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REPORT No. 39, 2013/2014, fall

ISSN 1103-467X
ISRN IML-R- -39-13/14- -SE+fall

A SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITY IN HYPERBOLIC n-SPACE

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ABSTRACT. In this paper we prove a quantitative version of the classical isoperimetric inequality in the hyperbolic space $\mathbb{H}^n$. The constant only depends on the dimension and an upper bound for the volume of the set.

1. INTRODUCTION

Quantitative versions of classical geometric inequalities, for example the isoperimetric inequality, have become an active field of research in recent years. Motivated by classical results of Bernstein and Bonnesen [4, 7] and further developments in [15, 18, 19], the optimal result for the Euclidean space $\mathbb{R}^n$ was finally established in [17]. Different proofs (by optimal mass transport, respectively a selection principle) are contained in [14, 9]. In the classical Euclidean case the optimal result states that if E is a set of finite measure in $\mathbb{R}^n$ then

$$\delta(E) \geq c(n)\alpha(E)^2 \tag{1.1}$$

holds true. Thereby, the *rescaled isoperimetric deficit* (also called isoperimetric gap) is defined as

$$\delta(E) := \frac{P(E) - P(B)}{P(B)},$$

where $B \subset \mathbb{R}^n$ is a ball with $|B| = |E|$. Here $|E|$ denotes the volume of the set E and $P(E)$ stands for the perimeter of E. The other quantity in (1.1) is the so called *Fraenkel asymmetry index*, which is given by

$$\alpha(E) := \min \frac{|E\Delta B|}{|B|},$$

where the minimum is taken over all balls $B \subset \mathbb{R}^n$ of volume $|E|$. Extensions of the stability estimate (1.1) to the case of the Gaussian isoperimetric inequality [8], to the Almgren higher codimension isoperimetric inequality [2, 5], to the case of sets in the sphere $S^n \subset \mathbb{R}^{n+1}$ [6], and finally to non-local perimeter functionals [13] have been recently established. Further extensions to other isoperimetric problems have been obtained [3, 1, 10].

The motivation for the present paper stems from the classical isoperimetric inequality for the hyperbolic space $\mathbb{H}^n$, which was established by Schmidt in [22]. The inequality ensures the isoperimetric property of balls, i.e. if $E \subset \mathbb{H}^n$ is a measurable set and $\mathbf{B}_\varrho(x_o) \subset \mathbb{H}^n$ a geodesic ball with the same volume as E, then

$$\mathbf{P}(E) \geq \mathbf{P}(\mathbf{B}_\varrho(x_o)),$$

Date: August 22, 2014.

2000 *Mathematics Subject Classification.* 49Q20, 52A40.

Key words and phrases. hyperbolic space, quantitative isoperimetric inequality.

with equality if and only if E is a geodesic ball. Here, $\mathbf{P}(E)$ stands for the perimeter of E; see §2.3 for the definition. The hyperbolic counterpart to (1.1) would be the estimate

$$\frac{\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_\varrho)} \geq c\left(\frac{\boldsymbol{\alpha}(E)}{\mathbf{V}(\mathbf{B}_\varrho)}\right)^2, \tag{1.2}$$

where the non rescaled isoperimetric gap is given by $\mathbf{D}(E) = \mathbf{P}(E) - \mathbf{P}(\mathbf{B}_\varrho)$, and the hyperbolic Fraenkel asymmetry index is defined via the hyperbolic volume $\mathbf{V}$ (see §2.3) as follows

$$\boldsymbol{\alpha}(E) := \min \mathbf{V}(E \Delta \mathbf{B}_\varrho(x_o)). \tag{1.3}$$

Again the minimum is taken over all geodesic balls $\mathbf{B}_\varrho(x_o) \subset \mathbb{H}^n$ with $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_\varrho)$. We note that (1.2) is invariant under the isometry group of $\mathbb{H}^n$, while (1.1) is invariant under isometries of $\mathbb{R}^n$ and moreover scaling invariant (i.e. invariant under homotheties). This is the reason why we defined the isoperimetric gap and the Fraenkel asymmetry index in an un-rescaled way. At this stage, one of the known methods of proof from [17, 14, 9] could easily be adapted to the present situation, yielding the inequality (1.2) with a constant $c = c(n, \varrho)$, blowing up as $\varrho \downarrow 0$ and $\varrho \to \infty$. One of the difficulties in proving a result with a constant only depending on n are sets E with small volume sparsely distributed over $\mathbb{H}^n$. These sets do not allow the reduction to the Euclidean case. The same holds true for sets with large volumes.

In the present paper we use instead of the Fraenkel asymmetry index from (1.3), which measures the L^1-distance between the set E and an optimal geodesic ball of the same volume, an L^2-*oscillation index*, which measures the oscillations of the outer normal to a set E relative to the outer normal of geodesic balls $\mathbf{B}_\varrho$ with the same volume as E. Roughly speaking, this index measures the distance between the distributional derivatives of the characteristic function χ_E and of $\chi_{\mathbf{B}_\varrho(x_o)}$, where $\mathbf{B}_\varrho(x_o)$ is an optimal geodesic ball of the same volume as E. The precise construction is given in §2.6. The heuristic definition is a follows: Let $\mathbf{B}_\varrho(x_o)$ be a geodesic ball with volume $\mathbf{V}(E)$, and denote by π the nearest point retraction of $\mathbb{H}^n$ onto $\partial\mathbf{B}_\varrho(x_o)$. We compare the outer unit normal $\nu_E(x) \in T_x\mathbb{H}^n$ with the outer unit normal $\nu_{\mathbf{B}_\varrho(x_o)}(\pi(x)) \in T_{\pi(x)}\mathbb{H}^n$ to $\partial\mathbf{B}_\varrho(x_o)$ in $\pi(x)$, by using the parallel transport $\Pi_{\pi(x),x}\colon T_{\pi(x)}\mathbb{H}^n \to T_x\mathbb{H}^n$ between the two tangent spaces. The L^2-*oscillation index of E with respect to* $\mathbf{B}_\varrho(x_o)$ is then defined by

$$\boldsymbol{\beta}(E; x_o) := \left[\tfrac{1}{2}\int_{\partial E} \left|\nu_E(x) - \Pi_{\pi(x),x}(\nu_{\mathbf{B}_\varrho(x_o)}(\pi(x)))\right|_h^2 d\sigma_h(x)\right]^{\frac{1}{2}}, \tag{1.4}$$

and the L^2-*oscillation index of E* is defined as the minimum over all balls $\mathbf{B}_\varrho(x_o) \subset \mathbb{H}^n$ with $\mathbf{V}(\mathbf{B}_\varrho(x_o)) = \mathbf{V}(E)$, that is

$$\boldsymbol{\beta}(E) := \min_{x_o \in \mathbb{H}^n} \boldsymbol{\beta}(E; x_o). \tag{1.5}$$

As already mentioned above, in some sense $\boldsymbol{\beta}(E)$ measures the L^2-distance of the distributional derivatives of the characteristic function of the set E and of the characteristic function of an optimal geodesic ball in (1.4) of the same volume as E. It could also be interpreted as an *Excess functional* between ∂E and $\partial\mathbf{B}_\varrho(x_o)$. It is worthwhile to note that the Fraenkel asymmetry index and the L^2-oscillation index are related by the following Poincaré type inequality (see Lemma 2.10):

$$\boldsymbol{\beta}^2(E) \geq \mathbf{D}(E) + \frac{c(n)}{\sinh^{n+1}\varrho}\,\boldsymbol{\alpha}^2(E). \tag{1.6}$$

The main result of the present paper now reads as follows:

Theorem 1.1. *For any $R_o > 0$ there exists a constant $c(n, R_o) > 0$ such that for any set $E \subset \mathbb{H}^n$ of finite perimeter with volume $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_\varrho)$ for some $\varrho \in (0, R_o]$, the following inequality holds*

$$\mathbf{D}(E) := \mathbf{P}(E) - \mathbf{P}(\mathbf{B}_\varrho) \geq c(n, R_o)\boldsymbol{\beta}^2(E). \tag{1.7}$$

The inequality (1.7) is the analogue for the hyperbolic space of a similar inequality for the Euclidean space $\mathbb{R}^n$ established in [16]. The counterpart for the sphere S^n was shown to hold in [6]. The definition of the L^2-oscillation index was first given in [16]. Observe, that the main estimate (1.7), the Poincaré type inequality (1.6) and Lemma 2.1 (i.e. elementary facts about the isoperimetric profile) yield the analogue of the classical quantitative isoperimetric inequality in the hyperbolic space as stated in (1.2) with a constant depending on n and R_o. We note that (1.7) is stronger than (1.2), since by (1.6) we have also the reversed inequality $\mathbf{D}(E) \leq \boldsymbol{\beta}^2(E)$.

Finally a few words concerning the methods of proof are in order. As it is well known from Geometric measure theory, regularity for a sequence of (almost) minimizing perimeter sets can be established whenever the sequence converges to a smooth set. This means, that for large indices the sets are smooth. Now, if the sequence arises from the contradiction hypothesis of a specific quantitative geometric inequality, for example as in the case considered here, it is likely that the sequence consists of almost minimizers to the perimeter and converges in the limit to a geodesic ball. The regularity then implies that the sets (at least for large indices) must be radial graphs over the boundary sphere. Therefore, the contradiction would follow, once a quantitative version of the geometric inequality holds true for radial graphs. In the context of quantitative isoperimetric problems this strategy was set up in [9]. Extensions to Almgren's higher co-dimension case and quantitative isoperimetric inequalities involving the L^2-oscillation index – for the Euclidean space and for the sphere – can be found in [5, 16, 6]. To summarize, the starting point for the proof of Theorem 1.1 is a Fuglede-type stability result aimed to establish (1.7) for sets $E \subset S^n$ whose boundary can be written as a radial graph over the boundary of a ball $\mathbf{B}_\varrho(x_o)$ with the same volume; this is achieved in Theorem 4.1. However, to deduce (1.7) for radial graphs with a constant depending only on n and an upper bound for the volume, one carefully has to keep track on the dependencies of the appearing constants in the proof. The main difficulty arises when passing from the special situation of radial graphs to arbitrary sets. In principle one has to overcome the missing scaling invariance (which is used in $\mathbb{R}^n$ in order to renormalize to sets of volume one) and also the missing compactness of the space (which can be used at certain stages to yield compactness of certain sequences). In the hyperbolic space both properties are not available, and therefore we need to change significantly the strategies developed in [9, 16, 5].

To explain one of the major difficulties, we note that the L^2-oscillation index can be re-written (see (2.26)) in the form

$$\boldsymbol{\beta}^2(E) = \mathbf{P}(E) - (n-1) \max_{x_o \in \mathbb{H}^n} \int_E \coth \mathbf{d}_{x_o}(x)\, d\mu_h(x),$$

where $\mathbf{d}_{x_o}(x)$ denotes the distance between x and x_o. Hence, we have to establish estimates for the singular integral

$$\int_E \coth \mathbf{d}_{x_o}(x)\, d\mu_h(x) \tag{1.8}$$

and its maximum with respect to x_o. These estimates must be independent of the volume of the set E, in the sense that the constants can only depend on an upper bound for the volume. Moreover, the slicing procedure, which is necessary in order to reduce to a situation with bounded sets, is much more delicate in the hyperbolic space. For instance, it turns out that the right analogues to the standard Euclidean hyperplanes $\{x_j = t\} \subset \mathbb{R}^n$ are not – as one might expect – totally geodesic submanifolds of $\mathbb{H}^n$, but the *hyperplanes* $\{x \in B_1 : \operatorname{Arsinh} \frac{2x_k}{1-|x|^2} = t\}$ in the Poincaré disk type model. All of this requires new technically involved ideas and strategies; cf. the proofs of the Continuity Lemma 2.9, the Slicing Lemma 3.3 and the final proof in § 6.

For example, in the contradiction argument used to deduce (1.7) for general sets from the case of a radial graph, it is crucial that the constants are independent of the volume of E, in the sense that they only depend on the given upper bound for the volume. The most delicate part is the one when the volume gets small, i.e. $\mathbf{P}(E) \to 0$. In this case we can show that if $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_\varrho) \to 0$, $E \subset \mathbf{B}_{R\varrho}$ and $\varrho^{1-n}\mathbf{D}(E) \to 0$, then also $\varrho^{1-n}\boldsymbol{\beta}^2(E) \to 0$; cf. Lemma 6.1. From this one can conclude that the re-scaled sets $\varrho^{-1}E$ converge to the Euclidean ball $B_{\frac12}$. Since the re-scaled sets are almost minimizers of perturbed perimeter functionals (the functionals converge to the perimeter as $\varrho \downarrow 0$) we can conclude that the re-scaled sets are radial graphs over $\partial B_{\frac12}$. Transforming back to the hyperbolic space the application of the Fuglede type result is possible, yielding the contradiction.

Acknowledgements. We acknowledge the warm hospitality of the Institut Mittag-Leffler in the Fall 2013 during the program "Evolutionary problems", where parts of this paper were written.

2. Preliminaries on hyperbolic space

2.1. **The Poincaré disk model.** In the Poincaré disk model for the hyperbolic n-space $\mathbb{H}^n$, with $n \in \mathbb{N}_{\ge 2}$, we consider the open unit ball $B^n \subset \mathbb{R}^n$, equipped with the Riemannian metric g, given by the coefficients $g_{ij}(x) = \big(\frac{2}{1-|x|^2}\big)^2\delta_{ij}$. More precisely, $g(x)$ is an inner product on the tangent spaces $T_x\mathbb{H}^n \simeq \mathbb{R}^n$ for every $x \in \mathbb{H}^n$. The hyperbolic metric leads to the notion of the **hyperbolic distance**

$$\mathbf{d}(x,y) = \operatorname{Arcosh}\left[\frac{(1+|x|^2)(1+|y|^2) - 4x\cdot y}{(1-|x|^2)(1-|y|^2)}\right], \qquad \text{for any } x, y \in B^n.$$

For notational convenience, we introduce the abbreviation $\mathbf{d}_y(x) := \mathbf{d}(x,y)$. In particular, for the distance to the origin we have

$$\mathbf{d}_o(x) = \operatorname{Arcosh}\frac{1+|x|^2}{1-|x|^2} = \log\frac{1+|x|}{1-|x|} = 2\operatorname{Artanh}|x|, \qquad \text{for any } x \in B^n.$$

By $\mathbf{B}_\varrho(x_o)$ we denote the geodesic balls $\{x \in \mathbb{H}^n\colon \mathbf{d}(x,x_o) < \varrho\}$ in $\mathbb{H}^n$, and similarly by $\mathbf{S}_\varrho(x_o)$ the geodesic spheres $\{x \in \mathbb{H}^n : \mathbf{d}(x,x_o) = \varrho\}$. Occasionally, we will also use Euclidean balls, which we denote by $B_\varrho(x_o)$. Let G be the group of orientation preserving conformal automorphisms of B^n, where $n \ge 2$. Then G is the oriented isometry group of $\mathbb{H}^n$. For $x \in B^n$ we have orientation-reversing automorphisms

$$s_x(y) := -x^* + \Big(\frac{1}{|x|^2} - 1\Big)\frac{y + x^*}{|y + x^*|^2}, \quad \text{where} \quad x^* := \frac{x}{|x|^2},$$
$$\tau_x(y) := y - 2\frac{x\cdot y}{|x|^2}\,\frac{x}{|x|^2}.$$

This allows to define a smooth map $h\colon B^n \to G$ with the property $h_x(0) = x$ by defining

$$h_x := \begin{cases} \tau_x \circ s_x & \text{for } x \neq 0, \\ \text{id} & \text{for } x = 0. \end{cases} \tag{2.1}$$

2.2. **The Hyperboloid or Lorentz model.** Here, we consider $\mathbb{R}^{n+1}$ equipped with the **Lorentz metric**

$$x \cdot_h y := -x_o y_o + \sum_{i=1}^{n} x_i y_i,$$

for vectors $x = (x_0, x_1, \dots, x_n)$ and $y = (y_o, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$. The hyperbolic n-space is then given by the Hyperboloid

$$H^n := \{x \in \mathbb{R}^{n+1} \,:\, x \cdot_h x = -1,\ x_o > 0\}$$

with the Riemannian metric induced by the Lorentz metric $\cdot_h$ on the tangent spaces of H^n. We note that this defines inner products on the tangent spaces $T_x H^n$ even though $\cdot_h$ is not positive definite on $\mathbb{R}^{n+1}$. For $x = (x_o, x') \in H^n$ and $y = (y_o, y') \in H^n$, we use $x_o^2 = 1 + |x'|^2$, respectively $y_o^2 = 1 + |y'|^2$ together with Young's inequality to infer that

$$\begin{aligned} -x \cdot_h y &= \sqrt{1 + |x'|^2}\,\sqrt{1 + |y'|^2} - x' \cdot y' \\ &\leq \tfrac{1}{2}(1 + |x'|^2 + 1 + |y'|^2) - x' \cdot y' = 1 + \tfrac{1}{2}|x' - y'|^2. \end{aligned}$$

In the Hyperboloid model, the hyperbolic distance between two points $x, y \in H^n$ is given by

$$d_h(x, y) = \operatorname{Arcosh}(-x \cdot_h y) = \operatorname{Arcosh}(|x \cdot_h y|).$$

This provides us with the following estimate, which compares the hyperbolic distance with the Euclidean norm:

$$d_h(x, y) \leq \operatorname{Arcosh}\big(1 + \tfrac{1}{2}|x' - y'|^2\big). \tag{2.2}$$

Later on, in certain slicing arguments, the hyperbolic distance of points x in H^n to the *hyperbolic hyperplanes* $H^n \cap \{y_k = 0\}$ with $k \in \{1, \dots, n\}$ will play a central role. For a given $x \in H^n$, the nearest point to $H^n \cap \{y_k = 0\}$ is given by the vector $y \in H^n$ with $y_k = 0$ and $y_i = x_i / \sqrt{1 + x_k^2}$ for $i \neq k$. This leads us to the formula

$$d_h(x, H^n \cap \{y_k = 0\}) = \operatorname{Arcosh}\Big(\sqrt{1 + x_k^2}\Big) = |\operatorname{Arsinh}(x_k)| \tag{2.3}$$

for any $k \in \{1, \dots, n\}$. Moreover, we will consider the *nearest-point retractions*

$$\pi_t\colon H^n \cap \{y_k > t\} \to H^n \cap \{y_k = t\} \qquad \text{for } t \geq 0.$$

A straightforward calculation shows that these maps are given by

$$\pi_t(x_o, \dots, x_n) := (p_o, \dots, p_n) \qquad \text{with} \qquad \begin{cases} p_\ell = t, & \text{if } \ell = k. \\ p_\ell = \frac{\sqrt{1+t^2}}{\sqrt{1+x_k^2}} x_\ell & \text{if } \ell \neq k. \end{cases}$$

We check that π_t is a contraction by calculating

$$\begin{aligned} &\cosh[d_h(\pi_t(x), \pi_t(y))] - 1 \\ &\quad = \tfrac{1+t^2}{\sqrt{1+x_k^2}\sqrt{1+y_k^2}} \Big(x_0 y_0 - \sum_{\ell \neq k} x_\ell y_\ell \Big) - t^2 - 1 \\ &\quad = \tfrac{1+t^2}{\sqrt{1+x_k^2}\sqrt{1+y_k^2}} \Big(\cosh[d_h(x, y)] - 1 \Big) + (1 + t^2) \Big(\tfrac{x_k y_k + 1}{\sqrt{1+x_k^2}\sqrt{1+y_k^2}} - 1 \Big) \end{aligned}$$

$$\le \tfrac{1+t^2}{\sqrt{1+x_k^2}\sqrt{1+y_k^2}}\Big(\cosh[d_h(x,y)]-1\Big) \le \cosh[d_h(x,y)]-1,$$

where we used the Cauchy-Schwarz inequality, $|x_k|, |y_k| > |t|$ and the fact that $\cosh[d_h(x,y)]-1 \ge 0$. This implies that

$$\pi_t : H^n \cap \{x_k > t\} \to H^n \cap \{x_k = t\} \tag{2.4}$$

has Lipschitz constant ≤ 1 with respect to the Riemannian distance if $t \ge 0$. Analogously, the nearest-point retraction

$$\pi_s : H^n \cap \{x_k < s\} \to H^n \cap \{x_k = s\} \tag{2.5}$$

is a contraction for $s \le 0$.

2.3. **Sets of finite perimeter in hyperbolic space.** For a Borel set $E \subset B^n$ the **hyperbolic volume** μ_h and **surface measure** σ_h are defined by

$$\mathbf{V}(E) = \mu_h(E) = \int_E \Big(\frac{2}{1-|x|^2}\Big)^n d\mathcal{L}^n x \quad \text{and} \quad \sigma_h(E) = \int_E \Big(\frac{2}{1-|x|^2}\Big)^{n-1} d\mathcal{H}^{n-1} x.$$

Here, $\mathcal{L}^n$ stands for the Lebesgue measure on $\mathbb{R}^n$, while $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Instead of $d\mathcal{L}^n x$ we write as usual dx. There are different equivalent ways to define the total variation of real-valued functions $f\colon \mathbb{H}^n \to \mathbb{R}$. The first and most useful one goes as follows: A function $f \in L^1_{\rm loc}(\mathbb{H}^n)$ is said to belong to $\mathrm{BV}_{\rm loc}(\mathbb{H}^n)$ if and only if for any set $U \Subset \mathbb{H}^n$ the total variation $\|Df\|_h(U)$ is finite, that is

$$\|Df\|_h(U) := \sup\bigg\{ \int_U f \,\mathrm{div}^{\mathbb{H}} X \, d\mu_h \colon X \in C^1_o(U, \mathbb{R}^n),\ |X(x)|_h \le 1 \text{ for } x \in U \bigg\} < \infty.$$

A measurable set $E \subset \mathbb{H}^n$ has locally finite perimeter, if the corresponding characteristic function satisfies $\chi_E \in \mathrm{BV}_{\rm loc}(\mathbb{H}^n)$. Instead of $\|D\chi_E\|_h(U)$ we write $\mathbf{P}_U(E)$ for the **hyperbolic perimeter of** E **in** U. For such sets we have that if $\psi\colon \mathbb{H}^n \supset U \to W \subset \mathbb{R}^n$ is a local chart, then $\psi(E \cap U)$ has locally finite perimeter in W. By the Riesz representation theorem we know that there exists a $\|D\chi_E\|$-measurable vector field $\nu_E\colon \mathbb{H}^n \to T\mathbb{H}^n$ with $|\nu_E(x)|_h = 1$ for $\|D\chi_E\|$-almost every $x \in \mathbb{H}^n$, such that

$$\int_E \mathrm{div}^{\mathbb{H}} X \, d\mu_h = \int \langle X, \nu_E\rangle_h \, d\|D\chi_E\|$$

holds true whenever X is a smooth tangential vectorfield on $\mathbb{H}^n$ with compact support. This allows us to define for finite perimeter sets the so called **reduced boundary**. Given $x_o \in \mathbb{H}^n$ we choose a local orthonormal frame $\{X_i\}_{i=1,\dots,n}$ at x_o. Then, x_o is a point of the reduced boundary $\partial^* E$ if and only if the limits

$$\nu_E^j(x_o) := \lim_{\varrho\downarrow 0} \frac{\int_{\mathbf{B}_\varrho(x_o)} \langle \nu_E, X_j\rangle_h \, d\|D\chi_E\|}{\|D\chi_E\|(\mathbf{B}_\varrho(x_o))}, \quad j = 1, \dots, n,$$

exist and $\nu_E(x_o) := \sum_{j=1}^n \nu_E^j(x_o) X_j(x_o) \in T_{x_o}\mathbb{H}^n$ satisfies $|\nu_E(x_o)|_h = 1$. As can easily be inferred, the definition is independent of the choice of the orthonormal frame. De Giorgi's structure theorem states that the reduced boundary $\partial^* E$ is countably σ_h rectifiable and the total variation measure $\|D\chi_E\|_h$ is supported on $\partial^* E$, more precisely $\|D\chi_E\|_h = \sigma_h \llcorner \partial^* E$ holds true. Moreover, for sets E with locally finite perimeter one has the Gauss-Green theorem

$$\int_E \mathrm{div}^{\mathbb{H}} X \, d\mu_h = \int_{\partial^* E} \langle X, \nu_E\rangle_h \, d\sigma_h$$

whenever X is a tangential vector field on $\mathbb{H}^n$ with compact support. Finally, we note that for sets $E \subset B^n$ with a C^1-boundary, we have

$$\mathbf{P}_U(E) = \int_{\partial E\cap U} \Big(\frac{2}{1-|x|^2}\Big)^{n-1} d\,\mathcal{H}^{n-1}x.$$

For the Euclidean perimeter, respectively the volume of a set E, we employ the notations $P(E)$ and $|E|$, in contrast to the hyperbolic variants $\mathbf{P}(E)$ and $\mathbf{V}(E)$.

2.4. **Volume and perimeter of hyperbolic balls.** The hyperbolic sphere $\mathbf{S}_\varrho$ of radius $\varrho > 0$ around the origin is given by the set $\{x \in B^n : |x| = \tanh \frac{\varrho}{2}\}$. Therefore, we get for a general hyperbolic sphere with radius $\varrho > 0$

$$\mathbf{P}(\mathbf{S}_\varrho) = n\omega_n \tanh^{n-1} \tfrac{\varrho}{2} \Big(\frac{2}{1-\tanh^2 \frac{\varrho}{2}}\Big)^{n-1} = n\omega_n \sinh^{n-1} \varrho, \tag{2.6}$$

where ω_n denotes the (Euclidean) volume of the n-dimensional unit ball. Similarly, the **hyperbolic volume of a hyperbolic ball** of radius $\varrho > 0$ can be computed by

$$\mathbf{V}(\mathbf{B}_\varrho) = n\omega_n \int_0^{\tanh \frac{\varrho}{2}} \sigma^{n-1}\Big(\frac{2}{1-\sigma^2}\Big)^n d\sigma = n\omega_n \int_0^\varrho \sinh^{n-1} t\, dt. \tag{2.7}$$

The isoperimetric property of hyperbolic balls was established by E. Schmidt [22]. More precisely he proved that for any set $E \subset B^n$ of finite perimeter with $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$ and $\varrho_o > 0$ there holds

$$\mathbf{P}(\mathbf{B}_{\varrho_o}) \le \mathbf{P}(E), \tag{2.8}$$

with equality if and only if E is a hyperbolic ball with radius ϱ_o. For bounded radii, the relation between perimeter and volume of hyperbolic balls is comparable to the Euclidean case in the following sense.

Lemma 2.1. *Let $R_o > 0$. For any radius $\varrho \in (0, R_o]$, we have the estimates*

$$n\omega_n^{1/n}\mathbf{V}(\mathbf{B}_\varrho)^{\frac{n-1}{n}} \le \mathbf{P}(\mathbf{B}_\varrho) \le n\omega_n^{1/n}\big[\cosh(R_o)\mathbf{V}(\mathbf{B}_\varrho)\big]^{\frac{n-1}{n}}.$$

Proof. For the first inequality, we use $\cosh t \ge 1$ and the formulas (2.7), (2.6) for the volume and perimeter of a ball in order to estimate

$$\mathbf{V}(\mathbf{B}_\varrho) \le n\omega_n \int_0^\varrho \sinh^{n-1} t\, \cosh t\, dt = \omega_n \sinh^n \varrho = n^{-\frac{n}{n-1}}\omega_n^{-\frac{1}{n-1}}\mathbf{P}(\mathbf{B}_\varrho)^{\frac{n}{n-1}}.$$

This yields the first asserted estimate. For the second one, we use instead $\cosh t \le \cosh(R_0)$ in case $0 \le t \le \varrho \le R_0$, which yields similarly as above

$$\cosh(R_0)\mathbf{V}(\mathbf{B}_\varrho) \ge n\omega_n \int_0^\varrho \sinh^{n-1} t\, \cosh t\, dt = n^{-\frac{n}{n-1}}\omega_n^{-\frac{1}{n-1}}\mathbf{P}(B_\varrho)^{\frac{n}{n-1}}.$$

This finishes the proof of the lemma. □

For large radii however, the perimeter of a ball is asymptotically proportional to the volume. More precisely, we have the estimate

$$\tfrac{1}{n-1}\mathbf{P}(\mathbf{B}_\varrho) \ge \mathbf{V}(\mathbf{B}_\varrho), \tag{2.9}$$

which we can check using the elementary estimate $\sinh t \le \cosh t$ as follows.

$$\mathbf{V}(\mathbf{B}_\varrho) \le n\omega_n \int_0^\varrho \sinh^{n-2} t \cosh t\, dt = \tfrac{n}{n-1}\omega_n \sinh^{n-1} \varrho = \tfrac{1}{n-1}\mathbf{P}(\mathbf{B}_\varrho).$$

Lemma 2.2. *For every $n \in \mathbb{N}$ the function $Q_n\colon (0,\infty) \to (0,\infty)$, defined by*

$$Q_n(\varrho) := \frac{\mathbf{V}(\mathbf{B}_\varrho)}{\mathbf{P}(\mathbf{B}_\varrho)}$$

is increasing. Moreover, we have the estimate

$$Q_n(\varrho) < \frac{\sinh\varrho\cosh\varrho}{n\cosh^2\varrho - \sinh^2\varrho} \qquad \text{for any } \varrho > 0. \tag{2.10}$$

Proof. Recalling formulas (2.6) and (2.7) for the perimeter and volume, we know that (2.10) is equivalent to

$$\int_0^\varrho \sinh^{n-1} t\,dt < \frac{\sinh^n\varrho\cosh\varrho}{n\cosh^2\varrho - \sinh^2\varrho}\,. \tag{2.11}$$

Clearly, for $\varrho = 0$ both sides of (2.11) are equal to 0. In order to conclude the estimate for every $\varrho > 0$, we compute the derivative of the right-hand side and obtain that

$$\frac{d}{d\varrho}\bigg[\frac{\sinh^n\varrho\cosh\varrho}{n\cosh^2\varrho - \sinh^2\varrho}\bigg] = \sinh^{n-1}\varrho\bigg(1 + \frac{2\sinh^2\varrho}{(n\cosh^2\varrho - \sinh^2\varrho)^2}\bigg) > \sinh^{n-1}\varrho$$

holds true, which implies (2.11). Next, we compute

$$Q_n'(\varrho) = 1 - \frac{(n-1)\cosh\varrho}{\sinh^n\varrho}\int_0^\varrho \sinh^{n-1} t\,dt.$$

Using (2.11) we therefore find that

$$Q_n'(\varrho) > 1 - \frac{(n-1)\cosh^2\varrho}{n\cosh^2\varrho - \sinh^2\varrho} = 1 - \frac{(n-1)\cosh^2\varrho}{1 + (n-1)\cosh^2\varrho} > 0,$$

which shows that Q_n is increasing. □

Lemma 2.3. *There exists a constant $c = c(n) > 0$ such that for any $0 \le \varrho_1 < \varrho_2$ we have*

$$\frac{\cosh\varrho_2}{\sinh\varrho_2}\cdot\frac{\mathbf{V}(\mathbf{B}_{\varrho_2}) - \mathbf{V}(\mathbf{B}_{\varrho_1})}{\mathbf{P}(\mathbf{B}_{\varrho_2}) - \mathbf{P}(\mathbf{B}_{\varrho_1})} \ge c. \tag{2.12}$$

Proof. In the case $n = 2$ a computation using the exact formulae for the hyperbolic volume and perimeter together with elementary properties of $\sinh$, $\cosh$ and $\tanh$ yields the result with $c = \frac{1}{2}$. In the case $n \ge 3$ we will distinguish three different cases. We start with *the case* $0 \le \varrho_1 < \varrho_2 \le 1$. In this case we have $\sigma \le \sinh\sigma \le 2\sigma$ for $\sigma \in [\varrho_1, \varrho_2]$. Therefore, using (2.7) we find that

$$\mathbf{V}(\mathbf{B}_{\varrho_2}) - \mathbf{V}(\mathbf{B}_{\varrho_1}) = n\omega_n\int_{\varrho_1}^{\varrho_2}\sinh^{n-1}\sigma\,d\sigma \ge n\omega_n\int_{\varrho_1}^{\varrho_2}\sigma^{n-1}\,d\sigma = \omega_n\big(\varrho_2^n - \varrho_1^n\big).$$

On the other hand, now using (2.6) we obtain in a similar way

$$\begin{aligned}
\frac{\sinh\varrho_2}{\cosh\varrho_2}\big[\mathbf{P}(\mathbf{B}_{\varrho_2}) - \mathbf{P}(\mathbf{B}_{\varrho_1})\big] &= n(n-1)\omega_n\frac{\sinh\varrho_2}{\cosh\varrho_2}\int_{\varrho_1}^{\varrho_2}\sinh^{n-2}\sigma\cosh\sigma\,d\sigma\\
&\le n(n-1)\omega_n\sinh\varrho_2\int_{\varrho_1}^{\varrho_2}\sinh^{n-2}\sigma\,d\sigma\\
&\le 2^{n-1}n(n-1)\omega_n\varrho_2\int_{\varrho_1}^{\varrho_2}\sigma^{n-2}\,d\sigma\\
&= 2^{n-1}n\omega_n\varrho_2\big(\varrho_2^{n-1} - \varrho_1^{n-1}\big)\\
&\le 2^{n-1}n\omega_n\big(\varrho_2^n - \varrho_1^n\big),
\end{aligned}$$

so that (2.12) holds with $c = 1/(2^{n-1}n)$. Next, we consider *the case* $1 \le \varrho_1 < \varrho_2$. Here, we have $\sinh\sigma \ge \frac{1}{2}(1-e^{-2})e^{\sigma}$ for $\sigma \in [\varrho_1, \varrho_2]$. This leads us to

$$\begin{aligned}\mathbf{V}(\mathbf{B}_{\varrho_2}) - \mathbf{V}(\mathbf{B}_{\varrho_1}) &\ge \frac{n\omega_n(1-e^{-2})^{n-1}}{2^{n-1}} \int_{\varrho_1}^{\varrho_2} e^{(n-1)\sigma}\,d\sigma \\ &= \frac{n\omega_n(1-e^{-2})^{n-1}}{2^{n-1}(n-1)} \big(e^{(n-1)\varrho_2} - e^{(n-1)\varrho_1}\big).\end{aligned}$$

Moreover, using $\sinh\sigma \le \frac{1}{2}e^{\sigma}$ we obtain

$$\begin{aligned}\frac{\sinh\varrho_2}{\cosh\varrho_2}\big[\mathbf{P}(\mathbf{B}_{\varrho_2}) - \mathbf{P}(\mathbf{B}_{\varrho_1})\big] &\le n(n-1)\omega_n \sinh\varrho_2 \int_{\varrho_1}^{\varrho_2} \sinh^{n-2}\sigma\,d\sigma \\ &\le \frac{n(n-1)\omega_n}{2^{n-1}} e^{\varrho_2} \int_{\varrho_1}^{\varrho_2} e^{(n-2)\sigma}\,d\sigma \\ &= \frac{n(n-1)\omega_n}{2^{n-1}(n-2)} e^{\varrho_2}\big(e^{(n-2)\varrho_2} - e^{(n-2)\varrho_1}\big) \\ &\le \frac{n(n-1)\omega_n}{2^{n-1}(n-2)} \big(e^{(n-1)\varrho_2} - e^{(n-1)\varrho_1}\big),\end{aligned}$$

proving (2.12) with $c = (n-2)(1-e^{-2})^{n-1}/(n-1)^2$. In the *remaining case* $0 \le \varrho_1 < 1 < \varrho_2$ we split the integral and then use the arguments from the first two cases. This leads us to

$$\begin{aligned}\mathbf{V}(\mathbf{B}_{\varrho_2}) - \mathbf{V}(\mathbf{B}_{\varrho_1}) &= n\omega_n\bigg[\int_{\varrho_1}^{1} \sinh^{n-1}\sigma\,d\sigma + \int_{1}^{\varrho_2} \sinh^{n-1}\sigma\,d\sigma\bigg] \\ &\ge \omega_n\big(1-\varrho_1^n\big) + \frac{n\omega_n(1-e^{-2})^{n-1}}{2^{n-1}(n-1)}\big(e^{(n-1)\varrho_2} - e^{n-1}\big),\end{aligned}$$

and moreover

$$\begin{aligned}&\frac{\sinh\varrho_2}{\cosh\varrho_2}\big[\mathbf{P}(\mathbf{B}_{\varrho_2}) - \mathbf{P}(\mathbf{B}_{\varrho_1})\big] \\ &\quad= n(n-1)\omega_n\frac{\sinh\varrho_2}{\cosh\varrho_2}\bigg[\int_{\varrho_1}^{1} \sinh^{n-2}\sigma\cosh\sigma\,d\sigma + \int_{1}^{\varrho_2} \sinh^{n-2}\sigma\cosh\sigma\,d\sigma\bigg] \\ &\quad\le n(n-1)\omega_n\bigg[2\int_{\varrho_1}^{1} \sinh^{n-2}\sigma\,d\sigma + \sinh\varrho_2\int_{1}^{\varrho_2} \sinh^{n-2}\sigma\,d\sigma\bigg] \\ &\quad\le n(n-1)\omega_n\bigg[2^{n-1}\int_{\varrho_1}^{1} \sigma^{n-2}\,d\sigma + \frac{e^{\varrho_2}}{2^{n-1}}\int_{1}^{\varrho_2} e^{(n-2)\sigma}\,d\sigma\bigg] \\ &\quad\le 2^{n-1}n\omega_n\big(1-\varrho_1^n\big) + \frac{n(n-1)\omega_n}{2^{n-1}(n-2)}\big(e^{(n-1)\varrho_2} - e^{n-1}\big),\end{aligned}$$

from which (2.12) follows once again. □

2.5. **The barycenter.** The **hyperbolic barycenter** of a measurable set $E \subset B^n$ of positive volume is defined as the unique minimizer $p \in B^n$ of the function

$$B^n \ni p \mapsto \int_E \mathbf{d}^2(x,p)\,d\mu_h(x) = \int_E \mathbf{d}^2(x,p)\Big(\frac{2}{1-|x|^2}\Big)^n dx.$$

We note that as a consequence of the nonpositive sectional curvature of the hyperbolic space, the function $p \mapsto \mathbf{d}^2(x,p)$ is strictly convex and consequently, the above expression attains a unique minimizer $p \in B^n$. This minimizer is characterized by the necessary condition

$$0 = \int_E 2\mathbf{d}(x,p)\nabla_p[\mathbf{d}(x,p)]\,d\mu_h(x), \tag{2.13}$$

where ∇_p denotes the gradient with respect to the p-variable. In the origin, a straightforward calculation gives

$$\tfrac{1}{4}\nabla_p[\mathbf{d}(x,p)]\Big|_{p=0} = \nabla_p^{\mathbb{H}}[\mathbf{d}(x,p)]\Big|_{p=0} = -\frac{x}{2|x|},$$

which also has the geometric interpretation of the direction in the point $p=0$ of the unique geodesic that connects x and $p=0$ and is parametrized by arc-length. In conclusion, the necessary condition for a set E to have its barycenter in the origin is

$$0 = \int_E \mathbf{d}_o(x)\frac{x}{|x|}\,d\mu_h(x) = \int_E 2\,\mathrm{Artanh}\,|x|\,\frac{x}{|x|}\,d\mu_h(x). \tag{2.14}$$

Our next aim is to prove the continuity of the barycenter in a special situation that will be needed in the final proof of the quantitative isoperimetric inequality; see Lemma 2.6. But before, we need two auxiliary lemmas.

Lemma 2.4. *If $E \subset \mathbf{B}_{\varrho_o}$ with $\varrho_o > 0$ and $\mathbf{V}(E) > 0$, then every barycenter p_o of E is contained in $\mathbf{B}_{\varrho_o}$.*

Proof. If $p_o \notin \mathbf{B}_{\varrho_o}$, a straightforward calculation implies that $p_o \cdot \nabla_p[\mathbf{d}(x,p_o)] > 0$ for any $x \in E$. But this contradicts the necessary condition (2.13) for p_o to be the barycenter of E. □

Lemma 2.5. *Let $w \in \mathbb{H}^n \setminus \{0\}$. Then, for any increasing function $f\colon [0,\infty) \to \mathbb{R}$ there holds*

$$\int_{\mathbf{B}_\varrho(w)\setminus\mathbf{B}_\varrho} f(\mathbf{d}_o(x))\,d\mu_h(x) - \int_{\mathbf{B}_\varrho\setminus\mathbf{B}_\varrho(w)} f(\mathbf{d}_o(x))\,d\mu_h(x) \geq c\,\min\{\varrho,\mathbf{d}_o(w)\}^n\Big(f\big(\varrho + \tfrac{1}{2}d_o(w)\big) - f(\varrho)\Big)$$

for some constant $c = c(n)$.

Proof. We let

$$\mathbf{I} := \int_{\mathbf{B}_\varrho(w)\setminus\mathbf{B}_\varrho} f(\mathbf{d}_o(x))\,d\mu_h(x) - \int_{\mathbf{B}_\varrho\setminus\mathbf{B}_\varrho(w)} f(\mathbf{d}_o(x))\,d\mu_h(x).$$

Then, if $\mathbf{d}_o(w) \geq 4\varrho$ the left-hand side can be bounded from below by

$$\mathbf{I} \geq \mathbf{V}(\mathbf{B}_\varrho)\big(f(\mathbf{d}_o(w)-\varrho) - f(\varrho)\big) \geq c(n)\varrho^n\Big(f\big(\varrho + \tfrac{1}{2}\mathbf{d}_o(w)\big) - f(\varrho)\Big).$$

If $\mathbf{d}_o(w) < 4\varrho$, we let $\varepsilon := \frac{1}{2\varrho}\mathbf{d}_o(w)$ and

$$S_1 := \big(\mathbf{B}_\varrho \cap \mathbf{B}_{\varrho(1+\varepsilon)}(w)\big) \setminus \mathbf{B}_\varrho(w), \qquad S_2 := \big(\mathbf{B}_\varrho(w) \cap \mathbf{B}_{\varrho(1+\varepsilon)}\big) \setminus \mathbf{B}_\varrho.$$

Since $\mathbf{V}(S_1) = \mathbf{V}(S_2)$ as well as $f(\mathbf{d}_o(x_1)) \leq f(\varrho) \leq f(\mathbf{d}_o(x_2))$ for any $x_1 \in S_1$ and $x_2 \in S_2$ (note that f is increasing), we find that

$$\begin{aligned}\mathbf{I} &:= \int_{\mathbf{B}_\varrho(w)\setminus\mathbf{B}_\varrho} f(\mathbf{d}_o(x))\,d\mu_h(x) - \int_{\mathbf{B}_\varrho\setminus\mathbf{B}_\varrho(w)} f(\mathbf{d}_o(x))\,d\mu_h(x)\\ &\geq \int_{\mathbf{B}_\varrho(w)\setminus\mathbf{B}_{\varrho(1+\varepsilon)}} f(\mathbf{d}_o(x))\,d\mu_h(x) - \int_{\mathbf{B}_\varrho\setminus\mathbf{B}_{\varrho(1+\varepsilon)}(w)} f(\mathbf{d}_o(x))\,d\mu_h(x)\end{aligned}$$

$$\geq \big(f(\varrho(1+\varepsilon)) - f(\varrho)\big)\,\mathbf{V}\big(\mathbf{B}_\varrho \setminus \mathbf{B}_{\varrho(1+\varepsilon)}(w)\big).$$

In the last line we used the fact that $\mathbf{V}\big(\mathbf{B}_\varrho(w)\setminus\mathbf{B}_{\varrho(1+\varepsilon)}\big) = \mathbf{V}\big(\mathbf{B}_\varrho\setminus\mathbf{B}_{\varrho(1+\varepsilon)}(w)\big)$. Letting $z := -\frac{w}{|w|}\tanh\big((1-\frac{\varepsilon}{2})\frac{\varrho}{2}\big)$, we can further estimate the volume from below by

$$\mathbf{V}\big(\mathbf{B}_\varrho \setminus \mathbf{B}_{\varrho(1+\varepsilon)}(w)\big) \geq \mathbf{V}\big(\mathbf{B}_{\varepsilon\varrho/2}(z)\big) \geq c(n)\varepsilon^n\varrho^n,$$

since we have $\mathbf{d}_o(z) = (1-\frac{\varepsilon}{2})\varrho = \varrho - \frac{1}{4}\mathbf{d}_o(w)$ and $\mathbf{d}(z,w) = \mathbf{d}_o(z)+\mathbf{d}_o(w) \geq \big(1+\frac{3}{2}\varepsilon\big)\varrho$, which implies the inclusion $\mathbf{B}_{\varepsilon\varrho/2}(z) \subset \mathbf{B}_\varrho \setminus \mathbf{B}_{\varrho(1+\varepsilon)}(w)$. Inserting this above yields

$$\mathbf{I} \geq c(n)\varepsilon^n\varrho^n\big(f(\varrho(1+\varepsilon)) - f(\varrho)\big) = c(n)\mathbf{d}_o(w)^n\big(f\big(\varrho + \tfrac{1}{2}\mathbf{d}_o(w)\big) - f(\varrho)\big),$$

proving the assertion of the lemma. □

Lemma 2.6 (**Continuity of barycenters**). *Assume that $G_k \subset \mathbb{H}^n$ is a sequence of sets of finite perimeter satisfying $\mathbf{B}_{\varrho_k(1-\delta_k)}(x_k) \subset G_k \subset \mathbf{B}_{\varrho_k(1+\delta_k)}(x_k)$, for some $\varrho_k, \delta_k > 0$ with $\varrho_k \downarrow 0$ and $\delta_k \downarrow 0$ as $k \to \infty$, and for arbitrary points $x_k \in \mathbb{H}^n$. Then for the barycenters p_k of G_k there holds*

$$\frac{\mathbf{d}(p_k, x_k)}{\varrho_k} \to 0 \qquad \text{as } k \to \infty.$$

Proof. Since the statement is invariant under hyperbolic isometries, we may assume that $x_k = 0$ for every $k \in \mathbb{N}$. We argue by contradiction assuming that there exists $\varepsilon \in (0,1)$, such that (after passing to a subsequence) we have

$$\mathbf{d}_o(p_k) \geq 2\varepsilon\varrho_k \quad \text{for all } k. \tag{2.15}$$

By Lemma 2.4 we know that $p_k \in \mathbf{B}_{\varrho_k(1+\delta_k)}$ and therefore $p_k \to 0$ as $k \to \infty$. Since p_k is the barycenter of G_k, we have that

$$\frac{1}{\varrho_k^{n+2}}\int_{G_k} \mathbf{d}^2_{p_k}(x)\,d\mu_h(x) \leq \frac{1}{\varrho_k^{n+2}}\int_{G_k} \mathbf{d}^2_o(x)\,d\mu_h(x). \tag{2.16}$$

Moreover, using the annular type assumption for the sets G_k we find for $k \gg 1$ that

$$\begin{aligned}
\frac{1}{\varrho_k^{n+2}}\bigg|\int_{\mathbf{B}_{\varrho_k}} \mathbf{d}^2_{p_k}(x)\,d\mu_h(x) - \int_{G_k} \mathbf{d}^2_{p_k}(x)\,d\mu_h(x)\bigg| &\leq \frac{1}{\varrho_k^{n+2}}\int_{\mathbf{B}_{\varrho_k}\Delta G_k} \mathbf{d}^2_{p_k}(x)\,d\mu_h(x)\\
&\leq \frac{4}{\varrho_k^n}\mathbf{V}(\mathbf{B}_{\varrho_k}\Delta G_k) \leq \frac{4}{\varrho_k^n}\mathbf{V}\big(\mathbf{B}_{\varrho_k(1+\delta_k)} \setminus \mathbf{B}_{\varrho_k(1-\delta_k)}\big)\\
&= \frac{4}{\varrho_k^n}\int_{\varrho_k(1-\delta_k)}^{\varrho_k(1+\delta_k)} \mathbf{P}(\mathbf{B}_\varrho)\,d\varrho \leq \frac{8n\omega_n\delta_k}{\varrho_k^{n-1}}\sinh^{n-1}(\varrho_k(1+\delta_k))\\
&\leq 16n\omega_n\delta_k.
\end{aligned}$$

Similarly, we get

$$\frac{1}{\varrho_k^{n+2}}\bigg|\int_{\mathbf{B}_{\varrho_k}} \mathbf{d}^2_o(x)\,d\mu_h(x) - \int_{G_k} \mathbf{d}^2_o(x)\,d\mu_h(x)\bigg| \leq 16n\omega_n\delta_k.$$

Therefore, combining the last two inequalities with (2.16) and recalling that $\delta_k \downarrow 0$ in the limit $k \to \infty$, we infer that

$$\lim_{k\to\infty}\frac{1}{\varrho_k^{n+2}}\int_{\mathbf{B}_{\varrho_k}} \mathbf{d}^2_{p_k}(x)\,d\mu_h(x) \leq \lim_{k\to\infty}\frac{1}{\varrho_k^{n+2}}\int_{\mathbf{B}_{\varrho_k}} \mathbf{d}^2_o(x)\,d\mu_h(x). \tag{2.17}$$

Here, we possibly have to pass to a subsequence such that the limits on both sides exist. We define $w_k := T_k(0)$, where T_k is a hyperbolic isometry with $T_k(p_k) = 0$. Then, (2.15) is equivalent to $\mathbf{d}_o(w_k) \geq 2\varepsilon\varrho_k$. Moreover, we can re-write (2.17) as

$$\lim_{k\to\infty} \mathbf{I}_k \leq 0, \tag{2.18}$$

where

$$\mathbf{I}_k := \frac{1}{\varrho_k^{n+2}} \bigg[\int_{\mathbf{B}_{\varrho_k}(w_k)\setminus \mathbf{B}_{\varrho_k}} \mathbf{d}_o^2(x)\, d\mu_h(x) - \int_{\mathbf{B}_{\varrho_k}\setminus \mathbf{B}_{\varrho_k}(w_k)} \mathbf{d}_o^2(x)\, d\mu_h(x) \bigg].$$

From Lemma 2.5 we know that

$$\mathbf{I}_k \geq \frac{c(n)\,(2\varepsilon\varrho_k)^n}{\varrho_k^{n+2}} \Big(\big(\varrho_k + \tfrac12 \mathbf{d}_0(w_k)\big)^2 - \varrho_k^2 \Big) \geq c(n)\,\varepsilon^{n+1},$$

which contradicts (2.18). □

2.6. **The L^2-oscillation index.** We consider a set $E \subset B^n$ with finite perimeter and a hyperbolic ball $\mathbf{B}_\varrho(y)$ with the same volume $\mathbf{V}(\mathbf{B}_\varrho(y)) = \mathbf{V}(E)$. The deviation of E from being a hyperbolic ball could be measured by a *hyperbolic version of the Fraenkel asymmetry index*, which is defined by

$$\boldsymbol{\alpha}(E) := \min_{y\in\mathbb{H}^n} \mathbf{V}(E\Delta \mathbf{B}_\varrho(y)). \tag{2.19}$$

However, we shall use a stronger asymmetry index, the so called *L^2-oscillation index*, which will be introduced in the following. As explained in § 2.3 we write $\partial^* E$ for the reduced boundary of E and $\nu_E(x)$ for the outer unit normal to $\partial^* E$ in a point $x \in \partial^* E$. The idea is to compare $\nu_E(x) \in T_x\mathbb{H}^n$ with the outer unit normal $\nu_{\mathbf{B}_\varrho(y)}(\pi(x)) \in T_{\pi(x)}\mathbb{H}^n$ to $\partial\mathbf{B}_\varrho(y)$ in $\pi(x)$, where π stands for the nearest-point retraction onto $\partial \mathbf{B}_\varrho(y)$. Writing

$$\Pi_{\pi(x),x} : T_{\pi(x)}\mathbb{H}^n \to T_x\mathbb{H}^n$$

for the parallel transport between the two tangent spaces involved in the construction, we thus define the **L^2-oscillation index of E with respect to $\mathbf{B}_\varrho(y)$** by

$$\boldsymbol{\beta}(E;y) := \bigg[\tfrac12 \int_{\partial^* E} \big| \nu_E(x) - \Pi_{\pi(x),x}(\nu_{\mathbf{B}_\varrho(y)}(\pi(x))) \big|_h^2 d\sigma_h(x) \bigg]^{\frac12}. \tag{2.20}$$

Now, the **L^2-oscillation index of E** is defined as the minimum over all geodesic balls $\mathbf{B}_\varrho(y) \subset \mathbb{H}^n$ with $\mathbf{V}(\mathbf{B}_\varrho(y)) = \mathbf{V}(E)$, that is

$$\boldsymbol{\beta}(E) := \min_{y\in\mathbb{H}^n} \boldsymbol{\beta}(E;y).$$

In order to compute the above integral, we fix $x \in \partial^* E$ and let $c\colon \mathbb{R} \to \mathbb{H}^n$ be the unit speed geodesic connecting $c(0) = y$ with $c(\mathbf{d}_y(x)) = x$, where $\mathbf{d}_y(x) := \mathbf{d}(x,y)$ as before. By definition of the nearest point retraction $\pi(x)$, we have $c(\varrho) = \pi(x)$. Furthermore, from the Gauß Lemma, we know $\nu_{\mathbf{B}_\varrho(y)}(\pi(x)) = c'(\varrho)$. Since the tangent vectors are parallel along the geodesic, we conclude

$$\Pi_{\pi(x),x}(\nu_{\mathbf{B}_\varrho(y)}(\pi(x))) = \Pi_{\pi(x),x}(c'(\varrho)) = c'(\mathbf{d}_y(x)) = \nabla^{\mathbb{H}}\mathbf{d}_y(x).$$

The last identity is due to the fact that the hyperbolic gradient $\nabla^{\mathbb{H}}\mathbf{d}_y(x)$ and the tangent vector $c'(\mathbf{d}_y(x))$ both have length one and are orthogonal to the level set of the function $\mathbf{d}_y$. Plugging the above identity into (2.20), we arrive at

$$\boldsymbol{\beta}(E;y) = \bigg[\tfrac12 \int_{\partial^* E} \big| \nu_E(x) - \nabla^{\mathbb{H}}\mathbf{d}_y(x) \big|_h^2 d\sigma_h(x) \bigg]^{\frac12}$$

$$= \left[\int_{\partial^* E} 1 - \left\langle \nu_E(x), \nabla^{\mathbb{H}} \mathbf{d}_y(x) \right\rangle_h \, d\sigma_h(x) \right]^{\frac{1}{2}}. \tag{2.21}$$

We square the last equality and employ the Gauß-Green theorem for Caccioppoli sets. This allows us to derive the following **alternative definition of the L^2-oscillation index**

$$\boldsymbol{\beta}^2(E; y) = \mathbf{P}(E) - \boldsymbol{\gamma}(E; y), \tag{2.22}$$

where

$$\boldsymbol{\gamma}(E; y) := \int_E \Delta^{\mathbb{H}} \mathbf{d}_y(x) \, d\mu_h(x) = (n-1) \int_E \coth \mathbf{d}_y(x) \, d\mu_h(x). \tag{2.23}$$

The last identity follows from the well-known formula for the hyperbolic Laplacian of the hyperbolic distance function

$$\Delta^{\mathbb{H}} \mathbf{d}_y(x) = (n-1) \coth \mathbf{d}_y(x). \tag{2.24}$$

Taking the maximum over all points $y \in \mathbb{H}^n$ we define

$$\boldsymbol{\gamma}(E) := \max_{y \in \mathbb{H}^n} \boldsymbol{\gamma}(E; y), \tag{2.25}$$

so that

$$\boldsymbol{\beta}^2(E) = \mathbf{P}(E) - \boldsymbol{\gamma}(E). \tag{2.26}$$

At this point we also note that in the case of a geodesic ball in $\mathbb{H}^n$ we have

$$\boldsymbol{\gamma}(\mathbf{B}_\varrho(x_o)) = \mathbf{P}(\mathbf{B}_\varrho(x_o)). \tag{2.27}$$

Definition 2.7. A set of finite perimeter $E \subset \mathbb{H}^n$ is centered at a point $y_o \in B^n$ if

$$\boldsymbol{\beta}^2(E) = \boldsymbol{\beta}^2(E; y_o).$$

Lemma 2.8. *If $E \subset \mathbf{B}_{\varrho_o}$ with $\varrho_o > 0$ and $\mathbf{V}(E) > 0$, then every center y_o of E is contained in $\mathbf{B}_{\varrho_o}$.*

Proof. We assume for contradiction $y_o \notin \mathbf{B}_{\varrho_o}$. As noted above in the proof of Lemma 2.4, in this case we have $y_o \cdot \nabla_y[\mathbf{d}(x, y_o)] > 0$ for any $x \in E$. Hence, we get

$$y_o \cdot \nabla_y[\boldsymbol{\gamma}(E; y_o)] = (n-1) \int_E \coth'[\mathbf{d}(x, y_o)] \, y_o \cdot \nabla_y[\mathbf{d}(x, y_o)] \, d\mu_h(x) < 0,$$

which contradicts the maximality of y_o. □

Lemma 2.9 (Continuity of centers with respect to the L^1-topology). *For every $\varepsilon \in (0, 1]$ and $r_o > 0$ there exists $\delta \equiv \delta(n, \varepsilon, r_o) > 0$ such that there holds: If $G \subset \mathbf{B}_{r_o}$ is a set of finite perimeter satisfying $\mathbf{V}(G \Delta \mathbf{B}_{\varrho_o}(p_o)) < \delta \sinh^n \varrho_o$, with $\varrho_o \in (0, r_o]$, then $\mathbf{d}(y_G, p_o) < \varepsilon \varrho_o$ holds true for any center y_G of G.*

Proof. We argue by contradiction assuming that there exist $\varepsilon \in (0, 1]$, $r_o > 0$ and sets $G_k \subset \mathbf{B}_{r_o}$ of finite perimeter and $p_k \in \mathbb{H}^n$ and radii $\varrho_k \in (0, r_o]$, $k \in \mathbb{N}$, such that

$$\frac{\mathbf{V}(G_k \Delta \mathbf{B}_{\varrho_k}(p_k))}{\sinh^n \varrho_k} \to 0 \quad \text{as } k \to \infty, \qquad \text{and} \qquad \mathbf{d}(y_k, p_k) \geq 2\varepsilon \varrho_k \ \text{ for all } k, \tag{2.28}$$

where y_k is a center of G_k for $k \in \mathbb{N}$. By $(2.28)_1$ we know that $p_k \in \mathbf{B}_{r_o}$ for $k \gg 1$ and Lemma 2.8 ensures us that $y_k \in \mathbf{B}_{r_o}$ for any $k \in \mathbb{N}$. Extracting a (not relabeled) subsequence we can also assume that $\varrho_k \to \varrho_o \in [0, r_o]$, $p_k \to p_o$ and $y_k \to y_o$. Since y_k is a center for G_k, by the maximality of $\boldsymbol{\gamma}(G_k; \cdot)$ we have that

$$\int_{G_k} \coth \mathbf{d}_{p_k}(x) \, d\mu_h(x) \leq \int_{G_k} \coth \mathbf{d}_{y_k}(x) \, d\mu_h(x). \tag{2.29}$$

We now define

$$\mathbf{I}_k := \frac{1}{\sinh^{n-1}\varrho_k}\bigg|\int_{G_k}\coth \mathbf{d}_{p_k}(x)\,d\mu_h(x) - \int_{\mathbf{B}_{\varrho_k}(p_k)}\coth \mathbf{d}_{p_k}(x)\,d\mu_h(x)\bigg|$$

and choose $r > 0$ such that $\mathbf{V}(\mathbf{B}_r(p_k)) = \mathbf{V}(G_k \Delta \mathbf{B}_{\varrho_k}(p_k))$. Taking into account that $\coth \mathbf{d}_{p_k}(x)$ increases when $\mathbf{d}_{p_k}(x)$ decreases and applying Lemma 2.1 we get

$$\begin{aligned}\mathbf{I}_k &\le \frac{1}{\sinh^{n-1}\varrho_k}\int_{G_k\Delta\mathbf{B}_{\varrho_k}(p_k)}\coth \mathbf{d}_{p_k}(x)\,d\mu_h(x)\\ &\le \frac{1}{\sinh^{n-1}\varrho_k}\int_{\mathbf{B}_r(p_k)}\coth \mathbf{d}_{p_k}(x)\,d\mu_h(x)\\ &= \frac{\mathbf{P}(\mathbf{B}_r(p_k))}{(n-1)\sinh^{n-1}\varrho_k} \le \frac{n\omega_n^{\frac{1}{n}}}{n-1}\bigg(\frac{\cosh r\,\mathbf{V}(G_k\Delta\mathbf{B}_{\varrho_k}(p_k))}{\sinh^n\varrho_k}\bigg)^{\frac{n-1}{n}}.\end{aligned}$$

Next we consider the right-hand side of (2.29). Similarly as before, we estimate

$$\begin{aligned}\mathbf{II}_k &:= \frac{1}{\sinh^{n-1}\varrho_k}\bigg|\int_{G_k}\coth \mathbf{d}_{y_k}(x)\,d\mu_h(x) - \int_{\mathbf{B}_{\varrho_k}(p_k)}\coth \mathbf{d}_{y_k}(x)\,d\mu_h(x)\bigg|\\ &\le \frac{n\omega_n^{\frac{1}{n}}}{n-1}\bigg(\frac{\cosh r\mathbf{V}(G_k\Delta\mathbf{B}_{\varrho_k}(p_k))}{\sinh^n\varrho_k}\bigg)^{\frac{n-1}{n}}.\end{aligned}$$

Taking into account $(2.28)_1$ and the fact that $\cosh r \le \cosh r_o$ for any k, we deduce that $\mathbf{I}_k, \mathbf{II}_k \to 0$ in the limit $k \to \infty$. Inserting this into (2.29), we conclude that

$$\lim_{k\to\infty}\frac{1}{\sinh^{n-1}\varrho_k}\int_{\mathbf{B}_{\varrho_k}(p_k)}\big[\coth \mathbf{d}_{p_k}(x)\,d\mu_h(x) - \coth \mathbf{d}_{y_k}(x)\big]\,d\mu_h(x) \le 0 \tag{2.30}$$

holds true. In this step we possibly have to pass to a subsequence such that the limit exists. We now distinguish between the cases $\varrho_o > 0$ and $\varrho_o = 0$. In the **case $\boldsymbol{\varrho_o > 0}$**, by passing to the limit $k \to \infty$ we conclude from (2.30) that we have

$$\int_{\mathbf{B}_{\varrho_o}(p_o)}\coth \mathbf{d}_{p_o}(x)\,d\mu_h(x) \le \int_{\mathbf{B}_{\varrho_o}(p_o)}\coth \mathbf{d}_{y_o}(x)\,d\mu_h(x).$$

Applying the Gauß-Green theorem to both sides of the previous inequality we infer that

$$\int_{\mathbf{S}_{\varrho_o}(p_o)} 1\,d\sigma_h(x) \le \int_{\mathbf{S}_{\varrho_o}(p_o)}\Pi_{\pi(x),x}(\nu_{\mathbf{B}_r(p_o)}(\pi(x)))\cdot\Pi_{\pi(x),x}(\nu_{\mathbf{B}_r(y_o)}(\pi(x)))\,d\sigma_h(x).$$

But this inequality can only hold if $y_o = p_o$, which gives the desired contradiction to $(2.28)_2$.

In the **Case $\boldsymbol{\varrho_o = 0}$** we choose hyperbolic isometries T_k with $T_k(y_k) = 0$ and set $w_k := T_k(p_k)$. Then, (2.30) can be rewritten as

$$\lim_{k\to\infty}\mathbf{J}_k \le 0, \tag{2.31}$$

where

$$\mathbf{J}_k := \frac{1}{\sinh^{n-1}\varrho_k}\bigg[\int_{\mathbf{B}_{\varrho_k}\setminus\mathbf{B}_{\varrho_k}(w_k)}\coth \mathbf{d}_o(x)d\mu_h(x) - \int_{\mathbf{B}_{\varrho_k}(w_k)\setminus\mathbf{B}_{\varrho_k}}\coth \mathbf{d}_o(x)d\mu_h(x)\bigg].$$

By Lemma 2.5 (applied with $f \equiv -\coth$) and $(2.28)_2$ we infer that

$$\mathbf{J}_k \ge \frac{c\,\varepsilon^n\varrho_k^n}{\sinh^{n-1}\varrho_k}\Big(\coth\varrho_k - \coth\big(\varrho_k(1+\varepsilon)\big)\Big)$$

for some constant $c = c(n)$. Using the convexity of coth for large values of $k \in \mathbb{N}$ we can further estimate

$$\mathbf{J}_k \geq \frac{c\,\varepsilon^{n+1}\varrho_k^{n+1}}{\sinh^{n-1}\varrho_k \sinh^2[\varrho_k(1+\varepsilon)]} \geq c(n)\,\varepsilon^{n+1}.$$

But this contradicts (2.31) and thus finishes the proof of the lemma. □

Lemma 2.10. *There exists a constant $c = c(n) > 0$ such that for any set $E \subset \mathbb{H}^n$ of finite perimeter with $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$ for some $\varrho_o > 0$ there holds*

$$\boldsymbol{\beta}(E)^2 \geq \mathbf{D}(E) + \frac{c(n)}{\sinh^{n+1}\varrho_o}\,\boldsymbol{\alpha}(E)^2.$$

Proof. Without loss of generality we may assume that $E \subset \mathbb{H}^n$ is centered at the origin. Using (2.27), i.e. $\mathbf{P}(\mathbf{B}_{\varrho_o}) = \boldsymbol{\gamma}(\mathbf{B}_{\varrho_o})$, we can rewrite the L^2-oscillation index in (2.26) as

$$\boldsymbol{\beta}(E)^2 = \mathbf{P}(E) - \boldsymbol{\gamma}(E) = \mathbf{D}(E) + \boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E). \tag{2.32}$$

In the sequel we will estimate the difference $\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E)$ from below. By the definition of $\boldsymbol{\gamma}$ in (2.23), (2.25) and the fact that E is centered in the origin, we find

$$\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E) = (n-1)\bigg[\int_{\mathbf{B}_{\varrho_o}\setminus E} \coth \mathbf{d}_o(x)\,d\mu_h(x) - \int_{E\setminus \mathbf{B}_{\varrho_o}} \coth \mathbf{d}_o(x)\,d\mu_h(x)\bigg].$$

Since $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$, we have

$$\mathbf{V}(\mathbf{B}_{\varrho_o} \setminus E) = \mathbf{V}(E \setminus \mathbf{B}_{\varrho_o}) =: a.$$

Next, we choose radii $0 \leq r < \varrho_o < R$ such that $\mathbf{V}(\mathbf{B}_R \setminus \mathbf{B}_{\varrho_o}) = a = \mathbf{V}(\mathbf{B}_{\varrho_o} \setminus \mathbf{B}_r)$; this means that we have

$$\int_{\varrho_o}^{R} \sinh^{n-1}\sigma\,d\sigma = \frac{a}{n\omega_n} = \int_r^{\varrho_o} \sinh^{n-1}\sigma\,d\sigma.$$

Since $\coth \mathbf{d}_o(x)$ decreases when $\mathbf{d}_o(x)$ increases, and $\mathbf{V}(\mathbf{B}_{\varrho_o} \setminus E) = \mathbf{V}(\mathbf{B}_{\varrho_o} \setminus \mathbf{B}_r)$ by the particular choice of r, we may conclude that

$$\int_{\mathbf{B}_{\varrho_o}\setminus E} \coth \mathbf{d}_o(x)\,d\mu_h(x) \geq \int_{\mathbf{B}_{\varrho_o}\setminus \mathbf{B}_r} \coth \mathbf{d}_o(x)\,d\mu_h(x),$$

and by the same argument also

$$\int_{E\setminus \mathbf{B}_{\varrho_o}} \coth \mathbf{d}_o(x)\,d\mu_h(x) \leq \int_{\mathbf{B}_R\setminus \mathbf{B}_{\varrho_o}} \coth \mathbf{d}_o(x)\,d\mu_h(x).$$

Inserting these inequalities in the above expression for $\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E)$ and using the co-area formula we arrive at

$$\begin{aligned}
\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E) &\geq (n-1)\bigg[\int_{\mathbf{B}_{\varrho_o}\setminus \mathbf{B}_r} \coth \mathbf{d}_o(x) d\mu_h(x) - \int_{\mathbf{B}_R\setminus \mathbf{B}_{\varrho_o}} \coth \mathbf{d}_o(x) d\mu_h(x)\bigg] \\
&= n(n-1)\omega_n\bigg[\int_r^{\varrho_o} \sinh^{n-2}\sigma \cosh\sigma\,d\sigma - \int_{\varrho_o}^{R} \sinh^{n-2}\sigma \cosh\sigma\,d\sigma\bigg] \\
&= n\omega_n\big[2\sinh^{n-1}\varrho_o - (\sinh^{n-1}R + \sinh^{n-1}r)\big].
\end{aligned}$$

We now define the function

$$F(\varrho) := n\omega_n \int_{\varrho_o}^{\varrho} \sinh^{n-1}\sigma\,d\sigma \qquad \text{for } \varrho > 0.$$

Noting that $\varrho_o = F^{-1}(0)$, $R = F^{-1}(a)$ and $r = F^{-1}(-a)$ we can rewrite the last inequality in the following form:

$$\begin{aligned}\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(E) \ge n\omega_n \Big[& 2\sinh^{n-1}(F^{-1}(0)) \\ & - \Big(\sinh^{n-1}(F^{-1}(a)) + \sinh^{n-1}(F^{-1}(-a))\Big)\Big].\end{aligned} \tag{2.33}$$

Since $\sinh^{n-1}\circ F^{-1}$ is strictly concave, for any $t,s\in[-a,a]$ we have that

$$\sinh^{n-1}(F^{-1}(t)) + \sinh^{n-1}(F^{-1}(s)) \le 2\sinh^{n-1}\big(F^{-1}\big(\tfrac{t+s}{2}\big)\big) - c(n,R)|t-s|^2,$$

where

$$\begin{aligned} c(n,R) &= \tfrac14 \inf_{t\in[-a,a]} (-\sinh^{n-1}\circ F^{-1})''(t) = \frac{n-1}{4n^2\omega_n^2} \inf_{t\in[-a,a]} \frac{1}{\sinh^{n+1}\circ F^{-1}(t)} \\ &= \frac{n-1}{4n^2\omega_n^2 \sinh^{n+1}\circ F^{-1}(a)} = \frac{n-1}{4n^2\omega_n^2\sinh^{n+1} R}\,. \end{aligned}$$

To conclude the desired estimate we need to control the constant $c(n,R)$ in terms of ϱ_o. We first observe that $F(R) = \mathbf{V}(\mathbf{B}_R\setminus\mathbf{B}_{\varrho_o}) \le \mathbf{V}(\mathbf{B}_{\varrho_o}) = -F(0)$. Now, we distinguish the cases $\varrho_o \le \frac12$ and $\varrho_o > \frac12$. In the **case** $\boldsymbol{\varrho_o \le \frac12}$ we use the fact that $\sigma\le\sinh\sigma$ for any $\sigma\ge0$ and $\sinh\sigma\le 2\sigma$ for any $\sigma\in[0,1]$, so that

$$\begin{aligned}\omega_n\big(R^n - \varrho_o^n\big) &= n\omega_n\int_{\varrho_o}^R \sigma^{n-1}\,d\sigma \le n\omega_n\int_{\varrho_o}^R \sinh^{n-1}\sigma\,d\sigma = F(R) \le -F(0) \\ &= n\omega_n\int_0^{\varrho_o}\sinh^{n-1}\sigma\,d\sigma \le 2^{n-1}n\omega_n\int_0^{\varrho_o}\sigma^{n-1}\,d\sigma = 2^{n-1}\omega_n\,\varrho_o^n.\end{aligned}$$

This implies that $R\le 2\varrho_o\le 1$ and hence

$$\sinh R \le 2R \le 4\varrho_o \le 4\sinh\varrho_o.$$

In the **case** $\boldsymbol{\varrho_o > \frac12}$ we use that $\sinh\sigma \ge \frac12 e^\sigma(1-e^{-1})$ for $\sigma\ge\frac12$, so that

$$\begin{aligned} &\frac{n\omega_n}{n-1}\Big(\frac{1-e^{-1}}{2}\Big)^{n-1}\big(e^{(n-1)R} - e^{(n-1)\varrho_o}\big) \\ &\qquad = n\omega_n 2^{1-n}(1-e^{-1})^{n-1}\int_{\varrho_o}^R e^{(n-1)\sigma}\,d\sigma \\ &\qquad \le n\omega_n\int_{\varrho_o}^R \sinh^{n-1}\sigma\,d\sigma = F(R)\le -F(0) \\ &\qquad \le n\omega_n\int_0^{\varrho_o} e^{(n-1)\sigma}\,d\sigma = \frac{n\omega_n}{n-1}\big(e^{(n-1)\varrho_o}-1\big).\end{aligned}$$

From this inequality we conclude that

$$\sinh R \le e^R \le \frac{4e^{\varrho_o+1}}{e-1} \le \frac{8e^2}{(e-1)^2}\sinh\varrho_o.$$

Hence, in any case, we have established that there exists a constant $\kappa(n)$ such that

$$c(n,R) \ge \frac{\kappa(n)}{\sinh^{n+1}\varrho_o} \qquad \text{for all } \varrho_o>0.$$

We use the preceding lower bound for the constant $c(n,R)$ in the inequality expressing the uniform concavity of $\sinh^{n-1}\circ F^{-1}$ with $t=a$ and $s=-a$. The resulting inequality we afterwards insert into (2.33). In this way we obtain

$$\boldsymbol{\gamma}(\mathbf{B}_{\varrho_o})-\boldsymbol{\gamma}(E)\ge \frac{c(n)}{\sinh^{n+1}\varrho_o}\,(2a)^2=\frac{c(n)}{\sinh^{n+1}\varrho_o}\,\mathbf{V}^2(E\Delta B_{\varrho_o}).$$

Inserting the preceding inequality in (2.32) and recalling the definition (2.19) of the Fraenkel asymmetry index (note that $\mathbf{V}(E)=\mathbf{V}(\mathbf{B}_{\varrho_o})$) we end up with

$$\boldsymbol{\beta}(E)^2\ge \mathbf{D}(E)+\frac{c(n)}{\sinh^{n+1}\varrho_o}\,\mathbf{V}^2(E\Delta B_{\varrho_o})\ge \mathbf{D}(E)+\frac{c(n)}{\sinh^{n+1}\varrho_o}\,\boldsymbol{\alpha}^2(E).$$

This proves the claim. □

3. Slicing in the hyperbolic space

3.1. **Slicing with non-standard hyperplanes.** We fix $k\in\{1,\dots,n\}$. We note that by (2.3) and the properties of the stereographic projection

$$\mathbb{H}^n\ni x\mapsto St(x):=\Big(\frac{1+|x|^2}{1-|x|^2},\frac{2x}{1-|x|^2}\Big)\in H^n,$$

the hyperbolic distance of a point $x\in\mathbb{H}^n$ to the hyperbolic hyperplane $\mathbb{H}^n\cap\{x_k=0\}$ is given by

$$\mathrm{dist}_{\mathbb{H}}(x,\{x_k=0\})=\big|\operatorname{Arsinh}\big(\tfrac{2x_k}{1-|x|^2}\big)\big|.$$

This follows easily from the fact that the stereographic projection is a global isometry between $\mathbb{H}^n$ and H^n and the corresponding formula (2.3) in H^n. This motivates the definition of slices $\mathcal{S}_t$ as the level sets of the function

$$\mathbb{H}^n\ni x\mapsto \delta_k(x):=\operatorname{Arsinh}\tfrac{2x_k}{1-|x|^2}\in\mathbb{R}.$$

To be more precise, for $k\in\{1,\dots,n\}$ we define the level sets

$$\mathcal{S}_{k,t}:=\{x\in\mathbb{H}^n\colon \delta_k(x)=t\},\qquad t\in\mathbb{R}.$$

We note that

$$\delta_k(x)=\operatorname{sgn}(x_k)\,\mathrm{dist}_{\mathbb{H}}(x,\{x_k=0\}).$$

From the properties of the distance function, we readily infer

$$|\nabla\delta_k(x)|=\frac{2}{1-|x|^2}\,\big|\nabla^{\mathbb{H}}\delta_k(x)\big|_h=\frac{2}{1-|x|^2}.$$

Hence, for a Borel set $E\subset\mathbb{H}^n$ the coarea formula implies

$$\begin{aligned}\mathbf{V}(E)&=\int_E\Big(\frac{2}{1-|x|^2}\Big)^n dx\\&=\int_E\Big(\frac{2}{1-|x|^2}\Big)^{n-1}|\nabla\delta_k(x)|\,dx\\&=\int_{-\infty}^{\infty}\int_{E\cap\delta_k^{-1}\{t\}}\Big(\frac{2}{1-|x|^2}\Big)^{n-1}d\mathcal{H}^{n-1}x\,dt.\end{aligned}$$

This allows us to express the hyperbolic volume of E by the $(n-1)$-dimensional hyperbolic surface area of the slices $E\cap\delta_k^{-1}\{t\}$ of E in the following way:

$$\mathbf{V}(E)=\int_{-\infty}^{\infty}\sigma_h(E\cap\mathcal{S}_{k,t})\,dt. \tag{3.1}$$

Moreover, we can express the hyperbolic distance to the origin in terms of the functions δ_k by the following computation:

$$\cosh^2(\mathbf{d}_o(x)) = \Big(\frac{1+|x|^2}{1-|x|^2}\Big)^2 = 1 + \frac{4|x|^2}{(1-|x|^2)^2} = 1 + \sum_{k=1}^n \sinh^2(\delta_k(x))$$

for any $x \in \mathbb{H}^n$. This can be re-written in the form

$$\sinh^2(\mathbf{d}_o(x)) = \sum_{k=1}^n \sinh^2(\delta_k(x)). \tag{3.2}$$

The following result will ensure that our way of slicing a set will not increase its perimeter.

Lemma 3.1. *The nearest-point retractions*

$$\pi_t \colon \mathbb{H}^n \to \mathbb{H}^n \cap \{\delta_k(x) \le t\} \qquad \text{for } t \ge 0$$

and

$$\pi_s \colon \mathbb{H}^n \to \mathbb{H}^n \cap \{\delta_k(x) \ge s\} \qquad \text{for } s \le 0$$

have Lipschitz constant at most 1 *with respect to the hyperbolic metric.*

Proof. It suffices to prove that the restrictions

$$\pi_t\big|_{\{\delta_k > t\}} \colon \mathbb{H}^n \cap \{\delta_k(x) > t\} \to \mathbb{H}^n \cap \{\delta_k(x) = t\}$$

and

$$\pi_s\big|_{\{\delta_k < s\}} \colon \mathbb{H}^n \cap \{\delta_k(x) < s\} \to \mathbb{H}^n \cap \{\delta_k(x) = s\}$$

have Lipschitz constant ≤ 1. But this is a consequence of the corresponding statements (2.4) and (2.5) in the hyperboloid model H^n and the fact that the stereographic projection is a global isometry between $\mathbb{H}^n$ and H^n. □

In order to apply a slicing argument with the above defined slices, we first need to center the sets around zero. This is done in the following

Lemma 3.2. *Assume that* $E \subset \mathbb{H}^n$ *satisfies* $\mathbf{V}(E) < \infty$. *Then there is a hyperbolic isometry* $g \colon \mathbb{H}^n \to \mathbb{H}^n$ *such that*

$$\mathbf{V}\big(g(E) \cap \{x_k > 0\}\big) = \mathbf{V}\big(g(E) \cap \{x_k < 0\}\big) = \tfrac{1}{2}\mathbf{V}(E) \tag{3.3}$$

holds for any $k \in \{1, \ldots, n\}$.

Proof. We inductively establish the existence of an isometry $g_\ell \colon \mathbb{H}^n \to \mathbb{H}^n$ for $\ell \in \{0, n-1\}$ with the property

$$\mathbf{V}\big(g_\ell(E) \cap \{x_k > 0\}\big) = \tfrac{1}{2}\mathbf{V}(E) \qquad \text{for any } k \in \{1, \ldots, \ell\}. \tag{3.4}$$

For $\ell = 0$, we let $g_o := id$ and observe that the statement holds trivially. We therefore assume that (3.4) is already established for some $\ell \in \{0, n-1\}$. For parameters $r \in (-1, 1)$, we construct isometries $h_r \colon \mathbb{H}^n \to \mathbb{H}^n$ with the properties

$$h_r(0) = re_{\ell+1} \qquad \text{and} \qquad Dh_r(0)e_i = (1-r^2)e_i \quad \text{for all } i \in \{1, \ldots, n\}, \tag{3.5}$$

where we identified the tangent spaces $T_x\mathbb{H}^n$ with $\mathbb{R}^n$ for convenience and e_i stands for the Euclidean basis in $\mathbb{R}^n$. These mappings can be explicitly written down in terms of (2.1) as $h_r \equiv h_{re_{\ell+1}}$. As can be inferred from their definition we have

$$h_r(\{x_k > 0\}) = \{x_k > 0\} \qquad \text{for any } k \ne \ell + 1. \tag{3.6}$$

Combining (3.4) and (3.6), we deduce

$$\mathbf{V}\big(h_r \circ g_\ell(E) \cap \{x_k > 0\}\big) = \mathbf{V}\big(g_\ell(E) \cap h_r^{-1}\{x_k > 0\}\big) = \tfrac{1}{2}\mathbf{V}(E) \tag{3.7}$$

for any $k \in \{1, \dots, \ell\}$ and any $r \in (-1, 1)$. Moreover, by the conformal invariance of the hyperbolic volume we have

$$\mathbf{V}\big(h_r \circ g_\ell(E) \cap \{x_{\ell+1} > 0\}\big) = \mathbf{V}\big(g_\ell(E) \cap h_r^{-1}\{x_{\ell+1} > 0\}\big).$$

We now consider the continuous function $(-1, 1) \ni r \mapsto \mathbf{V}(h_r \circ g_\ell(E) \cap \{x_{\ell+1} > 0\})$, which converges to $\mathbf{V}(E)$ as $r \uparrow 1$ and to 0 as $r \downarrow -1$. Therefore, by the intermediate value theorem, we can find a parameter $r \in (-1, 1)$ for which there holds

$$\mathbf{V}\big(h_r \circ g_\ell(E) \cap \{x_{\ell+1} > 0\}\big) = \tfrac{1}{2}\mathbf{V}(E). \tag{3.8}$$

Letting $g_{\ell+1} := h_r \circ g_\ell$ with $r \in (-1, 1)$ chosen in (3.8), the identities (3.7) and (3.8) yield the claim (3.4) for $\ell + 1$ instead of ℓ. We can therefore achieve (3.4) inductively for any $\ell \in \{1 \dots, n\}$. This implies the desired assertion (3.3) with the isometry $g := g_n$. □

3.2. **Reduction to bounded sets.** In this section we show by a slicing argument that one can reduce the general setting to a situation where the sets are bounded. Roughly speaking, any set E with given volume can be truncated in such a way that the L^2-oscillation index $\boldsymbol{\beta}$ does not decrease to much, while the isoperimetric deficit does not increase too much. The precise result is as follows:

Lemma 3.3. *For every $R_o > 0$ and $n \geq 2$ there are constants $C = C(n, R_o) > 0$ and $R_1 = R_1(n, R_o) > 1$ with the following property. For any $r \in (0, R_o)$ and any set $E \subset \mathbb{H}^n$ of finite hyperbolic perimeter with volume $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_r)$ we can find $\widetilde{E} \subset \mathbf{B}_{rR_1} \cap E \subset \mathbb{H}^n$ satisfying*

$$\mathbf{V}(\widetilde{E}) \geq \mathbf{V}(\mathbf{B}_r)\Big(1 - \frac{C\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_r)}\Big), \qquad \mathbf{P}(\widetilde{E}) \leq \mathbf{P}(E), \tag{3.9}$$

$$\mathbf{D}(\widetilde{E}) \leq (1 + C)\mathbf{D}(E), \tag{3.10}$$

and

$$\boldsymbol{\beta}(E)^2 \leq \boldsymbol{\beta}(\widetilde{E})^2 + (1 + C)\mathbf{D}(E). \tag{3.11}$$

Proof. We let $C > 0$ be a constant that will be chosen in the course of the proof in dependence on n and R_o. If $\mathbf{D}(E) > \frac{1}{C}\mathbf{P}(\mathbf{B}_r)$ the statement of the lemma becomes trivial, since we may simply choose $\widetilde{E} = \mathbf{B}_r$, so that

$$\boldsymbol{\beta}(E)^2 \leq \mathbf{P}(E) = \mathbf{P}(\mathbf{B}_r) + \mathbf{D}(E) \leq (1 + C)\mathbf{D}(E),$$

i.e. estimate (3.11) holds true. Moreover, $(3.9)_2$ and (3.10) follow by the isoperimetric property of geodesic balls, and $(3.9)_1$ is trivially satisfied. Therefore, it remains to consider the case when the deficit is small in the following sense:

$$\mathbf{D}(E) \leq \tfrac{1}{C}\mathbf{P}(\mathbf{B}_r). \tag{3.12}$$

By Lemma 3.2 we may assume that

$$\mathbf{V}(E \cap \{x_k > 0\}) = \mathbf{V}(E \cap \{x_k < 0\}) = \tfrac{1}{2}\mathbf{V}(E) \tag{3.13}$$

holds for any $k \in \{1, \dots, n\}$, since otherwise we replace E by $g(E)$ with g being the isometry constructed in Lemma 3.2. Now, for some fixed $k \in \{1, \dots, n\}$ and $t \in \mathbb{R}$ we define the sub-level sets

$$E_t^- := \{x \in E : \delta_k(x) < t\} \quad \text{and} \quad v_E(t) := \sigma_h(E \cap \mathcal{S}_{k,t}),$$

with δ_k and $\mathcal{S}_{k,t}$ as defined in § 3.1. Then, for a.e. $t \in \mathbb{R}$ we have that the inequalities

$$\mathbf{P}(E_t^-) \leq \mathbf{P}\big(E; \{\delta_k(x) < t\}\big) + v_E(t)$$

and

$$\mathbf{P}(E\setminus E_t^-)\le \mathbf{P}\big(E;\{\delta_k(x)>t\}\big)+v_E(t)$$

hold true. Combining both inequalities yields that

$$\begin{aligned}v_E(t)&\ge \tfrac12\big[\mathbf{P}(E_t^-)+\mathbf{P}(E\setminus E_t^-)-\mathbf{P}(E)\big]\\&=\tfrac12\big[\mathbf{P}(E_t^-)+\mathbf{P}(E\setminus E_t^-)-\mathbf{P}(\mathbf{B}_r)-\mathbf{D}(E)\big],\end{aligned}\tag{3.14}$$

for a.e. $t\in\mathbb{R}$, where in the last line we used the definition of the isoperimetric deficit. Next, we define the function $g\colon\mathbb{R}\to[0,1]$ by

$$g(t):=\frac{\mathbf{V}(E_t^-)}{\mathbf{V}(\mathbf{B}_r)}=\frac{1}{\mathbf{V}(\mathbf{B}_r)}\int_{-\infty}^t v_E(\tau)\,d\tau,$$

where the last identity is a consequence of the coarea formula (3.1). In order not to overburden the notation we write $g(t)$, but we keep in mind that $g(t)$ arises from a slicing procedure with respect to the k-th slicing function δ_k. Later on when defining the set $\widetilde{E}$ we use the notion $g_k(t)$ which essentially would be the correct abbreviation. Recalling that $\mathbf{V}(\mathbf{B}_r)=\mathbf{V}(E)$ we deduce that $g\le 1$ and from (3.3) we know that $g(0)=\frac12$. Moreover, g is a non-decreasing, continuous function and differentiable a.e. with

$$g'(t)=\frac{1}{\mathbf{V}(\mathbf{B}_r)}\frac{d}{dt}\int_{-\infty}^t v_E(\tau)\,d\tau=\frac{v_E(t)}{\mathbf{V}(\mathbf{B}_r)}\qquad\text{for a.e. } t\in\mathbb{R}.$$

Now, we let $-\infty\le a<b\le\infty$ be the two numbers satisfying $\{t\in\mathbb{R}:0<g(t)<1\}=(a,b)$. We define strictly increasing functions $\boldsymbol{v},\boldsymbol{p}\colon\mathbb{R}^+\to\mathbb{R}^+$ by letting $\boldsymbol{v}(\varrho):=\mathbf{V}(\mathbf{B}_\varrho)$ and $\boldsymbol{p}(\varrho):=\mathbf{P}(\mathbf{B}_\varrho)$. Moreover, we define $\varphi\colon\mathbb{R}^+\to\mathbb{R}^+$ by $\varphi:=\boldsymbol{p}\circ\boldsymbol{v}^{-1}$. Then, by the isoperimetric inequality in the hyperbolic space we conclude that for any finite perimeter set $F\subset\mathbb{H}^n$ and any geodesic ball $\mathbf{B}_\varrho\subset\mathbb{H}^n$ with $\mathbf{V}(\mathbf{B}_\varrho)=\mathbf{V}(F)$ we have

$$\mathbf{P}(F)\ge\mathbf{P}(\mathbf{B}_\varrho)=\varphi(\mathbf{V}(\mathbf{B}_\varrho))=\varphi(\mathbf{V}(F)).\tag{3.15}$$

From the last inequality and the definition of g we now find for any $t\in(a,b)$ that

$$\mathbf{P}(E_t^-)\ge\varphi\big(\mathbf{V}(E_t^-)\big)=\varphi\big(g(t)\mathbf{V}(\mathbf{B}_r)\big)$$

and

$$\mathbf{P}(E\setminus E_t^-)\ge\varphi\big(\mathbf{V}(E\setminus E_t^-)\big)=\varphi\big(\mathbf{V}(E)-\mathbf{V}(E_t^-)\big)=\varphi\big((1-g(t))\mathbf{V}(\mathbf{B}_r)\big).$$

Combining this with (3.14) and taking into account the fact that $\mathbf{P}(\mathbf{B}_r)=\varphi(\mathbf{V}(\mathbf{B}_r))$ (which follows from the definition of φ) we arrive at

$$\begin{aligned}v_E(t)&\ge\tfrac12\big[\varphi\big(g(t)\mathbf{V}(\mathbf{B}_r)\big)+\varphi\big((1-g(t))\mathbf{V}(\mathbf{B}_r)\big)-\varphi\big(\mathbf{V}(\mathbf{B}_r)\big)-\mathbf{D}(E)\big]\\&=\tfrac12\big[\psi_r(g(t))-\mathbf{D}(E)\big]\\&=\tfrac14\psi_r(g(t))+\tfrac14\big[\psi_r(g(t))-2\mathbf{D}(E)\big],\end{aligned}\tag{3.16}$$

where we have defined the function $\psi_r\colon[0,1]\to\mathbb{R}^+$ by

$$\psi_r(s):=\varphi\big(s\boldsymbol{v}(r)\big)+\varphi\big((1-s)\boldsymbol{v}(r)\big)-\varphi\big(\boldsymbol{v}(r)\big).$$

We obviously have that ψ_r is continuous and $\psi_r(0)=0=\psi_r(1)$. Moreover, since

$$\varphi'(s)=\frac{n-1}{\tanh(\boldsymbol{v}^{-1}(s))}\tag{3.17}$$

is decreasing, we have that φ is concave and hence also ψ_r is a concave function. Moreover, by symmetry (note that ψ_r is symmetric with respect to $s=\frac12$) we know that ψ_r takes its maximum in $s=\frac12$. Now we choose the constant $C=C(n,R_o)$ as

$$C:=\frac{2n\mathbf{P}(\mathbf{B}_{R_o})}{\psi_{R_o}(\frac12)}.$$

In view of Lemma 6.3 from the Appendix and (3.12) this choice implies in particular

$$\frac{\mathbf{P}(\mathbf{B}_r)}{\psi_r(\frac12)}\le\frac{\mathbf{P}(\mathbf{B}_{R_o})}{\psi_{R_o}(\frac12)}=\frac{C}{2n}\le\frac{\mathbf{P}(\mathbf{B}_r)}{2n\mathbf{D}(E)}. \tag{3.18}$$

In particular, this implies that $\psi_r(\frac12)\ge 2n\mathbf{D}(E)\ge 2\mathbf{D}(E)$. Hence, we can choose $a<s_k<0<t_k<b$ such that on the one hand $g(s_k)<\frac12<g(t_k)$ and on the other hand

$$\psi_r(g(s_k))=2\mathbf{D}(E)=\psi_r(g(t_k)).$$

In turn, we used that (3.13) holds true, which ensures that indeed $s_k<0<t_k$. Then, by the concavity of ψ_r, the continuity and monotonicity of $t\mapsto g(t)$ we have that

$$\psi_r(g(t))\ge 2\mathbf{D}(E)\qquad\forall\, t\in[s_k,t_k],$$

which shows that the second term on the right-hand side of (3.16) is non-negative. Therefore we find that

$$v_E(t)\ge\tfrac14\psi_r(g(t))\qquad\forall\, t\in[s_k,t_k]. \tag{3.19}$$

Taking into account that $g'(t)=v_E(t)/\mathbf{V}(\mathbf{B}_r)$ for a.e. $t\in\mathbb{R}$ we obtain that

$$g'(t)\ge\frac{\psi_r(g(t))}{4\mathbf{V}(\mathbf{B}_r)}\qquad\text{for a.e. } t\in[s_k,t_k] \tag{3.20}$$

holds true. We now define

$$H(t,r):=\int_0^t\frac{ds}{\psi_r(s)}\qquad\text{for } t\in[0,1].$$

Note that for fixed $r>0$ the map $t\mapsto H(t,r)$ is $C^1((0,1))\cap C^0([0,1])$ and

$$H(1,r)=\int_0^1\frac{ds}{\psi_r(s)}\in(0,\infty).$$

Here we used Lemma 6.4 from the Appendix. From the definition of H and (3.20) we infer

$$\frac{d}{dt}H(g(t),r)=\frac{g'(t)}{\psi_r(g(t))}\ge\frac{1}{4\mathbf{V}(\mathbf{B}_r)}\qquad\text{for a.e. } t\in(s_k,t_k),$$

which after integration over (s_k,t_k) implies the following bound for the difference t_k-s_k:

$$t_k-s_k\le 4\mathbf{V}(\mathbf{B}_r)\big(H(g(t_k),r)-H(g(s_k),r)\big)\le 4\mathbf{V}(\mathbf{B}_r)\,H(1,r)\,.$$

Keeping in mind that $s_k<0<t_k$ and applying Lemma 2.1, we further deduce

$$\begin{aligned}\max\{|s_k|,t_k\}&\le t_k-s_k\le c(n)[\mathbf{P}(\mathbf{B}_r)]^{\frac{n}{n-1}}\,H(1,r)\\&\le c(n)r\,\frac{\sinh R_o}{R_o}\mathbf{P}(B_{R_o})\,H(1,R_o)\\&\le c(n,R_o)r.\end{aligned} \tag{3.21}$$

For the second last step we used Lemma 6.3, which ensures that $r\mapsto\mathbf{P}(B_r)H(1,r)$ is non-decreasing, and the monotonicity of $r\mapsto\frac{\sinh r}{r}$ for $r>0$. Next, we use the concavity of ψ_r and the fact that $\psi_r(0)=0=\psi_r(1)$ to conclude that $\psi_r(s)\ge 2s\,\psi_r(\frac12)$ for any

$s \in [0, \frac{1}{2}]$ and $\psi_r(s) \geq 2(1-s)\,\psi_r(\frac{1}{2})$ for any $s \in [\frac{1}{2}, 1]$. In particular, taking $s = g(s_k)$ in the first inequality and $s = g(t_k)$ in the second one we find that

$$g(s_k) \leq \frac{\psi_r(g(s_k))}{2\psi_r(\frac{1}{2})} = \frac{\mathbf{D}(E)}{\psi_r(\frac{1}{2})} \tag{3.22}$$

and

$$g(t_k) \geq 1 - \frac{\psi_r(g(t_k))}{2\psi_r(\frac{1}{2})} = 1 - \frac{\mathbf{D}(E)}{\psi_r(\frac{1}{2})} \tag{3.23}$$

holds true. At this stage we are in a position to define the set $\widetilde{E}$. For any slicing direction $k \in \{1, \dots, n\}$ by the construction above – using the function $g_k \equiv g$ – we find $s_k < 0 < t_k$ with $t_k - s_k$ uniformly bounded as in (3.21) such that $g_k(s_k), g_k(t_k)$ fulfill (3.22) respectively (3.23). Now, $\widetilde{E}$ is defined as follows:

$$\widetilde{E} := E \cap \bigcap_{k=1}^{n} \{x \in \mathbb{H}^n : s_k < \delta_k(x) < t_k\}.$$

From the bound (3.21) we infer that any $x \in \widetilde{E}$ satisfies $|\delta_k(x)| \leq c(n, R_o)r$ and therefore, the formula (3.2) implies

$$\sinh^2\big(\mathbf{d}_o(x)\big) \leq n \sinh^2(c(n, R_0)r) \leq \sinh^2\big(\sqrt{n}\,c(n, R_0)r\big).$$

This yields the assertion $\widetilde{E} \subset \mathbf{B}_{rR_1}$ if we let $R_1 = \sqrt{n}\,c(n, R_o)$. Next, we note that since by Lemma 3.1 the nearest-point retractions $\pi_{t_k} : \mathbb{H}^n \to \{x \in \mathbb{H}^n : \delta_k(x) \geq t_k\}$ and $\pi_{s_k} : \mathbb{H}^n \to \{x \in \mathbb{H}^n : \delta_k(x) \leq s_k\}$ have Lipschitz constant ≤ 1 – note that $s_k < 0 < t_k$ holds true – we conclude that

$$\mathbf{P}(\widetilde{E}) \leq \mathbf{P}(E). \tag{3.24}$$

Moreover, from the definition of g_k and inequalities $(3.22)_k$ and $(3.23)_k$ we obtain

$$\begin{aligned}
\mathbf{V}(\widetilde{E}) &\geq \mathbf{V}(E) - \sum_{k=1}^{n} \mathbf{V}(E^-_{s_k}) - \sum_{k=1}^{n} \mathbf{V}(E \setminus E^-_{t_k}) \\
&= \mathbf{V}(\mathbf{B}_r)\Big[1 - \sum_{k=1}^{n} g_k(s_k) - \sum_{k=1}^{n} (1 - g_k(t_k))\Big] \\
&\geq \mathbf{V}(\mathbf{B}_r)\Big[1 - \frac{2n\mathbf{D}(E)}{\psi_r(\frac{1}{2})}\Big] \\
&\geq \mathbf{V}(\mathbf{B}_r)\Big[1 - \frac{C\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_r)}\Big],
\end{aligned} \tag{3.25}$$

where the last estimate follows from (3.18). This proves the first assertion in (3.9). Next, we turn our attention to the proof of (3.10). To this end, we write

$$\mathbf{D}(\widetilde{E}) = \mathbf{P}(\widetilde{E}) - \varphi\big(\mathbf{V}(\widetilde{E})\big)$$

and estimate the last term using the volume bound (3.25) together with the monotonicity and the concavity of φ. This leads us to

$$\begin{aligned}
\varphi(\mathbf{V}(\widetilde{E})) &\geq \varphi\big(\mathbf{V}(\mathbf{B}_r)[1 - C\,\mathbf{D}(E)/\mathbf{P}(\mathbf{B}_r)]\big) \\
&\geq \varphi\big(\mathbf{V}(\mathbf{B}_r)\big)\big[1 - C\,\mathbf{D}(E)/\mathbf{P}(\mathbf{B}_r)\big] = \mathbf{P}(\mathbf{B}_r) - C\,\mathbf{D}(E).
\end{aligned} \tag{3.26}$$

Joining the last two estimates and keeping in mind (3.24), we deduce the bound

$$\mathbf{D}(\widetilde{E}) \leq \mathbf{P}(E) - \mathbf{P}(\mathbf{B}_r) + C\mathbf{D}(E) = (1 + C)\mathbf{D}(E),$$

which proves the claim (3.10). It remains to show that (3.11) holds true. For this we observe that from (2.22) we get

$$\boldsymbol{\beta}(E)^2 - \boldsymbol{\beta}(\widetilde{E})^2 = \mathbf{P}(E) - \mathbf{P}(\widetilde{E}) + \boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E).$$

By $y_{\widetilde{E}}$, we denote one of the centers of $\widetilde{E}$. Using (2.23) and the fact that $\widetilde{E} \subset E$ we find

$$\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E) \le \boldsymbol{\gamma}(\widetilde{E}; y_{\widetilde{E}}) - \boldsymbol{\gamma}(E; y_{\widetilde{E}}) = -\int_{E\setminus\widetilde{E}} \frac{n-1}{\tanh \mathbf{d}_{y_{\widetilde{E}}}(x)}\, d\mu_h(x) \le 0.$$

This implies

$$\boldsymbol{\beta}(E)^2 - \boldsymbol{\beta}(\widetilde{E})^2 \le \mathbf{P}(E) - \mathbf{P}(\widetilde{E}) = \mathbf{D}(E) + \mathbf{P}(\mathbf{B}_r) - \mathbf{P}(\widetilde{E}).$$

At this point we use the isoperimetric inequality (3.15) and (3.26) to infer that

$$\mathbf{P}(\widetilde{E}) \ge \varphi(\mathbf{V}(\widetilde{E})) \ge \mathbf{P}(\mathbf{B}_r) - C\,\mathbf{D}(E).$$

Using this in the second last inequality we conclude that

$$\boldsymbol{\beta}(E)^2 - \boldsymbol{\beta}(\widetilde{E})^2 \le (1 + C)\mathbf{D}(E)$$

holds true. This finally proves (3.11) and finishes the proof of the lemma. □

4. FUGLEDE'S THEOREM IN HYPERBOLIC SPACE

Theorem 4.1. *For any $n \in \mathbb{N}$ and $R_o > 0$, there is a constant $\varepsilon_o \in [0, \frac{1}{2}]$, only depending on n and R_o, such that the following holds. Consider any nearly spherical set $E \subset \mathbb{H}^n$ in hyperbolic n-space with barycenter in the origin and satisfying the volume constraint*

$$\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o}), \qquad \textit{for some } \varrho_o \in (0, R_o],$$

whose boundary is given by a graph over S^{n-1} in the form

$$X\colon S^{n-1} \to \partial E,\ X(\omega) = \omega \tanh\big(\tfrac{\varrho_o}{2}(1 + u(\omega)\big) \tag{4.1}$$

for a Lipschitz function $u\colon S^{n-1} \to \mathbb{R}$ with

$$\|u\|_{W^{1,\infty}(S^{n-1})} \le \varepsilon_o. \tag{4.2}$$

Then we have the estimate

$$\frac{\mathbf{D}(E)}{\mathbf{P}(B_{\varrho_o})} = \frac{\mathbf{P}(E) - \mathbf{P}(B_{\varrho_o})}{\mathbf{P}(B_{\varrho_o})} \ge c_1(n, R_o)\|u\|^2_{W^{1,2}(S^{n-1})}.$$

Proof. Step 1: A formula for the volume of E. A parametrization of E is given by

$$\widetilde{X} : S^{n-1} \times [0, \varrho_o] \to E, \qquad \widetilde{X}(\omega, \varrho) = \omega \tanh\big(\tfrac{\varrho}{2}(1 + u(\omega))\big).$$

For fixed $(\omega, \varrho) \in S^{n-1} \times [0, \varrho_o]$ we introduce the notation $\mathbf{t} := \tanh\big(\frac{\varrho}{2}(1 + u(\omega))\big)$. In order to compute the Jacobian of $\widetilde{X}$, we fix an orthonormal basis $\tau_1, \ldots, \tau_{n-1}$ of $T_\omega S^{n-1}$. Then we choose $(\tau_1, 0), \ldots, (\tau_{n-1}, 0), e_{n+1}$ as orthonormal basis of $T_{\omega,\varrho}(S^{n-1} \times (0, \varrho_o))$ and $\tau_1, \ldots, \tau_{n-1}, \omega$ as orthonormal basis of $T_{X(\omega,\varrho)}\mathbb{H}^n \simeq \mathbb{R}^n$. With respect to these choices, the Jacobi matrix of $\widetilde{X}$ reads

$$D\widetilde{X}(\omega, \varrho) = \begin{pmatrix} \mathbf{t} & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{t} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{t} & 0 \\ \frac{\varrho(1-\mathbf{t}^2)}{2}\partial_{\tau_1} u & \cdots & \cdots & \frac{\varrho(1-\mathbf{t}^2)}{2}\partial_{\tau_{n-1}} u & \frac{1+u}{2}(1 - \mathbf{t}^2) \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Consequently, the Jacobian of $\widetilde{X}$ is given by

$$J\widetilde{X}(\omega,\varrho)=\frac{1+u}{2}\,\mathbf{t}^{n-1}(1-\mathbf{t}^2).$$

The volume of E can thus be re-written as

$$\begin{aligned}\mathbf{V}(E)&=\int_E\Big(\frac{2}{1-|x|^2}\Big)^n\,dx\\&=\int_0^{\varrho_o}\int_{S^{n-1}}\Big(\frac{2}{1-\mathbf{t}^2}\Big)^n\frac{1+u}{2}\,\mathbf{t}^{n-1}(1-\mathbf{t}^2)\,d\mathcal{H}^{n-1}\,d\varrho\\&=\int_0^{\varrho_o}\int_{S^{n-1}}(1+u)\Big(\frac{2\mathbf{t}}{1-\mathbf{t}^2}\Big)^{n-1}\,d\mathcal{H}^{n-1}\,d\varrho.\end{aligned}$$

Recalling the definition of $\mathbf{t}$, we compute

$$\frac{2\mathbf{t}}{1-\mathbf{t}^2}=2\sinh(\tfrac{\varrho}{2}(1+u(\omega)))\cosh(\tfrac{\varrho}{2}(1+u(\omega)))=\sinh(\varrho(1+u(\omega))). \tag{4.3}$$

Joining the last two formulae, we arrive at

$$\begin{aligned}\mathbf{V}(E)&=\int_0^{\varrho_o}\int_{S^{n-1}}(1+u)\sinh^{n-1}(\varrho(1+u))\,d\mathcal{H}^{n-1}\,d\varrho\\&=\int_{S^{n-1}}\int_0^{\varrho_o(1+u)}\sinh^{n-1}(\sigma)\,d\sigma\,d\mathcal{H}^{n-1}.\end{aligned}$$

Step 2: A consequence of the volume constraint. Using the preceding formula for the volume of a nearly spherical set, we can rewrite the volume constraint in the form

$$0=\int_{S^{n-1}}\big[F(\varrho_o(1+u))-F(\varrho_o)\big]\,d\mathcal{H}^{n-1},$$

if we abbreviate

$$F(\varrho):=\int_0^{\varrho}\sinh^{n-1}(\sigma)\,d\sigma.$$

The Taylor expansion of the function F yields

$$\begin{aligned}0&=\int_{S^{n-1}}\big[F'(\varrho_o)\varrho_o u+\tfrac12F''(\varrho_o)\varrho_o^2u^2\big]\,d\mathcal{H}^{n-1}+R_1\\&=\varrho_o\sinh^{n-1}(\varrho_o)\int_{S^{n-1}}\big[u+\tfrac{n-1}{2}\varrho_o\coth(\varrho_o)u^2\big]\,d\mathcal{H}^{n-1}+R_1.\end{aligned} \tag{4.4}$$

For the estimate of the remainder R_1, we compute for any $\varrho\in[\frac{\varrho_o}{2},2\varrho_o]$

$$\begin{aligned}F'''(\varrho)&=(n-1)\sinh^{n-1}(\varrho)\big[(n-1)+(n-2)\sinh^{-2}(\varrho)\big]\\&\le c(n)\frac{\sinh^{n-1}(2\varrho_o)}{\varrho_o^2}=c(n)\frac{\cosh^{n-1}(\varrho_o)\sinh^{n-1}(\varrho_o)}{\varrho_o^2}\\&\le c(n,R_o)\frac{\sinh^{n-1}(\varrho_o)}{\varrho_o^2}\,.\end{aligned}$$

Since $|u|\le\varepsilon_o\le\frac12$ by assumption, we can use this bound to estimate the remainder in Taylor's formula by

$$\begin{aligned}|R_1|&\le c(n,R_o)\varrho_o\sinh^{n-1}(\varrho_o)\int_{S^{n-1}}|u|^3\,d\mathcal{H}^{n-1}\\&\le c(n,R_o)\varrho_o\sinh^{n-1}(\varrho_o)\varepsilon_o\|u\|^2_{L^2(S^{n-1})}\,.\end{aligned}$$

Joining this with (4.4), we arrive at

$$\left|\int_{S^{n-1}} u\,d\mathcal{H}^{n-1}\right| \le \tfrac{n-1}{2}\varrho_o\coth(\varrho_o)\int_{S^{n-1}} u^2\,d\mathcal{H}^{n-1} + c(n,R_o)\varepsilon_o\|u\|^2_{L^2(S^{n-1})}. \tag{4.5}$$

Step 3: Consequences of the barycenter constraint. Since the barycenter of E equals zero by assumption, the condition (2.14) is fulfilled. Using once more the parametrization $\widetilde{X}: S^{n-1}\times[0,\varrho_o]\to E$ introduced above, we re-write this condition as follows.

$$\begin{aligned}
0 &= \int_E 2\,\mathrm{Artanh}\,|x|\,\frac{x}{|x|}\Big(\frac{2}{1-|x|^2}\Big)^n dx\\
&= \int_0^{\varrho_o}\int_{S^{n-1}} \varrho(1+u)\,\omega\,\Big(\frac{2}{1-\mathbf{t}^2}\Big)^n\frac{1+u}{2}\mathbf{t}^{n-1}(1-\mathbf{t}^2)\,d\mathcal{H}^{n-1}d\varrho\\
&= \int_0^{\varrho_o}\int_{S^{n-1}} \varrho(1+u)^2\Big(\frac{2\mathbf{t}}{1-\mathbf{t}^2}\Big)^{n-1}\omega\,d\mathcal{H}^{n-1}d\varrho\\
&= \int_0^{\varrho_o}\int_{S^{n-1}} \varrho(1+u)^2\sinh^{n-1}(\varrho(1+u))\,\omega\,d\mathcal{H}^{n-1}d\varrho,
\end{aligned}$$

where we used again the abbreviation $\mathbf{t} := \tanh(\frac{\varrho}{2}(1+u))$ and in the last line, we exploited once more the identity (4.3). The Taylor expansion of the function $f(x) := \varrho x^2\sinh^{n-1}(\varrho x)$ in $x_o = 1$ yields

$$\begin{aligned}
&\varrho(1+u(\omega))^2\sinh^{n-1}(\varrho(1+u(\omega)))\\
&\qquad= \varrho\sinh^{n-1}(\varrho) + \tfrac{d}{d\varrho}\big(\varrho^2\sinh^{n-1}(\varrho)\big)u(\omega) + \widetilde{R}_2(\omega,\varrho).
\end{aligned}$$

For the bound of $\widetilde{R}_2$, we use the facts $\varrho\le\sinh\varrho$ and $\varrho\le\varrho_o\le R_o$ in order to estimate

$$\begin{aligned}
|f''(x)| &\le c(n,R_o)\varrho\sinh^{n-1}(\varrho x) \le c(n,R_o)\varrho\sinh^{n-1}(2\varrho_o)\\
&= c(n,R_o)\varrho\sinh^{n-1}(\varrho_o)\cosh^{n-1}(\varrho_o) \le c(n,R_o)\varrho\sinh^{n-1}(\varrho_o)
\end{aligned}$$

for any $x\in[\frac12,2]$ and $\varrho\in[0,\varrho_o]$. Consequently, the remainder in Taylor's formula is bounded by

$$\widetilde{R}_2(\omega,\varrho) \le c(n,R_o)\varrho\sinh^{n-1}(\varrho_o)u^2(\omega)$$

and its integral by

$$\begin{aligned}
R_2 &:= \int_0^{\varrho_o}\int_{S^{n-1}} |\widetilde{R}_2(\omega,\varrho)|\,d\mathcal{H}^{n-1}(\omega)\,d\varrho\\
&\le c(n,R_o)\int_0^{\varrho_o}\varrho\sinh^{n-1}(\varrho_o)\,d\varrho\int_{S^{n-1}} u^2(\omega)\,d\mathcal{H}^{n-1}(\omega)\\
&\le c(n,R_o)\varrho_o^2\sinh^{n-1}(\varrho_o)\varepsilon_o\|u\|_{L^2}.
\end{aligned}$$

For this estimate, we used once more the assumption $|u|\le\varepsilon_o\le\frac12$. Collecting the estimates and using the fact $\int_{S^{n-1}}\omega\,d\mathcal{H}^{n-1}=0$, we deduce

$$\int_0^{\varrho_o}\tfrac{d}{d\varrho}\big(\varrho^2\sinh^{n-1}(\varrho)\big)\,d\varrho\int_{S^{n-1}} u\omega\,d\mathcal{H}^{n-1} \le c(n,R_o)\varrho_o^2\sinh^{n-1}(\varrho_o)\varepsilon_o\|u\|_{L^2}.$$

We have thus established the following estimate for the first order Fourier coefficients of u

$$\int_{S^{n-1}} u\omega_\ell\,d\mathcal{H}^{n-1} \le c(n,R_o)\varepsilon_o\|u\|_{L^2(S^{n-1})} \qquad \text{for } \ell=1,\dots,n. \tag{4.6}$$

Step 4: A lower bound for the perimeter of E. We compute the Jacobi matrix of $X: S^{n-1}\to\partial E$ with respect to an orthonormal basis $\tau_1,\dots,\tau_{n-1}$ of $T_\omega S^{n-1}\subset\mathbb{R}^n$ and the orthonormal basis $\tau_1, \dots, \tau_{n-1}, \omega$ of $T_{X(\omega,\varrho)}\mathbb{H}^n \simeq \mathbb{R}^n$. Using the short-hand notation $\mathbf{t} := \tanh(\frac{\varrho_o}{2}(1+u))$, we infer

$$DX(\omega) = \begin{pmatrix} \mathbf{t} & 0 & \cdots & 0 \\ 0 & \mathbf{t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{t} \\ \frac{\varrho_o(1-\mathbf{t}^2)}{2}\partial_{\tau_1}u & \cdots & \cdots & \frac{\varrho_o(1-\mathbf{t}^2)}{2}\partial_{\tau_{n-1}}u \end{pmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

For the Jacobian of X we therefore get

$$\begin{aligned} \big[JX(\omega)\big]^2 &= \mathbf{t}^{2(n-1)} + \mathbf{t}^{2(n-2)}(1-\mathbf{t}^2)^2 \tfrac{\varrho_o^2}{4}|\nabla_\tau u|^2 \\ &= \mathbf{t}^{2(n-1)}\Big[1 + \big(\tfrac{1-\mathbf{t}^2}{2\mathbf{t}}\big)^2 \varrho_o^2 |\nabla_\tau u|^2\Big]. \end{aligned} \tag{4.7}$$

Consequently, the perimeter of E can be expressed as

$$\begin{aligned} \mathbf{P}(E) &= \int_{\partial E} \Big(\frac{2}{1-|x|^2}\Big)^{n-1} d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} \Big(\frac{2}{1-\mathbf{t}^2}\Big)^{n-1} \mathbf{t}^{n-1} \sqrt{1 + \big(\tfrac{1-\mathbf{t}^2}{2\mathbf{t}}\big)^2 \varrho_o^2 |\nabla_\tau u|^2}\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} \sinh^{n-1}(\varrho_o(1+u)) \sqrt{1 + \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o(1+u))}}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.8}$$

The last step is a consequence of (4.3) with $\varrho = \varrho_o$. Next, we will estimate the square root appearing in the above integral. Using the elementary inequality $\sqrt{1+x} \ge 1 + \frac12 x - \frac18 x^2$ for $x \ge 0$, which follows from Taylor's formula, we first deduce

$$\begin{aligned} \sqrt{1 + \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o(1+u))}} &\ge 1 + \tfrac12 \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o(1+u))} - \tfrac18 \tfrac{\varrho_o^4|\nabla_\tau u|^4}{\sinh^4(\varrho_o(1+u))} \\ &\ge 1 + \tfrac12 \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o(1+u))} - 2\varepsilon_o^2 |\nabla_\tau u|^2, \end{aligned}$$

where here, we used $\sinh(\varrho_o(1+u)) \ge \sinh(\varrho_o/2) \ge \varrho_o/2$ and $\|u\|_{W^{1,\infty}} \le \varepsilon_o \le \frac12$ in the last step. On the right-hand side, we want to replace $\sinh^2(\varrho_o(1+u))$ by $\sinh^2(\varrho_o)$. To this end, we use the assumption $u \ge -\varepsilon_o \ge -\frac12$ in order to estimate

$$\begin{aligned} \bigg|\frac{\varrho_o^2}{\sinh^2(\varrho_o(1+u))} - \frac{\varrho_o^2}{\sinh^2(\varrho_o)}\bigg| &= \varrho_o^2 \bigg|\frac{\sinh^2(\varrho_o) - \sinh^2(\varrho_o(1+u))}{\sinh^2(\varrho_o(1+u))\sinh^2(\varrho_o)}\bigg| \\ &\le \frac{\varrho_o^2}{\sinh^4(\frac{\varrho_o}{2})} \bigg| \int_0^1 \frac{d}{d\tau} \sinh^2(\varrho_o(1+\tau u))\, d\tau \bigg| \\ &\le \frac{2\varrho_o^3 |u| \sinh(2\varrho_o)\cosh(2\varrho_o)}{\sinh^4(\frac{\varrho_o}{2})} \le c(R_o)|u| \le c(R_o)\varepsilon_o, \end{aligned}$$

provided $\varrho_o \le R_o$. In order to check that the constant depends only on R_o it suffices to observe that the right-hand side stays bounded as $\varrho_o \to 0$. Combining the two preceding estimates and keeping in mind $|u| \le \varepsilon_o$, we deduce

$$\sqrt{1 + \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o(1+u))}} \ge 1 + \tfrac12 \tfrac{\varrho_o^2|\nabla_\tau u|^2}{\sinh^2(\varrho_o)} - c(R_o)\varepsilon_o |\nabla_\tau u|^2.$$

We plug this into (4.8), with the result

$$\begin{aligned}\mathbf{P}(E) \geq \int_{S^{n-1}} \sinh^{n-1}(\varrho_o(1+u))\Big[1+\tfrac{\varrho_o^2|\nabla_\tau u|^2}{2\sinh^2(\varrho_o)}\Big]\,d\mathcal{H}^{n-1}\\ - c(n,R_o)\sinh^{n-1}(\varrho_o)\varepsilon_o\|\nabla_\tau u\|^2_{L^2(S^{n-1})}.\end{aligned} \tag{4.9}$$

Moreover, from (2.6) we know

$$\mathbf{P}(\mathbf{B}_{\varrho_o}) = \int_{S^{n-1}} \sinh^{n-1}(\varrho_o)\,d\mathcal{H}^{n-1}.$$

Subtracting the two preceding formulae yields

$$\begin{aligned}\mathbf{P}(E) - \mathbf{P}(\mathbf{B}_{\varrho_o}) &\geq \int_{S^{n-1}} \big[\sinh^{n-1}(\varrho_o(1+u)) - \sinh^{n-1}(\varrho_o)\big]\,d\mathcal{H}^{n-1}\\ &\quad + \tfrac{\varrho_o^2}{2\sinh^2(\varrho_o)}\int_{S^{n-1}} \sinh^{n-1}(\varrho_o(1+u))\,|\nabla_\tau u|^2\,d\mathcal{H}^{n-1}\\ &\quad - c(n,R_o)\sinh^{n-1}(\varrho_o)\varepsilon_o\|\nabla_\tau u\|^2_{L^2}\\ &=: I + II + III.\end{aligned} \tag{4.10}$$

For the estimate of I, we calculate the Taylor expansion of $\sinh^{n-1}$. Abbreviating $\mathbf{s}_o := \sinh(\varrho_o)$ and $\mathbf{c}_o := \cosh(\varrho_o)$, we thereby get

$$\begin{aligned}&\sinh^{n-1}(\varrho_o(1+u)) - \sinh^{n-1}(\varrho_o)\\ &\quad = (n-1)\mathbf{s}_o^{n-2}\mathbf{c}_o\varrho_o u + \tfrac{n-1}{2}\mathbf{s}_o^{n-3}[(n-1)\mathbf{c}_o^2 - 1]\varrho_o^2u^2 + \widetilde{R}_3(\omega)\\ &\quad = \mathbf{s}_o^{n-1}\Big((n-1)\frac{\mathbf{c}_o\varrho_o}{\mathbf{s}_o}u + \tfrac{n-1}{2}\Big[(n-1)\frac{\mathbf{c}_o^2}{\mathbf{s}_o^2} - \frac{1}{\mathbf{s}_o^2}\Big]\varrho_o^2u^2\Big) + \widetilde{R}_3(\omega).\end{aligned} \tag{4.11}$$

Keeping in mind $|u| \leq \varepsilon_o \leq \frac12$, we can estimate the remainder by

$$\begin{aligned}|\widetilde{R}_3(\omega)| &\leq \tfrac{n-1}{6}\sinh^{n-1}(2\varrho_o)\cosh(2\varrho_o)\Big[\tfrac{(n-1)^2}{\sinh(\varrho_o/2)} + \tfrac{(n-2)(n-3)}{\sinh^3(\varrho_o/2)}\Big]\varrho_o^3u^3\\ &\leq c(n,R_o)\sinh^{n-1}(\varrho_o)u^3.\end{aligned}$$

For the last estimate it suffices to check that the quotient of both sides stays bounded as $\varrho_o \to 0$. Integrating the two preceding estimates over S^{n-1}, we deduce

$$\begin{aligned}\tfrac{1}{\sinh^{n-1}(\varrho_o)}\,I &\geq (n-1)\varrho_o\coth(\varrho_o)\int_{S^{n-1}} u\,d\mathcal{H}^{n-1}\\ &\quad + \tfrac{n-1}{2}\Big[(n-1)\coth^2(\varrho_o) - \frac{1}{\sinh^2(\varrho_o)}\Big]\varrho_0^2\int_{S^{n-1}} u^2\,d\mathcal{H}^{n-1}\\ &\quad - c(n,R_o)\varepsilon_o\|u\|^2_{L^2}.\end{aligned} \tag{4.12}$$

For the estimate of II, we observe that (4.11) implies in particular

$$\sinh^{n-1}(\varrho_o(1+u)) \geq \sinh^{n-1}(\varrho_o)(1 - c(n,R_o)\varepsilon_o)$$

and therefore,

$$\tfrac{1}{\sinh^{n-1}(\varrho_o)}\,II \geq \tfrac{\varrho_o^2}{2\sinh^2(\varrho_o)}\int_{S^{n-1}} |\nabla_\tau u|^2\,d\mathcal{H}^{n-1} - c(n,R_o)\varepsilon_o\|\nabla_\tau u\|^2_{L^2}. \tag{4.13}$$

Joining the estimates (4.12) and (4.13) with (4.10) we deduce

$$\frac{\mathbf{P}(E) - \mathbf{P}(\mathbf{B}_{\varrho_o})}{\sinh^{n-1}(\varrho_o)} \geq (n-1)\varrho_o\coth(\varrho_o)\int_{S^{n-1}} u\,d\mathcal{H}^{n-1}$$

$$+ \tfrac{n-1}{2}\Big[(n-1)\coth^2(\varrho_o) - \frac{1}{\sinh^2(\varrho_o)}\Big]\varrho_0^2 \int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1}$$
$$+ \tfrac{\varrho_o^2}{2\sinh^2(\varrho_o)} \int_{S^{n-1}} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} - c(n,R_o)\varepsilon_o \|u\|^2_{W^{1,2}}\,.$$

Next, we estimate the first integral on the right-hand side using the consequence (4.5) of the volume constraint. We thereby arrive at

$$\frac{\mathbf{P}(E)-\mathbf{P}(B_{\varrho_o})}{\sinh^{n-1}(\varrho_o)} \ge \tfrac{\varrho_o^2}{2\sinh^2(\varrho_o)}\bigg(\int_{S^{n-1}} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} - (n-1)\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1}\bigg)$$
$$- c(n,R_o)\varepsilon_o\|u\|^2_{W^{1,2}}.$$

Since the expression $\varrho_o/\sinh(\varrho_o)$ is decreasing in $\varrho_o > 0$, we conclude

$$\frac{\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_{\varrho_o})} = \frac{\mathbf{P}(E)-\mathbf{P}(B_{\varrho_o})}{n\omega_n \sinh^{n-1}(\varrho_o)}$$
$$\ge c_o\bigg[\int_{S^{n-1}} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} - (n-1)\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} - c(n,R_o)\varepsilon_o\|u\|^2_{W^{1,2}}\bigg], \tag{4.14}$$

where $c_o := \frac{R_o^2}{2n\omega_n\sinh^2(R_o)}$.

Step 5: Fourier expansion of u. The final step consists of expanding u in terms of the spherical harmonics $Y_{j,\ell} \in L^2(S^{n-1})$. More precisely, we denote by $Y_{j,\ell}$ the eigenfunctions of the spherical Laplacian for the eigenvalue $j(j+n-2)$, i.e. the solutions to

$$-\Delta_{S^{n-1}} Y_{j,\ell} = j(j+n-2)Y_{j,\ell} \qquad \text{for } j\in\mathbb{N}_0 \text{ and } \ell = 1,\dots,m_j.$$

The corresponding Fourier series of u reads

$$u = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} a_{j,\ell}Y_{j,\ell} \qquad \text{with} \qquad a_{j,\ell} = \int_{S^{n-1}} uY_{j,\ell}\, d\mathcal{H}^{n-1}.$$

In particular, we have $m_0 = 1$ and $Y_{0,1} \equiv 1/\sqrt{n\omega_n}$ so that the above formula implies

$$a_0 := a_{0,1} = \frac{1}{\sqrt{n\omega_n}} \int_{S^{n-1}} u\, d\mathcal{H}^{n-1}.$$

Moreover, for $j = 1$ there are $m_1 = n$ spherical harmonics, which are given by $Y_{1,\ell}(x) = x_\ell/\sqrt{\omega_n}$ for $\ell = 1,\dots,n$. The first order Fourier coefficients of u are thus given by

$$a_{1,\ell} = \frac{1}{\sqrt{\omega_n}} \int_{S^{n-1}} u\, x_\ell\, d\mathcal{H}^{n-1}.$$

Using the orthonormality of the spherical harmonics, the L^2-norms of u and ∇u can be expressed by the Fourier coefficients as follows.

$$\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} a_{j,\ell}^2\,,$$
$$\int_{S^{n-1}} |\nabla u|^2\, d\mathcal{H}^{n-1} = -\int_{S^{n-1}} u\Delta_{S^{n-1}}u\, d\mathcal{H}^{n-1} = \sum_{j=1}^{\infty}\sum_{\ell=1}^{m_j} j(j+n-2)\, a_{j,\ell}^2\,.$$

We therefore have

$$\|u\|^2_{W^{1,2}} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} [j(j+n-2)+1]\, a_{j,\ell}^2\,.$$

The constraints on the volume and on the barycenter of u now provide us with estimates for the first and zero order Fourier coefficients of u. More precisely, from (4.5) and (4.2) we infer the bound

$$a_0^2 \le c(n, R_o)\varepsilon_o^2 \|u\|_{L^2}^2 \le c(n, R_o)\varepsilon_o \|u\|_{W^{1,2}}^2, \tag{4.15}$$

while (4.6) and (4.2) imply

$$a_{1,\ell}^2 \le c(n, R_o)\varepsilon_o^2 \|u\|_{L^2}^2 \le c(n, R_o)\varepsilon_o \|u\|_{W^{1,2}}^2 \qquad \text{for } \ell = 1, \dots, n. \tag{4.16}$$

In terms of the Fourier coefficients, the bound (4.14) reads

$$\frac{\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_{\varrho_o})} \ge c_o \Big[\sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} j(j+n-2) a_{j,\ell}^2 - (n-1) \sum_{j=0}^{\infty} \sum_{\ell=1}^{m_j} a_{j,\ell}^2 - c(n, R_o)\varepsilon_o \|u\|_{W^{1,2}}^2 \Big].$$

In order to estimate the right-hand side further, we use the elementary inequality $j(j+n-2) - (n-1) \ge \frac{1}{2}[j(j+n-2)+1]$ that holds for any $j \ge 2$, together with the estimates (4.15) and (4.16). This leads us to

$$\begin{aligned}
\frac{\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_{\varrho_o})} &\ge c_o \Big[\tfrac{1}{2} \sum_{j=2}^{\infty} \sum_{\ell=1}^{m_j} [j(j+n-2)+1] a_{j,\ell}^2 - c(n, R_o)\varepsilon_o \|u\|_{W^{1,2}}^2 \Big] \\
&= c_o \Big[\tfrac{1}{2} \|u\|_{W^{1,2}}^2 - \tfrac{1}{2} a_0^2 - \tfrac{n}{2} \sum_{\ell=1}^{n} a_{1,\ell}^2 - c(n, R_o)\varepsilon_o \|u\|_{W^{1,2}}^2 \Big] \\
&\ge c_o \big(\tfrac{1}{2} - c(n, R_o)\varepsilon_o \big) \|u\|_{W^{1,2}}^2 .
\end{aligned}$$

We can thus choose $\varepsilon_o > 0$, only depending on n and R_o so small that

$$\frac{\mathbf{D}(E)}{\mathbf{P}(\mathbf{B}_{\varrho_o})} \ge \tfrac{c_o}{4} \|u\|_{W^{1,2}}^2 = c_1 \|u\|_{W^{1,2}(S^{n-1})}^2$$

with $c_1 = \frac{c_o}{4} = \frac{R_o^2}{8n\omega_n \sinh^2(R_o)}$. This completes the proof of the theorem. □

The next lemma ensures that the $W^{1,2}$-norm of u is comparable to the L^2-oscillation index of E.

Lemma 4.2. *Let $R_o > 0$ and suppose that $E \subset \mathbb{H}^n$ is a nearly spherical set (as in (4.1)) with volume $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$ for some $\varrho_o \in (0, R_o]$. Furthermore, suppose that the representing function $u \in W^{1,\infty}(S^{n-1})$ from the global graph representation is Lipschitz continuous with $\|u\|_{W^{1,\infty}(S^{n-1})} \le \frac{1}{2}$. Then there exists a constant $\tilde{c} = \tilde{c}(n, R_o) \ge 1$ such that*

$$\tfrac{1}{\tilde{c}} \sinh^{n-1} \varrho_o \, \|u\|_{W^{1,2}}^2 \le \boldsymbol{\beta}(E, 0)^2 \le \tilde{c} \sinh^{n-1} \varrho_o \, \|u\|_{W^{1,2}}^2 .$$

Proof. We set $\mathbf{s} := \sinh[\varrho_o(1 + u(\omega))]$, $\mathbf{c} := \cosh[\varrho_o(1 + u(\omega))]$ and $\mathbf{s}_o := \sinh \varrho_o$, $\mathbf{c}_o := \cosh \varrho_o$. With these abbreviations, the unit outer normal vector to ∂E in the point $x = X(\omega)$ is given by

$$\nu_E(x) = \frac{\mathbf{s}\omega - \varrho_o \nabla_\omega u(\omega)}{(1 + \mathbf{c}) \sqrt{\mathbf{s}^2 + \varrho_o^2 |\nabla_\omega u(\omega)|^2}},$$

and since $\nabla^{\mathbb{H}} \mathbf{d}_o(x)$ points in radial direction and has hyperbolic length one, we have

$$\nabla^{\mathbb{H}} \mathbf{d}_o(x) = \frac{\omega}{1 + \mathbf{c}}.$$

These expressions allow us to compute the integrand in (2.21) as follows:

$$\begin{aligned}1-\big\langle \nu_E(x),\nabla^{\mathbb{H}}\mathbf{d}_o(x)\big\rangle_h &= 1-\frac{\mathbf{s}}{\sqrt{\mathbf{s}^2+\varrho_o^2|\nabla_\omega u(\omega)|^2}}\\ &= \frac{1}{\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}}\Big[\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}-1\Big]\\ &= \frac{\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}{\Big[\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}+1\Big]\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}}.\end{aligned}$$

Since $\|u\|_{W^{1,\infty}(S^{n-1})}\le\frac12$ we have

$$\frac{\varrho_o}{\mathbf{s}}\le\frac{\varrho_o}{\sinh\frac{\varrho_o}{2}}\le 2$$

and therefore

$$1-\big\langle \nu_E(x),\nabla^{\mathbb{H}}\mathbf{d}_o(x)\big\rangle_h\le 4|\nabla_\omega u(\omega)|^2.$$

On the other hand, using $\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2\le 1$ and hence $\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}\le\sqrt2$ we get the following bound from below:

$$1-\big\langle \nu_E(x),\nabla^{\mathbb{H}}\mathbf{d}_o(x)\big\rangle_h\ge\frac{\varrho_o^2}{4\mathbf{s}^2}|\nabla_\omega u(\omega)|^2\ge c(R_o)|\nabla_\omega u(\omega)|^2.$$

Integrating the first of the two preceding inequalities with respect to ω over S^{n-1}, taking into account the formula for the n-Jacobian of X from (4.7) and $\|\nabla_\omega u\|_{L^\infty}\le\frac12$, we find

$$\begin{aligned}\boldsymbol{\beta}^2(E;0) &= \int_{\partial^*E}\big(1-\big\langle \nu_E(x),\nabla^{\mathbb{H}}\mathbf{d}_o(x)\big\rangle_h\big)\,d\sigma_h\le 4\int_{S^{n-1}}|\nabla_\omega u(\omega)|^2 JX\,d\sigma_h\\ &\le 4\int_{S^{n-1}}|\nabla_\omega u(\omega)|^2\mathbf{s}^{n-1}\sqrt{1+\varrho_o^2/\mathbf{s}^2|\nabla_\omega u(\omega)|^2}\,d\omega\\ &\le 4\sqrt2\sinh^{n-1}(2\varrho_o)\,\|\nabla_\omega u\|_{L^2}^2\le c(n,R_o)\sinh^{n-1}\varrho_o\,\|\nabla_\omega u\|_{L^2}^2.\end{aligned}$$

Since a similar estimate from below holds true, also with a constant depending on n and R_o, the lemma is proved. □

Corollary 4.3. *Let $R_o>0$. Under the assumptions of Theorem 4.1 there exists a constant $c_2=c_2(n,R_o)>0$ such that the following inequality holds true*

$$\mathbf{P}(E)-\mathbf{P}(\mathbf{B}_{\varrho_o})\ge c_2\boldsymbol{\beta}(E)^2.$$

5. A PERTURBED PERIMETER FUNCTIONAL

In this section we deal with functionals of the type

$$\mathbf{F}(G):=\mathbf{P}(G)+\frac{\Lambda\cosh\varrho_o}{\sinh\varrho_o}\big|\mathbf{V}(G)-\mathbf{V}(\mathbf{B}_{\varrho_o})\big|+C_o|\boldsymbol{\beta}^2(G)-\varepsilon^2|,\tag{5.1}$$

where $\Lambda\ge1$, $\varrho_o>0$, $C_o\in[0,1]$ and $\varepsilon\ge0$. These functionals are well defined for sets $G\subset\mathbb{H}^n$ of finite perimeter.

Lemma 5.1. *Let $\mathbf{F}$ be as in (5.1) with $C_o=0$. Then, there exists a constant $\Lambda_o(n)\ge1$ such that any minimizer of the functional $\mathbf{F}_{\varrho_o}\equiv\mathbf{F}$ for some given $\varrho_o>0$ is a geodesic ball $\mathbf{B}_{\varrho_o}(x_o)$, provided $\Lambda>\Lambda_o(n)$.*

Proof. Let $E \subset \mathbb{H}^n$ be a minimizer of the functional $\mathbf{F}_{\varrho_o}$. By the minimality of E we get

$$\mathbf{P}(E) \le \mathbf{F}_{\varrho_o}(E) \le \mathbf{F}_{\varrho_o}(\mathbf{B}_{\varrho_o}) = \mathbf{P}(B_{\varrho_o}). \tag{5.2}$$

From the isoperimetric property of geodesic balls from (2.8) we conclude that $\mathbf{V}(E) \le \mathbf{V}(\mathbf{B}_{\varrho_o})$. If $\mathbf{V}(E) < \mathbf{V}(\mathbf{B}_{\varrho_o})$, then there would exist $\varrho < \varrho_o$ such that $\mathbf{V}(\mathbf{B}_\varrho) = \mathbf{V}(E)$. By the minimality of E we have

$$\mathbf{P}(E) + \frac{\Lambda \cosh \varrho_o}{\sinh \varrho_o}\big[\mathbf{V}(\mathbf{B}_{\varrho_o}) - \mathbf{V}(\mathbf{B}_\varrho)\big] \le \mathbf{P}(\mathbf{B}_{\varrho_o}).$$

By the isoperimetric property of spheres (2.8) we have that $\mathbf{P}(\mathbf{B}_\varrho) \le \mathbf{P}(E)$, so that

$$\frac{\cosh \varrho_o}{\sinh \varrho_o} \cdot \frac{\mathbf{V}(\mathbf{B}_{\varrho_o}) - \mathbf{V}(\mathbf{B}_\varrho)}{\mathbf{P}(\mathbf{B}_{\varrho_o}) - \mathbf{P}(\mathbf{B}_\varrho)} \le \frac{1}{\Lambda},$$

which by Lemma 2.3 is impossible, provided we choose $\Lambda > c(n)^{-1}$, where $c(n)$ denotes the constant from Lemma 2.3. Therefore, $\mathbf{V}(E) < \mathbf{V}(\mathbf{B}_{\varrho_o})$ cannot hold, so that we must have $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$. Then, the isoperimetric property of balls yields that $\mathbf{P}(\mathbf{B}_{\varrho_o}) \le \mathbf{P}(E)$. Together with (5.2) we conclude that E must have the same perimeter as $\mathbf{B}_{\varrho_o}$, as well as the same volume. The uniqueness part of the isoperimetric inequality now implies that E must be a geodesic ball of radius ϱ_o, and this completes the proof of the Lemma. □

Our next aim is to ensure that the functional $\mathbf{F}$ from (5.1) is lower semi-continuous. Therefore, we first have to establish the continuity of the functional $\boldsymbol{\gamma}$ from (2.23).

Lemma 5.2. *Whenever G_k, G are measurable sets in $\mathbb{H}^n$ with $G_k \to G$ in $L^1(\mathbb{H}^n)$ there holds*

$$\lim_{k\to\infty} \boldsymbol{\gamma}(G_k) = \boldsymbol{\gamma}(G),$$

i.e the functional $\boldsymbol{\gamma}$ defined in (2.23) is continuous with respect to L^1-convergence.

Proof. We choose centers y_k of G_k and y of G. By the maximality of the center we have

$$\boldsymbol{\gamma}(G_k; y) \le \boldsymbol{\gamma}(G_k; y_k) = \boldsymbol{\gamma}(G_k).$$

Together with $\lim_{k\to\infty} \boldsymbol{\gamma}(G_k; y) = \boldsymbol{\gamma}(G; y)$, which is a consequence of the L^1-convergence $G_k \to G$ and the dominated convergence theorem, we obtain the lower semi-continuity of $\boldsymbol{\gamma}$, i.e.

$$\liminf_{k\to\infty} \boldsymbol{\gamma}(G_k) \ge \boldsymbol{\gamma}(G).$$

To prove the upper semi-continuity we first choose radii $r_k > 0$ in such a way that $\mathbf{V}(\mathbf{B}_{r_k}(y_k)) = \mathbf{V}(G_k \setminus G)$ and observe that

$$\begin{aligned}\boldsymbol{\gamma}(G_k) &\le \boldsymbol{\gamma}(G) + \int_{G_k\setminus G} \frac{n-1}{\tanh \mathbf{d}_{y_k}(x)}\, d\mu_h \le \boldsymbol{\gamma}(G) + \int_{\mathbf{B}_{r_k}(y_k)} \frac{n-1}{\tanh \mathbf{d}_{y_k}(x)}\, d\mu_h \\ &= \boldsymbol{\gamma}(G) + \mathbf{P}(\mathbf{B}_{r_k}) \le \boldsymbol{\gamma}(G) + n\omega_n^{\frac{1}{n}} \big(\cosh r_k\, \mathbf{V}(G_k \setminus G)\big)^{\frac{n-1}{n}}.\end{aligned}$$

Here we used in the last line Lemma 2.1. Since $G_k \to G$ in $L^1(\mathbb{H}^n)$ and hence $\cosh r_k \to 1$ and $\mathbf{V}(G_k \setminus G) \to 0$ we conclude the upper semi-continuity, i.e.

$$\limsup_{k\to\infty} \boldsymbol{\gamma}(G_k) \le \boldsymbol{\gamma}(G).$$

Together with the lower semi-continuity this establishes the continuity of $\boldsymbol{\gamma}$ with respect to L^1-convergence. □

The previous Lemma allows us to prove the lower semi-continuity of the penalized functionals $\mathbf{F}$.

Lemma 5.3. *The functional* $\mathbf{F}$ *defined in* (5.1) *is lower semi-continuous with respect to* L^1*-convergence, i.e.*

$$\mathbf{F}(G) \le \liminf_{k\to\infty} \mathbf{F}(G_k)$$

whenever G_k, G *are sets in* $\mathbb{H}^n$ *of finite perimeter with* $G_k \to G$ *in* $L^1(\mathbb{H}^n)$.

Proof. Consider subsets G_k and G in $\mathbb{H}^n$ as in the statement. Passing to a subsequence we may assume without loss of generality that

$$\liminf_{k\to\infty} \mathbf{F}(G_k) = \lim_{k\to\infty} \mathbf{F}(G_k)$$

and $\lim_{k\to\infty} \mathbf{P}(G_k) = \alpha$. By the lower semi-continuity of the perimeter with respect to L^1-convergence we have $\alpha \ge \mathbf{P}(G)$. Using (2.26), Lemma 5.2 and the assumption $0 \le C_o \le 1$ we infer

$$\begin{aligned}
&\lim_{k\to\infty} \mathbf{F}(G_k)\\
&\quad= \lim_{k\to\infty} \bigg[\mathbf{P}(G_k) + \frac{\Lambda\cosh\varrho_o}{\sinh\varrho_o}\big|\mathbf{V}(G_k) - \mathbf{V}(\mathbf{B}_{\varrho_o})\big| + C_o\big|\mathbf{P}(G_k) - \boldsymbol{\gamma}(G_k) - \varepsilon^2\big|\bigg]\\
&\quad= \alpha + \frac{\Lambda\cosh\varrho_o}{\sinh\varrho_o}\big|\mathbf{V}(G) - \mathbf{V}(\mathbf{B}_{\varrho_o})\big| + C_o\big|\alpha - \boldsymbol{\gamma}(G) - \varepsilon^2\big|\big]\\
&\quad\ge \mathbf{F}(G) + (1-C_o)(\alpha - \mathbf{P}(G)) \ge \mathbf{F}(G),
\end{aligned}$$

proving the lower semi-continuity of the functional $\mathbf{F}$. □

Definition 5.4 (Quasi-minimizers of the perimeter)**.** For a given radius $r_1 > 0$ and a given constant $K \ge 1$ we say that a set $G \subset \mathbb{H}^n$ of finite perimeter is a perimeter (K, r_1)-*quasi-minimizer* if

$$\mathbf{P}(G; \mathbf{B}_r(x)) \le K\,\mathbf{P}(D; \mathbf{B}_r(x)), \tag{5.3}$$

whenever $D \subset \mathbb{H}^n$ is a finite perimeter set such that $G\Delta D \Subset \mathbf{B}_r(x)$ for some ball $\mathbf{B}_r(x)$ with radius $0 < r \le r_1$. Here, $\mathbf{P}(G; \mathbf{B}_r(x)) \equiv \mathbf{P}_{\mathbf{B}_r(x)}(G)$ denotes the perimeter of G in $\mathbf{B}_r(x)$. □

Lemma 5.5. *Let* $R > 1$ *and* $\mathbf{F}$ *be a functional of the form* (5.1) *with* $\Lambda \ge 1$, $\varrho_o > 0$, $C_o \in (0, \frac{1}{8}]$ *and* $\varepsilon \ge 0$. *Then there exists* $r_1 = r_1(\Lambda) \in (0, \frac{1}{8}]$ *such that any set* $G \subset \mathbb{H}^n$ *of finite perimeter that minimizes* $\mathbf{F}$ *amongst finite perimeter sets* E *satisfying the constraint* $E \subset \overline{\mathbf{B}}_R$, *is a perimeter* $(K, r_1 \min\{1, \varrho_o\})$*-quasi-minimizer with* $K = 3$.

Proof. We abbreviate $\hat\varrho_o := \min\{1, \varrho_o\}$. Next, we consider a finite perimeter set $D \subset \mathbb{H}^n$ satisfying $G\Delta D \Subset \mathbf{B}_{r\hat\varrho_o}(x)$ for some ball $\mathbf{B}_{r\hat\varrho_o}(x)$, with $r \le 1$. In the following we distinguish between three different cases.

Case 1: $\mathbf{B}_{r\hat\varrho_o}(x) \cap \overline{\mathbf{B}}_R = \varnothing$. In this case estimate (5.3) trivially holds since $G \cap \mathbf{B}_{r\hat\varrho_o}(x) = \varnothing$.

Case 2: $\mathbf{B}_{r\hat\varrho_o}(x) \cap \overline{\mathbf{B}}_R \neq \varnothing$ *and* $D \subset \overline{\mathbf{B}}_R$. Here, we use the $\mathbf{F}$-minimality of G and find that

$$\mathbf{P}(G) \le \mathbf{P}(D) + \frac{\Lambda\cosh\varrho_o}{\sinh\varrho_o}|\mathbf{V}(G) - \mathbf{V}(D)| + C_o|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)|. \tag{5.4}$$

By Lemma 2.1 we have

$$Q_n(r\hat\varrho_o) := \frac{\mathbf{V}(\mathbf{B}_{r\hat\varrho_o})}{\mathbf{P}(\mathbf{B}_{r\hat\varrho_o})} \le \frac{\sinh(r\hat\varrho_o)}{n}.$$

Moreover, from Lemma 2.2 we know that Q_n is increasing. From the isoperimetric property of hyperbolic balls (2.8) we conclude that there holds:

$$\begin{aligned}\frac{\cosh\varrho_o}{\sinh\varrho_o}\,|\mathbf{V}(G)-\mathbf{V}(D)| &\le \frac{\cosh\varrho_o}{\sinh\varrho_o}\,\mathbf{V}(G\Delta D) \le \frac{Q_n(r\hat\varrho_o)\cosh\varrho_o}{\sinh\varrho_o}\,\mathbf{P}(G\Delta D)\\ &\le \frac{\sinh(r\hat\varrho_o)\cosh\varrho_o}{n\,\sinh\varrho_o}\big[\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))+\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x))\big].\end{aligned}$$

Using the convexity of $\sinh$ and distinguishing between the cases $\varrho_o\le 1$ and $\varrho_o>1$, we can further estimate

$$\frac{\sinh(r\hat\varrho_o)\cosh\varrho_o}{n\,\sinh\varrho_o}\le\frac{r\sinh(\hat\varrho_o)\cosh\varrho_o}{n\,\sinh\varrho_o}\le\frac{r}{n}\cosh 1\le r.$$

Provided we choose $r\le r_1:=\frac{1}{4\Lambda}$, we thereby get

$$\frac{\cosh\varrho_o}{\sinh\varrho_o}\,|\mathbf{V}(G)-\mathbf{V}(D)|\le\tfrac{1}{4\Lambda}\big[\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))+\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x))\big].$$

In order to estimate the last term on the right-hand side of (5.4) we first consider the case $\boldsymbol\beta(G)\le\boldsymbol\beta(D)$. Denoting by y_G a center of G we obtain (using the definition of the center)

$$\begin{aligned}|\boldsymbol\beta^2(D)-\boldsymbol\beta^2(G)| &\le \boldsymbol\beta^2(D;y_G)-\boldsymbol\beta^2(G;y_G)\\ &=\int_{\partial^* D}\big[1-\big\langle\nu_D(x),\Pi_{\pi(x),x}(\nu_{\mathbf{B}_r(y_G)}(\pi(x)))\big\rangle\big]\,d\sigma_h(x)\\ &\quad-\int_{\partial^* G}\big[1-\big\langle\nu_G(x),\Pi_{\pi(x),x}(\nu_{\mathbf{B}_r(y_G)}(\pi(x)))\big\rangle\big]\,d\sigma_h(x)\\ &\le 2\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x)).\end{aligned}$$

On the other hand, if $\boldsymbol\beta(G)>\boldsymbol\beta(D)$ we use the same argument to conclude

$$|\boldsymbol\beta^2(D)-\boldsymbol\beta^2(G)|\le 2\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x)).$$

In any case we therefore have that

$$|\boldsymbol\beta^2(D)-\boldsymbol\beta^2(G)|\le 2\big[\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x))+\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))\big]$$

holds true. Inserting the preceding inequalities into (5.4) and re-absorbing the terms with $\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))$ from the right-hand side into the left we arrive at

$$\big(1-\tfrac14-2C_o\big)\,\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))\le\big(1+\tfrac14+2C_o\big)\,\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x)).$$

Since $0\le C_o\le\frac18$ we finally arrive at

$$\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))\le 3\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x)),$$

whenever $D\subset\overline{B}_R$ is a finite perimeter set such that $G\Delta D\Subset B_{r\hat\varrho_o}(x)$ with $0<r\le r_1$. This proves estimate (5.3) in the second case.

Case 3: $\mathbf{B}_{r\hat\varrho_o}(x)\cap\overline{\mathbf{B}}_R\neq\varnothing$ *and* $D\setminus\overline{\mathbf{B}}_R\neq\varnothing$. In this case we first consider the set $D\cap\mathbf{B}_R$. From the second case we observe that

$$\mathbf{P}(G;\mathbf{B}_{r\hat\varrho_o}(x))\le 3\mathbf{P}(D\cap\mathbf{B}_R;\mathbf{B}_{r\hat\varrho_o}(x)).$$

Moreover, we can estimate

$$\mathbf{P}(D\cap\mathbf{B}_R;\mathbf{B}_{r\hat\varrho_o}(x))\le\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x))+\mathbf{P}(\mathbf{B}_R)-\mathbf{P}(D\cup\mathbf{B}_R)\le\mathbf{P}(D;\mathbf{B}_{r\hat\varrho_o}(x)),$$

where we first used the fact $D\Delta\mathbf{B}_R\Subset\mathbf{B}_{r\hat\varrho_o}(x)$ and then the isoperimetric property of $\mathbf{B}_R$. Joining the two preceding estimates finally proves the claim. □

The following result has been proved in [20, Theorem 5.2] in the context of metric spaces. The Euclidean version is due to David and Semmes [11].

Theorem 5.6. *Suppose that $G \subset \mathbb{H}^n$ is a perimeter (K, r_1)-quasi-minimizer. Then, up to modifying G in a set of measure zero, the topological boundary of G coincides with the reduced boundary, i.e. $\partial G = \partial^* G$. Moreover G and $\mathbb{H}^n \setminus G$ are locally porous in the sense that there exists a constant $C > 1$, depending only on n and K such that for every $x \in \partial G$ and $0 < r < r_1$ there are points $y, z \in \mathbf{B}_r(x)$ for which*

$$\mathbf{B}_{r/C}(y) \subset G \qquad \text{and} \qquad \mathbf{B}_{r/C}(z) \subset \mathbb{H}^n \setminus G$$

hold true.

Definition 5.7 (Almost perimeter minimizing sets)**.** For a given radius $r_2 > 0$ and a constant $K > 0$ we say that a finite perimeter set $G \subset \mathbb{H}^n$ is a (K, r_2)-*almost minimizer of the perimeter* if

$$\mathbf{P}(G) \le \mathbf{P}(D) + K\,\mathbf{V}(G\Delta D) \tag{5.5}$$

holds true whenever $D \subset \mathbb{H}^n$ is a set of finite perimeter such that $G\Delta D \Subset \mathbf{B}_r(x)$ for some geodesic ball $\mathbf{B}_r(x)$ with radius $r \in (0, r_2]$. □

Lemma 5.8. *Let $R > 1$ and $\Lambda \ge 1$, $C_o \in (0,1)$, $r_o > 0$. Then, there exist constants $\delta_o(n, r_o) > 0$, $r_2(n, r_o) > 0$ and $K = K(n, \Lambda, C_o)$ such that the following holds true: Assume that the set $G \subset \overline{\mathbf{B}}_R$ minimizes the functional $\mathbf{F}$ from (5.1) with parameters $(\Lambda, C_o, \varrho_o)$ under all comparison sets $D \subset \overline{\mathbf{B}}_R$, where $2\varrho_o \le r_o$. Moreover, assume that G satisfies the almost ball property*

$$\mathbf{B}_{\varrho_o(1-\delta_o)}(x_o) \subset G \subset \mathbf{B}_{\varrho_o(1+\delta_o)}(x_o) \tag{5.6}$$

for some $x_o \in \mathbb{H}^n$. Then G is $\big(K\big(1 + \frac{1}{\varrho_o}\big), r_2\varrho_o\big)$-almost perimeter minimizing.

Proof. Let $\delta_o, r_2 > 0$ be arbitrary numbers satisfying $2r_2 + \delta_o < \frac12$. We will choose δ_o and r_2 in the course of the proof in dependence on n and r_o in a universal way. Note that $G \subset \mathbf{B}_{2\varrho_o}(x_o) \subset \mathbf{B}_{r_o}(x_o)$. We consider $D \subset \mathbb{H}^n$ satisfying $G\Delta D \Subset \mathbf{B}_{r\varrho_o}(y)$ for some ball $\mathbf{B}_{r\varrho_o}(y)$ with $0 < r \le r_2$, and distinguish between four cases.

Case 1: $\mathbf{B}_{r\varrho_o}(y) \subset \mathbf{B}_{\varrho_o(1-\delta_o)}(x_o)$. In this case, the annulus type assumption (5.6) implies

$$G\Delta D \Subset \mathbf{B}_{r\varrho_o}(y) \subset \mathbf{B}_{\varrho_o(1-\delta_o)}(x_o) \subset G.$$

But $G\Delta D \Subset G$ implies that $\mathbf{P}(G) \le \mathbf{P}(D)$, i.e. (5.5) holds true without volume term on the right-hand side.

Case 2: $\mathbf{B}_{r\varrho_o}(y) \cap \mathbf{B}_{\varrho_o(1+\delta_o)}(x_o) = \varnothing$. In this case we have $(G\Delta D) \cap \mathbf{B}_{\varrho_o(1+\delta_o)}(x_o) = \varnothing$ and therefore also in this case $\mathbf{P}(G) \le \mathbf{P}(D)$ holds true.

Case 3: $\mathbf{B}_{r\varrho_o}(y) \setminus \mathbf{B}_{\varrho_o(1-\delta_o)}(x_o) \ne \varnothing$, $\mathbf{B}_{r\varrho_o}(y) \cap \mathbf{B}_{\varrho_o(1+\delta_o)}(x_o) \ne \varnothing$ and $D \subset \overline{B}_R$. Due to the minimality of G and (2.26) we get

$$\begin{aligned}
\mathbf{P}(G) &\le \mathbf{P}(D) + \frac{\Lambda \cosh \varrho_o}{\sinh \varrho_o}|\mathbf{V}(G) - \mathbf{V}(D)| + C_o|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)| \\
&\le \mathbf{P}(D) + C_o|\mathbf{P}(G) - \mathbf{P}(D)| + \frac{\Lambda \cosh \varrho_o}{\sinh \varrho_o}\mathbf{V}(G\Delta D) + C_o|\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D)|.
\end{aligned}$$

Distinguishing between the cases $\mathbf{P}(G) \le \mathbf{P}(D)$ and $\mathbf{P}(D) < \mathbf{P}(G)$ to re-absorb the term containing $\mathbf{P}(G)$ from the right into the left hand side we infer that

$$\mathbf{P}(G) \le \mathbf{P}(D) + \tfrac{\Lambda}{1-C_o}\Big(2 + \tfrac{1}{\varrho_o}\Big)\mathbf{V}(G\Delta D) + \tfrac{C_o}{1-C_o}|\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D)|. \tag{5.7}$$

Now, we observe that the annulus assumption (5.6) and $\delta_o \le 1$ imply that

$$\begin{aligned}\mathbf{V}(G\Delta \mathbf{B}_{\varrho_o}(x_o)) &\le \mathbf{V}\big(\mathbf{B}_{\varrho_o(1+\delta_o)}(x_o)\setminus \mathbf{B}_{\varrho_o(1-\delta_o)}(x_o)\big)\\ &\le 2n\omega_n\delta_o\varrho_o\sinh^{n-1}(2\varrho_o)\\ &\le n\omega_n\delta_o\sinh^n(2\varrho_o)\\ &\le 2^n n\omega_n\delta_o\cosh^n\varrho_o\sinh^n\varrho_o\\ &\le 2^n n\omega_n\delta_o \mathrm{e}^{n\varrho_o}\sinh^n\varrho_o\\ &\le 2^n n\omega_n\delta_o\mathrm{e}^{2nr_o}\sinh^n\varrho_o\,.\end{aligned}$$

For the comparison set D we similarly obtain, using $r \le r_2 \le 1$,

$$\begin{aligned}\mathbf{V}(D\Delta \mathbf{B}_{\varrho_o}(x_o)) &\le \mathbf{V}\big(\mathbf{B}_{r\varrho_o}(y)\cup(G\Delta\mathbf{B}_{\varrho_o}(x_o))\big)\\ &\le n\omega_n r\varrho_o\sinh^{n-1}(r\varrho_o)+\mathbf{V}(G\Delta\mathbf{B}_{\varrho_o}(x_o))\\ &\le n\omega_n\big(r_2+2^n\delta_o e^{2nr_o}\big)\sinh^n\varrho_o.\end{aligned}$$

We note that the comparison set D is contained in $\mathbf{B}_{\varrho_o(1+\delta_o+2r_2)}(x_o) \subset \mathbf{B}_{r_o}(x_o)$. In Lemma 2.9 we fix $\varepsilon := \frac14$ and denote by $\delta = \delta(n, \frac14, r_o) > 0$ the corresponding constant. At this stage we choose $\delta_o, r_2 \in (0, \frac12]$ in dependence on n and r_o small enough in order to guarantee that

$$\mathbf{V}(G\Delta\mathbf{B}_{\varrho_o}(x_o)) < \delta\sinh^n\varrho_o \qquad\text{and}\qquad \mathbf{V}(D\Delta\mathbf{B}_{\varrho_o}(x_o)) < \delta\sinh^n\varrho_o.$$

Hence, the hypotheses of Lemma 2.9 are fulfilled, and therefore we conclude that

$$\mathbf{d}_o(y_G, x_o) \le \tfrac14\varrho_o \quad\text{and}\quad \mathbf{d}_o(y_D, x_o) \le \tfrac14\varrho_o.$$

Moreover, since $\mathbf{B}_{r\varrho_o}(y)\setminus\mathbf{B}_{\varrho_o(1-\delta_o)}(x_o) \ne \varnothing$ we obtain for any $x \in G\Delta D$ that

$$\mathbf{d}_{y_G}(x) = \mathbf{d}(x, y_G) \ge \mathbf{d}_o(x, x_o) - \mathbf{d}_o(y_G, x_o) \ge \varrho_o(1-\delta_o-2r) - \tfrac14\varrho_o \ge \tfrac14\varrho_o.$$

Recalling the definition of $\boldsymbol{\gamma}$ in (2.25) – in particular the fact that the maximum of $\boldsymbol{\gamma}$ is attained in a center of the set – and using the last inequality we find

$$\begin{aligned}\boldsymbol{\gamma}(G)-\boldsymbol{\gamma}(D) &\le (n-1)\bigg[\int_G \coth\mathbf{d}_{y_G}(x)\,d\mu_h(x) - \int_D\coth\mathbf{d}_{y_G}(x)\,d\mu_h(x)\bigg]\\ &\le (n-1)\int_{G\Delta D}\coth\mathbf{d}_{y_G}(x)\,d\mu_h(x) \le (n-1)\coth\big(\tfrac14\varrho_o\big)\,\mathbf{V}(G\Delta D).\end{aligned}$$

Analogously, we can bound the difference $\gamma(D)-\gamma(G)$ from above, which together with the preceding inequality yields the estimate

$$|\boldsymbol{\gamma}(G)-\boldsymbol{\gamma}(D)| \le (n-1)\coth\big(\tfrac14\varrho_o\big)\,\mathbf{V}(G\Delta D).$$

Inserting the preceding inequality into (5.7) we arrive at

$$\begin{aligned}\mathbf{P}(G) &\le \mathbf{P}(D) + \tfrac{1}{1-C_o}\Big[\Lambda\big(2+\tfrac{1}{\varrho_o}\big)+C_o(n-1)\coth\big(\tfrac14\varrho_o\big)\Big]\mathbf{V}(G\Delta D)\\ &\le \mathbf{P}(D)+K\big(1+\tfrac{1}{\varrho_o}\big)\mathbf{V}(G\Delta D)\end{aligned}$$

with $K := \frac{2\Lambda+4C_o(n-1)}{1-C_o}$. This proves the claimed almost minimizing property of G when compared to sets $D \subset \overline{\mathbf{B}}_R$. It therefore only remains to consider

Case 4: $\mathbf{B}_{r\varrho_o}(y)\setminus\mathbf{B}_{\varrho_o(1-\delta_o)}(x_o) \ne \varnothing$, $\mathbf{B}_{r\varrho_o}(y)\cap\mathbf{B}_{\varrho_o(1+\delta_o)}(x_o) \ne \varnothing$ *and* $D\setminus\overline{\mathbf{B}}_R \ne \varnothing$. In this case, we use $D\cap\overline{\mathbf{B}}_R$ as a comparison set, for which case 3 is applicable. We thereby know

$$\mathbf{P}(G) \le \mathbf{P}\big(D\cap\overline{\mathbf{B}}_R\big)+K\big(1+\tfrac{1}{\varrho_o}\big)\mathbf{V}\big(G\Delta(D\cap\overline{\mathbf{B}}_R)\big).$$

The first term on the right-hand side can be estimated by

$$\mathbf{P}\big(D\cap\overline{\mathbf{B}}_R\big)\le \mathbf{P}(D)+\mathbf{P}(\mathbf{B}_R)-\mathbf{P}(D\cup\overline{\mathbf{B}}_R)\le \mathbf{P}(D),$$

where the last inequality is a consequence of the isoperimetric property of the ball $\mathbf{B}_R$, which implies $\mathbf{P}(\mathbf{B}_R)\le\mathbf{P}(D\cup\overline{\mathbf{B}}_R)$. Furthermore, since $G\subset\overline{\mathbf{B}}_R$, we have

$$G\Delta(D\cap\overline{\mathbf{B}}_R)=\big[G\setminus D\big]\cup\big[(D\cap\overline{\mathbf{B}}_R)\setminus G\big]\subset G\Delta D.$$

Joining the three preceding formulae, we arrive at

$$\mathbf{P}(G)\le\mathbf{P}(D)+K\big(1+\tfrac{1}{\varrho_o}\big)\mathbf{V}(G\Delta D),$$

also in this last case. Combining all four cases, we conclude that G is a $\big(K\big(1+\frac{1}{\varrho_o}\big), r_2\varrho_o\big)$-almost minimizer of the perimeter in $\mathbb{H}^n$ with a constant $K=\frac{2\Lambda+4C_o(n-1)}{1-C_o}$ and a radius $r_2=r_2(n,r_o)>0$. This proves the claim. □

Finally, we state a regularity result for almost minimizers of the perimeter. Since the arguments are standard, we skip the details and refer to [23, 24, 21, 12, 6].

Theorem 5.9. *Suppose that $G_k\subset\mathbb{H}^n$ is a sequence of (K,r_o)-almost minimizers of the perimeter for some constant $K>0$ and some radius $r_o>0$ satisfying*

$$\sup_{k\in\mathbb{N}}\mathbf{P}(G_k)<\infty\qquad\textit{and}\qquad \chi_{G_k}\to\chi_{\mathbf{B}_{\varrho_o}}\ \textit{in } L^1(\mathbb{H}^n),$$

where $\varrho_o>0$. Then, there exists $k_o\in\mathbb{N}$ such that for any $k\ge k_o$ the set G_k is a nearly spherical set whose boundary is given by a graph over S^{n-1} in the form (4.1) with a representing function $u_k\in C^{1,\frac12}(S^{n-1})$. Furthermore, we have $u_k\to 0$ in $C^{1,\alpha}(S^{n-1})$ for every $\alpha\in(0,\frac12)$.

6. Proof of the main result

The aim of this section is to give the proof of Theorem 1.1. We start with

6.1. **A technical lemma.**

Lemma 6.1. *We consider a sequence of finite perimeter sets $E_k\subset\mathbb{H}^n$ satisfying the volume constraint $\mathbf{V}(E_k)=\mathbf{V}(\mathbf{B}_{r_k})$ for radii $r_k\to 0$ as $k\to\infty$. Moreover, we assume that the sets E_k are contained in geodesic balls $\mathbf{B}_{2Rr_k}$ for some $R>\frac12$. Under these requirements the assumption*

$$r_k^{1-n}\mathbf{D}(E_k)=\frac{\mathbf{P}(E_k)-\mathbf{P}(\mathbf{B}_{r_k})}{r_k^{n-1}}\to 0\qquad \textit{as } k\to\infty \tag{6.1}$$

implies that

$$r_k^{1-n}\boldsymbol{\beta}^2(E_k)\to 0\qquad\textit{as } k\to\infty.$$

Proof. We begin with the observation that

$$E_k\subset\mathbf{B}_{2Rr_k}=\{x\in B^n:|x|<\tanh(Rr_k)\}\subset B_{Rr_k}.$$

Then, we consider the rescaled sets $F_k:=\frac{1}{r_k}E_k\subset B_R$, which satisfy

$$P(F_k)=r_k^{1-n}P(E_k)\le(2r_k)^{1-n}\mathbf{P}(E_k).$$

Here $P(E)$ denotes the Euclidean perimeter of the set E. From (6.1), we thus infer

$$\limsup_{k\to\infty}P(F_k)\le\lim_{k\to\infty}(2r_k)^{1-n}\mathbf{P}(\mathbf{B}_{r_k})$$

$$= \lim_{k\to\infty} \frac{n\omega_n \sinh^{n-1}(r_k)}{(2r_k)^{n-1}} = \frac{n\omega_n}{2^{n-1}} = P(B_{\frac12}), \tag{6.2}$$

from which we deduce that $P(F_k)$ is bounded independently from $k \in \mathbb{N}$. After extracting a subsequence, we can thus assume $F_k \to F_\infty$ in $L^1(\mathbb{H}^n)$ for a set $F_\infty \subset B_R$. We claim that

$$F_\infty = B_{\frac12}(x_o) \qquad \text{for some } x_o \in B_{R-\frac12}. \tag{6.3}$$

For the proof of this claim, we observe first that

$$\begin{aligned} |F_k| &= \tfrac{1}{r_k^n}|E_k| = \tfrac{1}{(2r_k)^n}\mathbf{V}(E_k) + \tfrac{1}{r_k^n}|E_k| - \tfrac{1}{(2r_k)^n}\mathbf{V}(E_k) \\ &= \frac{n\omega_n}{(2r_k)^n}\int_0^{r_k} \sinh^{n-1} t\, dt + \mathbf{I}_k, \end{aligned}$$

with the obvious meaning of $\mathbf{I}_k$. For $\mathbf{I}_k$ we obtain, since $E_k \subset B_{Rr_k}$, that there holds

$$\begin{aligned} |\mathbf{I}_k| &= \tfrac{1}{r_k^n}\big||E_k| - 2^{-n}\mathbf{V}(E_k)\big| = \frac{1}{r_k^n}\bigg|\int_{E_k} 1 - \frac{1}{(1-|x|^2)^n}\, dx\bigg| \\ &\le \omega_n R^n \sup_{B_{Rr_k}} \bigg|1 - \frac{1}{(1-|x|^2)^n}\bigg| \xrightarrow[k\to\infty]{} 0. \end{aligned}$$

Because of the L^1-convergence $F_k \to F_\infty$, the last two formulae imply that there holds

$$|F_\infty| = \lim_{k\to\infty} \frac{n\omega_n}{(2r_k)^n}\int_0^{r_k} \sinh^{n-1} t\, dt = \frac{\omega_n}{2^n} = |B_{\frac12}|.$$

Next, we use the lower semicontinuity of the Euclidean perimeter with respect to L^1-convergence and the bound (6.2) for the estimate

$$P(F_\infty) \le \liminf_{k\to\infty} P(F_k) \le P(B_{\frac12}) \le P(F_\infty),$$

where here, the last estimate follows from the isoperimetric inequality in Euclidean space. But this means that we have equality in the isoperimetric inequality, which can only hold if $F_\infty = B_{\frac12}(x_o)$ for some $x_o \in \mathbb{R}^n$. Note that $R > \frac12$ is necessary, since $F_\infty \subset B_R$. Note also that $x_o \in B_{R-\frac12}$. This establishes the assertion (6.3). Next, we wish to show

$$\frac{\boldsymbol{\gamma}(E_k; r_k y)}{r_k^{n-1}} - 2^{n-1}\boldsymbol{\gamma}_{\mathbb{E}}(\tfrac{1}{r_k}E_k; y) \longrightarrow 0 \tag{6.4}$$

for every $y \in B_R$ as $k \to \infty$, where

$$\boldsymbol{\gamma}_{\mathbb{E}}(\tfrac{1}{r_k}E_k; y) = \int_{\frac{1}{r_k}E_k} \frac{n-1}{|x-y|}\, dx$$

is the function corresponding to $\boldsymbol{\gamma}$ from (2.23) in the Euclidean case, which was introduced in [16]. For the proof of (6.4), we observe that on B_{Rr_k}, the hyperbolic metric $g_{\alpha\beta}$ is arbitrarily close to $2\delta_{\alpha\beta}$ and therefore

$$\frac{\mathbf{d}(r_k y, r_k z)}{2|r_k y - r_k z|} \longrightarrow 1 \qquad \text{as } k \to \infty \tag{6.5}$$

uniformly in $y, z \in B_R$. Then, using the definition of $\boldsymbol{\gamma}$ from (2.23) we compute by a change of variables

$$\frac{\boldsymbol{\gamma}(E_k; r_k y)}{r_k^{n-1}} = r_k^{-n}\int_{E_k} \frac{(n-1)r_k}{\tanh \mathbf{d}(x, r_k y)}\Big(\frac{2}{1-|x|^2}\Big)^n\, dx$$

$$= \int_{\frac{1}{r_k}E_k} \frac{n-1}{2|y-z|} \frac{2|r_k y - r_k z|}{\tanh \mathbf{d}(r_k z, r_k y)} \Big(\frac{2}{1-|r_k z|^2}\Big)^n \, dz.$$

Because of (6.5), the integrand converges uniformly to $2^{n-1}\frac{n-1}{|y-z|}$, proving the claim (6.4). From the L^1-convergence $\frac{1}{r_k}E_k \to F_\infty$ and the absolute continuity of the integral, we deduce moreover

$$\lim_{k\to\infty} \boldsymbol{\gamma}_{\mathbb{E}}(\tfrac{1}{r_k}E_k; x_o) = \boldsymbol{\gamma}_{\mathbb{E}}(F_\infty; x_o) = \boldsymbol{\gamma}_{\mathbb{E}}(B_{\frac12}).$$

Combining this with (6.4), we obtain

$$\lim_{k\to\infty} r_k^{1-n}\boldsymbol{\gamma}(E_k; r_k x_o) = 2^{n-1}\boldsymbol{\gamma}_{\mathbb{E}}(F_\infty; x_o) = 2^{n-1}\boldsymbol{\gamma}_{\mathbb{E}}(B_{\frac12}).$$

Analogously, since also $\frac{1}{r_k}\mathbf{B}_{r_k} \to B_{\frac12}$ in L^1 (the convergence has to be understood in the sense that $r_k^{-1}\{|x| < \tanh(\frac12 r_k)\} \to B_{\frac12}$ in $L^1(\mathbb{R}^n)$) we infer

$$\lim_{k\to\infty} r_k^{1-n}\boldsymbol{\gamma}(\mathbf{B}_{r_k}) = \lim_{k\to\infty} r_k^{1-n}\boldsymbol{\gamma}(\mathbf{B}_{r_k}; 0) = 2^{n-1}\boldsymbol{\gamma}_{\mathbb{E}}(B_{\frac12}).$$

Joining the two preceding formulae with (6.1) and recalling the definition of $\boldsymbol{\beta}$, we arrive at the claim

$$\frac{\boldsymbol{\beta}^2(E_k)}{r_k^{n-1}} \le \frac{\boldsymbol{\beta}^2(E_k; r_k x_o)}{r_k^{n-1}} = \frac{\mathbf{P}(E_k) - \mathbf{P}(\mathbf{B}_{r_k})}{r_k^{n-1}} - \frac{\gamma(E_k; r_k x_o) - \gamma(\mathbf{B}_{r_k})}{r_k^{n-1}} \xrightarrow[k\to\infty]{} 0,$$

proving the assertion of the Lemma. □

6.2. **Reduction to sets with small isoperimetric gap.** First, we observe that we may assume that E is centered at the origin, i.e. $\boldsymbol{\beta}(E) = \boldsymbol{\beta}(E; 0)$; otherwise, we send the center to the origin by an isometry. This is possible, since all quantities in the quantitative isoperimetric inequality are invariant under isometries.

In the following, we show that it is enough to prove that there exists a constant $\chi_o = \chi_o(n, R_o) > 0$ such that whenever $E \subset \mathbb{H}^n$ satisfies $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$ for some $\varrho_o \in (0, R_0)$, $\boldsymbol{\beta}(E) = \boldsymbol{\beta}(E; 0)$ and $\mathbf{D}(E) \le \chi_o \mathbf{P}(\mathbf{B}_{\varrho_o})$ then the quantitative isoperimetric inequality

$$\mathbf{D}(E) \ge C_1 \boldsymbol{\beta}(E)^2 \tag{6.6}$$

holds true with a constant $C_1 = C_1(\chi_o)$. Indeed, for a set $E \subset \mathbb{H}^n$ of finite perimeter with $\mathbf{V}(E) = \mathbf{V}(\mathbf{B}_{\varrho_o})$, $\boldsymbol{\beta}(E) = \boldsymbol{\beta}(E; 0)$ and $\mathbf{D}(E) > \chi_o \mathbf{P}(B_{\varrho_o})$ we have

$$\boldsymbol{\beta}(E)^2 \le 2\mathbf{P}(E) = 2(\mathbf{D}(E) + \mathbf{P}(\mathbf{B}_{\varrho_o})) < 2\big(1 + \chi_o^{-1}\big)\mathbf{D}(E),$$

so that the quantitative isoperimetric inequality holds with $C_1 := [2(1 + \chi_o^{-1})]^{-1}$. Therefore, it remains to establish (6.6).

6.3. **The contradiction assumption.** We argue by contradiction, assuming that (6.6) fails to hold. Then, there exists a sequence $\varrho_k \in (0, R_o)$ and a sequence of sets with finite perimeter $E_k \subset \mathbb{H}^n$, with $\mathbf{V}(E_k) = \mathbf{V}(\mathbf{B}_{\varrho_k})$ and $\boldsymbol{\beta}(E_k) = \boldsymbol{\beta}(E_k; 0)$ satisfying

$$\mathbf{D}(E_k) < C_1 \boldsymbol{\beta}^2(E_k) \quad \text{and} \quad \frac{\mathbf{D}(E_k)}{\mathbf{P}(\mathbf{B}_{\varrho_k})} \to 0 \quad \text{as } k \to \infty. \tag{6.7}$$

6.4. **Reduction to bounded sets.** By $C = C(n, R_o) > 0$ and $R_1 = R_1(n, R_o) > 1$ we denote the constants from Lemma 3.3. The application of the Lemma yields that for k large, there exist sets $\widetilde{E}_k \subset \mathbf{B}_{R_1 \varrho_k} \cap E_k$, such that

$$\mathbf{V}(\widetilde{E}_k) \ge \mathbf{V}(\mathbf{B}_{\varrho_k}) \bigg[1 - \frac{C\,\mathbf{D}(E_k)}{\mathbf{P}(\mathbf{B}_{\varrho_k})}\bigg], \qquad \mathbf{P}(\widetilde{E}_k) \le \mathbf{P}(E_k), \tag{6.8}$$

$$\mathbf{D}(\widetilde{E}_k) \le (1 + C)\mathbf{D}(E_k), \tag{6.9}$$

and

$$\boldsymbol{\beta}^2(E_k) \le \boldsymbol{\beta}^2(\widetilde{E}_k) + (1 + C)\mathbf{D}(E_k). \tag{6.10}$$

Combining this with the contradiction assumption $(6.7)_1$, we infer

$$\mathbf{D}(E_k) \le C_1 \boldsymbol{\beta}^2(\widetilde{E}_k) + C_1(1 + C)\mathbf{D}(E_k).$$

Our choice of χ_o and thereby of $C_1 = C_1(\chi_o)$ in (6.19) below implies in particular that $C_1(1 + C) \le \frac{1}{2}$. Therefore, the last term in the preceding formula can be re-absorbed into the right-hand side, and we arrive at

$$\mathbf{D}(E_k) \le 2C_1 \boldsymbol{\beta}^2(\widetilde{E}_k). \tag{6.11}$$

Since $\varrho_k \in (0, R_o)$ we can extract a (not relabeled) subsequence satisfying $\varrho_k \to \varrho_o \in [0, R_o]$. From now on we have to distinguish between the cases where $\varrho_o > 0$ and $\varrho_o = 0$.

6.5. **The case $\boldsymbol{\varrho_o > 0}$.** We start with the simple observation that

$$\lim_{k\to\infty} \mathbf{V}(E_k \setminus \widetilde{E}_k) = 0 \tag{6.12}$$

holds true. Indeed, from $(6.8)_1$, $\varrho_k \le R_o$ and $(6.7)_2$ we obtain

$$\mathbf{V}(E_k \setminus \widetilde{E}_k) = \mathbf{V}(E_k) - \mathbf{V}(\widetilde{E}_k) \le C\,\mathbf{V}(\mathbf{B}_{R_o}) \frac{\mathbf{D}(E_k)}{\mathbf{P}(\mathbf{B}_{\varrho_k})} \to 0 \quad \text{as } k \to \infty.$$

6.5.1. *Convergence to a ball.* From $(6.8)_2$ and $(6.7)_2$ we conclude that $\mathbf{P}(\widetilde{E}_k)$ is uniformly bounded, and moreover that $\widetilde{E}_k \subset \mathbf{B}_{R_1 R_o}$ for any $k \ge 1$. Therefore by compactness we infer that up to a (not relabeled) subsequence we have $\widetilde{E}_k \to E_\infty$ in $L^1(\mathbb{H}^n)$ for some set $E_\infty \subset \overline{\mathbf{B}}_{R_1 \varrho_o}$. Therefore, on the one hand we have that

$$\mathbf{V}(E_\infty) = \lim_{k\to\infty} \mathbf{V}(\widetilde{E}_k) \le \lim_{k\to\infty} \mathbf{V}(E_k) = \lim_{k\to\infty} \mathbf{V}(\mathbf{B}_{\varrho_k}) = \mathbf{V}(\mathbf{B}_{\varrho_o})$$

and on the other hand, using $(6.8)_1$ and $(6.7)_2$:

$$\mathbf{V}(E_\infty) = \lim_{k\to\infty} \mathbf{V}(\widetilde{E}_k) \ge \lim_{k\to\infty} \mathbf{V}(\mathbf{B}_{\varrho_k}) \bigg[1 - \frac{C\,\mathbf{D}(E_k)}{\mathbf{P}(\mathbf{B}_{\varrho_k})}\bigg] = \mathbf{V}(\mathbf{B}_{\varrho_o}).$$

Together we conclude that $\mathbf{V}(E_\infty) = \mathbf{V}(\mathbf{B}_{\varrho_o})$. Now, by the lower semicontinuity of the perimeter with respect to L^1-convergence and by $(6.8)_2$ and $(6.7)_2$ we have

$$\mathbf{P}(E_\infty) \le \liminf_{k\to\infty} \mathbf{P}(\widetilde{E}_k) \le \liminf_{k\to\infty} \mathbf{P}(E_k) = \mathbf{P}(\mathbf{B}_{\varrho_o}).$$

On the other hand, by the optimal isoperimetric inequality from (2.8) we must have $\mathbf{P}(E_\infty) \ge \mathbf{P}(\mathbf{B}_{\varrho_o})$ such that in conclusion $\mathbf{P}(E_\infty) = \mathbf{P}(\mathbf{B}_{\varrho_o})$. But this means that in the isoperimetric inequality we must have equality, so that $E_\infty = \mathbf{B}_{\varrho_o}(x_o)$. Using (6.12) and $\widetilde{E}_k \to \mathbf{B}_{\varrho_o}(x_o)$ in L^1 we conclude that also the sets E_k converge to $\mathbf{B}_{\varrho_o}(x_o)$.

Lemma 5.2 implies $\boldsymbol{\gamma}(\widetilde{E}_k) \to \boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}(x_o))$. Therefore, taking into account $(6.8)_2$, $(6.7)_2$ and $\varrho_k \to \varrho_o > 0$, we find in the limit $k \to \infty$ that

$$\varepsilon_k^2 := \boldsymbol{\beta}^2(\widetilde{E}_k) = \mathbf{P}(\widetilde{E}_k) - \boldsymbol{\gamma}(\widetilde{E}_k) \le \mathbf{P}(E_k) - \boldsymbol{\gamma}(\widetilde{E}_k) \to \mathbf{P}(\mathbf{B}_{\varrho_o}) - \boldsymbol{\gamma}(\mathbf{B}_{\varrho_o}) = 0.$$

Joining this with (6.10) and the contradiction assumption $(6.7)_2$, we infer

$$\boldsymbol{\beta}^2(E_k) \le \boldsymbol{\beta}^2(\widetilde{E}_k) + (1+C)\mathbf{D}(E_k) \to 0$$

as $k \to \infty$.

6.5.2. *Penalization.* For $\Lambda > \Lambda_o(n)$, where $\Lambda_o(n)$ is the constant from Lemma 5.1, and $C_o \in (0, \frac{1}{8}]$ we define the penalized functionals

$$\mathbf{F}_k(G) := \mathbf{P}(G) + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k} \big|\mathbf{V}(G) - \mathbf{V}(\mathbf{B}_{\varrho_k})\big| + C_o\, |\boldsymbol{\beta}^2(G) - \varepsilon_k^2|.$$

From Lemma 5.3 we know that the functionals $\mathbf{F}_k$ are lower semicontinuous with respect to L^1-convergence of sets in $\mathbb{H}^n$ and therefore for any $k \in \mathbb{N}$ there exists a set of finite perimeter $G_k \subset \overline{\mathbf{B}}_{R_1 \varrho_k}$ minimizing the functional $\mathbf{F}_k$ amongst all sets of finite perimeter contained in $\overline{\mathbf{B}}_{R_1 \varrho_k}$. Note that the sets G_k could have their barycenter away from the origin. By the minimality of G_k – note that $\boldsymbol{\beta}(\mathbf{B}_{\varrho_k}) = 0$ – we have

$$\mathbf{P}(G_k) \le \mathbf{F}_k(G_k) \le \mathbf{F}_k(\mathbf{B}_{\varrho_k}) = \mathbf{P}(\mathbf{B}_{\varrho_k}) + C_o\, \varepsilon_k^2\,,$$

i.e. $\mathbf{P}(G_k)$ is uniformly bounded and moreover $G_k \subset \overline{\mathbf{B}}_{R_1 \varrho_k} \subset \mathbf{B}_{R_1 R_o}$. Therefore we may extract a (not relabeled) subsequence satisfying $G_k \to G_\infty$ in $L^1(\mathbb{H}^n)$ for some $G_\infty \subset \overline{\mathbf{B}}_{R_1 \varrho_o}$. Next, we define the functionals

$$\widetilde{\mathbf{F}}_k(G) := \mathbf{P}(G) + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k} \big|\mathbf{V}(G) - \mathbf{V}(\mathbf{B}_{\varrho_k})\big|.$$

By our choice of Λ, Lemma 5.1 is applicable to the functional $\widetilde{\mathbf{F}}_k$ and therefore we conclude that $\mathbf{B}_{\varrho_k}$ is a minimizer of $\widetilde{\mathbf{F}}_k$ in the class of sets of finite perimeter contained in $\overline{\mathbf{B}}_{R_1 \varrho_k}$. Hence, by the minimality of G_k, (6.8) and the contradiction assumption $(6.7)_1$ we obtain

$$\begin{aligned}
\widetilde{\mathbf{F}}_k(G_k) + C_o\, |\boldsymbol{\beta}^2(G_k) - \varepsilon_k^2| &= \mathbf{F}_k(G_k) \le \mathbf{F}_k(\widetilde{E}_k) \\
&= \mathbf{P}(\widetilde{E}_k) + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k} \big|\mathbf{V}(\widetilde{E}_k) - \mathbf{V}(\mathbf{B}_{\varrho_k})\big| \\
&\le \mathbf{P}(E_k) + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k} C \frac{\mathbf{V}(\mathbf{B}_{\varrho_k})}{\mathbf{P}(\mathbf{B}_{\varrho_k})} \mathbf{D}(E_k) \\
&\le \mathbf{P}(E_k) + \Lambda C \cosh R_o \frac{\mathbf{V}(\mathbf{B}_{\varrho_k})}{[\mathbf{P}(\mathbf{B}_{\varrho_k})]^{\frac{n}{n-1}}} \mathbf{D}(E_k) \\
&\le \mathbf{P}(\mathbf{B}_{\varrho_k}) + (1 + \Lambda C)\mathbf{D}(E_k),
\end{aligned}$$

where we used the isoperimetric inequality from Lemma 2.1 and the definition of the isoperimetric deficit in the last step. Next, we employ the estimate (6.11) and the minimality of $\mathbf{B}_{\varrho_k}$ with respect to $\widetilde{\mathbf{F}}_k$ in order to deduce

$$\begin{aligned}
\widetilde{\mathbf{F}}_k(G_k) + C_o\, |\boldsymbol{\beta}^2(G_k) - \varepsilon_k^2| &< \mathbf{P}(\mathbf{B}_{\varrho_k}) + 2C_1(1 + \Lambda C)\varepsilon_k^2 \\
&= \widetilde{\mathbf{F}}_k(\mathbf{B}_{\varrho_k}) + 2C_1(1 + \Lambda C)\varepsilon_k^2 \\
&\le \widetilde{\mathbf{F}}_k(G_k) + 2C_1(1 + \Lambda C)\varepsilon_k^2.
\end{aligned} \tag{6.13}$$

But this shows that

$$C_o|\boldsymbol{\beta}^2(G_k) - \varepsilon_k^2| \le 2C_1(1+\Lambda C)\,\varepsilon_k^2. \tag{6.14}$$

Therfore, by distinguishing between the cases $\varepsilon_k^2 \le \boldsymbol{\beta}^2(G_k)$ and $\varepsilon_k^2 > \boldsymbol{\beta}^2(G_k)$, we deduce

$$\varepsilon_k^2 \le \frac{C_o}{C_o - 2C_1(1+\Lambda C)}\,\boldsymbol{\beta}^2(G_k), \tag{6.15}$$

provided $C_o > 2C_1(1+\Lambda C)$. A few words concerning the dependencies of the constants are in order. Recalling the definition of the constant C_1, we have

$$2C_1(1+\Lambda C) = \frac{1+\Lambda C}{1+\frac{1}{\chi_o}}. \tag{6.16}$$

But this implies that χ_o can be chosen small enough to ensure the condition from before. Since the constant C from Lemma 3.3 depends on n and R_o, this leads to the dependencies $\chi_o = \chi_o(n, R_o, \Lambda, C_o)$.

Next, we observe that (6.14) certainly implies that $\boldsymbol{\beta}^2(G_k) \to 0$ as $k \to \infty$. Now, using the lower semicontinuity of the perimeter with respect to L^1-convergence and the minimizing property of G_k we find that

$$\begin{aligned}\widetilde{\mathbf{F}}_\infty(G_\infty) &:= \mathbf{P}(G_\infty) + \frac{\Lambda\cosh\varrho_o}{\sinh\varrho_o}\big|\mathbf{V}(G_\infty) - \mathbf{V}(\mathbf{B}_{\varrho_o})\big| \\ &\le \liminf_{k\to\infty}\mathbf{F}_k(G_k) \le \liminf_{k\to\infty}\mathbf{F}_k(\mathbf{B}_{\varrho_o}) = \mathbf{P}(\mathbf{B}_{\varrho_o}) = \widetilde{\mathbf{F}}_\infty(\mathbf{B}_{\varrho_o}).\end{aligned}$$

Since by Lemma 5.1 geodesic balls $\mathbf{B}_{\varrho_o}(x_o)$ are the only minimizers of $\widetilde{\mathbf{F}}_\infty$ we conclude that G_∞ is a ball $\mathbf{B}_{\varrho_o}(x_o) \subset \overline{\mathbf{B}}_{R_1\varrho_o}$.

6.5.3. *Almost ball property of G_k.* First, recall that G_k is $\mathbf{F}_k$-minimizing amongst sets contained in $\overline{\mathbf{B}}_{R_1\varrho_k}$, where the $\mathbf{F}_k$ are defined according to the choice $(\Lambda, \varrho_k, C_o)$. Recall that R_1 depends only on n and R_o. From Lemma 5.5 we conclude that there exists a radius $r_1 = r_1(\Lambda) \in (0, \frac{1}{8}]$ such that G_k is a $(3, r_1\min\{1, \varrho_k\})$-quasi minimizer of the perimeter for any $k \in \mathbb{N}$. Then, Theorem 5.6 ensures, after modifying the set G_k on a set of measure zero, that $\partial G_k = \partial^* G_k$.

Now, let $\delta_o = \delta_o(n, 2R_1R_o) > 0$ be the constant from Lemma 5.8, i.e. we take $r_o = 2R_1R_o$ so that $2\varrho_k \le 2R_o \le 2R_1R_o = r_o$ holds true. Due to the dependencies of R_1 upon n and R_o we have $\delta_o = \delta_o(n, R_o)$. In the following we prove that for any $\delta \in (0, \delta_o)$ there exists $k_o = k_o(\delta) \in \mathbb{N}$ so that the inclusion

$$\mathbf{B}_{\varrho_k(1-\delta)}(x_o) \subset G_k \subset \mathbf{B}_{\varrho_k(1+\delta)}(x_o) \tag{6.17}$$

holds true for any $k \ge k_o$. We argue by contradiction assuming that there exists $0 < \delta < \min\{\delta_o, r_1\}$ so that for infinitely many $k \in \mathbb{N}$ we can find $x_k \in \partial G_k$ with

$$x_k \notin \mathbf{B}_{\varrho_k(1+\delta)}(x_o) \setminus \mathbf{B}_{\varrho_k(1-\delta)}(x_o).$$

Then, setting $\hat\varrho_k := \min\{1, \varrho_k\}$, by Theorem 5.6 there exists a constant $C = C(n) > 1$ (note that $K \equiv 3$) and $y_k, z_k \in \mathbf{B}_{\delta\varrho_k/2}(x_k)$ such that

$$\mathbf{B}_{\delta\hat\varrho_k/(2C)}(y_k) \subset G_k \qquad\text{and}\qquad \mathbf{B}_{\delta\hat\varrho_k/(2C)}(z_k) \subset \mathbb{H}^n \setminus G_k.$$

If $x_k \in \mathbb{H}^n \setminus \mathbf{B}_{\varrho_k(1+\delta)}(x_o)$ we further have the inclusion $\mathbf{B}_{\delta\hat\varrho_k/(2C)}(y_k) \subset \mathbb{H}^n \setminus \mathbf{B}_{\varrho_k}(x_o)$ and hence $\mathbf{B}_{\delta\hat\varrho_k/(2C)}(y_k) \subset G_k \setminus \mathbf{B}_{\varrho_k}(x_o)$. On the other hand, if $x_k \in \mathbf{B}_{\varrho_k(1-\delta)}(x_o)$, then the inclusion $\mathbf{B}_{\delta\hat\varrho_k/(2C)}(z_k) \subset \mathbf{B}_{\varrho_k}(x_o)$ holds and hence $\mathbf{B}_{\delta\hat\varrho_k/(2C)}(z_k) \subset \mathbf{B}_{\varrho_k}(x_o) \setminus G_k$. Therefore we either have

$$\mathbf{V}(G_k \setminus \mathbf{B}_{\varrho_k}(x_o)) \ge \mathbf{V}(\mathbf{B}_{\delta\hat\varrho_k/(2C)}) \quad\text{or}\quad \mathbf{V}(\mathbf{B}_{\varrho_k}(x_o) \setminus G_k) \ge \mathbf{V}(\mathbf{B}_{\delta\hat\varrho_k/(2C)}) \tag{6.18}$$

for infinitely many $k \in \mathbb{N}$. But this contradicts the fact that $\mathbf{V}(G_k \Delta \mathbf{B}_{\varrho_k}(x_o)) \to 0$ and thus proves the claim.

6.5.4. *Regularity.* From (6.17) and Lemma 5.8 we conclude that there exists $r_2 = r_2(n, R_o) > 0$ and $K = K(n, \Lambda, C_o)$ such that G_k is a $\big(K(1+\frac{1}{\varrho_k}), r_2\varrho_k\big)$-almost minimizer on $\mathbb{H}^n$ of the perimeter for k large. This together with the fact that G_k is converging in $L^1(\mathbb{H}^n)$ to $\mathbf{B}_{\varrho_o}(x_o)$ allows us, first to translate their barycenter into x_o by a hyperbolic isometry, and then translate the barycenters into the origin. Subsequently, we apply Theorem 5.9 which yields for k large enough that the set G_k admits a spherical graph representation with respect to the ball $\mathbf{B}_{\varrho_o}$ of class $C^{1,\frac12}$, i.e. there exist functions $v_k \in C^{1,\frac12}(S^{n-1})$ such that

$$\partial G_k = \big\{\omega \tanh\big[\tfrac{\varrho_o}{2}(1+v_k(\omega))\big] : \omega \in S^{n-1}\big\}$$

and $\|v_k\|_{W^{1,\infty}(S^{n-1})} \to 0$ as $k \to \infty$. We now let $\varrho_k' > 0$ be such that $\mathbf{V}(\mathbf{B}_{\varrho_k'}) = \mathbf{V}(G_k)$. Since $G_k \to \mathbf{B}_{\varrho_o}$ we have that $\varrho_k' \to \varrho_o$. Letting $u_k = \frac{\varrho_o}{\varrho_k'}(1+v_k) - 1$ we therefore have $\|u_k\|_{W^{1,\infty}(B^n)} \to 0$ as $k \to \infty$ and moreover

$$\partial G_k = \big\{\omega \tanh\big[\tfrac{\varrho_k'}{2}(1+u_k(\omega))\big] : \omega \in S^{n-1}\big\}$$

so that G_k is also nearly spherical with respect to $\mathbf{B}_{\varrho_k'}$.

6.5.5. *The final contradiction.* Since $\varrho_k \to \varrho_o$ and $\varrho_k' \to \varrho_o$ we have that $\frac{\varrho_k'}{\varrho_k} \to 1$. Therefore, applying Lemma 2.3, and taking into account that $\frac{\cosh \varrho_k'}{\sinh \varrho_k'} \le 2\frac{\cosh \varrho_k}{\sinh \varrho_k}$ for k large, we obtain

$$\begin{aligned}\big|\mathbf{P}(\mathbf{B}_{\varrho_k}) - \mathbf{P}(\mathbf{B}_{\varrho_k'})\big| &\le \frac{c(n)\cosh \varrho_k}{\sinh \varrho_k}\big|\mathbf{V}(\mathbf{B}_{\varrho_k}) - \mathbf{V}(G_k)\big| \\ &\le \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k}\big|\mathbf{V}(\mathbf{B}_{\varrho_k}) - \mathbf{V}(G_k)\big|,\end{aligned}$$

provided we choose $\Lambda \ge c(n)$. Using this inequality together with the first estimate in (6.13) and (6.15) we find that

$$\begin{aligned}\mathbf{D}(G_k) &= \big[\mathbf{P}(G_k) - \mathbf{P}(\mathbf{B}_{\varrho_k})\big] + \big[\mathbf{P}(\mathbf{B}_{\varrho_k}) - \mathbf{P}(\mathbf{B}_{\varrho_k'})\big] \\ &\le \big[\mathbf{P}(G_k) - \mathbf{P}(\mathbf{B}_{\varrho_k})\big] + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k}\big|\mathbf{V}(\mathbf{B}_{\varrho_k}) - \mathbf{V}(G_k)\big| \\ &= \widetilde{\mathbf{F}}_k(G_k) - \mathbf{P}(\mathbf{B}_{\varrho_k}) \le 2C_1(1+\Lambda C)\,\varepsilon_k^2 \\ &\le \frac{2C_oC_1(1+\Lambda C)}{C_o - 2C_1(1+\Lambda C)}\,\boldsymbol{\beta}^2(G_k).\end{aligned}$$

But this yields the desired contradiction provided we choose C_o and χ_o in such a way that

$$\frac{2C_oC_1(1+\Lambda C)}{C_o - 2C_1(1+\Lambda C)} = \frac{C_o\frac{1+\Lambda C}{1+1/\chi_o}}{C_o - \frac{1+\Lambda C}{1+1/\chi_o}} < c_2,$$

where $c_2 = c_2(n, R_o)$ denotes the constant from Corollary 4.3. But this is the same as first fixing $C_o \in (0, \frac18]$ universally and then choosing χ_o small enough such that

$$\frac{1+\Lambda C}{1+1/\chi_o} < \tfrac12 \min\{c_2, C_o\}. \tag{6.19}$$

Since $C = C(n, R_o)$, $\Lambda = \Lambda(n)$ and $c_2 = c_2(n, R_o)$ also χ_o depends only on n and R_o, and consequently, $C_1 = C_1(\chi_o)$ has the same dependencies. This finishes the proof in the case $0 < \varrho_o \le R_o < \infty$.

6.6. **The case $\varrho_o = 0$.** The case $\varrho_o = 0$ is more difficult than the case $\varrho_o > 0$. However, since some parts of the proof are similar, we will only indicate the differences with respect to the case $\varrho_o > 0$.

6.6.1. *Changes with respect to the case $\varrho_o > 0$.* Since $\varrho_k \to 0$, we conclude from (6.7)$_2$ and (6.9) that $\varrho_k^{1-n}\mathbf{D}(\widetilde{E}_k) \to 0$. Denoting by $\widetilde{\varrho}_k$ the radius of the hyperbolic ball with the same volume as $\widetilde{E}_k$, i.e. $\mathbf{V}(\mathbf{B}_{\widetilde{\varrho}_k}) = \mathbf{V}(\widetilde{E}_k)$, we infer that

$$\lim_{k\to\infty} \frac{\widetilde{\varrho}_k}{\varrho_k} = 1, \tag{6.20}$$

so that also $\widetilde{\varrho}_k^{1-n}\mathbf{D}(\widetilde{E}_k) \to 0$. Note that the sets $\widetilde{E}_k$ are contained in the geodesic balls $\mathbf{B}_{2R_1\widetilde{\varrho}_k}$ for large k, with $R_1 > 1$. This allows us to apply Lemma 6.1 to the sequence of sets $\widetilde{E}_k$, yielding that $\widetilde{\varrho}_k^{1-n}\varepsilon_k^2 \equiv \widetilde{\varrho}_k^{1-n}\boldsymbol{\beta}^2(\widetilde{E}_k) \to 0$, which – apart from the fact that we consider here the re-scaled version – was the first conclusion of § 6.5.1. The second one, i.e. the assertion that the corresponding quantities $\varrho_k^{1-n}\boldsymbol{\beta}^2(E_k)$ converge to 0, is a consequence of (6.10), since

$$\varrho_k^{1-n}\boldsymbol{\beta}^2(E_k) \le \Big(\frac{\widetilde{\varrho}_k}{\varrho_k}\Big)^{n-1} \widetilde{\varrho}_k^{1-n}\boldsymbol{\beta}^2(\widetilde{E}_k) + (1+C)\varrho_k^{1-n}\mathbf{D}(E_k) \to 0.$$

Now, the beginning of § 6.5.2 works as before and we obtain estimates (6.13) and (6.15) with exactly the same arguments. Only the conclusion of this Section is now different. We let ϱ_k' be such that $\mathbf{V}(\mathbf{B}_{\varrho_k'}) = \mathbf{V}(G_k)$. In the following we will show that

$$\lim_{k\to\infty} \frac{\varrho_k'}{\varrho_k} = 1. \tag{6.21}$$

From (6.13), taking into account that $\mathbf{V}(G_k) = \mathbf{V}(\mathbf{B}_{\varrho_k'})$ we get

$$\mathbf{P}(G_k) - \mathbf{P}(\mathbf{B}_{\varrho_k}) + \frac{\Lambda \cosh \varrho_k}{\sinh \varrho_k}|\mathbf{V}(\mathbf{B}_{\varrho_k'}) - \mathbf{V}(\mathbf{B}_{\varrho_k})| < 2C_1(1+\Lambda C)\varepsilon_k^2.$$

In the case $\varrho_k' > \varrho_k$ we have

$$\frac{\mathbf{V}(\mathbf{B}_{\varrho_k'}) - \mathbf{V}(\mathbf{B}_{\varrho_k})}{\sinh^n \varrho_k} < \frac{2C_1(1+\Lambda C)\varepsilon_k^2}{\Lambda \sinh^{n-1}\varrho_k} \le C\,\widetilde{\varrho}_k^{1-n}\varepsilon_k^2 \to 0,$$

as $k \to \infty$, where we used the fact $\widetilde{\varrho}_k \le \varrho_k$ for the last estimate. This implies the upper bound $\limsup_{k\to\infty} \frac{\varrho_k'}{\varrho_k} \le 1$. In the remaining case $\varrho_k' < \varrho_k$ we apply Lemma 2.3 to infer from the second last inequality (recall that $\mathbf{P}(G_k) \ge \mathbf{P}(\mathbf{B}_{\varrho_k'})$) that

$$0 \le \frac{\mathbf{P}(\mathbf{B}_{\varrho_k}) - \mathbf{P}(\mathbf{B}_{\varrho_k'})}{\sinh^{n-1}\varrho_k} < \frac{2C_1(1+\Lambda C)}{\sinh^{n-1}\varrho_k}\varepsilon_k^2 \le C\,\widetilde{\varrho}_k^{1-n}\varepsilon_k^2 \to 0$$

as $k \to \infty$. But this implies the remaining lower bound $\liminf_{k\to\infty} \frac{\varrho_k'}{\varrho_k} \ge 1$ and thereby the claim (6.21).

Because of (6.21) and $\varrho_k \to 0$, we may assume that $\sinh \varrho_k' \le 2\varrho_k$ holds for every $k \in \mathbb{N}$. From Lemma 2.10 and (6.14), we therefore get the estimate

$$\boldsymbol{\alpha}(G_k)^2 \le \frac{\sinh^{n+1}\varrho_k'}{c(n)}\,\boldsymbol{\beta}(G_k)^2 \le \frac{2^{n+1}\varrho_k^{2n}}{c(n)}\,\varrho_k^{1-n}\boldsymbol{\beta}(G_k)^2$$

$$\le \frac{2^{n+1}\varrho_k^{2n}}{c(n)}\bigg(1+\frac{2C_1(1+\Lambda C)}{C_o}\bigg)\varrho_k^{1-n}\varepsilon_k^2 \le \frac{2^{n+2}\varrho_k^{1-n}\varepsilon_k^2}{c(n)}\,\varrho_k^{2n},$$

by our choice of χ_o in (6.19) and $2C_1 = [1+1/\chi_o]^{-1}$. We choose geodesic balls $\mathbf{B}_{\varrho_k'}(\mathfrak{x}_k)$ such that $\boldsymbol{\alpha}(G_k) = \mathbf{V}(G_k \Delta \mathbf{B}_{\varrho_k'}(\mathfrak{x}_k))$; note that G_k has the volume $\mathbf{V}(\mathbf{B}_{\varrho_k'})$. Taking (6.21) into account, we find that

$$\begin{aligned}(6.22)\qquad \lim_{k\to\infty}\frac{\mathbf{V}(G_k\Delta\mathbf{B}_{\varrho_k}(\mathfrak{x}_k))}{\varrho_k^n} &\le \lim_{k\to\infty}\frac{\boldsymbol{\alpha}(G_k)}{\varrho_k^n} + \lim_{k\to\infty}\frac{\mathbf{V}(\mathbf{B}_{\varrho_k'}\Delta\mathbf{B}_{\varrho_k})}{\varrho_k^n}\\ &\le c(n)\lim_{k\to\infty}\sqrt{\varrho_k^{1-n}\varepsilon_k^2} + \lim_{k\to\infty}\frac{|\mathbf{V}(\mathbf{B}_{\varrho_k'})-\mathbf{V}(\mathbf{B}_{\varrho_k})|}{\varrho_k^n} = 0.\end{aligned}$$

In § 6.5.3 we proceed almost as before. We only have to adjust the center x_o, i.e. for the set G_k we take the ball $\mathbf{B}_{\varrho_k}(\mathfrak{x}_k)$ instead of $\mathbf{B}_{\varrho_k}(x_o)$. In particular, we show similar to (6.17) that for any given $0<\delta<\min\{\delta_o, r_1\}$ there exists $k_o = k_o(\delta)\in\mathbb{N}$ such that the inclusion

$$(6.23)\qquad \mathbf{B}_{\varrho_k(1-\delta)}(\mathfrak{x}_k)\subset G_k\subset \mathbf{B}_{\varrho_k(1+\delta)}(\mathfrak{x}_k)$$

holds true for $k\ge k_o$. The proof follows § 6.5.3 until (6.18). Now, if the first of the two inequalities holds (the other case is similar), we get for some $0<\delta<\min\{\delta_o,r_1\}$ that

$$c(n)\Big(\frac{\delta}{2C}\Big)^n \le \frac{\mathbf{V}\big(G_k\setminus \mathbf{B}_{\varrho_k}(\mathfrak{x}_k)\big)}{\varrho_k^n},$$

and therefore the contradiction follows with (6.22).

Finally, as in § 6.5.4 an application of Lemma 5.8 implies that there exist $r_2 = r_2(n) > 0$ (note that the dependence on R_o does not appear, since we can assume that $\varrho_k\le 1$ for all k) and $K = K(n,\Lambda,C_o)$ such that the sets G_k are $\big(K(1+\frac{1}{\varrho_k}), r_2\varrho_k\big)$-almost minimizers of the perimeter in $\mathbb{H}^n$. Next, we observe that we can find a sequence $\delta_k\downarrow 0$ so that (6.23) holds with δ replaced by δ_k, for any $k\in\mathbb{N}$. Therefore, we can apply Lemma 2.6 to the sequence G_k and the barycenters p_k of G_k. This lemma implies that for any given $0<\delta<\min\{\delta_o,r_1\}$, there exists $k_1 = k_1(\delta)$ with $\mathbf{d}(p_k,\mathfrak{x}_k)\le\delta\varrho_k$ for $k\ge k_1$. Therefore, by applying a hyperbolic isometry, we can achieve that the sets G_k have their barycenters in the origin and satisfy

$$(6.24)\qquad \mathbf{B}_{\varrho_k(1-2\delta)}\subset G_k\subset\mathbf{B}_{\varrho_k(1+2\delta)}$$

for every $k\ge\max\{k_0,k_1\}$. This is equivalent to the conclusion (6.17) of § 6.5.3 with $x_o = 0$.

6.6.2. *The final conclusion in the case* $\varrho_o = 0$. We now define rescaled sets $H_k := \frac{1}{\varrho_k}G_k$. In view of (6.23) we have that the rescaled sets H_k are converging in Hausdorff distance to the Euclidean ball $B_{\frac12}$. Moreover, for $k\gg 1$ the sets H_k are Euclidean $(2^{n+1}K, \frac12 r_2)$-almost minimizers of the functional

$$H\mapsto \int_{\partial^* H}\bigg(\frac{2}{1-\varrho_k^2|x|^2}\bigg)^{n-1} d\mathcal{H}^{n-1}.$$

Applying the analogous result to [24] to the above functionals, i.e. the Euclidean version of Theorem 5.9 for a sequence of almost minimizers of the above functionals (note that the integrands converge uniformly in C^∞ to the Euclidean area functional), we conclude that the sets H_k converge in $C^{1,\alpha}$ to $B_{\frac12}$ for any $0<\alpha<\frac12$. This means that there exist functions w_k converging to zero in $C^{1,\alpha}(S^{n-1})$, such that ∂H_k is a radial graph in the sense that

$$\partial H_k = \left\{\omega\left(\tfrac{1}{2} + w_k(\omega)\right) : \omega \in S^{n-1}\right\}.$$

Setting now $v_k := \frac{2}{\varrho_k} \operatorname{Artanh}\left[\varrho_k\left(\frac{1}{2} + w_k\right)\right] - 1$, also the functions v_k converge to 0 in $C^{1,\alpha}(S^{n-1})$ and we have

$$\partial G_k = \left\{\omega \tanh\left[\tfrac{\varrho_k}{2}(1 + v_k(\omega))\right] : \omega \in S^{n-1}\right\}.$$

Finally, setting $u_k = \frac{\varrho_k}{\varrho_k'}(1 + v_k) - 1 \to 0$, we see that ∂G_k can be written in the form

$$\partial G_k = \left\{\omega \tanh\left[\tfrac{\varrho_k'}{2}(1 + u_k(\omega))\right] : \omega \in S^{n-1}\right\}.$$

Having arrived at this stage the contradiction can be obtained exactly as in § 6.5.5.

6.7. **Appendix: Elementary facts in hyperbolic space.** Here we give the proofs of certain facts we used in the course of the slicing lemma. We use the notation from the proof of Lemma 3.3 without any further explanation.

Lemma 6.2. *The function*

$$(0, \infty) \ni y \mapsto \frac{y\varphi'(y)}{\varphi(y)}$$

is strictly increasing.

Proof. Because of the strict monotonicity of $\mathbf{v}(r)$ it is equivalent to show that

$$h(r) := \frac{\mathbf{v}(r)\varphi'(\mathbf{v}(r))}{\varphi(\mathbf{v}(r))}$$

is increasing in $r > 0$. Using the identities

$$\varphi(\mathbf{v}(r)) = \mathbf{p}(r) = n\omega_n \sinh^{n-1} r \qquad \text{and} \qquad \varphi'(\mathbf{v}(r)) = (n-1)\coth r$$

(cf. (3.17)), we compute that $h'(r) > 0$ is equivalent to

$$\mathbf{v}(r) \sinh^{n-1} r (n \cosh^2 r - \sinh^2 r) < \mathbf{v}'(r) \cosh r \sinh^n r,$$

which, taking into account that $\mathbf{v}'(r) = \mathbf{p}(r)$, is equivalent to (2.10). □

Lemma 6.3. *The expression*

$$\frac{\mathbf{p}(r)}{\psi_r(s)}$$

is increasing in $r > 0$ for every $s \in (0, 1)$.

Proof. We begin by calculating

$$\frac{\psi_r(s)}{\mathbf{p}(r)} = \frac{\varphi(s\mathbf{v}(r))}{\varphi(\mathbf{v}(r))} + \frac{\varphi((1-s)\mathbf{v}(r))}{\varphi(\mathbf{v}(r))} - 1.$$

It therefore suffices to prove that the function

$$f(y) := \frac{\varphi(sy)}{\varphi(y)}, \qquad y > 0,$$

is decreasing for every $s \in [0, 1]$. A straightforward calculation yields that $f'(y) < 0$ is equivalent to

$$\frac{sy\varphi'(sy)}{\varphi(sy)} < \frac{y\varphi'(y)}{\varphi(y)}. \tag{6.25}$$

But Lemma 6.2 implies that the right-hand side is increasing in y. Since $s \in (0, 1)$ and $y > 0$, we infer the asserted estimate (6.25) and thereby complete the proof of the lemma. □

Lemma 6.4. *For any $r > 0$ we have*

$$\int_0^1 \frac{1}{\psi_r(s)}\,ds < \infty.$$

Proof. We first note that it is enough to prove the integrability of $1/\psi_r$ in a neighborhood of the singular points 0 and 1. Therefore, we consider $s \in [0, \frac{1}{2}]$. By the mean value theorem there exist $\xi_1 \in [0, s]$ and $\xi_2 \in [1-s, 1]$ such that there holds:

$$\begin{aligned}
\psi_r(s) &= \varphi(s\mathbf{v}(r)) + \varphi((1-s)\mathbf{v}(r)) - \varphi(\mathbf{v}(r)) \\
&= s\left[\frac{d}{ds}\Big|_{s=\xi_1}\varphi(s\mathbf{v}(r)) - \frac{d}{ds}\Big|_{s=\xi_2}\varphi(s\mathbf{v}(r))\right] \\
&= s\mathbf{v}(r)\left[\frac{n-1}{\tanh(\mathbf{v}^{-1}(\xi_1\mathbf{v}(r)))} - \frac{n-1}{\tanh(\mathbf{v}^{-1}(\xi_2\mathbf{v}(r)))}\right] \\
&\geq s\mathbf{v}(r)\left[\frac{n-1}{\tanh(\mathbf{v}^{-1}(s\mathbf{v}(r)))} - \frac{n-1}{\tanh(\mathbf{v}^{-1}((1-s)\mathbf{v}(r)))}\right] \\
&= \frac{(n-1)s\mathbf{v}(r)}{\tanh(\mathbf{v}^{-1}(s\mathbf{v}(r)))}\left[1 - \frac{\tanh(\mathbf{v}^{-1}(s\mathbf{v}(r)))}{\tanh(\mathbf{v}^{-1}((1-s)\mathbf{v}(r)))}\right].
\end{aligned}$$

We now choose $s_o \in [0, \frac{1}{2}]$ in dependence of r small enough to have

$$\tanh(\mathbf{v}^{-1}(s_o\mathbf{v}(r))) \leq \tfrac{1}{2}\tanh(\mathbf{v}^{-1}((1-s_o)\mathbf{v}(r))).$$

Then, for $s \in (0, s_o]$ we find that

$$\begin{aligned}
\psi_r(s) &\geq \frac{(n-1)s\mathbf{v}(r)}{\tanh(\mathbf{v}^{-1}(s\mathbf{v}(r)))}\left[1 - \frac{\tanh(\mathbf{v}^{-1}(s_o\mathbf{v}(r)))}{\tanh(\mathbf{v}^{-1}((1-s_o)\mathbf{v}(r)))}\right] \\
&\geq \frac{(n-1)s\mathbf{v}(r)}{2\tanh(\mathbf{v}^{-1}(s\mathbf{v}(r)))} \geq \frac{(n-1)s\mathbf{v}(r)}{2\sinh(\mathbf{v}^{-1}(s\mathbf{v}(r)))}.
\end{aligned} \tag{6.26}$$

Next, we note that for $0 \leq \sigma \leq 1$ we have $\cosh\sigma \leq 2$ and therefore we have

$$\sinh^{n-1}\sigma\cosh\sigma \leq 2\sinh^{n-1}\sigma.$$

Integrating both sides with respect to σ over $(0, t)$, we obtain for $0 \leq t \leq 1$ that

$$\sinh^n t \leq 2n\int_0^t \sinh^{n-1}\sigma\,d\sigma = \tfrac{2}{\omega_n}\mathbf{v}(t).$$

Assuming that $\mathbf{v}^{-1}(s\mathbf{v}(r)) \leq 1$, an assumption which can be imposed after possibly reducing the value of s_o, we can use the preceding estimate in (6.26) to infer that for all $s \in (0, s_o]$

$$\frac{1}{\psi_r(s)} \leq \frac{2^{\frac{n+1}{n}}\mathbf{v}(\mathbf{v}^{-1}(s\mathbf{v}(r)))^{\frac{1}{n}}}{(n-1)\omega_n^{\frac{1}{n}}\,s\mathbf{v}(r)} = \frac{2^{\frac{n+1}{n}}}{(n-1)\omega_n^{\frac{1}{n}}\,(s\mathbf{v}(r))^{\frac{n-1}{n}}} = c(n,r)\,s^{-\frac{n-1}{n}}.$$

But this ensures that the integral $\int_0^{s_o} 1/\psi_r\,ds$ is finite and by symmetry we also have that $\int_{1-s_o}^1 1/\psi_r\,ds = \int_0^{s_o} 1/\psi_r\,ds < \infty$. This finishes the proof of the lemma. □

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