2(\mathbb{R}^n)$. In particular, it was shown in [L] (the reader is recommended to also see the paper [P] by Preunkert) that

$$\lim_{t\to 0^+} \sqrt{\frac{\pi}{t}} \int_{\mathbb{R}^n\setminus B} T_t\chi_B(x)\,dx = P(B), \tag{1.1}$$

whenever B is a Euclidean ball and $P(B)$ the perimeter of B in $\mathbb{R}^n$. Moreover, the inequality

$$\sqrt{\frac{\pi}{t}} \int_{\mathbb{R}^n\setminus E} T_t\chi_E(x)\,dx \leq P(E)$$

holds for every $t \geq 0$ and for all subsets E of $\mathbb{R}^n$ with finite measure.

The authors in [MP2] pursued the investigation of the relation between the perimeter of a set and the short-time behavior of the heat semigroup as described in (1.1). They observed, by considering the measure-theoretic properties of the reduced boundary of a set, that equality (1.1) is actually valid for all sets of finite perimeter. In addition, the finiteness of the limit on the left hand side in (1.1) is enough to characterize sets of finite perimeter in $\mathbb{R}^n$. Similar chracterizations have also been obtained in [B], [D], and [P-P], where more general convolution kernels than the classical Gauss–Weierstrass kernel were considered. On the other hand, the approach taken in [MP2] has more geometric flavor.

In the present paper, we study characterizations of sets of finite perimeter and hence, by the co-area formula, all functions in the BV-class, in terms of the heat semigroup. The recent results in [JK] demonstrate that under some curvature assumptions on the metric measure space, if the space is also doubling in measure and supports a 2-Poincaré inequality, a (global) isoperimetric inequality follows. Adding this to the discussion above, it is reasonable to ask whether the notion of a set of finite perimeter in a metric measure space (first defined in [Mr]), using L^1-approximations by Lipschitz functions, is connected to the behavior of heat extension of the characteristic function of the set. In this paper we show that a Ledoux-type characterization of sets of finite perimeter in terms of the heat extension of the characteristic function of such sets is possible if the measure on the space is doubling and supports a 1-Poincaré inequality.

1.2. **De Giorgi-type characterization.** In the Euclidean setting the original definition of sets of finite perimeter in terms of heat extension, is due to De Giorgi [DeG]. In Section 5 we consider a characterization of total variation and BV functions. Such a characterization is related to the definition of sets of finite perimeter considered by De Giorgi [DeG] making use of a regularization procedure based on the heat kernel. Indeed, for a function $u \in L^1(\mathbb{R}^n)$ the limit

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} |\nabla T_t u(x)| \, dx \tag{1.2}$$

exists and is finite if, and only if, the distributional gradient of u is an $\mathbb{R}^n$-valued measure Du with finite total variation $|Du|(\mathbb{R}^n)$. Moreover, the limit in (1.2) equals $|Du|(\mathbb{R}^n)$. This result has been generalized to the setting of Riemannian manifolds in [CM], with the restriction that the Ricci curvature of a manifold is bounded from below. We refer also to the result in [MP], where further bounds on the geometry of the manifold were assumed. We refer also to a recent paper [GP] where a more general condition on the Ricci curvature has been considered, that is the Ricci tensor can be splitted into a sum of two terms, one which is bounded from below and the second one belonging to a suitable Kato class. We mention, in passing, that in the setting of Carnot groups the authors of [BMP] showed that the aforementioned result is valid in a weaker sense, namely that both the limit inferior and superior are comparable to the total variation of u, but it is not known whether equality holds.

In Section 5, we shall give a generalization of the preceding result to metric measure spaces as discussed above by imposing an additional assumption on the Dirichlet form $\mathcal{E}$ associated with X and defined in terms of the Cheeger differentiable structure. We shall assume that the Dirichlet form satisfies the Bakry–Émery condition $\mathrm{BE}(K,\infty)$ (see Definition 5.1), for some $K \in \mathbb{R}$. Providing the analogous self-improvement property of the $\mathrm{BE}(K,\infty)$ condition as obtained by Savaré in a recent paper [S], we obtain in Proposition 5.2 a metric space version of the De Giorgi characterization of the total variation of a BV function. We point out here, however, that the condition $\mathrm{BE}(K,\infty)$ is not satisfied by some Carnot groups, for example Heisenberg groups, and so our discussion does not overlap with that of [BMP].

1.3. **Organization of the paper.** We have organized our paper as follows. In Section 2 we recall the tools needed for our analysis in metric measure spaces as well as the basic properties of the heat semigroup and BV functions in this setting. Our main results are then stated in Theorem 3.1 and Theorem 4.1 in Section 3 and Section 4, respectively. In Section 5 we consider a characterization of total variation and BV functions. In the appendix, see Section 6, we gather together properties of the Bakry–Émery condition needed in Section 5.

Acknowledgement. The research was mostly carried out during the stay of the three authors at Institut Mittag–Leffler, Sweden, in Fall 2013, and finalized while N.M. and M.M. were visiting N.S. at the University of Cincinnati, USA, in Spring 2014. The authors wish to thank these both institutions for the support and kind hospitality.

N.M. was supported by the Academy of Finland and the Väisälä Foundation. The research of M.M. was supported partially by the PRIN 2012 and partially by FAR grant of the Department of Mathematics and Computer Science of University of Ferrara. N.S. was partially funded by NSF grant #DMS-1200915.

2. Basic concepts

In this section we recall the basic concepts that allow for nonsmooth analysis on metric measure spaces.

2.1. **Standing assumptions.** *Throughout the paper we will assume that (X,d,μ) is a complete metric measure space equipped with a Borel regular doubling measure μ supporting a 1-Poincaré inequality.*

Recall that a Borel-regular measure μ is doubling if there exists a constant $c \geq 1$ such that for every ball $B_r(x) := B(x,r) = \{y \in X : \ d(x,y) < r\}$, $x \in X$ and $r > 0$,

$$0 < \mu(B_{2r}(x)) \leq c_D \mu(B_r(x)) < \infty, \qquad \forall x \in X,\ r > 0;$$

we shall denote by c_D the minimal constant verifying the previous condition. Moreover, (X,d,μ) supports a 1-Poincaré inequality if there exist constants $c > 0$ and $\lambda \geq 1$ such that for any $u \in \mathrm{Lip}(X)$ (real-valued Lipschitz continuous function on X), the inequality

$$\int_{B_r(x)} |u(y) - u_{B_r(x)}|\, d\mu(y) \leq c_P r \int_{B_{\lambda r}(x)} \mathrm{lip}(u)(y)\, d\mu(y)$$

holds, where $\mathrm{lip}(u)$ is the local Lipschitz constant of u defined as

$$\mathrm{lip}(u)(y) := \liminf_{\varrho \to 0^+} \sup_{z \in B_\varrho(y)} \frac{|u(y) - u(z)|}{d(y,z)};$$

we shall denote by c_P the minimal constant verifying the Poincaré inequality. We write the integral average of a function u over a ball $B_r(x)$ as $u_{B_r(x)}$. Let us mention in passing that, by the doubling property, for every $B_R(x) \subset X$ and $y \in B_R(x)$ and for $0 < r \leq R < \infty$ the inequality

$$\frac{\mu(B_R(x))}{\mu(B_r(y))} \leq C \left(\frac{R}{r}\right)^{q_\mu}, \tag{2.1}$$

holds, where C is a positive constant depending only on c_D and $q_\mu = \log_2 c_D$. In what follows, q_μ denotes a counterpart of dimension related to the measure μ on X. In what follows, and unless otherwise stated, C denotes a positive constant whose exact value is not important and it may change even within a line. A concentric α-dilate, $\alpha > 1$, of a ball $B = B(x.r)$ is written as αB.

2.2. **Differentiable structure.** An upper gradient for an extended real-valued function u on X is a Borel function $g : X \to [0, \infty]$ such that

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds \tag{2.2}$$

for every nonconstant compact rectifiable curve $\gamma : [0, l_\gamma] \to X$. This notion is due to [HK]. We say that g is a p–weak upper gradient of u if (2.2) holds for p–almost every curve, see [KM]. If u has an upper gradient in $L^p(X)$, then there exists a unique minimal p–weak upper gradient $g_u \in L^p(X)$ of u, where $g_u \leq g$ μ-a.e. for every p–weak upper gradient $g \in L^p(X)$ of u.

Under our standing assumptions on X, we have also the Cheeger differentiable structure available (see [C]) that allow us to define a linear gradient operator for Lipschitz functions. There exists a countable measurable partition U_α of X, and Lipschitz coordinate charts $X^\alpha = (X_1^\alpha, \dots, X_{k_\alpha}^\alpha) : X \to \mathbb{R}^{k_\alpha}$ such that $\mu(U_\alpha) > 0$ for each α, and $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$. Moreover, for all α the charts $(X_1^\alpha, \dots, X_{k_\alpha}^\alpha)$ are linearly independent on U_α and $1 \leq k_\alpha \leq N$, where N is a constant depending on c_D, c_P, and λ, and in addition for any Lipschitz function $u : X \to \mathbb{R}$ there is an associated unique (up to a set of zero μ-measure) measurable function $D_\alpha u : U_\alpha \to \mathbb{R}^{k_\alpha}$ for which the following Taylor-type approximation

$$u(x) = u(x_0) + D_\alpha u(x_0) \cdot (X_\alpha(x) - X_\alpha(x_0)) + o(d(x, x_0))$$

holds for μ-a.e. $x_0 \in U_\alpha$. In particular, for $x \in U_\alpha$ there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^{k_\alpha}$ equivalent to the Euclidean norm $|\cdot|$, such that $g_u(x) = \|D_\alpha u(x)\|_x$ for almost every $x \in U_\alpha$. Moreover, it is possible to show that there exists a constant $c > 1$ such that $c^{-1} g_u(x) \leq |Du(x)| \leq c g_u(x)$ for all Lipschitz functions u and μ-a.e. $x \in X$. By $Du(x)$ we mean $D_\alpha u(x)$ whenever $x \in U_\alpha$. Indeed, one can choose the cover such that $U_\alpha \cap U_\beta$ is empty whenever $\alpha \neq \beta$.

For the definition of the Sobolev spaces $N^{1,p}(X)$ we will follow [Sh]. Since we assume X to satisfy a 1-Poincaré inequality, the Sobolev space $N^{1,p}(X)$, $1 \leq p < \infty$, can also be defined as the closure of the collection of Lipschitz functions on X in the $N^{1,p}$-norm defined as $\|u\|_{1,p}^p = \|u\|_{L^p(X)}^p + \|g_u\|_{L^p(X)}^p$. The space $N^{1,p}(X)$ equipped with the $N^{1,p}$-norm is a Banach space and a lattice [Sh]. By [FHK], the Cheeger differentiable structure extends to all functions in $N^{1,p}(X)$.

2.3. **Semigroup associated with $\mathcal{E}$.** In the metric space setting, there is a standard way to construct a semigroup by using the Dirichlet forms approach. The best way to construct it is to use the L^2–theory of (bilinear) Dirichlet forms and then apply a classical result asserting that such semigroup can be extrapolated to any L^p, $1 \le p \le \infty$.

We start with the Dirichlet form $\mathcal{E} : L^2(X) \times L^2(X) \to [-\infty, \infty]$ defined in terms of the Cheeger differentiable structure by

$$\mathcal{E}(u,v) = \int_X Du(x) \cdot Dv(x)\, d\mu(x)$$

with domain $D(\mathcal{E}) := N^{1,2}(X)$ (if u or v does not belong to $N^{1,2}(X)$, then $\mathcal{E}(u,v) = \infty$). This bilinear form is an example of a regular and strongly local Dirichlet form as defined in [FOT]. The associated infinitesimal generator A acts on a dense subspace $D(A)$ of $N^{1,2}(X)$ so that for each $u \in D(A)$ and for every $v \in N^{1,2}(X)$,

$$\int_X vAu\, d\mu = -\mathcal{E}(u,v).$$

The operator A is dissipative in the sense that

$$\int_X uAu\, d\mu = -\mathcal{E}(u,u) \le 0$$

and is merely the Laplacian Δ when $X = \mathbb{R}^n$, the Cheeger differentiable structure is the standard Euclidean structure, and μ is the Lebesgue measure.

We also point out that under our standing assumptions, the metric induced by the form

$$d_{\mathcal{E}}(x,y) = \sup\left\{f(x) - f(y) \colon f \in N^{1,2}(X),\ |Df| \le 1\ \mu\text{-a.e.}\right\}$$

is bi-Lipschitz equivalent to the original metric d.

Remark 2.1. *Associated with the Dirichlet form $\mathcal{E}$ and its infinitesimal generator A there is a Markov semigroup $(T_t)_{t>0}$ acting on $L^2(X)$ with the following properties (we refer to* [FOT], [KRS], *or* [MMS] *for properties (1) through (7)):*

(1) *$T_t \circ T_s = T_{t+s}$ for all $t, s > 0$;*
(2) *since A is symmetric in $L^2(X)$, then T_t is self adjoint in $L^2(X)$, that is*

$$\int_X T_t f g\, d\mu = \int_X f T_t g\, d\mu,$$

for all $f, g \in L^2(X)$;
(3) *$T_t f \to f$ in $L^2(X)$ when $t \to 0$;*
(4) *since A is dissipative, T_t is a contraction in $L^2(X)$, i.e. $\|T_t f\|_{L^2(X)} \le \|f\|_{L^2(X)}$ for all $f \in L^2(X)$ and $t > 0$;*
(5) *if $f \in D(A)$, then $\frac{1}{t}(T_t f - f) \to Af$ in $L^2(X)$ as $t \to 0$;*

(6) *for all* $t > 0$ *and* $f \in L^2(X)$, *in a weak sense*

$$\partial_t T_t f = A T_t f;$$

(7) $(T_t)_{t>0}$ *is a Markovian semigroup, that is if* $0 \leq f \leq 1$, *then* $0 \leq T_t f \leq 1$;

(8) $(T_t)_{t>0}$ *can be extended to* $L^\infty(X)$ *by first considering positive functions* $f \geq 0$ *and sequences* $f_n \in L^2(X)$, $f_n \nearrow f$ *and setting*

$$T_t f = \lim_{n\to\infty} T_t f_n,$$

we refer to [FOT, p. 56];

(9) *since the measure* μ *is doubling,* $(T_t)_{t>0}$ *is stochastically complete :* $T_t 1 = 1$ [St1, Theorem 4] *(see also* [G] *for stochastic completeness in the manifold setting);*

(10) *Since* $|T_t f| \leq T_t|f|$ μ*-a.e. for* $f \in L^1(X) \cap L^\infty(X)$, *and since* $L^1(X) \cap L^\infty(X)$ *is dense in* $L^1(X)$, $(T_t)_t$ *can be extended to a contraction semigroup on* $L^1(X)$.

A measurable function $p\colon \mathbb{R} \times X \times X \to [0, \infty]$ is called a *heat kernel* or *transition function* on X associated with the semigroup $(T_t)_{t>0}$ if

$$T_t f(x) = \int_X p(t, x, y) f(y)\, d\mu(y),$$

for every $f \in L^2(X)$ and for every $t > 0$. For $t \leq 0$ we have $p(t, x, y) = 0$, and $p(t, x, y) = p(t, y, x)$ by the symmetry of the semigroup. The existence of a heat kernel is a direct consequence of the linearity of the operator $f \mapsto T_t f$ together with the L^∞-boundedness of such an operator (Markovian property) and the Riesz representation theorem, see for instance Sturm [St2, Proposition 2.3]. Standard regularity arguments on doubling metric measure spaces admitting a 2-Poincaré inequality imply that the map $x \mapsto p(t, x, y)$ is Hölder continuous for any $(t, y) \in (0, \infty) \times X$.

Under our standing assumptions on X, we have the following estimates for a heat kernel uniformly for all $x, y \in X$ and all $t > 0$, we refer to [St3, Corollary 4.10]. There are positive constants C, C_1, C_2 such that

$$p(t, x, y) \geq \frac{C^{-1} e^{-\frac{d(x,y)^2}{C_1 t}}}{\sqrt{\mu(B_{\sqrt{t}}(x))}\sqrt{\mu(B_{\sqrt{t}}(y))}}, \tag{2.3}$$

$$p(t, x, y) \leq \frac{C e^{-\frac{d(x,y)^2}{C_2 t}}}{\sqrt{\mu(B_{\sqrt{t}}(x))}\sqrt{\mu(B_{\sqrt{t}}(y))}}. \tag{2.4}$$

2.4. **BV functions and sets of finite perimeter.** Following [Mr], we say that a function $u \in L^1(X)$ is of bounded variation ($u \in \mathrm{BV}(X)$) if

$$\|Du\|(X) = \inf\left\{\liminf_{j\to\infty} \int_X g_{u_j}\, d\mu : u_j \in \mathrm{Lip}_{\mathrm{loc}}(X),\ u_j \to u \text{ in } L^1_{\mathrm{loc}}(X)\right\}$$

is finite, where g_{u_j} is the minimal 1-weak upper gradient of u_j. Moreover, a Borel set $E \subset X$ is said to have finite perimeter if $\chi_E \in \mathrm{BV}(X)$. We denote the perimeter measure of E by $P(E) = \|D\chi_E\|(X)$. By replacing X with an open set F we may define $\|Du\|(F)$ and we shall write the perimeter measure of E with respect to F as $P(E, F) = \|D\chi_E\|(F)$.

Strictly speaking, the definition given in [Mr] considers $\mathrm{lip}(u_j)$ instead of g_{u_j}. However, under our standing assumptions of doubling property and 1-Poincaré inequality, $\mathrm{lip}(u_j) = g_{u_j}$ μ-a.e. in X (see [C]).

An equivalent definition can be also given by way of the Cheeger differentiable structure by replacing a 1-weak upper gradient of u_j with its Cheeger derivative. We shall say that u has bounded total Cheeger variation if $\|D_c u\|(X) < \infty$. A set with Cheeger finite perimeter is a μ-measurable set E such that $\|D_c\chi_E\|(X) < \infty$. By the results contained in [C], it follows that these two definitions are equivalent, in the sense that $C^{-1}\|D_c u\| \le \|Du\| \le C\|D_c u\|$.

It follows from a 1-Poincaré inequality that for each $u \in \mathrm{BV}(X)$

$$\int_{B_r(x)} |u(y) - u_{B_r(x)}|\, d\mu(y) \le c_P r \|Du\|(B_{2\lambda r}(x)).$$

The factor of 2 in the ball on the right-hand side follows from the fact that weak limits of measures might charge the boundary of $\lambda B_r(x)$. In particular, if E is a set of finite perimeter and $u = \chi_E$ we recover the following form of the relative isoperimetric inequality

$$\mu(B_r(x) \cap E)\frac{\mu(B_r(x) \setminus E)}{\mu(B_r(x))} \le c_P r P(E, B_{2\lambda r}(x)). \tag{2.5}$$

Lastly, we recall that for any $u \in \mathrm{BV}(X)$ and Borel set $E \subset X$ the co-area formula

$$\int_{-\infty}^{\infty} P(\{x \in X\colon u(x) > t\}, E)\, dt = \|Du\|(E) \tag{2.6}$$

holds, and for the proof, we refer to [Mr, Proposition 4.2].

3. A characterization of BV functions

In the setting of metric measure spaces satisfying the standing assumptions listed in Section 2, we now give a new characterization of BV functions in terms of the near-diagonal parts of the Korevaar–Schoen type energy [KS]. The product measure $\mu \otimes \mu$ in the space $X \times X$ shall be written as $\mu(x, y)$.

We point out here that in the more general setting of topological spaces X, with $X \times X$ equipped with a nonnegative symmetric Radon measure Γ locally finite outside the diagonal, Maz'ya proved in [M] a conductor inequality for compactly supported continuous functions f on X for which

$$\langle f\rangle_{p,\Gamma}^p := \int_{X\times X} |f(x) - f(y)|^p\, d\Gamma(x, y) < \infty.$$

It was shown in [M, Theorem 2, Section 4] that, when $1 \le p \le q < \infty$ and $a > 1$, the following co-area–type integral can be majorized by the energy $\langle f\rangle_{p,\Gamma}$

$$\left(\int_0^{\infty} \mathrm{cap}_{p,\Gamma}(\overline{M}_{at}, M_t)^{q/p}\, dt^q\right)^{p/q} \le C\langle f\rangle_{p,\Gamma}^p, \tag{3.1}$$

where $M_t = \{x \in X\colon |f(x)| > t\}$ and the capacity is obtained via minimization of $\langle\varphi\rangle_{p,\Gamma}^p$ over all admissible functions φ. With $p = q = 1$, in general the constant C in (3.1) depends on a, and blows up in the order of $(a-1)^{-1}$ as $a \to 1$, see [M].

If we now take Γ_a to be defined by

$$d\Gamma_a(x,y) = \frac{\chi_{\Delta^{a-1}}(x,y)\, d\mu(x,y)}{(a-1)\sqrt{\mu(B_{a-1}(x))}\sqrt{\mu(B_{a-1}(y))}},$$

we obtain a near-diagonal energy

$$\langle f\rangle_{1,\Gamma_a} = \frac{1}{a-1}\int_{\Delta^{a-1}} \frac{|f(x)-f(y)|}{\sqrt{\mu(B_{a-1}(x))}\sqrt{\mu(B_{a-1}(y))}}\, d\mu(x,y),$$

with the constant C in (3.1) now independent of a. Here Δ^ε, $\varepsilon > 0$, denotes the ε-neighborhood of the diagonal in $X \times X$, i.e.

$$\Delta^\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}.$$

We point out here that when $p = q = 1$, the above Maz'ya-type inequality, (3.1), in this setting, is equivalent to the the following generalization of the co-area inequality (with $\varepsilon = a - 1$)

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\infty \left(\int_{M_t}\int_{B_\varepsilon(x)\setminus M_t} \frac{d\mu(y)d\mu(x)}{\sqrt{\mu(B_\varepsilon(x))}\sqrt{\mu(B_\varepsilon(y))}}\right) dt \le C\, \limsup_{\varepsilon\to 0}\langle f\rangle_{1,\Gamma_{1+\varepsilon}}$$

associated with the near-diagonal Korevaar-Schoen energy.

In this section we will show that in our more specialized setting of metric measure spaces, the family (with respect to $\varepsilon > 0$) of above energies $\langle f\rangle_{1,\Gamma_{1+\varepsilon}}$ corresponding to a given function f has a finite limit infimum as $\varepsilon \to 0$ if and only if f is in the BV class. The proof will also show that for small $\varepsilon > 0$ the above energy is controlled by a constant multiple of the BV-energy of f. This also provides a connection between the energy studied by Maz'ya [M] in our, more specialized, setting and the BV-energy.

Let us now formulate the first main theorem of this section.

Theorem 3.1. *Suppose that $u \in L^1(X)$. Then $u \in \mathrm{BV}(X)$ if, and only if,*

$$\liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Delta^\varepsilon} \frac{|u(x)-u(y)|}{\sqrt{\mu(B_\varepsilon(x))}\sqrt{\mu(B_\varepsilon(y))}}\, d\mu(x,y) < \infty.$$

Proof. Let us assume first that $u \in \mathrm{BV}(X)$ and fix $\varepsilon > 0$. Then we can find a sequence of points in X, $\{x_i\}_{i\in\mathbb{N}}$, such that

$$X = \bigcup_{i\in\mathbb{N}} B_\varepsilon(x_i), \quad \text{and} \quad B_{\varepsilon/2}(x_i)\cap B_{\varepsilon/2}(x_j) = \emptyset \quad \text{whenever} \quad i \ne j,$$

and such that the covering has bounded overlap

$$\sum_{i\in\mathbb{N}} \chi_{B_{4\lambda\varepsilon}(x_i)}(x) \le c_O. \tag{3.2}$$

Since for $x \in B_\varepsilon(x_i)$ and $y \in B_\varepsilon(x)$ we have that $B_\varepsilon(x_i) \subset B_{2\varepsilon}(x)$ and $B_\varepsilon(x_i) \subset B_{4\varepsilon}(y)$, we get, by the doubling property of μ,

$$\mu(B_\varepsilon(x_i)) \le \mu(B_{4\varepsilon}(y)) \le c_D^2 \mu(B_\varepsilon(y)),$$

$$\mu(B_\varepsilon(x_i)) \le \mu(B_{2\varepsilon}(x)) \le c_D \mu(B_\varepsilon(x)).$$

We obtain

$$\begin{aligned}
&\int_{\Delta^\varepsilon} \frac{|u(x)-u(y)|}{\sqrt{\mu(B_\varepsilon(x))}\sqrt{\mu(B_\varepsilon(y))}}\, d\mu(x,y)\\
\le& c_O c_D^{\frac{3}{2}} \sum_{i\in\mathbb{N}} \frac{1}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} d\mu(x) \int_{B_\varepsilon(x)} |u(x)-u(y)|\, d\mu(y)\\
\le& c_O c_D^{\frac{3}{2}} \sum_{i\in\mathbb{N}} \frac{1}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} d\mu(x) \int_{B_\varepsilon(x)} |u(x)-u_{B_\varepsilon(x_i)}|\, d\mu(y)\\
&+ c_O c_D^{\frac{3}{2}} \sum_{i\in\mathbb{N}} \frac{1}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} d\mu(x) \int_{B_\varepsilon(x)} |u(y)-u_{B_\varepsilon(x_i)}|\, d\mu(y)\\
=& c_O c_D^{\frac{3}{2}} (I_1 + I_2).
\end{aligned}$$

Since $B_\varepsilon(x) \subset B_{2\varepsilon}(x_i)$ for $x \in B_\varepsilon(x_i)$ we get, by a 1-Poincaré inequality,

$$\begin{aligned}
I_1 =& \sum_{i\in\mathbb{N}} \frac{1}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} \mu(B_\varepsilon(x))|u(x)-u_{B_\varepsilon(x_i)}|\, d\mu(x)\\
\le& \sum_{i\in\mathbb{N}} \frac{\mu(B_{2\varepsilon}(x_i))}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} |u(x)-u_{B_\varepsilon(x_i)}|\, d\mu(x)\\
\le& \varepsilon c_P c_D \sum_{i\in\mathbb{N}} \|Du\|(B_{2\lambda\varepsilon}(x_i)) \le \varepsilon c_O c_P c_D \|Du\|(X).
\end{aligned}$$

We treat the term I_2 in a similar fashion

$$\begin{aligned}
I_2 \le& \sum_{i\in\mathbb{N}} \frac{1}{\mu(B_\varepsilon(x_i))} \int_{B_\varepsilon(x_i)} d\mu(x) \int_{B_{2\varepsilon}(x_i)} |u(y)-u_{B_\varepsilon(x_i)}|\, d\mu(y)\\
=& \sum_{i\in\mathbb{N}} \int_{B_{2\varepsilon}(x_i)} |u(y)-u_{B_\varepsilon(x_i)}|\, d\mu(y)\\
\le& \sum_{i\in\mathbb{N}} \int_{B_{2\varepsilon}(x_i)} |u(y)-u_{B_{2\varepsilon}(x_i)}|\, d\mu(y)\\
&+ \sum_{i\in\mathbb{N}} |u_{B_\varepsilon(x_i)} - u_{B_{2\varepsilon}(x_i)}|\mu(B_{2\varepsilon}(x_i))\\
\le& 2\varepsilon c_P(1+c_D) \sum_{i\in\mathbb{N}} \|Du\|(B_{4\lambda\varepsilon}(x_i)) \le 2\varepsilon c_P(1+c_D)c_O\|Du\|(X),
\end{aligned}$$

where we have used a 1-Poincaré inequality and the fact that

$$\begin{aligned}\mu(B_{2\varepsilon}(x_i))|u_{B_\varepsilon(x_i)} - u_{B_{2\varepsilon}(x_i)}| \leq & c_D \int_{B_\varepsilon(x_i)} |u(x) - u_{B_{2\varepsilon}(x_i)}| \, d\mu(x) \\ \leq & c_D \int_{B_{2\varepsilon}(x_i)} |u(x) - u_{B_{2\varepsilon}(x_i)}| \, d\mu(x) \\ \leq & 2\varepsilon c_D c_P \|Du\|(B_{4\lambda\varepsilon}(x_i)).\end{aligned}$$

This completes the proof of the claim that if $u \in \mathrm{BV}(X)$ then the limit infimum is finite (in fact, we have obtained that the lim sup is finite).

For the converse statement, we assume that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Delta^\varepsilon} \frac{|u(x) - u(y)|}{\sqrt{\mu(B_\varepsilon(x))}\sqrt{\mu(B_\varepsilon(y))}} \, d\mu(x, y) < \infty.$$

For every $\varepsilon > 0$, let us define a positive and finite measure on X as

$$d\mu_\varepsilon(x) = \widetilde{u}_{B_\varepsilon}(x) \, d\mu(x),$$

where for each $x \in X$ we set

$$\widetilde{u}_{B_\varepsilon}(x) = \frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \frac{|u(x) - u(y)|}{\varepsilon} \, d\mu(y).$$

Let $\{B_i^\varepsilon = B_\varepsilon(x_i)\}$ be a family of balls on X with bounded overlapping

$$\sum_{i \in \mathbb{N}} \chi_{B_{5\varepsilon}(x_i)}(x) \leq c_O$$

and $(\varphi_i^\varepsilon)_i$ be a partition of unity of X with respect to this family of balls. We refer to [HKT] for the properties of these functions φ_i^ε. We define

$$u_\varepsilon(x) = \sum_{i=1}^{\infty} u_{B_i^\varepsilon} \varphi_i^\varepsilon(x)$$

for every $x \in X$. The function u_ε is Lipschitz continuous, since it is locally a finite sum of Lipschitz functions, and the sequence $(u_\varepsilon)_\varepsilon$ converges to u in $L^1(X)$ as $\varepsilon \to 0$ ([HKT, Lemma 5.3.(2)]).

We need an estimate on the Lipschitz constant of u_ε. Suppose that $x, y \in B_i^\varepsilon$ and let $J = \{j : 2B_i^\varepsilon \cap 2B_j^\varepsilon \neq \emptyset\}$. By the doubling property, balls B_i^ε have bounded overlap and so the cardinality of J is bounded by a constant depending only on the doubling constant c_D, that is $\#J \leq c_\#$ with $c_\# = c_\#(c_D)$. In addition $B_j^\varepsilon \subset 5B_i^\varepsilon$. By the properties of the partition of unity, we have exactly as in [HKT] that for any $x, y \in B_i^\varepsilon$

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(y)| &= \left| \sum_{j \in J} (u_{B_j^\varepsilon} - u_{B_i^\varepsilon})(\varphi_j^\varepsilon(x) - \varphi_j^\varepsilon(y)) \right| \\ &\leq C \frac{d(x, y)}{\varepsilon} \sum_{j \in J} |u_{B_i^\varepsilon} - u_{B_j^\varepsilon}|.\end{aligned}$$

For every $j \in J$ we also have that

$$\begin{aligned}|u_{B_i^\varepsilon} - u_{B_j^\varepsilon}| \leq & \frac{1}{\mu(B_i^\varepsilon)\mu(B_j^\varepsilon)} \int_{B_i^\varepsilon} \int_{B_j^\varepsilon} |u(y) - u(x)| \, d\mu(y) d\mu(x) \\ \leq & \frac{1}{\mu(B_i^\varepsilon)\mu(B_j^\varepsilon)} \int_{B_i^\varepsilon} \int_{B_{6\varepsilon}(x)} |u(y) - u(x)| \, d\mu(y) \, d\mu(x) \\ \leq & \frac{\varepsilon c_D^3}{\mu(B_i^\varepsilon)} \mu_{6\varepsilon}(B_i^\varepsilon),\end{aligned}$$

where in the last inequality we have used the fact that $B_j^\varepsilon \subset B_{6\varepsilon}(x)$ whenever $x \in B_i^\varepsilon$. In other words, we have proved that

$$\operatorname{lip}(u_\varepsilon)(x) \leq \frac{c_D^3 c_\#}{\mu(B_i^\varepsilon)} \mu_{6\varepsilon}(B_i^\varepsilon)$$

for every $x \in B_i^\varepsilon$. We therefore obtain

$$\begin{aligned}\int_X \operatorname{lip}(u_\varepsilon)(x) \, d\mu(x) \leq \sum_{i \in \mathbb{N}} \int_{B_i^\varepsilon} \operatorname{lip}(u_\varepsilon)(x) \, d\mu(x) \leq \sum_{i=1}^\infty c_D^3 c_\# \mu_{6\varepsilon}(B_i^\varepsilon) \\ \leq c_D^3 c_\# c_O \mu_{6\varepsilon}(X).\end{aligned}$$

Now, by the assumption and the doubling property of the measure μ, we have

$$\liminf_{\varepsilon \to 0^+} \mu_\varepsilon(X) = \liminf_{\varepsilon \to 0^+} \int_X \tilde{u}_{B_\varepsilon}(x) \, d\mu(x) < \infty,$$

Hence we can find a sequence $\{\varepsilon_j\}_j$ going to 0 such that

$$\sup_{j \in \mathbb{N}} \mu_{\varepsilon_j}(X) < \infty,$$

and then the sequence of Lipschitz functions $u_j = u_{\varepsilon_j/6}$ converges to u in $L^1(X)$ and

$$\limsup_{j \to \infty} \int_X \operatorname{lip}(u_j)(x) \, d\mu(x) < \infty,$$

which implies that $u \in \mathrm{BV}(X)$. □

4. Sets of finite perimeter: Ledoux characterization

We shall make use of Theorem 3.1 to connect the sets of finite perimeter to the theory of heat semigroup in the sense of Ledoux [L]. The reader should also consult [P].

Theorem 4.1. *Let $E \subset X$ be μ-measurable and assume that E has finite measure. Then E is of finite perimeter, i.e. $\chi_E \in \mathrm{BV}(X)$ if, and only if,*

$$\liminf_{t \to 0^+} \frac{1}{\sqrt{t}} \int_{X \setminus E} T_t \chi_E(x) \, d\mu(x) < \infty.$$

Proof. Note first that

$$\begin{aligned}\int_X &|T_t\chi_E(x)-\chi_E(x)|\,d\mu(x)\\ &=\int_X\left|\int_X p(t,x,y)(\chi_E(y)-\chi_E(x))\,d\mu(y)\right|\,d\mu(x)\\ &=\int_E\left|-\int_{X\setminus E}p(t,x,y)\,d\mu(y)\right|\,d\mu(x)\\ &\qquad+\int_{X\setminus E}\left|\int_E p(t,x,y)\,d\mu(y)\right|\,d\mu(x)\\ &=2\int_E\int_{X\setminus E}p(t,x,y)\,d\mu(x)d\mu(y)\\ &=2\int_X\int_X\chi_{X\setminus E}(x)p(t,x,y)\chi_E(y)\,d\mu(x)d\mu(y)\\ &=2\int_{X\setminus E}T_t\chi_E(x)\,d\mu(x).\end{aligned}$$

Hence by the fact that $T_tf\to f$ in $L^1(X)$ as $t\to 0$ for each $f\in L^1(X)$, we have that

$$\lim_{t\to 0^+}\int_{X\setminus E}T_t\chi_E(x)\,d\mu(x)=0.$$

Suppose now that

$$\liminf_{t\to 0^+}\frac{1}{\sqrt{t}}\int_X|T_t\chi_E(x)-\chi_E(x)|\,d\mu(x)<\infty,$$

and we want to show that the set E has finite perimeter. By the symmetry of the heat kernel

$$\begin{aligned}2\int_E\int_{X\setminus E}p(t,x,y)\,d\mu(y)\,d\mu(x)&\ge\int_E\int_{B_{\sqrt{t}}(x)\setminus E}p(t,x,y)\,d\mu(y)d\mu(x)\\ &\qquad+\int_{X\setminus E}\int_{B_{\sqrt{t}}(x)\cap E}p(t,x,y)\,d\mu(y)\,d\mu(x)\\ &=\int_{\Delta_{\sqrt{t}}}p(t,x,y)|\chi_E(y)-\chi_E(x)|\,d\mu(y)\,d\mu(x).\end{aligned}$$

By estimate (2.3) for the kernel $p(t,x,y)$,

$$\begin{aligned}&\frac{1}{\sqrt{t}}\int_E\int_{X\setminus E}p(t,x,y)\,d\mu(y)d\mu(x)\\ &\qquad\ge\frac{C}{\sqrt{t}}\int_{\Delta_{\sqrt{t}}}\frac{|\chi_E(x)-\chi_E(y)|}{\sqrt{\mu(B_{\sqrt{t}}(x))}\sqrt{\mu(B_{\sqrt{t}}(y))}}\,d\mu(y)d\mu(x),\end{aligned}$$

and so we have

$$\liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \int_{\Delta^{\sqrt{t}}} \frac{|\chi_E(x)-\chi_E(y)|}{\sqrt{\mu(B_{\sqrt{t}}(x))}\sqrt{\mu(B_{\sqrt{t}}(y))}}\, d\mu(y)d\mu(x) < \infty.$$

By Theorem 3.1, we conclude that $\chi_E \in \mathrm{BV}(X)$.

For the converse, suppose first that $E \subset X$ is open with $\mu(E)$ finite, $\mu(\partial E) = 0$, and that $\chi_E \in \mathrm{BV}(X)$. For each $t > 0$ and each nonnegative integer k let us define the sets

$$E_0^t = \{x \in E : \ d(x, X \setminus E) \le \sqrt{t}\},$$
$$E_k^t = \{x \in E : \ 2^{k-1}\sqrt{t} < d(x, X \setminus E) \le 2^k\sqrt{t}\}, \quad \text{for} \quad k \ge 1,$$

and for each $k \ge 0$ and $x \in E_k^t$ we write the set $X \setminus E$ as

$$X \setminus E = (B_{2^{k+1}\sqrt{t}}(x) \setminus E) \cup (X \setminus (B_{2^{k+1}\sqrt{t}}(x) \cup E)) =: A_k^1 \cup A_k^2.$$

By (2.4),

$$\begin{aligned}
\frac{1}{\sqrt{t}} \int_{X\setminus E} T_t\chi_E(x)\, d\mu(x) &= \frac{1}{\sqrt{t}} \int_E \int_{X\setminus E} p(t,x,y)\, d\mu(y)d\mu(x) \\
&\le \frac{C}{\sqrt{t}} \sum_{k=0}^{\infty} \int_{E_k^t} \int_{X\setminus E} \frac{e^{-\frac{d(x,y)^2}{C_2 t}}\, d\mu(y)d\mu(x)}{\sqrt{\mu(B_{\sqrt{t}}(x)}\sqrt{\mu(B_{\sqrt{t}}(y)}} \\
&\le \frac{C}{\sqrt{t}} \sum_{k=0}^{\infty} \left(\int_{E_k^t} \int_{A_k^1} \frac{e^{-\frac{d(x,y)^2}{C_2 t}}\, d\mu(y)d\mu(x)}{\sqrt{\mu(B_{\sqrt{t}}(x)}\sqrt{\mu(B_{\sqrt{t}}(y)}} \right. \\
&\qquad \left. + \int_{E_k^t} \int_{A_k^2} \frac{e^{-\frac{d(x,y)^2}{C_2 t}}\, d\mu(y)d\mu(x)}{\sqrt{\mu(B_{\sqrt{t}}(x)}\sqrt{\mu(B_{\sqrt{t}}(y)}} \right) \\
&=: \frac{C}{\sqrt{t}} \sum_{k=0}^{\infty} (I_k^1 + II_k^2).
\end{aligned}$$

Let us estimate the preceding terms separately, starting with the term I_k^1. Note that when $x \in E_k^t$, if there is a point $y \in X \setminus E$, then we must have $d(x,y) \ge 2^{k-1}\sqrt{t}$. Therefore, by the doubling condition and (2.1),

$$\begin{aligned}
I_k^1 &\le C \int_{E_k^t} \int_{A_k^1} \frac{e^{-C4^k}}{\mu(B_{\sqrt{t}}(x))}\, d\mu(y)d\mu(x) \\
&\le C \int_{E_k^t} 2^{kq_\mu} e^{-C4^k} \frac{\mu(B_{2^{k+1}\sqrt{t}}(x) \setminus E)}{\mu(B_{2^{k+1}\sqrt{t}}(x))}\, d\mu(x).
\end{aligned}$$

We can cover each set E_k^t by balls B_i^k, $i = 1, \ldots, N_k$, centered at some point of E_k^t and of radius $2^{k+1}\sqrt{t}$, such that the dilated balls $2\lambda B_i^k$ have bounded overlap, (3.2), the bound c_O of the overlap depending solely on the doubling constant of the measure μ. By the above and by the doubling property of μ, now we have

$$I_k^1 \le C2^{kq_\mu}e^{-C4^k}\sum_{i=1}^{N_k}\frac{\mu(B_i^k\setminus E)}{\mu(B_i^k)}\mu(B_i^k\cap E).$$

By (2.5), we now have

$$\begin{aligned}I_k^1 &\le C2^{kq_\mu}e^{-C4^k}\sum_{i=1}^{N_k}2^{k+1}\sqrt{t}P(E,\lambda B_i^k)\\ &\le C2^{k(q_\mu+1)}e^{-C4^k}\sqrt{t}P(E).\end{aligned}$$

The second term II_k^2 is treated as follows. For every $k\ge 0$ and $j\ge 1$, $j\in\mathbb{N}$, we set

$$A_k^2=\bigcup_{j\ge 1}\overline{B}_{2^{k+j}\sqrt{t}}(x)\setminus(B_{2^{k+j-1}\sqrt{t}}(x)\cup E)=:\bigcup_{j\ge 1}A_{k,j}^2.$$

Hence, we have, with an analogous cover B_i^k, $i=1,\cdots,N_k$ as above,

$$\begin{aligned}\mathrm{II}_k^2 &\le C\sum_{j=1}^{\infty}\int_{E_k^t}\int_{A_{k,j}^2}\frac{e^{-\frac{d(x,y)^2}{C_2t}}\,d\mu(y)d\mu(x)}{\sqrt{\mu(B_{\sqrt{t}}(x)}\sqrt{\mu(B_{\sqrt{t}}(y)}}\\
&\le C\sum_{j=1}^{\infty}\int_{E_k^t}\int_{A_{k,j}^2}c_D^{k+j}e^{-C4^{k+j}}\frac{d\mu(y)d\mu(x)}{\mu(B_{2^{k+j}\sqrt{t}}(x))}\\
&\le C\sum_{i=1}^{N_k}\sum_{j=1}^{\infty}\int_{B_i^k}c_D^{k+j}e^{-C4^{k+j}}\frac{\mu(B_{2^{k+j}\sqrt{t}}(x)\setminus[B_{2^{k+j-1}\sqrt{t}}(x)\cup E])}{\mu(B_{2^{k+j}\sqrt{t}}(x))}\,d\mu(x)\\
&\le C\sum_{i=1}^{N_k}\sum_{j=1}^{\infty}\int_{B_i^k}c_D^{k+j}e^{-C4^{k+j}}\frac{\mu(2^jB_i^k\setminus[2^{-j-2}B_i^k\cup E])}{\mu(2^jB_i^k)}\,d\mu(x)\\
&\le C\sum_{i=1}^{N_k}\left(\sum_{j=1}^{\infty}c_D^{k+j}e^{-C4^{k+j}}\frac{\mu(2^jB_i^k\setminus[2^{-j-2}B_i^k\cup E])}{\mu(2^jB_i^k)}\right)\mu(B_i^k\cap E).\end{aligned}$$

The sum

$$\sum_{j=1}^{\infty}c_D^{k+j}e^{-C4^{k+j}}\frac{\mu(2^jB_i^k\setminus[2^{-j-2}B_i^k\cup E])}{\mu(2^jB_i^k)}$$

is convergent, with the value dominated by the first term in the series, that is,

$$\sum_{j=1}^{\infty}c_D^{k+j}e^{-C4^{k+j}}\frac{\mu(2^jB_i^k\setminus[2^{-j-2}B_i^k\cup E])}{\mu(2^jB_i^k)}\le Cc_D^k\,e^{-C4^k}\frac{\mu(B_i^k\setminus E)}{\mu(B_i^k)}.$$

Observe that $c_D \geq 1$. Hence by (2.5) again,

$$\begin{aligned} II_k^2 &\leq C c_D^k e^{-C4^k} \sum_{i=1}^{N_k} \frac{\mu(B_i^k \setminus E)}{\mu(B_i^k)} \mu(B_i^k \cap E) \\ &\leq C c_D^k e^{-C4^k} \sum_{i=1}^{N_k} 2^{k+1} \sqrt{t} P(E, \lambda B_i^k) \\ &\leq C c_D^k 2^k e^{-C4^k} \sqrt{t} P(E). \end{aligned}$$

Putting the preceding estimates together we obtain the desired estimate

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_{X \setminus E} T_t \chi_E(x)\, d\mu(x) &\leq \frac{C}{\sqrt{t}} \sum_{k=0}^{\infty} (2^{kq_\mu} + c_D^k) 2^k e^{-C4^k} \sqrt{t} P(E) \\ &\leq C P(E) < \infty. \end{aligned}$$

Now let E be a general set of finite perimeter. There is a sequence $(f_k)_k$ of Lipschitz functions in $\mathrm{Lip}_c(X)$ with $0 \leq f_k \leq 1$ such that $f_k \to \chi_E$ in $L^1(X)$ and $\lim_{k\to\infty} |Df_k|(X) = P(E)$. Given such an approximating sequence, for each k we can choose, with the aid of the co-area formula (2.6), a superlevel set of f_k written as $E_k = \{x \in X\colon f_k(x) > t_k\}$, $1/2 \leq t_k \leq 1$, which is open, such that

(1) $\mu(\partial E_k) = 0$, and
(2) $\mu(E_k \Delta E) \to 0$ as $k \to \infty$.

Here the symmetric difference between the sets E_k and E is written as $E_k \Delta E = (E_k \setminus E) \cup (E \setminus E_k)$. By (2.6), we have in particular by a suitable choice of t_k that

$$\begin{aligned} P(E) &= \lim_{k\to\infty} \int_0^1 P(\{x \in X\colon f_k(t) > t\})\, dt \\ &\geq \lim_{k\to\infty} \int_{1/2}^1 P(\{x \in X\colon f_k(t) > t\})\, dt \\ &\geq \frac{1}{2} \lim_{k\to\infty} P(\{x \in X\colon f_k(t) > t_k\}) = \frac{1}{2} \lim_{k\to\infty} P(E_k). \end{aligned}$$

Now from the above arguments we see that

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_{X\setminus E} T_t \chi_E(x)\, d\mu(x) &= \frac{1}{\sqrt{t}} \int_E \int_{X\setminus E} p(t,x,y)\, d\mu(y) d\mu(x) \\ &= \frac{1}{\sqrt{t}} \int_X \int_X \chi_E(y) \chi_{X\setminus E}(x) p(t,x,y)\, d\mu(y) d\mu(x) \\ &= \frac{1}{\sqrt{t}} \lim_{k\to\infty} \int_X \int_X \chi_{E_k}(y) \chi_{X\setminus E_k}(x) p(t,x,y)\, d\mu(y) d\mu(x) \\ &= \frac{1}{\sqrt{t}} \lim_{k\to\infty} \int_{E_k} \int_{X\setminus E_k} p(t,x,y)\, d\mu(y) d\mu(x) \\ &\leq C \lim_{k\to\infty} P(E_k) \leq 2C P(E). \end{aligned}$$

Thus, also the converse statement follows and the proof is complete. □

5. Total variation: De Giorgi characterization under Bakry–Émery condition

We close this paper by proving a metric space version of the De Giorgi characterization [DeG] of the total variation of a BV function. Our approach is based on recent works by Amborsio, Gigli, and Savaré [AGS] and is essentially a consequence of [S].

We refer to [CM, MP, GP, BMP] for the De Giorgi characterization on Riemannian manifolds and on Carnot groups, and to [AMP] for the same result but with the semigroup of a rather general second order elliptic operator on domains in Euclidean spaces. All of these results require a condition on the curvature of the spaces; these conditions are related to the Ricci curvature in the case of Riemannian manifolds. In [MP] the requirement was on a lower bound of Ricci curvature plus a technical requirement on a lower bound on the volume of balls. It was pointed out in [CM] that, thanks to the works of Bakry and Émery (see for instance [BE]), the same result can be obtained only with a lower bound condition on the Ricci curvature. In [GP] it has been shown that one can also require that the Ricci tensor can be split into two parts, one bounded below and one belonging to a Kato class. Also in this case the De Giorgi characterization of the total variation holds true.

We recall the definition of $\mathrm{BE}(K,\infty)$ condition as formulated in [BE, Ba].

Definition 5.1 (Bakry–Émery condition)**.** *The Dirichlet form $\mathcal{E}$ is said to satisfy the $\mathrm{BE}(K,\infty)$ condition, $K\in\mathbb{R}$, if for every $f\in D(A)$ such that $Af\in N^{1,2}(X)$, we have*

$$\frac{1}{2}\int_X |Df|^2 A\varphi\, d\mu - \int_X \varphi Df\cdot DAf\, d\mu \geq K\int_X \varphi |Df|^2\, d\mu, \tag{5.1}$$

whenever $\varphi\in D(A)\cap L^\infty(X)$ is nonnegative such that $A\varphi\in L^\infty(X)$.

In the appendix, we recall some consequences of the Bakry–Émery condition. These consequences are needed in the proof of the main result of this section, Proposition 5.2, and have been investigated for instance in [S]. The setting in [S] is rather general, and hence for the convenience of the reader we provide sketches of simplified proofs of these results in the appendix.

The main result of this section is the following.

Proposition 5.2. *Let $u\in L^1(X)$. Suppose that*

$$\limsup_{t\to 0^+}\int_X |DT_t u|\, d\mu < \infty, \tag{5.2}$$

then $u\in\mathrm{BV}(X)$. Moreover, whenever (5.2) holds and if the Dirichlet form $\mathcal{E}$ satisfies the $\mathrm{BE}(K,\infty)$ condition, we have

$$\|D_c u\|(X) = \lim_{t\to 0^+}\int_X |DT_t u|\, d\mu.$$

By the self-improvement property of the Bakry-Émery condition $\mathrm{BE}(K,\infty)$, we can easily prove Proposition 5.2. Of course, it would be interesting to obtain a similar result, or even the weaker version as in [BMP], without imposing the Bakry–Émery condition on $\mathcal{E}$.

Proof of Proposition 5.2. The claim that (5.2) implies that $u \in \mathrm{BV}(X)$ follows immediately from the definition of $\mathrm{BV}(X)$ upon noticing that as $u \in L^1(X)$ we must have $T_t u \to u$ in $L^1(X)$ as $t \to 0$ (see Remark 2.1(10)) and $T_t u \in N^{1,1}(X)$. Furthermore, it is also immediate that

$$\|D_c u\|(X) \le \limsup_{t\to 0^+} \int_X |DT_t u|\, d\mu.$$

Let $u \in \mathrm{BV}(X)$. We consider a sequence of Lipschitz functions $(f_j)_{j\in\mathbb{N}} \subset \mathrm{Lip}_c(X)$ such that $f_j \to u$ in $L^1(X)$ and

$$\|D_c u\|(X) = \lim_{j\to\infty} \int_X |Df_j|\, d\mu.$$

By the lower semicontinuity of the total variation together with Proposition 6.3 and the fact that $(T_t)_{t>0}$ is a contraction semigroup on $L^1(X)$ (see Remark 2.1(10)), we conclude

$$\begin{aligned}\|D_c u\|(X) &\le \liminf_{t\to 0^+} \int_X |DT_t u|\, d\mu \le \liminf_{t\to 0^+} \liminf_{j\to\infty} \int_X |DT_t f_j|\, d\mu \\ &\le \liminf_{t\to 0^+} \liminf_{j\to\infty} e^{-Kt} \int_X T_t |Df_j|\, d\mu \\ &\le \liminf_{t\to 0^+} \liminf_{j\to\infty} e^{-Kt} \int_X |Df_j|\, d\mu = \|D_c u\|(X),\end{aligned}$$

and hence the proof is complete. □

In conclusion, we note that by the analog of both the De Giorgi characterization and the Ledoux characterization of functions of bounded variation demonstrated in this paper, even in the general metric measure space setting (with the measure doubling and supporting a 1-Poincaré inequality), the behavior of sets of finite perimeter is intimately connected to the heat semigroup.

6. Appendix

In this appendix we gather together properties associated with the Bakry–Émery condition in Section 5. In the metric setting a related condition, guaranteed by the logarithmic Sobolev inequality, was first studied in [KRS]. The results in this appendix are from [S] adapted to our purposes.

In the sequel, the following Pazy convolution operator will play an important role:

$$\beta_\varepsilon f = \frac{1}{\varepsilon} \int_0^\infty T_s f \varrho(s/\varepsilon)\, ds.$$

Here $\varrho \in C_c^\infty(0,\infty)$ is a nonnegative convolution kernel with $\int_0^\infty \varrho(s)\,ds = 1$. We also define the measure operator A^* by setting

$$D(A^*) = \Big\{ f \in N^{1,2}(X) \colon \text{there exists } \nu \in \mathcal{M}(X) \text{ such that} \\ \mathcal{E}(f,v) = -\int_X v\,d\nu \text{ for all } v \in \mathrm{Lip}_c(X) \Big\},$$

and denote by A^*f the measure ν. Here $\mathcal{M}(X)$ denotes the set of finite Borel measures on X. The integration by parts formula

$$\int_X Df \cdot D\varphi\,d\mu = -\int_X \varphi\,dA^*f$$

can be extended from functions $\varphi \in \mathrm{Lip}_c(X)$ to functions $\varphi \in N^{1,2}(X) \cap L^\infty(X)$ if on the right hand side we consider the representative $\tilde\varphi \in N^{1,2}(X)$ of φ, since A^*f does not charge sets of zero Sobolev 2-capacity. We refer to [MMS] for more details.

We consider the function class

$$\mathbb{D}_\infty = \{ f \in D(A) \cap \mathrm{Lip}_b(X) \colon Af \in N^{1,2}(X) \};$$

as we shall see in Proposition 6.1, if $f \in \mathbb{D}_\infty$, then $|Df|^2 \in D(A^*)$. We shall denote by $\Gamma_2^*(f)$ the measure

$$\Gamma_2^*(f) = \frac{1}{2} A^*|Df|^2 - (Df \cdot DAf)\,\mu = \Gamma_2^\perp(f) + \gamma_2(f)\,\mu,$$

where $\Gamma_2^\perp(f) \perp \mu$ is the singular part of the measure $\Gamma_2^*(f)$ and $\gamma_2(f)$ its absolute continuous component.

We list in the following proposition the main properties of functions in $\mathbb{D}_\infty$ needed in the proof of Proposition 5.2. We refer to [S] for the details of the proof, but we provide a sketch of the proof for the convenience of the reader.

Proposition 6.1. *Suppose that $f \in \mathbb{D}_\infty$ and that $\mathcal{E}$ satisfies the* $\mathrm{BE}(K,\infty)$ *condition. Then*

(1) $|Df|^2 \in N^{1,2}(X)$*;*
(2) $|Df|^2 \in D(A^*)$ *and for any nonnegative* $\varphi \in \mathrm{Lip}_c(X)$*,*

$$\frac{1}{2}\int_X \varphi dA^*|Df|^2\,d\mu - \int_X \varphi Df \cdot DAf\,d\mu \ge K \int_X \varphi |Df|^2\,d\mu;$$

(3) *we have*

$$\int_X \varphi |D|Df|^2|^2\,d\mu \le 4\int_X \varphi\left(\gamma_2(f) - K|Df|^2\right)\,d\mu$$

for any nonnegative $\varphi \in \mathrm{Lip}_c(X)$*.*

Proof. Let us prove (1). Since $f \in \mathbb{D}_\infty$, condition $\mathrm{BE}(K,\infty)$ implies that

$$\int_X |Df|^2 A\varphi\,d\mu \ge 2\int_X (K|Df|^2 + Df \cdot DAf)\varphi\,d\mu$$

for any positive $\varphi \in D(A)$. Then, by setting $u = |Df|^2$ and $u_\varepsilon = \beta_\varepsilon u$, taking into account the fact that for any $\varphi \in D(A)$, since A commutes with the Pazy convolution β_ϵ, there holds

$$\int_X u_\varepsilon A\varphi \, d\mu = \int_X uA\beta_\epsilon \varphi \, d\mu.$$

We obtain

$$\begin{aligned}\int_X |Du_\epsilon|^2 \, d\mu &= -\int_X u_\epsilon A u_\epsilon \, d\mu = -\int_X uA\beta_\epsilon u_\epsilon \, d\mu \\ &\leq -2\int_X (K|Df|^2 + Df \cdot DAf)\beta_\epsilon u_\epsilon \, d\mu.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, we arrive at the conclusion that $|Df|^2 \in N^{1,2}(X)$ with

$$\int_X |D|Df|^2|^2 \, d\mu \leq -2\int_X (K|Df|^2 + Df \cdot DAf)|Df|^2 \, d\mu.$$

Let us then prove (2). Using the $\mathrm{BE}(K, \infty)$ condition and by approximating any $\mathrm{Lip}_c(X)$ function with functions in $D(A)$, we can deduce that

$$\int_X Au_\varepsilon \varphi \, d\mu \geq -\int_X g\beta_\epsilon \varphi \, d\mu = -\int_X \beta_\varepsilon g\varphi \, d\mu$$

for any $\varphi \in \mathrm{Lip}_c(X)$, where we have defined

$$g = -2(K|Df|^2 + Df \cdot DAf).$$

Passing to the limit as $\varepsilon \to 0$, we define

$$\mathcal{L}_f(\varphi) := -\int_X D|Df|^2 \cdot D\varphi \, d\mu + \int_X g\varphi \, d\mu$$

which is a positive linear functional defined on $\mathrm{Lip}_c(X)$. If in the estimate

$$0 \leq \int_X (Au_\varepsilon + \beta_\epsilon g)\varphi \, d\mu$$

we take an increasing sequence of Lipschitz functions φ_n such that $0 \leq \varphi_n \leq 1$, $\varphi_n \equiv 1$ on $B_n(x_0)$ and $\varphi_n \equiv 0$ on $X \setminus B_{n+1}(x_0)$ for a fixed point x_0, by the dominated convergence theorem,

$$0 \leq \int_X (Au_\varepsilon + \beta_\varepsilon g)\varphi_n \, d\mu \leq \int_X (Au_\varepsilon + \beta_\varepsilon g) \, d\mu = \int_X \beta_\varepsilon g d\mu = \int_X g \, d\mu.$$

We have obtained

$$|\mathcal{L}_f(\varphi)| \leq \|\varphi\|_\infty \int_X g \, d\mu,$$

and then $\mathcal{L}_f$ can be represented by a positive measure $\lambda \in \mathcal{M}(X)$. This proves that $|Df|^2 \in D(A^*)$ with

$$A^*|Df|^2 = \lambda - g\mu.$$

The positivity of the measure

$$\frac{1}{2}A^*|Df|^2 - (Df \cdot DAf + K|Df|^2)\mu$$

follows by the positivity of λ.

The proof of (3) is rather technical and does not simplify in our setting, so we refer to [S, Theorem 3.4]. □

The Bakry–Émery condition in Definition 5.1 has equivalent formulations, we refer to [W] for the Riemannian case and to [AGS, S] for the metric space setting. We recall here some of those equivalent formulations that we shall use.

Proposition 6.2. *The following are equivalent:*

(1) $\mathcal{E}$ *satisfies the* $\mathrm{BE}(K,\infty)$ *condition;*

(2) *for any* $f \in N^{1,2}(X)$*, any* $t > 0$*, and for every nonnegative* $\varphi \in \mathrm{Lip}_c(X)$

$$\int_X \varphi |DT_t f|^2 \, d\mu \leq e^{-2Kt} \int_X \varphi T_t |Df|^2 \, d\mu; \tag{6.1}$$

(3) *for any* $f \in L^2(X)$*, any* $t > 0$*, and for every nonnegative* $\varphi \in \mathrm{Lip}_c(X)$

$$\frac{e^{2Kt} - 1}{K} \int_X \varphi |DT_t f|^2 \, d\mu \leq \int_X \varphi (T_t f^2 - (T_t f)^2) \, d\mu. \tag{6.2}$$

Proof. Let us show that (1) implies (2). Let $f \in N^{1,2}(X)$ and $\varphi \in D(A) \cap L^\infty(X)$ such that $A\varphi \in L^\infty(X)$ and $\varphi \geq 0$ be fixed. We define

$$G_\varphi(s) = \int_X T_s \varphi |DT_{t-s} f|^2 \, d\mu.$$

Taking derivatives, and by $\mathrm{BE}(K,\infty)$, we get

$$\begin{aligned} G'_\varphi(s) =& \int_X AT_s\varphi |DT_{t-s} f|^2 \, d\mu - 2 \int_X T_s \varphi DT_{t-s} f \cdot DAT_{t-s} f \, d\mu \\ \geq& 2K \int_X T_s \varphi |DT_{t-s} f|^2 \, d\mu = 2K G_\varphi(s). \end{aligned}$$

If for some $s > 0$, $G_\varphi(s) = 0$, then if φ is not identically zero, $T_s \varphi > 0$ and we must have $|DT_{t-s} f| = 0$ at μ-a.e. Hence $T_{t-s} f$ is constant and thus f is also constant. In this case (2) is trivially satisfied. On the other hand, if $G_\varphi(s) > 0$, we can integrate and conclude that

$$G_\varphi(t) \geq G_\varphi(0) e^{2Kt},$$

which is condition (2).

Let us show that (2) implies (3). By introducing the function

$$G_\varphi(s) = \int_X T_s \varphi (T_{t-s} f)^2 \, d\mu,$$

taking derivatives, and taking into account (2), we obtain

$$\begin{aligned} G'_\varphi(s) =& \int_X AT_s \varphi (T_{t-s} f)^2 \, d\mu - 2 \int_X T_s \varphi T_{t-s} f A T_{t-s} f \, d\mu = 2 \int_X \varphi T_s |DT_{t-s} f|^2 \, d\mu \\ \geq& 2e^{2Ks} \int_X \varphi |DT_s T_{t-s} f|^2 \, d\mu = 2e^{2Ks} \int_X \varphi |DT_t f|^2 \, d\mu. \end{aligned}$$

By integrating with respect to s, we obtain condition (3).

Let us now assume (3) and let us take $f \in D(A)$ such that $Af \in N^{1,2}(X)$ and $\varphi \in D(A) \cap L^\infty(X)$ with $A\varphi \in L^\infty$. We can define the function

$$F_\varphi(t) = \int_X \varphi(T_t f^2 - (T_t f)^2)\, d\mu.$$

The second order Taylor expansion formula implies

$$F_\varphi(t) = 2t \int_X \varphi |Df|^2\, d\mu + t^2 \left(\int_X |Df|^2 A\varphi\, d\mu + 2\int_X \varphi Df \cdot DAf\, d\mu \right) + o(t^2).$$

In the same way, we obtain that

$$\begin{aligned} G_\varphi(t) &= \frac{e^{2Kt}-1}{K} \int_X \varphi |DT_t f|^2\, d\mu \\ &= 2t \int_X \varphi |Df|^2 d\mu + 2t^2 \left(K \int_X \varphi |Df|^2\, d\mu + 2 \int_X \varphi Df \cdot DAf\, d\mu \right) + o(t^2). \end{aligned}$$

Then, since

$$\begin{aligned} 0 \leq & F_\varphi(t) - G_\varphi(t) \\ =& 2t^2 \left(\int_X A\varphi |Df|^2\, d\mu - 2\int_X \varphi Df \cdot DAf\, d\mu - 2K \int_X \varphi |Df|\, d\mu \right) + o(t^2) \end{aligned}$$

we obtain condition (1). □

Under the above hypotheses, we can prove the following self-improving result of the Bakry–Émery condition. The result is essentially contained in a monograph by Deuschel and Strook [DS, Lemma 6.2.39] in the Riemannian setting. The self-improvement of the Bakry–Émery condition has also been obtained by Savaré in [S, Corollary 3.5] in very general setting. We provide a proof here for the sake of completeness.

Proposition 6.3. *Suppose that the Dirichlet form $\mathcal{E}$ satisfies the* $\mathrm{BE}(K,\infty)$ *condition for some $K \in \mathbb{R}$. Then for any $f \in N^{1,2}(X)$, any $t > 0$, and every nonnegative $\varphi \in \mathrm{Lip}_c(X)$, we have*

$$\int_X \varphi |DT_t f|\, d\mu \leq e^{-Kt} \int_X \varphi T_t |Df|\, d\mu.$$

Proof. We start by considering $f \in \mathbb{D}_\infty$; we point out that by (6.2), $\mathbb{D}_\infty$ is dense in $N^{1,2}(X)$. Indeed, $\mathrm{Lip}_b(X)$ is dense in $N^{1,2}(X)$ and taking approximations of functions $f \in N^{1,2}(X) \cap \mathrm{Lip}_b(X)$ in terms of the semigroup, we get that $T_t f \in \mathbb{D}_\infty$ for any $t > 0$ by (6.2) and

$$AT_t f = T_{t/2} A T_{t/2} f \in D(A) \subset N^{1,2}(X).$$

Here we also used the fact that by the Bakry–Émery condition (6.2), if $f \in \mathrm{Lip}_b(X)$ then $T_t f \in \mathrm{Lip}_b(X)$.

Let us fix $\delta > 0$ and set

$$u_t^\delta(x) := \sqrt{|DT_t f(x)|^2 + \delta^2} - \delta. \tag{6.3}$$

We also fix a nonnegative function $\varphi \in N^{1,2}(X) \cap L^\infty(X)$ and introduce the function

$$G_\delta(s) = \int_X T_s\varphi u^\delta_{t-s}\, d\mu.$$

If we take the derivative, we obtain:

$$\begin{aligned}
G'{}_\delta(s) &= \int_X AT_s\varphi u^\delta_{t-s}\, d\mu - \int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta} DT_{t-s}f \cdot DAT_{t-s}f\, d\mu \\
&= -\int_X \frac{1}{2(u^\delta_{t-s}+\delta)} DT_s\varphi \cdot D|DT_{t-s}f|^2\, d\mu - \int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta} DT_{t-s}f \cdot DAT_{t-s}f\, d\mu \\
&= \int_X \frac{T_s\varphi}{2(u^\delta_{t-s}+\delta)}\, dA^*|DT_{t-s}f|^2 - \frac{1}{4}\int_X \frac{T_s\varphi}{(u^\delta_{t-s}+\delta)^3} |D|DT_{t-s}f|^2|^2\, d\mu \\
&\qquad - \int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta} DT_{t-s}f \cdot DAT_{t-s}f\, d\mu \\
&= \int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta}\, d\Gamma_2^*(T_{t-s}f) - \frac{1}{4}\int_X \frac{T_s\varphi}{(u^\delta_{t-s}+\delta)^3} |D|DT_{t-s}f|^2|^2\, d\mu \\
&\geq \int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta} \gamma_2(T_{t-s}f)\, d\mu \\
&\qquad - \int_X \frac{T_s\varphi}{(u^\delta_{t-s}+\delta)^3} |DT_{t-s}f|^2 \Big(\gamma_2(T_{t-s}f) - K|DT_{t-s}f|^2\Big)\, d\mu,
\end{aligned}$$

where in the last line we have used the properties contained in Proposition 6.1. We have

$$\begin{aligned}
G'_\delta(s) \geq &\int_X \frac{T_s\varphi}{(u^\delta_{t-s}+\delta)^3} \Big(\delta^2\gamma_2(T_{t-s}f) - K\delta^2|DT_{t-s}f|^2 \\
&\qquad + K|DT_{t-s}f|^2(u^\delta_{t-s}+\delta)^2\Big)\, d\mu \\
\geq &K\int_X \frac{T_s\varphi}{u^\delta_{t-s}+\delta} |DT_{t-s}f|^2\, d\mu \geq K\int_X T_s\varphi u^\delta_{t-s}\, d\mu,
\end{aligned}$$

where we used the fact that $\gamma_2(f) - K|Df|^2 \geq 0$ μ-a.e. by Proposition 6.1. We thus have $G'_\delta(s) \geq KG_\delta(s)$, and by integrating this over $(0,t)$, we arrive at

$$\int_X \varphi(\sqrt{|DT_tf|^2+\delta^2} - \delta)\, d\mu \leq e^{-Kt} \int_X T_t\varphi(\sqrt{|Df|^2+\delta^2} - \delta)\, d\mu.$$

Passing to the limit $\delta \to 0$ and using the fact that the semigroup is self-adjoint, we finally obtain the desired inequality. □

References

[AGS] L. Ambrosio, N. Gigli, and G. Savaré. *Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds,* Preprint 2012, 1–61.

[AMP] L. Angiuli, M. Miranda jr., D. Pallara, and F. Paronetto. *BV functions and parabolic initial boundary value problems on domains,* Ann. Mat. Pura Appl. (4) **188** (2009), 297–331.

[Ba] D. Bakry. *L'hypercontractivité et son utilisation en théorie des semigroupes,* in Lectures on probability theory (Saint-Flour, 1992), Lecture Notes in Math., 1581, Springer, Berlin, 1994, 1–114.

[BE] D. Bakry and M. Émery. *Diffusions hypercontractives,* in Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., 1123, Springer, Berlin, 1985, 177–206.

[BMP] M. Bramanti, M. Miranda Jr., and D. Pallara. *Two characterization of BV functions on Carnot groups via the heat semigroup,* Int. Math. Res. Not. IMRN **17** (2012), 3845–3876.

[B] H. Brezis. *How to recognize constant functions. A connection with Sobolev spaces,* Uspekhi Mat. Nauk **57** (2002), 59–74.

[CM] A. Carbonaro and G. Mauceri. *A note on bounded variation and heat semigroup on Riemannian manifolds,* Bull. Austral. Math. Soc. **76** (2007), 155–160.

[C] J. Cheeger. *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999), 428–517.

[D] J. Dávila. *On an open question about functions of bounded variation,* Calc. Var. Partial Differential Equations **15** (2002), 519–527.

[DeG] E. De Giorgi. *Selected papers,* Edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo. Springer-Verlag, Berlin, 2006 .

[DS] J.-D. Deuschel and D. W. Stroock. *Large Deviations,* Pure and Applied Mathematics, Academic Press, Boston, MA, **137** , 1989 .

[FHK] B. Franchi, P. Hajłasz, and P. Koskela. *Definitions of Sobolev classes on metric spaces,* Ann. Inst. Fourier (Grenoble) **49** (1999), 1903–1924.

[FOT] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes,* de Gruyter studies in mathematics, Walter de Gruyter and Co., Berlin **19** , 1994 .

[G] A. Grigor'yan. *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds,* Bull. Amer. Math. Soc. (N.S.) **36** (1999), 135–249.

[GP] B. Güneysu and D. Pallara. *Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below,* Preprint 2013, 1–26.

[HKT] T. Heikkinen, P. Koskela, and H. Tuominen. *Sobolev-type spaces from generalized Poincaré inequalities,* Studia Math. **181** (2007), 1–16.

[HK] J. Heinonen and P. Koskela. *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. **181** (1998), 1–61.

[JK] R. Jiang and P. Koskela. *Isoperimetric inequality from the Poisson equation via curvature,* Comm. Pure Appl. Math. **65** (2012), 1145–1168.

[KS] N.J. Korevaar and R.M. Schoen. *Sobolev spaces and harmonic maps for metric space targets,* Comm. Anal. Geom. **1** (1993), 561–659.

[KM] P. Koskela and P. MacManus. *Quasiconformal mappings and Sobolev spaces,* Studia Math. **131** (1998), 1–17.

[KRS] P. Koskela, K. Rajala, and N. Shanmugalingam. *Lipschitz continuity of Cheeger-harmonic functions in metric measure spaces,* J. Funct. Anal. **202** (2003), 147–173.

[L] M. Ledoux. *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space,* Bull. Sci. Math. **118** (1994), 485–510.

[MMS] N. Marola, M. Miranda jr., and N. Shanmugalingam. *Boundary measures, generalized Gauss–Green formulas, and mean value property in metric measure spaces,* Preprint 2013, 1–33.

[M] V. Mazýa. *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings,* J. Funct. Anal. **224** (2005), 408–430.

[Mr] M. Miranda jr. *Functions of bounded variation on "good" metric spaces,* J. Math. Pures Appl. **82** (2003), 975–1004.

[MP] M. Miranda Jr., D. Pallara, F. Paronetto, and M. Preunkert. *Heat semigroup and functions of bounded variation on Riemannian manifolds,* J. Reine Angew. Math. **613** (2007), 99–119.

[MP2] M. Miranda Jr., D. Pallara, F. Paronetto, and M. Preunkert. *Short-time heat flow and functions of bounded variation in* $\mathbb{R}^N$, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), 125–145.

[P-P] K. Pietruska-Pałuba. *Heat kernels on metric spaces and a characterisation of constant functions,* Manuscripta Math. **115** (2004), 389–399.

[P] M. Preunkert. *A semigroup version of the isoperimetric inequality,* Semigroup Forum **68** (2004), 233–245.

[S] G. Savaré. *Self-improvement of the Bakry–Émery condition and Wasserstein contraction of the heat flow in* $\mathrm{RCD}(K,\infty)$ *metric measure spaces,* Discrete Contin. Dyn. Syst. **34** (2014), 1641–1661.

[Sh] N. Shanmugalingam. *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces,* Rev. Mat. Iberoamericana **16** (2000), 243–279.

[St1] K.-T. Sturm. *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and* L^p*-Liouville properties,* J. Reine Angew. Math. **456** (1994), 173–196.

[St2] K.-T. Sturm. *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations,* Osaka J. Math. **32** (1995), 275–312.

[St3] K.-T. Sturm. *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality,* J. Math. Pures Appl. (9) **75** (1996), 273–297.

[W] F.Y. Wang. *Equivalent semigroup properties for the curvature–dimension condition,* Bull.Sci.Math. **135** (2011), 803–815.

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