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REPORT No. 42, 2013/2014, fall
ISSN 1103-467X
ISRN IML-R- -42-13/14- -SE+fall

A TIME DEPENDENT VARIATIONAL APPROACH TO IMAGE RESTORATION

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ABSTRACT. In this paper we introduce a purely variational approach to the gradient flows, naturally arising in image denoising models, yielding the existence of global parabolic minimizers, in the sense that

$$\int_0^T \left[\int_\Omega u\partial_t \varphi \, dx + \boldsymbol{F}(u) \right] dt \le \int_0^T \boldsymbol{F}(u+\varphi)\, dt,$$

whenever $T > 0$ and $\varphi \in C_0^\infty(\Omega \times (0,T))$. Our method applies to a wide class of non-parametric regression models in image restoration analysis, such as quantile, robust and logistic regression. A prototype functional $\boldsymbol{F}$ is the by now classical $\mathrm{TV}(L^2)$-functional (i.e. the pure TV-denoising case in image reconstruction) introduced by Rudin, Osher and Fatemi [32]:

$$\boldsymbol{F}(u) := \mathbf{TV}(u) + \tfrac{\kappa}{2} \int_\Omega |u - u_o|^2 \, dx,$$

where $u_o \colon \Omega \to [0,1]$ is a noisy, monochromatic image and $\kappa \gg 1$ a large penalization parameter. The evolutionary variational solutions are obtained as limits of maps, minimizing a convex variational functional in $n+1$ dimensions with domain $\Omega_T := \Omega \times (0,T)$. Our approach yields a new way of proving the existence of global weak solutions to the associated Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = \kappa(u - u_o) & \text{in } \Omega \times (0,\infty). \\ u = u_o & \text{on } \Omega \times \{0\} \cup \partial\Omega \times (0,\infty). \end{cases}$$

Our approach also applies in situations where the considered functionals do not allow the derivation of the associated parabolic equation. We are able to deal with Dirichlet and Neumann type boundary conditions on the lateral boundary, and furthermore with the gradient flow associated to functionals modeling image deblurring, such as

$$\boldsymbol{F}(u) = \mathbf{TV}(u) + \tfrac{\kappa}{2} \int_\Omega \big|\mathbf{K}[u] - u_o\big|^2 \, dx,$$

where $\mathbf{K} \colon L^1(\Omega) \to L^2(\Omega)$ is a bounded, linear, injective operator satisfying the DC-condition $\mathbf{K}[1] = 1$.

1. INTRODUCTION

In this paper we are concerned with the existence of solutions for the total variation flow in image processing, in the sense that we are aiming to construct solutions which inherit a certain minimizing property. The advantage of such *parabolic minimizers* or *variational solutions* stems from the fact that they might inherit better regularity properties due to their minimization property. Using this viewpoint we are able to give quite simple existence and uniqueness proofs for gradient flows associated to functionals in the pure denoising case of image reconstruction as well as functionals modeling deblurring. The method also applies to functionals without smoothness in the regression term.

1.1. **General Rudin-Osher-Fatemi models.** To recover from a noisy, monochromatic image, given by a function $u_o \colon \Omega \to [0,1]$ (assigning to each point $x \in \Omega$ of a bounded domain $\Omega \subset \mathbb{R}^n$ its gray-scale $u_o(x)$) the real image, Rudin & Osher & Fatemi suggested

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Date: November 12, 2014.
2000 *Mathematics Subject Classification.* 94A08, 68U10, 35K51, 35A01, 35A15.
Key words and phrases. total variation, image restoration, gradient flow, general regression functionals.

in [32] to take as the *solution of the image restoration problem* the (unique) unconstrained minimizer of the variational integral

$$\boldsymbol{F}(u) := \mathbf{TV}(u) + \tfrac{\kappa}{\zeta}\int_\Omega \big|u-u_o\big|^\zeta\,dx\,, \tag{1.1}$$

for some, typically large $\kappa \ge 1$ and some exponent $\zeta \ge 1$. Here, $\mathbf{TV}(u)$ stands for the total variation of u. The original model was formulated with $\zeta = 2$. In this by now classical model – usually called (ROF)- or $\mathrm{TV}(L^2)$-model – minimizers exist and are unique. Later on, Chan & Esedoḡlu [19] studied the model with $\zeta = 1$, in which minimizers might be non-unique. This approach is nowadays known in image restoration as $\mathrm{TV}(L^\zeta)$ denoising model. The regularization term in the functional (1.1) is given by the total variation $\mathbf{TV}(u)$ of the gradient measure Du, while the lower order term – called *regression term* – is given by the deviation of u from the noisy image u_o measured with respect to the L^ζ-norm. The basic idea behind these models is the mild regularizing effect of the total variation in regions without jumps in the grayscale, which makes the reconstruction less noisy than the original image u_o, while contour distortions in the image are preserved or even sharpened. The regression term

$$\boldsymbol{S}(u) := \tfrac{\kappa}{\zeta}\int_\Omega |u(x)-u_o(x)|^\zeta\,dx$$

works as a penalization and forces the reconstruction u to stay close to the original image. Since, by the maximum principle minimizers to (1.1) take their values in $[0,1]$, it is natural to consider the unconstrained minimization problem; see [17, Lemma 6.126]. It should be noted that no boundary conditions are imposed here, so that minimizers formally solve the following Neumann type boundary problem for the 1-Laplacian:

$$\begin{cases} \operatorname{div}\big(\frac{Du}{|Du|}\big) = \kappa|u-u_o|^{\zeta-2}(u-u_o) & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Later on more general models, such as anisotropic Rudin-Osher-Fatemi models [24] have been studied. Moreover, several regression models from statistics, such as quantile, robust and logistic regression, were incorporated into the context of image restoration. All models can be summarized under the roof of the following regression model (see [31]):

$$\boldsymbol{S}(u) := \int_\Omega S(x,u(x))\,dx, \tag{1.2}$$

where $S\colon \Omega\times\mathbb{R}\to[0,\infty)$ is a Carathéodory-integrand such that for almost every $x\in\Omega$ the partial map $\mathbb{R}\ni u\mapsto S(x,u)$ is convex. The associated regression model could be termed *generalized (ROF)-model*:

$$\boldsymbol{F}(u) := \mathbf{TV}(u) + \boldsymbol{S}(u) \;= \mathbf{TV}(u) + \int_\Omega S(x,u(x))\,dx\,. \tag{1.3}$$

Note, that with $S(x,u) := \frac{\kappa}{\zeta}|u-u_o(x)|^\zeta$ the pure $\mathrm{TV}(L^\zeta)$-denoising model is included as a special case. For more information on image processing and analysis we refer to the monographs [20, 33].

1.2. **Gradient flow to the general Rudin-Osher-Fatemi model.** Our motivation for this paper arises from recent literature for the total variation flow. Here we are interested in flows associated to models using the total variation as leading term and with a general convex lower order penalty term as defined in (1.3). More precisely, our first main concern is the study of the Cauchy-Dirichlet problem associated to general (ROF)-models as in (1.3), i.e. the initial boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div}\big(\frac{Du}{|Du|}\big) = \frac{\partial S}{\partial u}(\cdot,u) & \text{in } \Omega_T,\\ u = u_o & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{1.4}$$

for a given noisy monochromatic picture $u_o\colon \Omega \to [0,1]$. Here, Ω_T, $0 < T \le \infty$, denotes the space time cylinder $\Omega \times (0,T)$ and $\partial_{\mathrm{par}}\Omega_T$ stands for its parabolic boundary $\overline{\Omega} \times \{0\} \cup \partial\Omega \times (0,T)$. The precise formulation of the problem can be given in spaces of functions of bounded variation, i.e. for functions $u \in L^1_w(0,T;\mathrm{BV}(\Omega))$. For the definition we refer to §2.1. However, boundary values of BV-functions are a delicate issue, since the trace operator is not continuous with respect to the weak* convergence in $\mathrm{BV}(\Omega)$). One possibility to overcome this difficulty is to consider a slightly larger domain Ω^* compactly containing the bounded open set Ω, and to assume that the initial datum u_o is defined on Ω^*. The boundary condition $u = u_o$ on the lateral boundary $\partial\Omega \times (0,T)$ could then be re-formulated by requiring that $u(t) = u_o$ a.e. on $\Omega^* \setminus \Omega$ for all $t \in (0,T)$. To consider a larger reference domain is natural, since in general the total variation of minimizers will charge the boundary $\partial\Omega$ of Ω. The annular region $\Omega^* \setminus \Omega$ could be viewed as a picture frame around the original noisy picture u_o, and $u_o|_{\Omega^*\setminus\Omega}$ as the (constant in time) grayscale of the frame. The precise notion of boundary values is now as follows: Given $u_o \in \mathrm{BV}(\Omega^*)$ a function u belongs to $\mathrm{BV}_{u_o}(\Omega)$ if and only if $u \in \mathrm{BV}(\Omega^*)$ and $u = u_o$ a.e. on $\Omega^* \setminus \Omega$. With this respect the condition on the lateral boundary has to be understood in the sense that $u(t) \in \mathrm{BV}_{u_o}(\Omega^*)$ for a.e. $t \in (0,T)$. Instead of using a larger reference set Ω^*, one could use an integral representation formula for the total variation $\mathbf{TV}(u)$ containing a boundary penalty term. Such a formula is well known for Lipschitz domains Ω, which posses the extension property for the space $\mathrm{BV}(\Omega)$. Both approaches lead to a notion of weak solution to the Cauchy-Dirichlet problem. In this paper we prefer to use the definition of boundary values by introducing the larger reference domain, since the annulus type region has the nice interpretation of a picture frame. The precise set up and the definition of the boundary values are given in §2.1.

For the evolutionary problem associated to the general (ROF)-functionals from (1.3), we assume that the initial datum $u_o \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*)$ fulfills the requirement that its $\boldsymbol{S}$-energy is finite, i.e. that

$$\boldsymbol{S}(u_o) := \int_{\Omega^*} S(x, u_o(x))\,dx < \infty \tag{1.5}$$

holds true, which implies that the $\boldsymbol{F}$-energy of the acquired image u_o is finite, i.e.

$$\boldsymbol{F}(u_o) = \mathbf{TV}(u_o) + \int_{\Omega^*} \boldsymbol{S}(x, u_o(x))\,dx < \infty. \tag{1.6}$$

Here, of course we assume that u_o is defined on Ω^*, and S on $\Omega^* \times \mathbb{R}$. Moreover, $\mathbf{TV}(u_o)$ is the total variation of u_o on Ω^*.

At this stage we have all ingredients at hand to introduce the concept of *variational solutions to the Cauchy-Dirichlet problem* for the gradient flow associated to general (ROF)-functionals as in (1.3). For the definition of the involved function spaces we refer to §2.1.

Definition 1.1 (Variational Solutions for the Cauchy-Dirichlet problem)**.** Assume that the Cauchy-Dirichlet datum u_o fulfills (1.5) and the lower order Borel integrand is as in (1.3). We identify a map $u\colon \Omega^*_T \to \mathbb{R}$, $T \in (0,\infty)$ in the class

$$u \in L^1_w\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big) \cap C^0\big([0,T];L^2(\Omega^*)\big)$$

as a *variational solution on* Ω_T to the Cauchy-Dirichlet problem for the *total variation flow associated to the general regression model* $\boldsymbol{S}$ if and only if the variational inequality

$$\begin{aligned}\int_0^T \boldsymbol{F}(u(t))\,dt \le \int_0^T &\left[\int_{\Omega^*} \partial_t v(v-u)\,dx + \boldsymbol{F}(v(t))\right] dt\\ &- \tfrac12\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega^*)}\end{aligned} \tag{1.7}$$

holds true for any $v \in L^1_w(0,T;\mathrm{BV}_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$. A map $u\colon \Omega^*_\infty \to \mathbb{R}$ is termed *global variational solution* (or variational solution on Ω_∞) if

$$u \in L^1_w\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big) \cap C^0\big([0,T];L^2(\Omega^*)\big), \quad \text{for any } T > 0$$

and u is a variational solution on Ω_T for any $T \in (0,\infty)$. □

Here and throughout the rest of the paper we use the short hand notation $u(t) := u(\cdot,t)$. In the previous definition the time independent extension $v(x,t) := u_o(x)$ for $(x,t) \in \Omega^* \times (0,\infty)$ is an admissible comparison map in the variational inequality (1.7). This is guaranteed by (1.6). Therefore, we have that $\int_0^T \boldsymbol{F}(u(t))\,dt < \infty$ for any variational solution u. We should mention that the previous definition is strongly inspired by ideas of Lichnewsky & Temam [30], a paper dealing with variational solutions to the evolutionary minimal surface equation. Our main result concerning flows associated to general (ROF)-functionals is the following:

Theorem 1.2. *Suppose that $S(x,u)$ is a Carathédory integrand as in (1.2) and that the Cauchy-Dirichlet datum u_o fulfills the requirements of (1.5). Then, there exists a unique global variational solution u in the sense of Definition 1.1.* □

We emphasize that variational solutions are unique even if $\boldsymbol{F}$ is convex, but not strictly convex. Moreover, we have the following regularity properties of variational solutions.

Theorem 1.3. *Suppose that $S(x,u)$ is a Carathédory integrand as in (1.2) and that the Cauchy-Dirichlet datum u_o fulfills the requirements of (1.5). Then, any variational solution in the sense of Definition 1.1 on Ω_T with $T \in (0,\infty]$ satisfies*

$$\partial_t u \in L^2\big(\Omega_T^*\big) \quad \text{and} \quad u \in C^{0,\frac12}\big([0,\tau];L^2(\Omega^*)\big) \quad \forall\,\tau \in \mathbb{R}\cap(0,T].$$

Further, for the time derivative $\partial_t u$ there holds the quantitative bound

$$\int_0^T\int_{\Omega^*} |\partial_t u|^2\,dxdt \le \boldsymbol{F}(u_o).$$

Moreover, for any $t_1,t_2 \in \mathbb{R}$ with $0 \le t_1 < t_2 \le T$ the energy estimate

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} \boldsymbol{F}(u(t))\,dt \le \boldsymbol{F}(u_o) \tag{1.8}$$

holds true. □

As an easy consequence of (1.8) we obtain that variational solutions u belong to the space $L^\infty\big(0,T;\mathrm{BV}(\Omega^*)\big)$, in the sense that $\sup_{0\le t\le T}\mathbf{TV}(u(t)) < \infty$ holds true. Moreover, the proof of the Theorem 1.3 actually shows, that whenever $0 \le t_1 < t_2 \le T$ are such that $u(t_1), u(t_2) \in \mathrm{BV}(\Omega^*)$ then

$$\int_{t_1}^{t_2}\int_{\Omega^*} |\partial_t u|^2\,dxdt \le \boldsymbol{F}(u(t_2)) - \boldsymbol{F}(u(t_1)).$$

1.3. **TV-Deblurring.** Throughout this Section we assume that $\mathbf{K}\colon L^1(\Omega) \to L^2(\Omega)$ is bounded, injective, satisfying the DC-condition $\mathbf{K}[1] = 1$. In the time independent variational setting the corresponding energy functionals are of the type

$$\boldsymbol{F}(u) := \mathbf{TV}(u) + \tfrac{\kappa}{2}\int_\Omega \big|\mathbf{K}[u]-u_o\big|^2\,dx\,, \tag{1.9}$$

for $u_o \in L^2(\Omega)$. Again, κ is a large penalization factor. Functionals as in (1.9) are often called Tikhonov-functionals with total variation penalty. The spirit of our presentation has its origin in [1, 18], where the time independent variational problem was treated. We also refer to the monographs [20, §5.4.2] and [17, §6.3.3] for more details. As special cases these models contain linear blur operators as $\mathbf{K}[u] = k * u$, where k is an integral kernel depending on the model (cf. [17, Beispiel 6.127]). The injectivity of the blur is only needed to guarantee uniqueness of minimizers for the functional $\boldsymbol{F}$; see [20, §5.4.2].

Our goal here is to treat the associated evolutionary problem. We restrict our considerations to the model case with energy functionals as in (1.9), to keep the presentation simple, but emphasize that more general functionals could be treated. We assume that the initial datum satisfies

$$u_o \in L^2(\Omega) \cap \mathrm{BV}(\Omega)\,, \tag{1.10}$$

which certainly implies that its $\boldsymbol{F}$-energy is finite. As in the last section we could deal with a Dirichlet-boundary condition, by introducing a slightly larger reference set Ω^*. We will not do this here. Instead we introduce a concept of variational solutions allowing a Neumann-type boundary condition on the lateral boundary.

Definition 1.4 (Variational solutions, Neumann type boundary condition)**.** Let $\mathbf{K}$ be a linear blur as above and assume that the initial datum u_o satisfies (1.10). We identify a map $u\colon \Omega_T \to \mathbb{R}$, $T \in (0,\infty)$ in the class

$$u \in L^1_w\big(0,T;\mathrm{BV}(\Omega)\big) \cap C^0\big([0,T];L^2(\Omega)\big),$$

as a *variational solution on* Ω_T associated to the Cauchy-problem for the *total variation flow with deblurring operator* $\mathbf{K}$ *and Neumann boundary condition* if and only if the variational inequality

$$\begin{aligned}\int_0^T \boldsymbol{F}(u(t))\,dt \le \int_0^T \bigg[\int_\Omega \partial_t v(v-u)\,dx + \boldsymbol{F}(v(t))\bigg]\,dt \\ - \tfrac12\|(v-u)(T)\|^2_{L^2(\Omega)} + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega)}\end{aligned} \tag{1.11}$$

holds true for any $v \in L^1_w(0,T;\mathrm{BV}(\Omega))$ with $\partial_t v \in L^2(\Omega_T)$. Further, a map $u\colon \Omega_\infty \to \mathbb{R}$ is a *global variational solution* (or variational solution on Ω_∞) if

$$u \in L^1_w\big(0,T;\mathrm{BV}(\Omega)\big) \cap C^0\big([0,T];L^2(\Omega)\big), \quad \text{for any } T>0$$

and u is a variational solution on Ω_T for any $T \in (0,\infty)$. □

Also here the time independent extension $v(x,t) := u_o(x)$ for $(x,t) \in \Omega\times(0,\infty)$ is an admissible comparison map in the variational inequality (1.11), so that also in this model variational solutions have finite energy, i.e. that $\int_0^T \boldsymbol{F}(u(t))\,dt < \infty$ holds. This is implied by (1.10). We have the following existence result for variational solutions to the Cauchy-problem with a Neumann type boundary condition associated to $\mathbf{TV}$-deblurring functionals.

Theorem 1.5. *Suppose that* $\mathbf{K}$ *is a linear blur as in* (1.9) *and* u_o *satisfies* (1.10)*. Then, there exists a unique global variational solution* u *in the sense of Definition* (1.4)*.* □

Moreover, we have the following regularity properties of variational solutions.

Theorem 1.6. *Suppose that* $\mathbf{K}$ *is a linear blur as in* (1.9) *and* u_o *satisfies* (1.10)*. Then, any global variational solution* u *in the sense of Definition* (1.4) *on* Ω_∞ *satisfies*

$$\partial_t u \in L^2\big(\Omega_\infty\big) \quad \textit{and} \quad u \in C^{0,\frac12}\big([0,T];L^2(\Omega)\big) \quad \forall T>0.$$

Further, the time derivative $\partial_t u$ *satisfies the quantitative bound*

$$\int_0^\infty\int_\Omega |\partial_t u|^2\,dxdt \le \boldsymbol{F}(u_o)$$

and for any $0 \le t_1 < t_2 < \infty$ *the energy estimate*

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} \boldsymbol{F}(u(t))\,dt \le \boldsymbol{F}(u_o)$$

holds true. □

1.4. **Remarks on the history of the problem.** The first result concerning evolutionary problems for linear growth functionals goes back to [30]. In this paper the concept of *variational solutions* (or pseudo-solutions) was introduced for the evolutionary minimal surface equation. More general flows for functionals of linear growth have been considered in [27]. Solutions are constructed as limits of solutions to more regular problems. In case of the total variation flow one would consider as an approximation of $(1.4)_1$ (with $S \equiv 0$) the non-degenerate parabolic equation

$$\partial_t u - \varepsilon \Delta u - \operatorname{div}\left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}}\right) = 0$$

with $\varepsilon \in (0, 1]$. Subsequently, in a series of papers [3, 4, 5, 7, 8, 13] (see also the monograph [6]) different notions of weak solutions to the total variation flow have been introduced and used, to establish existence and uniqueness of global weak solutions on $\Omega \times (0, \infty)$ to the initial value problem with Dirichlet, respectively Neumann boundary conditions on the lateral boundary. The different notions have been introduced to treat more and more irregular initial data u_o, starting with $L^2(\Omega)$-data and the notion of *strong solutions*, passing over to $L^1(\Omega)$-data and the concept of *entropy solutions*, and finally to Radon measures of the form $\mu = h + \mu_s$, defined on the whole space $\mathbb{R}^n$ with an absolutely continuous part $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a singular part $\mu_s = \alpha \mathcal{H}^k \llcorner S$, $\alpha \geq 0$ and S a k-dimensional submanifold of $\mathbb{R}^n$ with $0 \leq k < n$, via approximation by L^1-data. Furthermore, evolutionary problems of the type

$$\partial_t u - \operatorname{div} \mathbf{a}(x, Du) = 0 \tag{1.12}$$

associated to vector-fields depending on (x, Du), i.e. of the type $\mathbf{a}(x, \xi) := D_\xi f(x, \xi)$, have been considered in [9]. Typical examples of such integrands f are given by the non-parametric area integrand $f(x, \xi) := \sqrt{1 + |\xi|^2}$ or more general Lagrangians of the type $f(x, \xi) = \sqrt{1 + x^2 + a_{ij}(x)\xi_i \xi_j}$. Finally, evolution equations with vector-fields depending on (u, Du), more precisely with coefficients of the form $\mathbf{a}(u, \xi) := D_\xi f(u, \xi)$, have been investigated in [10, 11, 12]. A prototype of these type of equations is given by the *relativistic heat equation*

$$\partial_t u - \nu \operatorname{div}\left(\frac{|u| Du}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}}\right) = 0.$$

The existence proofs in the before mentioned papers are based on the nonlinear semigroup theory, in particular they rely on techniques of completely accretive operators and the Crandall–Liggett's semigroup generation theorem. Finally, a different existence proof for the time dependent minimal surface equation was established in [36]. Finally, we refer to the monograph [2] for the treatment of gradient flows in abstract metric measure spaces.

The purpose of the present paper is twofold. On the one hand we follow the idea of Lichnevsky & Temam to regard solutions as variational solutions in the sense of Definition 1.1, respectively 1.4. On the other hand, we present a purely variational approach, which has its origins in a conjecture of De Giorgi [21] (see also [22]), concerning the existence of global weak solutions to the Cauchy problem for non-linear hyperbolic wave equations. To be more precise, De Giorgi suggested to construct such solutions as limits of minimizers of certain convex variational integrals on $\mathbb{R}^n \times (0, \infty)$. In [34] Serra & Tilli solved this conjecture for the Cauchy problem to wave equations of the type $\partial_t^2 u = \Delta u - |u|^{q-2} u$ on $\mathbb{R}^n \times (0, \infty)$ for arbitrary $q \geq 2$ (see also [37]).

In the present paper we use a similar approach to treat evolutionary problems from image restoration related to general Rudin-Osher-Fatemi functionals as defined in (1.5), respectively TV-deblurring functionals as in (1.9). Note that the *Classical Calculus of Variations* ensures the existence of minimizers to the Dirichlet respectively Neumann problem associated to such variational functionals $\boldsymbol{F}$. In the latter case one has to assume a mild regularity for the boundary $\partial\Omega$, yielding a Poincaré type inequality. Remarkably enough such a regularity assumption is not needed for the corresponding evolutionary problem; see Theorems 1.5 and 1.6.

In a recent paper [15] such a variational approach has been developed for evolutionary problems associated to variational integrands $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to [0,\infty]$ (here $N \in \mathbb{N}$ can be larger than 1, in which case the problem is vectorial) and for the corresponding variational functionals

$$\boldsymbol{F}(u) := \int_\Omega f(x,u,Du)\,dx.$$

The integrand f is assumed to be convex with respect to (u, Du) (for fixed x) and coercive, in the sense that a growth condition from below $f(x,u,Du) \ge \nu|Du|^p$ for some $p > 1$ holds true. The associated parabolic system formally is of the type

$$\partial_t u - \operatorname{div} D_\xi f(x,u,Du) = -D_u f(x,u,Du).$$

In general, due to the weak assumptions on f, this system is however no more meaningful. Nevertheless, variational solutions can be defined in the spirit of Definition 1.1. Prototype examples of functionals falling into this class, are functionals of non-standard p,q-growth (without any restriction for the gap $q-p$), or functionals with exponential growth. We refer to the recent papers [14, 16] concerning existence and regularity results of parabolic systems with non-standard growth in the diffusion part.

As already mentioned, the aim of the present paper is to develop a related theory for evolutionary problems for linear growth functionals. We restrict ourselves to splitting type functionals with leading term given by the total variation and a lower order perturbation, i.e. to those functionals arising naturally in image restoration problems.

1.5. **Methods of proof.** A few words concerning our methods are in order. For a time independent datum $u_o\colon \Omega \to \mathbb{R}$ and a time $T > 0$ we consider mappings $u\colon \Omega_T \to \mathbb{R}$ satisfying the initial condition $u(0) = u_o$ and either the Dirichlet boundary condition $u(t) = u_o$ on $\partial\Omega$, or we impose no boundary condition on $\partial\Omega$, in order to model the Neumann type condition on the lateral boundary. Then, for given $\varepsilon \in (0,1]$, we consider the following (strictly) convex variational integral

$$\mathcal{F}_\varepsilon(v) := \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \bigg[\tfrac12 \int_\Omega |\partial_t v|^2\,dx + \tfrac{1}{\varepsilon}\boldsymbol{F}(v(t)) \bigg]\,dt,$$

where $\boldsymbol{F}$ is defined in (1.5), respectively in (1.9). Classical methods form the Calculus of Variations for functionals with linear growth combined with a classical compactness result (for sequences of functions satisfying uniform bounds for the energy and the time derivative, see [35, Theorem 1]), essentially ensure the existence of unique minimizers u_ε. Formally, these minimizers solve in case of the functionals defined in (1.6) the elliptic equation

$$-\varepsilon\partial_{tt}u_\varepsilon + \partial_t u_\varepsilon - \operatorname{div}\big(\tfrac{Du_\varepsilon}{|Du_\varepsilon|}\big) = -\tfrac{\partial S}{\partial u}(x,u_\varepsilon),$$

and therefore it is natural to expect that $\mathcal{F}_\varepsilon$-minimizers u_ε converge in the limit $\varepsilon \downarrow 0$ to a solution u of the associated evolutionary problem (1.4). Clearly, this argument is purely heuristic. First of all, the derivation of the Euler-Lagrange equation to the functionals $\mathcal{F}_\varepsilon$ is in general not possible due to the weak assumption on S; secondly, one has to interpret the diffusion part in terms of the BV-theory (cf. [6] for this non-trivial task), and thirdly after all these efforts, it is not clear at all how to pass to the limit $\varepsilon \downarrow 0$. The main idea to overcome all these difficulties is to argue on the level of the functionals and minimizers. Here one could expect, that lower-semicontinuity arguments preserve the minimizing property of the approximating $\mathcal{F}_\varepsilon$-minimizers u_ε in the limit, and therefore the limit map u could inherit itself a certain minimization property. This is a subtle point, since we are dealing with a sequence of strictly convex variational functionals $\mathcal{F}_\varepsilon$ containing a weight, which concentrates at $t = 0$, and the corresponding sequence of (unique) minimizers u_ε. The link between the strictly convex variational functionals $\mathcal{F}_\varepsilon$ and the evolutionary character of the limit problem, is the concept of *variational solutions* from Definition 1.1, respectively 1.4. This notion of solution goes back to Lichnewsky & Temam [30] and it is the key tool that allows to stay on the level of minimizers, instead of passing to the Euler-Lagrange equation.

To conclude sub-convergence $u_\varepsilon \to u$ in a weak sense, one needs uniform a priori bounds for the energy – in this case the total variation – and the time derivative of the sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$. These estimates (compare (5.6) and (5.10)) are obtained in §5.3 by a direct comparison argument using a time mollification procedure (cf. §2.2). The regularization is necessary to obtain comparison maps sufficiently regular with respect to time. On the level of the corresponding Euler-Lagrange equation, these a priori estimates could be obtained by testing the equation formally with the time derivative $\partial_t u_\varepsilon$. As explained before, this is not possible, and therefore we use direct energy comparison arguments for $\mathcal{F}_\varepsilon$-minimizers. The derived a priori estimates together with a classical compactness result allow us to conclude strong sub-convergence in $L^1(\Omega_T)$, and this is sufficient to prove that the limit map is a variational solution.

The uniqueness of variational solutions follows in case of the general (ROF)-functionals form a comparison principle, whose proof is given in Lemma 4.2. The validity of such a comparison principle is a bit surprising, since we are dealing not necessarily with a strictly convex variational integral $\boldsymbol{F}$. However, as already observed in [27], the presence of the term involving the time derivative in the variational inequality can be used to gain the comparison principle. In case of the TV-deblurring functionals, the uniqueness follows from the strict convexity of the variational functionals, which is implied by the injectivity of the linear blur $\mathbf{K}$.

1.6. **Possible Extensions.** It is worth to note that the developed methods also allow the treatment of more general functionals of the type

$$\boldsymbol{F}(u) := \int_\Omega f(x, Du) + \int_\Omega S\big(x, \mathbf{K}[u](x)\big)\, dx,$$

for a Carathéodory function $f\colon \Omega\times\mathbb{R}^n \to [0,\infty)$ satisfying a linear growth condition with respect to the gradient variable Du, a lower order perturbation S as in (1.2), and some linear blur operator $\mathbf{K}\colon L^1(\Omega)\to L^1(\Omega)$, and under essentially the same hypotheses vectorial problems. In this paper we restrict ourselves to the model case of the pure total variation integrand $f(\xi) := |\xi|$, to keep the presentation simple.

Acknowledgments. Part of this work has been written at the Institute Mittag-Leffler (Djursholm, Sweden), that the three authors were visiting in Autumn 2013 during the program *Evolutionary Problems*. We would like to thank the staff at the Institute for their kind hospitality. Moreover, we would like to thank Juha Kinnunen for stimulating discussions on the subject. V. Bögelein has been supported by the DFG-Project BO3598/1-1 “Evolutionsgleichungen mit p, q-Wachstum” and P. Marcellini is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probalità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

2. Notations and Preliminaries

2.1. **Notations.** For $p\in[1,\infty]$ and an open set $\Omega\subset\mathbb{R}^n$, the spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Lebesgue, respectively Sobolev spaces. Moreover, by Ω_T, with $T\in(0,\infty)$ we denote the space-time cylinder $\Omega\times(0,T)$; when $T=\infty$ we write Ω_∞ for $\Omega\times(0,\infty)$. By $\mathrm{BV}(\Omega)$ we denote the space of functions $u\in L^1(\Omega)$ with finite *total variation*

$$\mathbf{TV}(u) := \sup\bigg\{\int_\Omega u\,\mathrm{div}\,\zeta\,dx : \zeta\in C^1_0(\Omega,\mathbb{R}^n),\ \|\zeta\|_{L^\infty(\Omega)}\le 1\bigg\} < \infty. \tag{2.1}$$

The norm in $\mathrm{BV}(\Omega)$ is defined by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \mathbf{TV}(u).$$

We refer to [25] as standard reference for functions of bounded variation. The concept of variational solutions makes use of the space

$$L^1_w(0,T;\mathrm{BV}(\Omega)).$$

This space consists of those maps $v\colon [0,T] \to \mathrm{BV}(\Omega)$ such that $v \in L^1(\Omega \times (0,T))$, and such that for any $\phi \in C^1_0(\Omega,\mathbb{R}^n)$ the maps $[0,T] \ni t \to \langle Dv(t), \phi\rangle$ are measurable (see (2.2)), and moreover that

$$\int_0^T \mathbf{TV}(v(t))\,dt < \infty$$

holds true. Moreover, $Dv(t) = \big(D_1 v(t), \dots, D_n v(t)\big)$ denotes the vector-valued Radon-measure, representing the distributional gradient of the BV-function $v(t)$, in the sense that there holds

$$\langle Dv(t), \phi\rangle = -\int_\Omega v(t)\,\mathrm{div}\,\phi\,dx \qquad \forall\,\phi \in C^1_0(\Omega,\mathbb{R}^n). \tag{2.2}$$

Note that the map $[0,T] \ni t \mapsto \mathbf{TV}(v(t))$ is measurable in view of (2.1). The following Lemma ensures that the limit of a sequence of functions in $L^1_w(0,T;\mathrm{BV}(\Omega))$ is itself an $L^1_w(0,T;\mathrm{BV}(\Omega))$-function. The precise statement is as follows:

Lemma 2.1. *Suppose that the sequence $u_j \in L^1_w(0,T;\mathrm{BV}(\Omega))$, $j \in \mathbb{N}$, satisfies*

$$\sup_{j\in\mathbb{N}} \int_0^T \mathbf{TV}(u_j(t))\,dt < \infty$$

and $u_j \to u$ in $L^1(\Omega_T)$. Then, $u \in L^1_w(0,T;\mathrm{BV}(\Omega))$.

Proof. By Fubini's theorem we have for a (not relabeled) subsequence and for a.e. $t \in (0,T)$ that $u_j(t) \to u(t)$ in $L^1(\Omega)$. Therefore we have for any $\phi \in C^1_0(\Omega,\mathbb{R}^n)$ that

$$\int_\Omega u(t)\,\mathrm{div}\,\phi\,dx = \lim_{j\to\infty} \int_\Omega u_j(t)\,\mathrm{div}\,\phi\,dx = -\lim_{j\to\infty} \langle Du_j(t), \phi\rangle\,, \tag{2.3}$$

i.e. for any $\phi \in C^1_0(\Omega,\mathbb{R}^n)$ the function

$$[0,T] \ni t \mapsto \int_\Omega u(t)\,\mathrm{div}\,\phi\,dx$$

is the pointwise limit of measurable functions, and therefore itself measurable. Next we use the lower semicontinuity of the total variation with respect to L^1-convergence on the time slices and Fatou's lemma to conclude that

$$\int_0^T \mathbf{TV}(u(t))\,dt \le \int_{t_1}^{t_2} \liminf_{j\to\infty} \mathbf{TV}(u_j(t))\,dt \le \liminf_{j\to\infty} \int_0^T \mathbf{TV}(u_j(t))\,dt < \infty.$$

This implies for a.e. $t \in (0,T)$ that $u(t) \in \mathrm{BV}(\Omega)$. In particular, the distributional gradient $Du(t)$ is represented by a vector-valued Radon measure. This allows us to re-write (2.3) in the form

$$\langle Du(t), \phi\rangle = \lim_{j\to\infty} \langle Du_j(t), \phi\rangle \qquad \forall\,\phi \in C^1_0(\Omega,\mathbb{R}^n)$$

establishing the measurability of $t \mapsto \langle Du(t), \phi\rangle$. Together with the before established bound $\int_0^T \mathbf{TV}(u(t))\,dt < \infty$ this proves the claim $u \in L^1_w(0,T;\mathrm{BV}(\Omega))$. ☐

It is well known that boundary values for $\mathrm{BV}(\Omega)$-functions are a delicate issue, since the trace operator is not anymore continuous with respect to the weak* convergence in $\mathrm{BV}(\Omega)$. For instance, a sequence of characteristic functions of finite perimeter sets converging to the characteristic function χ_Ω demonstrates the occurring difficulties. One possibility to overcome these difficulties is to consider a slightly larger domain Ω^* containing Ω on which the boundary values can be extended, and then to formulate the boundary condition in terms of the extension u_o by requiring that $u = u_o$ on $\Omega^* \setminus \Omega$. To consider a larger reference domain is natural, since in general the total variation of minimizers will charge the boundary $\partial\Omega$ of Ω. The precise set up in the case of Dirichlet boundary values is as follows: Let Ω and Ω^* be two bounded open subsets of $\mathbb{R}^n$ such that $\Omega \Subset \Omega^*$ and let $u_o \in \mathrm{BV}(\Omega^*)$ be given. Then we define the space $\mathrm{BV}_{u_o}(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = u_o$ almost everywhere in $\Omega^* \setminus \Omega$.

In the second part of the paper, i.e. when we deal with **TV**-deblurring models, we impose a Neumann boundary condition on the lateral boundary $\partial\Omega \times (0, T)$.

2.2. **Mollification in time.** Due to the lack of regularity with respect to time, variational solutions in the sense of Definition 1.1 respectively 1.4 are in general not admissible as comparison maps in (1.7) respectively (1.11). To overcome this problem one uses a mollification procedure with respect to time. Indeed, with the help of this mollification, we are able to show that the time derivative of a variational solution exists and belongs to L^2; see Theorems 1.3 and 1.6. The precise construction of the mollification is as follows. Let X be a Banach space and $v_o \in X$; in the application we will for instance have $X = L^r(\Omega)$ for $r \geq 1$ and the related parabolic space $L^r(0, T; L^r(\Omega))$. Later on, we need the property that for maps $v \in L^1_w(0, T; \mathrm{BV}(\Omega))$ the mollification with respect to time (note that in this case $v \in L^1(0, T; L^1(\Omega)) = L^1(\Omega_T)$) also belongs to $L^1_w(0, T; \mathrm{BV}(\Omega))$; see Lemma 2.6.

Now, we consider some $v \in L^r(0, T; X)$ for some $1 \leq r \leq \infty$, and define for $h \in (0, T]$ and $t \in [0, T]$ *the mollification in time* by

$$[v]_h(t) := e^{-\frac{t}{h}} v_o + \tfrac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(s)\, ds. \tag{2.4}$$

One of the basic features of this mollification is, that $[v]_h$ (formally) solves the ordinary differential equation

$$\partial_t [v]_h = -\tfrac{1}{h}\big([v]_h - v\big) \tag{2.5}$$

with initial condition $[v]_h(0) = v_o$. The basic properties of the mollification $[\,\cdot\,]_h$ are provided in the following Lemma, cf. [29, Lemma 2.2], or [15, Appendix B] for the proofs of the particular statements.

Lemma 2.2. *Suppose X is a separable Banach space and $v_o \in X$. If $v \in L^r(0, T; X)$ for some $r \geq 1$, then the mollification $[v]_h$ defined in (2.4) fulfills $[v]_h \in L^r(0, T; X)$ and for any $t_o \in (0, T]$ there holds*

$$\big\|[v]_h\big\|_{L^r(0,t_o;X)} \leq \|v\|_{L^r(0,t_o;X)} + \Big[\tfrac{h}{r}\Big(1 - e^{-\frac{t_o r}{h}}\Big)\Big]^{\frac{1}{r}} \|v_o\|_X .$$

In the case $r = \infty$ the bracket $[\dots]^{\frac{1}{r}}$ in the preceding inequality has to be interpreted as 1. Moreover, $[v]_h \to v$ in $L^r(0, T; X)$ as $h \downarrow 0$. Finally, if $v \in C^0([0, T]; X)$, then $[v]_h \in C^0([0, T]; X)$, $[v]_h(0) = v_o$, and moreover $[v]_h \to v$ in $C^0([0, T]; X)$ as $h \downarrow 0$. □

For maps $v \in L^r(0, T; X)$ with $\partial_t v \in L^r(0, T; X)$ we have the following assertion.

Lemma 2.3. *Let X be a separable Banach space and $r \geq 1$. Assume that $v \in L^r(0, T; X)$ with $\partial_t v \in L^r(0, T; X)$. Then, for the mollification in time defined by*

$$[v]_h(t) := e^{-\frac{t}{h}} v(0) + \tfrac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(s)\, ds \tag{2.6}$$

the time derivative can be computed by

$$\partial_t [v]_h(t) = \tfrac{1}{h} \int_0^t e^{\frac{s-t}{h}} \partial_s v(s)\, ds,$$

and, moreover we have that

$$\big\|\partial_t [v]_h\big\|_{L^r(0,T;X)} \leq \|v\|_{L^r(0,T;X)}$$

holds true. □

For the following lemma we refer to [15, Lemma 2.3]; note that the statement is formulated in a slightly different way, but the proof remains essentially the same. It ensures the convergence $S(\cdot,[v]_h) \to S(\cdot,v)$ in the limit $h \downarrow 0$. The precise statement is as follows.

Lemma 2.4. *Let $T>0$. Assume that $S\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathédory integrand as in (1.2) and that $v\in L^1(\Omega_T)$ with $S(\cdot,v)\in L^1(\Omega_T)$ and $v_o\in L^1(\Omega)$ with $S(\cdot,v_o)\in L^1(\Omega)$. Then, $S(\cdot,[v]_h)\in L^1(\Omega_T)$, and moreover*

$$\lim_{h\downarrow 0}\int_0^T\int_\Omega S(x,[v]_h(x,t))\,dxdt=\int_0^T\int_\Omega S(x,v(x,t))\,dxdt. \tag{2.7}$$

To treat $\mathbf{TV}$-deblurring functionals we need a similar result for the lower order term in (1.9). The precise statement is

Lemma 2.5. *Let $T>0$ and $\mathbf{K}$ be as in (1.9). Furthermore, let $v_o\in L^1(\Omega)$, $v\in L^1(\Omega_T)$ and $u_o\in L^2(\Omega)$, and define $[v]_h$ according to (2.4). Then, $|\mathbf{K}[[v]_h]-u_o|^2\in L^1(\Omega_T)$, and*

$$\big|\mathbf{K}[[v]_h(t)]-u_o\big|^2\le\big[|\mathbf{K}[v]-u_o|^2\big]_h(t),$$

where $\big[|\mathbf{K}[v]-u_o|^2\big]_h(t)$ is defined according to (2.4) with $|\mathbf{K}[v_o]-u_o|^2$ instead of v_o. Moreover, we have

$$\lim_{h\downarrow 0}\int_0^T\int_\Omega\big|\mathbf{K}\big[[v]_h(t)\big]-u_o\big|^2\,dxdt=\int_0^T\int_\Omega\big|\mathbf{K}[v(t)]-u_o\big|^2\,dxdt. \tag{2.8}$$

Proof. Note, that the assumptions guarantee that $|\mathbf{K}[v]-u_o|^2\in L^1(\Omega_T)$. We start with the following identity:

$$\mathbf{K}\big[[v]_h(t)\big]=\mathbf{K}\bigg[e^{-\frac{t}{h}}v_o+\tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}v(s)\,ds\bigg]=e^{-\frac{t}{h}}\mathbf{K}[v_o]+\tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}\mathbf{K}[v(s)]\,ds,$$

which is a consequence of the fact that $\mathbf{K}$ is bounded and linear. Now, by convexity we have

$$\begin{aligned}\big|\mathbf{K}\big[[v]_h(t)\big]-u_o\big|^2&=\bigg|e^{-\frac{t}{h}}\big(\mathbf{K}[v_o]-u_o\big)+\frac{1-e^{-\frac{t}{h}}}{h(1-e^{-\frac{t}{h}})}\int_0^t e^{\frac{s-t}{h}}\big(\mathbf{K}[v(s)]-u_o\big)\,ds\bigg|^2\\&\le e^{-\frac{t}{h}}\big|\mathbf{K}[v_o]-u_o\big|^2+\tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}\big|\mathbf{K}[v(s)]-u_o\big|^2\,ds.\end{aligned}$$

Here we used the fact that

$$\frac{1}{h(1-e^{-\frac{t}{h}})}\int_0^t e^{\frac{s-t}{h}}\,ds=1. \tag{2.9}$$

The preceding inequality implies on the one hand the uniform bound

$$\begin{aligned}&\int_0^T\int_\Omega\big|\mathbf{K}\big[[v]_h(t)\big]-u_o\big|^2\,dxdt\\&\qquad\le h\int_\Omega\big|\mathbf{K}[v_o]-u_o\big|^2\,dx+\int_0^T\int_\Omega\big|\mathbf{K}[v(t)]-u_o\big|^2\,dxdt,\end{aligned}$$

and on the other hand, by a variant of the dominated convergence theorem (note that the first term on the right-hand side converges to 0 in the limit $h\downarrow 0$), the claim (2.8). □

In the next Lemma we establish for functions $v\in L^1_w(0,T;\mathrm{BV}(\Omega))$ a result concerning the total variation in the spirit of Lemma 2.4, respectively 2.5.

Lemma 2.6. *Let $T>0$, $v_o\in \mathrm{BV}(\Omega)$ and $v\in L^1_w(0,T;\mathrm{BV}(\Omega))$. Then, $[v]_h$ as defined in* (2.4) *satisfies that $[v]_h\in L^1_w(0,T;\mathrm{BV}(\Omega))$. Moreover, $\mathbf{TV}\big([v]_h(t)\big)\le [\mathbf{TV}(v)]_h(t)$ for any $t\in(0,T)$ and*

$$\lim_{h\downarrow 0}\int_0^T \mathbf{TV}\big([v]_h(t)\big)\,dt=\int_0^T \big[\mathbf{TV}(v)\big]_h(t)\,dt. \tag{2.10}$$

Proof. By Lemma 2.2 (applied with $X=L^1(\Omega)$ and $r=1$) we conclude that $[v]_h\to v$ in $L^1(0,T,L^1(\Omega))$, so that $[v]_h(t)\to v(t)$ in $L^1(\Omega)$ for a.e. $t\in(0,T)$. We now consider a testing function $\phi\in C^1_0(\Omega,\mathbb{R}^n)$. Then, using Fubini's theorem, the definition of the mollification with respect to time and the definition of the distributional gradient of BV-functions we obtain for any $t\in[0,T|$ that

$$\begin{aligned}\int_\Omega [v]_h(t)\operatorname{div}\phi\,dx&=e^{-\frac{t}{h}}\int_\Omega v_o\operatorname{div}\phi\,dx+\tfrac1h\int_0^t e^{\frac{s-t}{h}}\int_\Omega v(s)\operatorname{div}\phi\,dx\,ds\\&=-\bigg[e^{-\frac{t}{h}}\,\langle Dv_o,\phi\rangle+\tfrac1h\int_0^t e^{\frac{s-t}{h}}\,\langle Dv(s),\phi\rangle\,ds\bigg]\end{aligned}$$

holds true. This implies in particular that $[0,T]\ni t\mapsto \int_\Omega [v]_h(t)\operatorname{div}\phi\,dx$ is measurable. Moreover, taking the supremum over all $\phi\in C^1_0(\Omega,\mathbb{R}^n)$ with $|\phi|\le 1$, we conclude the following bound for the total variation of $[v]_h(t)$, i.e. we have that

$$\mathbf{TV}\big([v]_h(t)\big)\le e^{-\frac{t}{h}}\mathbf{TV}(v_o)+\tfrac1h\int_0^t e^{\frac{s-t}{h}}\mathbf{TV}(v(s))\,ds=\big[\mathbf{TV}(v)\big]_h(t)$$

holds true, where $[\mathbf{TV}(v)]_h(t)$ is defined according to (2.4) with v_o replaced by $\mathbf{TV}(v_o)$. Therefore we can perform an integration by parts in the left-hand side (in order to express the integral by the distributional gradient represented by the Radon-measures $D_i[v]_h(t)$) and obtain that

$$\langle D[v]_h(t),\phi\rangle=e^{-\frac{t}{h}}\,\langle Dv_o,\phi\rangle+\tfrac1h\int_0^t e^{\frac{s-t}{h}}\,\langle Dv(s),\phi\rangle\,ds.$$

But this implies that for any $\phi\in C^1_0(\Omega,\mathbb{R}^n)$ the real-valued function $[0,T]\ni t\mapsto \langle D[v]_h(t),\phi\rangle$ is measurable with respect to t. Since $t\mapsto \mathbf{TV}(v(t))\in L^1(0,T)$ (note that this holds true by the hypothesis $v\in L^1_w(0,T;\mathrm{Bv}(\Omega))$) we obtain, using Lemma 2.2, the bound

$$\int_0^T \mathbf{TV}\big([v]_h(t)\big)\,dt\le h\mathbf{TV}(v_o)+\int_0^T \mathbf{TV}(v(t))\,dt<\infty.$$

This proves $[v]_h\in L^1_w(0,T;\mathrm{BV}(\Omega))$. Using the lower semi-continuity of the total variation with respect to L^1-convergence, Fatou's Lemma, and the last inequality we conclude that

$$\begin{aligned}\int_0^T \mathbf{TV}(v(t))\,dt&\le\int_0^T \liminf_{h\downarrow 0}\mathbf{TV}\big([v]_h(t)\big)\,dt\\&\le\liminf_{h\downarrow 0}\int_0^T \mathbf{TV}\big([v]_h(t)\big)\,dt\le\int_0^T \mathbf{TV}(v(t))\,dt\end{aligned}$$

holds true, proving the claim (2.10). This completes the proof of the Lemma. □

As an immediate consequence of Lemma 2.4, respectively 2.5 and Lemma 2.6 we obtain

Corollary 2.7. *Let $T>0$, $\boldsymbol{F}$ a (ROF)-functional as defined in* (1.3) *or a $\mathbf{TV}$-deblurring functional as in* (1.9)*, $v_o\in\mathrm{BV}(\Omega)$ with $\boldsymbol{F}(v_o)<\infty$ and $v\in L^1_w(0,T;\mathrm{BV}(\Omega))$ satisfying*

$$\int_0^T \boldsymbol{F}(v(t))\,dt<\infty.$$

Then, we have $[v]_h \in L^1_w(0,T;\mathrm{BV}(\Omega))$*, and moreover for any* $t \in (0,T)$ *the inequality* $\boldsymbol{F}([v]_h(t)) \le [\boldsymbol{F}(v)]_h(t)$ *holds true. Finally, we have that the following identity*

$$\lim_{h\downarrow 0}\int_0^T \boldsymbol{F}\big([v]_h(t)\big)\,dt = \int_0^T \boldsymbol{F}(v(t))\,dt$$

holds true. □

2.3. **Localizing the problem on a smaller cylinder.** Our goal in this section is to establish that a variational solution u defined on some cylinder Ω^*_T with $T \in (0,\infty)$ in the sense of Definition 1.1, is also a variational solution on any on any sub-cylinder $\Omega^*_{t_1,t_2} := \Omega^* \times (t_1,t_2)$ for any pair $0 \le t_1 < t_2 \le T$. To establish this *localizing principle*, we consider a function $v \in L^1_w(t_1,t_2;\mathrm{BV}_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_{t_1,t_2})$, and choose for fixed $\theta \in \big(0,\frac12(t_2-t_1)\big)$ the cut-off function

$$\zeta_\theta(t) := \begin{cases} 0 & \text{for } t\in[0,t_1],\\ \frac{1}{\theta}(t-t_1) & \text{for } t\in(t_1,t_1+\theta),\\ 1 & \text{for } t\in[t_1+\theta,t_2-\theta],\\ \frac{1}{\theta}(t_2-t) & \text{for } t\in(t_2-\theta,t_2),\\ 0 & \text{for } t\in[t_2,T]. \end{cases}$$

The comparison map is now defined by $\tilde v := \zeta_\theta v + (1-\zeta_\theta)[u]_h$, where $[u]_h$ is defined according to (2.4) with u_o instead of v_o. Here, we extended $\zeta_\theta v$ outside of $\Omega^*\times[t_1,t_2]$ by 0. To check that $\tilde v$ is indeed admissible, we recall from Lemma 2.6 that $[u]_h \in L^1_w(0,T;\mathrm{BV}_{u_o}(\Omega^*))$. Hence, $\zeta_\theta v, (1-\zeta_\theta)[u]_h \in L^1_w(0,T;\mathrm{BV}(\Omega^*))$, and therefore also $\tilde v \in L^1_w(0,T;\mathrm{BV}(\Omega^*))$. Moreover, $\partial_t\tilde v \in L^2(\Omega^*_T)$ since $\partial_t[u]_h \in L^2(\Omega^*_T)$; recall that (2.5) holds true, that $u \in C^0([0,T];L^2(\Omega^*))$ and therefore also that $[u]_h \in C^0([0,T];L^2(\Omega^*))$ by Lemma 2.2. The remaining properties, such as the boundary condition $\tilde v(x,t) = u_o(x)$ for a.e. $x\in(\Omega^*\setminus\Omega)$ and any $t\in(0,T)$ and the initial condition $\tilde v(x,0) = u_o(x)$ on Ω^*, are easy consequences of the corresponding properties of v and $[u]_h$ and the definition of $\tilde v$ as a convex combination. Therefore, we can use $\tilde v$ in the variational inequality (1.7). In the variational inequality we re-write the integral containing the time derivative in the following way:

$$\begin{aligned}
\partial_t\big(\zeta_\theta v + (1-\zeta_\theta)[u]_h\big)&\big(\zeta_\theta v + (1-\zeta_\theta)[u]_h) - u\big)\\
&= \zeta_\theta\partial_t v\big(\zeta_\theta(v-u) + (1-\zeta_\theta)([u]_h-u)\big)\\
&\quad + (1-\zeta_\theta)\partial_t[u]_h\big([u]_h - u + \zeta_\theta(v-[u]_h)\big)\\
&\quad + \zeta_\theta'\big(v-[u]_h\big)\big([u]_h - u + \zeta_\theta(v-[u]_h)\big)\\
&= \zeta_\theta^2\partial_t v(v-u) + (1-\zeta_\theta)\partial_t[u]_h\big([u]_h-u\big)\\
&\quad + \zeta_\theta(1-\zeta_\theta)\big[\partial_t v\big([u]_h-u\big) + \partial_t[u]_h\big(v-[u]_h\big)\big]\\
&\quad + \zeta_\theta'\big(v-[u]_h\big)\big([u]_h-u\big) + \zeta_\theta\zeta_\theta'\big(v-[u]_h\big)^2.
\end{aligned}$$

In the integral containing $\boldsymbol{F}(\tilde v(t))$ we use the convexity of $\boldsymbol{F}$ in the form

$$\boldsymbol{F}(\tilde v(t)) \le \zeta_\theta(t)\boldsymbol{F}(v(t)) + (1-\zeta_\theta(t))\boldsymbol{F}\big([u]_h(t)\big).$$

Using the preceding assertion in the variational inequality, we conclude that the following estimate holds true:

$$\begin{aligned}
\int_0^T \boldsymbol{F}(u(t))\,dt &\le \int_{t_1}^{t_2}\zeta_\theta(t)\bigg[\int_{\Omega^*}\partial_t v(v-u)\,dx + \boldsymbol{F}(v(t))\bigg]\,dt\\
&\quad + \int_0^T(1-\zeta_\theta(t))\bigg[\int_{\Omega^*}\partial_t[u]_h\big([u]_h-u\big)\,dx + \boldsymbol{F}\big([u]_h(t)\big)\bigg]\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\zeta_\theta(t)(1-\zeta_\theta(t))\int_{\Omega^*}\Big[\partial_t v\big([u]_h-u\big)+\partial_t[u]_h\big(v-[u]_h\big)\Big]\,dxdt\\
&+\int_{t_1}^{t_2}\zeta_\theta'(t)\int_{\Omega^*}\big(v-[u]_h\big)\big([u]_h-u\big)\,dxdt\\
&+\int_{t_1}^{t_2}\zeta_\theta(t)\zeta_\theta'(t)\int_{\Omega^*}\big|v-[u]_h\big|^2\,dxdt\\
&-\tfrac12\big\|\big(\zeta_\theta v+(1-\zeta_\theta)[u]_h-u\big)(T)\big\|^2_{L^2(\Omega^*)}\\
&+\tfrac12\big\|\big(\zeta_\theta v+(1-\zeta_\theta)[u]_h-u\big)(0)\big\|^2_{L^2(\Omega^*)}.
\end{aligned}\tag{2.11}
$$

Since $\zeta_\theta(T)=0$ and $\zeta_\theta(0)=0$ the two boundary terms in (2.11) simplify to

$$
-\tfrac12\big\|\big([u]_h-u\big)(T)\big\|^2_{L^2(\Omega^*)}+\tfrac12\big\|([u]_h-u)(0)\big\|^2_{L^2(\Omega^*)}.
$$

Now, we pass to the limit $\theta\downarrow 0$ (observe that the mixed term, i.e. the one containing $\zeta_\theta(t)(1-\zeta_\theta(t))$ vanishes as $\theta\downarrow 0$) and arrive at

$$
\begin{aligned}
&\int_{t_1}^{t_2}\boldsymbol{F}(u(t))\,dt\\
&\quad\le\int_{t_1}^{t_2}\bigg[\int_{\Omega^*}\partial_t v(v-u)\,dx+\boldsymbol{F}(v(t))\bigg]\,dt\\
&\qquad+\tfrac12\big\|\big(v-[u]_h\big)(t_1)\big\|^2_{L^2(\Omega^*)}-\tfrac12\big\|\big(v-[u]_h\big)(t_2)\big\|^2_{L^2(\Omega^*)}\\
&\qquad+\int_{(0,t_1)\cup(t_2,T)}\bigg[\int_{\Omega^*}\partial_t[u]_h([u]_h-u)\,dx+\big[\boldsymbol{F}([u]_h(t))-\boldsymbol{F}(u(t))\big]\bigg]\,dt\\
&\qquad-\tfrac12\big\|\big([u]_h-u\big)(T)\big\|^2_{L^2(\Omega^*)}+\tfrac12\big\|([u]_h-u)(0)\big\|^2_{L^2(\Omega^*)}\\
&\qquad+\int_{\Omega^*\times\{t_1\}}\big(v-[u]_h\big)\big([u]_h-u\big)\,dx-\int_{\Omega^*\times\{t_2\}}\big(v-[u]_h\big)\big([u]_h-u\big)\,dx.
\end{aligned}
$$

In the course of the proof we used the identity

$$
\lim_{\theta\downarrow 0}\int_{t_1}^{t_2}\int_{\Omega^*}\zeta_\theta\zeta_\theta'\big|v-[u]_h\big|^2\,dxdt=\tfrac12\big\|\big(v-[u]_h\big)(t_1)\big\|^2_{L^2(\Omega^*)}-\tfrac12\big\|\big(v-[u]_h\big)(t_2)\big\|^2_{L^2(\Omega^*)},
$$

which easily follows, since $v,[u]_h\in C^0([t_1,t_2];L^2(\Omega^*))$. Taking $\partial_t[u]_h([u]_h-u)\le 0$ into account, which holds true by (2.5), we discard the first term in the third line on the right-hand side. Next, we use the facts $[u]_h\to u$ in $L^\infty(0,T;L^2(\Omega^*))$ and $\boldsymbol{F}([u]_h)\to\boldsymbol{F}(u)$ in $L^1(\Omega^*_T)$, which follow from Lemma 2.2 and Corollary 2.7, and conclude that the remaining terms in the last two lines vanish in the limit $h\downarrow 0$. Moreover, for the terms in the second line we find $\|(v-[u]_h)(t_i)\|^2_{L^2(\Omega^*)}\to\|(v-u)(t_i)\|^2_{L^2(\Omega^*)}$ for $i\in\{1,2\}$. All together, this proves that u is a variational solution on the smaller cylinder $\Omega^*_{t_1,t_2}$, i.e. the variational inequality

$$
\begin{aligned}
\int_{t_1}^{t_2}\boldsymbol{F}(u(t))\,dt\le\int_{t_1}^{t_2}&\bigg[\int_{\Omega^*}\partial_t v(v-u)\,dx+\boldsymbol{F}(v(t))\bigg]\,dt\\
&-\tfrac12\|(v-u)(t_2)\|^2_{L^2(\Omega^*)}+\tfrac12\|(v-u)(t_1)\|^2_{L^2(\Omega^*)}
\end{aligned}
$$

holds true for any $0\le t_1<t_2\le T$ and any testing function $v\in L^1_w(t_1,t_2;\mathrm{BV}_{u_o}(\Omega))$ with $\partial_t v\in L^2(\Omega^*_{t_1,t_2})$.

Remark 2.8. The localization principle also holds for variational solutions in the sense of Definition 1.4. The proof from above works almost verbatim. One only has to replace the functional $\boldsymbol{F}$ from (1.6) by the functional defined in (1.9) and Ω^* by Ω.

2.4. **The initial condition.** Here we establish that variational solutions in the sense of Definition 1.1 fulfill the initial condition $u(0) = u_o$ on Ω^* in the usual L^2-sense. This follows from the fact that the difference $\|u(t) - u_o\|^2_{L^2(\Omega^*)}$ grows at most linearly with respect to $t > 0$, cf. the estimate (2.12) below.

Lemma 2.9. *Any variational solution u on Ω^*_T for some $T \in (0, \infty]$ in the sense of Definition 1.1 fulfills the initial condition $u(0) = u_o$ in the usual L^2-sense, that is*

$$\lim_{t\downarrow 0} \|u(t) - u_o\|^2_{L^2(\Omega^*)} = 0.$$

Proof. In §2.3 we have shown that u satisfies (1.7) on any sub-cylinder Ω^*_τ for $\tau \in (0, T)$. We test the minimality condition (1.7) on Ω^*_τ with the time independent extension of u_o to Ω^*_τ, i.e. with $v(t) \equiv u_o$, for $t \in (0, \tau]$. Note that by (1.6) v is admissible in (1.7). We obtain that

$$\int_0^\tau \boldsymbol{F}(u(t))\, dt + \tfrac{1}{2}\|u(\tau) - u_o\|^2_{L^2(\Omega^*)} \le \tau\, \boldsymbol{F}(u_o) < \infty \tag{2.12}$$

holds true for any $\tau \in (0, T)$. Here, we discard the non-negative energy term in the left-hand side, and then let $\tau \downarrow 0$ in the right-hand side. This proves the claim, that u satisfies the initial boundary condition $u(0) = u_o$ in the L^2-sense. □

Remark 2.10. Taking into account Remark 2.8, we conclude that also variational solutions in the sense of Definition 1.4 fulfill the initial condition $u(0) = u_o$ in the usual $L^2(\Omega)$-sense. The proof from above remains modulo the obvious replacement mechanism the same as for variational solutions in the sense of Definition 1.1. □

3. The time derivative

In the definition of variational solutions we do not assume that the time derivative exists in some sense. However, since the lateral boundary data are constant with respect to time, we can show that the time derivative $\partial_t u$ belongs to L^2. This fact is the main conclusion of Theorems 1.3 and 1.6, which we are going to prove in this section.

Proof of Theorems 1.3 and 1.6. We start with the proof of Theorem 1.3. We define $[u]_h$ according to (2.4) with u_o instead of v_o and let $t_1, t_2 \in \mathbb{R}$ with $0 \le t_1 < t_2 \le T$. From §2.3 we recall that u satisfies the variational inequality (1.7) on the sub-cylinder $\Omega^*_{t_1,t_2}$. Now, we consider $\tilde{u}(s) := u(s + t_1)$ for $s \in (0, t_2 - t_1)$. If at the initial value t_1 we have $u(t_1) \in \mathrm{BV}(\Omega^*)$ (which holds true for a.e. t_1 and in particular for $t_1 = 0$), we know that $\tilde{u}$ satisfies the variational inequality (1.7) on $\Omega^*_{t_2-t_1}$ with $u(t_1)$ instead of u_o. Moreover, from Corollary 2.7 we know that $v = [\tilde{u}]_h$ (defined according to (2.4) with $u(t_1)$ instead of v_o) is an admissible comparison map in the variational inequality for $\tilde{u}$ with $v(0) = \tilde{u}(0) = u(t_1)$. Now, we test (1.7) on $\Omega_{t_2-t_1}$ with $v = [u]_h$, and discard the negative term $-\frac{1}{2}\|([u]_h - u)(t_2 - t_1)\|^2_{L^2(\Omega^*)}$ on the right-hand side. Using Corollary 2.7 and (2.5), we arrive at

$$\begin{aligned}
-\int_0^{t_2-t_1} \int_{\Omega^*} \partial_t[\tilde{u}]_h \big([\tilde{u}]_h - \tilde{u}\big)\, dxdt &\le \int_0^{t_2-t_1} \big[\boldsymbol{F}\big([\tilde{u}]_h(t)\big) - \boldsymbol{F}(\tilde{u}(t))\big]\, dt \\
&\le \int_0^{t_2-t_1} \big[[\boldsymbol{F}(\tilde{u}(t))]_h - \boldsymbol{F}(\tilde{u}(t))\big]\, dt \\
&= -h \int_0^{t_2-t_1} \partial_t [\boldsymbol{F}(\tilde{u}(t))]_h\, dt \\
&= h\,\big(\boldsymbol{F}(\tilde{u}(0)) - [\boldsymbol{F}(\tilde{u}(t))]_h(t_2 - t_1)\big).
\end{aligned}$$

Using again (2.5) we can re-write the preceding inequality in the form

$$\int_0^{t_2-t_1} \int_{\Omega^*} \big|\partial_t[\tilde{u}]_h\big|^2\, dxdt \le \boldsymbol{F}(\tilde{u}(0)) - [\boldsymbol{F}(\tilde{u})]_h(t_2 - t_1) \le \boldsymbol{F}(\tilde{u}(0)).$$

The preceding estimate is uniform with respect to $h > 0$, and therefore ensures the existence of the time derivative $\partial_t \tilde{u} \in L^2(\Omega^*_\tau)$ together with the quantitative estimate

$$\int_0^{t_2-t_1} \int_{\Omega^*} |\partial_t \tilde{u}|^2 \, dxdt \le \boldsymbol{F}(u(t_1)).$$

Recalling that $[\boldsymbol{F}(\tilde{u})]_h(\tau) \to \boldsymbol{F}(\tilde{u}(\tau))$ for a.e. $\tau \in (0,T)$, we conclude from the second last inequality that

$$\int_0^{t_2-t_1} \int_{\Omega^*} |\partial_t \tilde{u}|^2 \, dxdt \le \boldsymbol{F}(\tilde{u}(0)) - \boldsymbol{F}(\tilde{u}(t_2 - t_1)).$$

Transforming back to u, we have shown that

$$\int_{t_1}^{t_2} \int_{\Omega^*} |\partial_t u|^2 \, dxdt \le \boldsymbol{F}(u(t_1)) - \boldsymbol{F}(u(t_2)) \tag{3.1}$$

holds true for a.e. $0 \le t_1 < t_2 \le T$. In particular, (3.1) holds true with $t_1 = 0$. Now, we choose $t_1 = 0$ and $T < \infty$ if $t_2 = T$ and otherwise, if $T = \infty$, we let $t_2 \to \infty$ to obtain the first assertion of the theorem. Moreover, for $t_1, t_2 \in \mathbb{R}$ with $0 \le t_1 < t_2 \le T$ we have

$$\begin{aligned}
\|u(t_2) - u(t_1)\|^2_{L^2(\Omega^*)} &= \int_{\Omega^*} \bigg| \int_{t_1}^{t_2} \partial_t u \, dt \bigg|^2 dx \\
&\le |t_2 - t_1| \int_{t_1}^{t_2} \int_{\Omega^*} |\partial_t u|^2 \, dxdt \\
&\le |t_2 - t_1| \boldsymbol{F}(u_o).
\end{aligned}$$

Choosing $t_1 = 0$ here, we find for any $t \in \mathbb{R} \cap (0,T]$ that

$$\int_{\Omega^*} |u(t)|^2 \, dx \le 2 \int_{\Omega^*} |u_o|^2 \, dx + 2 \int_{\Omega^*} |u(t) - u_o|^2 \, dx \le 2 \int_{\Omega} |u_o|^2 \, dx + 2t\boldsymbol{F}(u_o).$$

Therefore, we obtain

$$u \in C^{0,\frac{1}{2}}\big([0,\tau]; L^2(\Omega^*)\big) \quad \text{for any } \tau \in \mathbb{R} \cap (0,T].$$

At this point it remains to establish the estimate (1.8). Since we have already shown that $\partial_t u \in L^2(\Omega^*_T)$, we can re-write by partial integration the minimality condition (1.7) in the form

$$\int_0^\tau \boldsymbol{F}(u(t)) \, dt \le \int_0^\tau \bigg[\int_{\Omega^*} \partial_t u (v - u) \, dx + \boldsymbol{F}(v(t)) \bigg] dt,$$

for any $\tau \in \mathbb{R} \cap (0,T]$. Now, for $t_1, t_2 \in \mathbb{R}$ with $0 \le t_1 < t_2 \le \tau$ we define

$$\zeta_{t_1,t_2}(t) := \begin{cases} 1 & \text{if } t \in [0, t_1], \\ \frac{t_2 - t}{t_2 - t_1} & \text{if } t \in (t_1, t_2), \\ 0 & \text{if } t \in [t_2, \tau]. \end{cases}$$

We choose $v = u + \zeta_{t_1,t_2}([u]_h - u)$ as comparison function in the minimality condition on Ω^*_τ. We note that v is indeed admissible, i.e. that $v \in L^1_w(0,\tau; \mathrm{BV}_{u_o}(\Omega^*))$ and moreover $\partial_t v \in L^2(\Omega^*_\tau)$ (see the argument from the proof in §2.3 establishing that $\tilde{v}$ is admissible; a similar argument can be applied to the testing function v). Using also the convexity of $\boldsymbol{F}$ and Corollary 2.7, this procedure yields the inequality

$$\begin{aligned}
\int_0^{t_2} \boldsymbol{F}(u(t)) \, dt &\le \int_0^{t_2} \int_{\Omega^*} \zeta_{t_1,t_2} \partial_t u ([u]_h - u) \, dxdt \\
&\quad + \int_0^{t_2} \Big[(1 - \zeta_{t_1,t_2}) \boldsymbol{F}(u(t)) + \zeta_{t_1,t_2} [\boldsymbol{F}(u)]_h \Big] dt,
\end{aligned}$$

We re-arrange terms, use (2.5), and integrate by parts. This leads to

$$\begin{aligned}0 &\le \int_0^{t_2}\int_{\Omega^*} \zeta_{t_1,t_2}\partial_t u([u]_h - u)\,dxdt + \int_0^{t_2} \zeta_{t_1,t_2}\big[[\boldsymbol{F}(u)]_h - \boldsymbol{F}(u(t))\big]\,dt\\ &= -h\int_0^{t_2}\int_{\Omega^*} \zeta_{t_1,t_2}\partial_t u\partial_t[u]_h\,dxdt - h\int_0^{t_2} \zeta_{t_1,t_2}\partial_t[\boldsymbol{F}(u)]_h\,dt\\ &= -h\int_0^{t_2}\int_{\Omega^*} \zeta_{t_1,t_2}\partial_t u\partial_t[u]_h\,dxdt + h\int_0^{t_2} \zeta'_{t_1,t_2}[\boldsymbol{F}(u)]_h\,dt + h\boldsymbol{F}(u_o).\end{aligned}$$

Now, we divide both sides by $h > 0$ and pass to the limit $h \downarrow 0$. This leads us to

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \boldsymbol{F}(u)\,dt \le \boldsymbol{F}(u_o) - \int_0^{t_2}\int_{\Omega^*} \zeta_{t_1,t_2}|\partial_t u|^2\,dxdt \le \boldsymbol{F}(u_o),$$

and finishes the proof of Theorem 1.3.

For the proof of Theorem 1.6, where functionals of the type (1.9) are considered, exactly the same argument works. The only difference is, that one has to replace again Ω^* by Ω and the functional $\boldsymbol{F}$ from (1.6) by the functional defined in (1.9). □

4. Uniqueness

The uniqueness of variational solutions to the generalized (ROF)-model will be a direct consequence of the comparison principle from Lemma 4.2. The main idea of the proof is already contained in [27]. Before we present the comparison principle we state a useful property of BV-functions which can be deduced for instance from [26, Theorem 2.8 (iii)].

Lemma 4.1. *For functions $v, w \in \mathrm{BV}(\Omega)$ we have $\min\{v,w\}, \max\{v,w\} \in \mathrm{BV}(\Omega)$ and*

$$\mathbf{TV}(\min\{v,w\}) + \mathbf{TV}(\max\{v,w\}) \le \mathbf{TV}(v) + \mathbf{TV}(w).$$

Lemma 4.2 (Comparison principle)**.** *Let $u_o, \tilde{u}_o \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*)$ with $u_o \le \tilde{u}_o$ a.e. in Ω^* and $u, \tilde{u}$ be variational solutions in the sense of Definition 1.1 on Ω_T^* for some $T \in (0,\infty]$ with initial and lateral boundary values u_o and $\tilde{u}_o$, respectively. Then, we have*

$$u \le \tilde{u} \quad \text{a.e. in } \Omega_T.$$

Proof. We let $\tau \in \mathbb{R} \cap (0,T]$. Due to §2.3 we can take $v := \min\{u,\tilde{u}\}$ as comparison function in the variational inequality for u and $w := \max\{u,\tilde{u}\}$ as comparison function in the variational inequality for $\tilde{u}$ on the smaller space-time cylinder Ω_τ. This can easily be inferred as follows: The assumption $u, \tilde{u} \in L^1_w(0,T,\mathrm{BV}(\Omega^*))$ implies on the one hand that $v = \min\{u,\tilde{u}\} \in L^1(\Omega_T^*)$. By Fubini's theorem we therefore have that for any $\phi \in C_0^1(\Omega,\mathbb{R}^n)$ the partial map $t \mapsto \int_\Omega v(t)\operatorname{div}\phi\,dx$ is measurable with respect to t. On the other hand, since $v = \min\{u,\tilde{u}\}(t) \in \mathrm{BV}(\Omega)$ for a.e. $t \in (0,T)$ by Lemma 4.1, we also have that

$$t \mapsto \langle Dv(t), \phi\rangle = -\int_\Omega v(t)\operatorname{div}\phi\,dx$$

is measurable with respect to t. Finally, again Lemma 4.1 implies finiteness of the integral $\int_0^T \mathbf{TV}(v(t))\,dt < \infty$. This proves $v \in L^1_w(0,T;\mathrm{BV}(\Omega^*))$. A similar argument shows $w \in L^1_w(0,T;\mathrm{BV}(\Omega^*))$. The remaining properties, such as the boundary condition on the lateral boundary and the initial condition at $t = 0$, can easily be checked. By Theorem 1.3 we know that $\partial_t u, \partial_t\tilde{u} \in L^2(\Omega_T^*)$, and hence $\partial_t v, \partial_t w \in L^2(\Omega_T^*)$. Adding the two resulting inequalities we obtain

$$\begin{aligned}\int_0^\tau \big[\boldsymbol{F}(u(t)) + \boldsymbol{F}(\tilde{u}(t))\big]\,dt &\le \int_0^\tau \big[\boldsymbol{F}(v(t)) + \boldsymbol{F}(w(t))\big]\,dt\\ &\quad + \int_0^\tau\int_{\Omega^*} \big[\partial_t v(v-u) + \partial_t w(w-\tilde{u})\big]\,dxdt\end{aligned}$$

$$- \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega^*)} - \tfrac{1}{2}\|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega^*)}. \tag{4.1}$$

Here, we used that $v(0) = u_o$ and $w(0) = \tilde{u}_o$. In the following we estimate the terms on the right-hand side of (4.1). We start with the integral involving the functional $\boldsymbol{F}$. By Lemma 4.1 we have

$$\int_0^\tau \big[\boldsymbol{F}(v(t)) + \boldsymbol{F}(w(t))\big]\, dt \le \int_0^\tau \big[\boldsymbol{F}(u(t)) + \boldsymbol{F}(\tilde{u}(t))\big]\, dt.$$

Next, we treat the integral involving the time derivatives. On $\{(x,t) \in \Omega^*_\tau \colon u(x,t) \le \tilde{u}(x,t)\}$ we have

$$\partial_t v(v-u) + \partial_t w(w-\tilde{u}) = 0,$$

while on the complement, i.e. on the set $\{(x,t) \in \Omega^*_\tau \colon u(x,t) > \tilde{u}(x,t)\}$, we compute that

$$\partial_t v(v-u) + \partial_t w(w-\tilde{u}) = \partial_t \tilde{u}(\tilde{u}-u) + \partial_t u(u-\tilde{u}) = \tfrac{1}{2}\partial_t |u-\tilde{u}|^2$$

holds true. Joining the preceding identities, we arrive at

$$\begin{aligned} \int_0^\tau \int_{\Omega^*} \big[\partial_t v(v-u) + \partial_t w(w-\tilde{u})\big]\, dxdt &= \tfrac{1}{2}\int_0^\tau \int_{\Omega^*} \partial_t (u-\tilde{u})_+^2\, dxdt \\ &\le \tfrac{1}{2}\int_{\Omega^*\times\{\tau\}} (u-\tilde{u})_+^2\, dx. \end{aligned}$$

Next, we consider the $L^2(\Omega^*)$-terms, i.e. the two terms of the third line of (4.1). The definition of v and w easily imply that

$$-\tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega^*)} = -\tfrac{1}{2}\int_{\Omega^*\times\{\tau\}} (u-\tilde{u})_+^2\, dx$$

and

$$-\tfrac{1}{2}\|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega^*)} = -\tfrac{1}{2}\int_{\Omega^*\times\{\tau\}} (u-\tilde{u})_+^2\, dx.$$

Joining the preceding estimates with (4.1), we find that

$$\int_{\Omega^*\times\{\tau\}} (u-\tilde{u})_+^2\, dx \le 0.$$

Since $\tau \in \mathbb{R} \cap (0,T]$ was arbitrary, this proves the claim that $u \le \tilde{u}$ a.e. in Ω_T. □

Here, we prove that variational solutions in the sense of Definition 1.4 are unique, provided K is injective.

Lemma 4.3. *Suppose that K is injective. Then, for any $T \in (0,\infty]$ and any initial datum u_o as in (1.10), there is at most one variational solution in the sense of Definition 1.4.*

Proof. We let $\tau = T$ if $T < \infty$ and $\tau \in (0,\infty)$ if $T = \infty$. We assume that

$$u_1, u_2 \in L^1_w\big(0,\tau;\mathrm{BV}(\Omega)\big) \cap C^0\big([0,\tau];L^2(\Omega)\big),$$

are two different variational solutions to (1.11) with initial datum u_o. Adding the variational inequalities (1.11) for u_1 and u_2 on Ω_τ and taking into account $\|(v-u_i)(T)\|^2_{L^2(\Omega)} \ge 0$ for $i = 1,2$ we obtain for any $v \in L^1_w(0,\tau;\mathrm{BV}(\Omega))$ with $\partial_t v \in L^2(\Omega_\tau)$ that

$$\begin{aligned} &\int_0^\tau \big[\boldsymbol{F}(u_1(t)) + \boldsymbol{F}(u_2(t))\big]\, dxdt \\ &\qquad \le 2\int_0^\tau \int_\Omega \big[\partial_t v(v-w) + \boldsymbol{F}(v(t))\big]\, dxdt + \|v(0)-u_o\|^2_{L^2(\Omega)}. \end{aligned}$$

Here, we have abbreviated $w := \frac{u_1+u_2}{2}$. In the preceding inequality we are allowed to take the comparison map $v = w$. To check that w is admissible is straightforward; see the proof of Lemma 4.2 for a similar argument. Note that any variational solution has its time derivative in $L^2(\Omega_T)$ by Theorem 1.6. Since $w(0) = u_o$ we obtain that

$$\tfrac{1}{2}\int_0^\tau \big[\boldsymbol{F}(u_1(t)) + \boldsymbol{F}(u_2(t))\big]\,dt \le \int_0^\tau \boldsymbol{F}(w(t))\,dt < \tfrac{1}{2}\int_0^\tau \big[\boldsymbol{F}(u_1(t)) + \boldsymbol{F}(u_2(t))\big]\,dt.$$

Here we used the strict convexity of $\boldsymbol{F}$ for the last inequality. Note that if $u_1 \neq u_2$ we must have $\mathbf{K}[u_1] \neq \mathbf{K}[u_2]$ in the sense of $L^2(\Omega_T)$, which guarantees the strict convexity of $\boldsymbol{F}$. This is the point where the injectivity assumption for $\mathbf{K}$ comes into the play. Since the last inequality yields a contradiction we conclude the uniqueness of variational solutions. □

5. Proof of Theorem 1.2

In this section we provide the proof of Theorem 1.2. Recall that $\Omega \Subset \Omega^*$ are two bounded, open subsets of $\mathbb{R}^n$ and that $u_o \in \mathrm{BV}(\Omega^*) \cap L^2(\Omega^*)$ fulfills (1.5) and therefore has finite $\boldsymbol{F}$-energy; see (1.6).

5.1. A sequence of minimizers to a variational functional on Ω_T^*.

In this chapter we let $T \in (0, \infty)$ and consider for $\varepsilon \in (0, 1]$ variational integrals of the form

$$\mathcal{F}_\varepsilon(v) := \int_0^T e^{-\frac{t}{\varepsilon}} \bigg[\frac{1}{2}\int_{\Omega^*} |\partial_t v|^2\,dx + \tfrac{1}{\varepsilon}\boldsymbol{F}(v(t))\bigg]\,dt.$$

In order to deal with the existence problem associated to these functionals we first introduce a suitable function space, in which the minimization will be achieved. We define

$$\mathcal{K}_\varepsilon := \big\{v \in L^1_w\big(0, T; \mathrm{BV}(\Omega^*)\big) : \partial_t v \in L^2(\Omega_T^*)\big\}$$

and let

$$\|v\|_{\mathcal{K}_\varepsilon} := \int_0^T \|v(t)\|_{\mathrm{BV}(\Omega^*)}\,dt + \|\partial_t v\|_{L^2(\Omega_T^*)}.$$

We note that

$$e^{-\frac{T}{\varepsilon}}\|v\|_{\mathcal{K}_\varepsilon} \le \int_0^T e^{-\frac{t}{\varepsilon}}\|v(t)\|_{\mathrm{BV}(\Omega^*)}\,dt + \bigg[\int_0^T\int_{\Omega^*} e^{-\frac{t}{\varepsilon}}|\partial_t v|^2\,dxdt\bigg]^{\frac{1}{2}} \le \|v\|_{\mathcal{K}_\varepsilon}.$$

The subclass $\mathcal{K}_{\varepsilon,u_o}$ consists of those functions $v \in \mathcal{K}_\varepsilon$ satisfying $u = u_o$ almost everywhere on $(\Omega^* \setminus \Omega) \times (0, T)$ and $v(0) = u_o$. Note, that the L^2-bound $\|\partial_t v\|_{L^2(\Omega_T^*)} < \infty$ already implies that $v \in C^{0,\frac{1}{2}}([0, T]; L^2(\Omega^*))$, and this allows us to define the initial condition $v(0) = u_o$ in the usual L^2-sense. Note also that the time independent extension of $u_o \in \mathrm{BV}(\Omega^*) \cap L^2(\Omega^*)$ to Ω_T^*, i.e. $v(t) = u_o$ for $t \in (0, T]$, belongs to $\mathcal{K}_{\varepsilon,u_o}$. Actually, we have $\|v\|_{\mathcal{K}_\varepsilon} = T\|u_o\|_{\mathrm{BV}(\Omega^*)}$.

At this stage it is worth to remark, that the hypothesis $\partial_t v \in L^2(\Omega_T^*)$ can be used to derive an L^1-bound for v in terms of $\partial_t v$ and u_o. Indeed, for functions $v \in \mathcal{K}^*_{\varepsilon,u_o}$ we have for any $t \in [0, T]$ that

$$\begin{aligned}
\|v(t)\|_{L^1(\Omega^*)} &\le \|v(t) - u_o\|_{L^1(\Omega^*)} + \|u_o\|_{L^1(\Omega^*)} \\
&= \int_{\Omega^*}\bigg|\int_0^t \partial_\tau v(\tau)\,d\tau\bigg|\,dx + \|u_o\|_{L^1(\Omega^*)} \\
&\le |\Omega_T^*|^{\frac{1}{2}}\|\partial_t v\|_{L^2(\Omega_T^*)} + \|u_o\|_{L^1(\Omega^*)},
\end{aligned}$$

holds true. Integrating the preceding inequality with respect to $t \in (0, T)$ we obtain

$$\|v\|_{L^1(\Omega_T^*)} \le T\big[|\Omega_T^*|^{\frac{1}{2}}\|\partial_t v\|_{L^2(\Omega_T^*)} + \|u_o\|_{L^1(\Omega^*)}\big]. \tag{5.1}$$

Next, we define the subclass of mappings in $\mathcal{K}_{\varepsilon,u_o}$ with finite $\mathcal{F}_\varepsilon$-energy. We shall denote this class by $\mathcal{K}^*_{\varepsilon,u_o}$, i.e. we define

$$\mathcal{K}^*_{\varepsilon,u_o} := \{u \in \mathcal{K}_{\varepsilon,u_o} : \mathcal{F}_\varepsilon(u) < \infty\}\ .$$

We note that this class of mappings is non-empty since the time independent extension of u_o to Ω_T^* belongs to $\mathcal{K}^*_{\varepsilon,u_o}$. Actually, by (1.6) we have $\mathcal{F}_\varepsilon(v) \le \boldsymbol{F}(u_o) < \infty$. The following Lemma ensures that minimizers of $\mathcal{F}_\varepsilon$ exist. More precisely we have

Lemma 5.1. *For any $\varepsilon \in (0,1]$, the functional $\mathcal{F}_\varepsilon$ admits a unique minimizer $u_\varepsilon \in \mathcal{K}^*_{\varepsilon,u_o}$.*

Proof. Using (5.1) we have for maps $v \in \mathcal{K}^*_{\varepsilon,u_o}$ that

$$\begin{aligned}\|v\|_{\mathcal{K}_\varepsilon} &\le \int_0^T \mathbf{TV}(v(t))\,dt + \big(1+T|\Omega_T^*|^{\frac12}\big)\|\partial_t v\|_{L^2(\Omega_T^*)} + T\|u_o\|_{L^1(\Omega_T^*)}\\ &\le \big(1+T|\Omega_T^*|^{\frac12}\big)\bigg[\int_0^T \mathbf{TV}(v(t))\,dt + \|\partial_t v\|^2_{L^2(\Omega_T^*)} + 1\bigg] + T\|u_o\|_{L^1(\Omega_T^*)}\\ &\le 2e^{\frac{T}{\varepsilon}}\big(1+T|\Omega_T^*|^{\frac12}\big)\big[\mathcal{F}_\varepsilon(v)+1\big] + T\|u_o\|_{L^1(\Omega_T^*)}.\end{aligned}$$

In the last line we used that S is non-negative by assumption. Now, we consider a minimizing sequence $u_j \in \mathcal{K}^*_{\varepsilon,u_o}$, $j\in\mathbb{N}$, i.e.

$$\lim_{j\to\infty}\mathcal{F}_\varepsilon(u_j) = \inf_{u\in\mathcal{K}^*_{\varepsilon,u_o}}\mathcal{F}_\varepsilon(u) \le \mathcal{F}_\varepsilon(u_o) = \boldsymbol{F}(u_o)\big(1-\mathrm{e}^{-\frac{T}{\varepsilon}}\big).$$

Without loss of generality we can assume that $\mathcal{F}_\varepsilon(u_j) \le \boldsymbol{F}(u_o)$. From the last inequality we therefore obtain

$$\|u_j\|_{\mathcal{K}_\varepsilon} \le 4e^{\frac{T}{\varepsilon}}\big(1+T|\Omega_T^*|^{\frac12}\big)\big[\boldsymbol{F}(u_o)+1\big] + T\|u_o\|_{L^1(\Omega_T^*)},$$

i.e. the minimizing sequence $(u_j)_{j\in\mathbb{N}}$ is uniformly bounded with respect to $\|\cdot\|_{\mathcal{K}_\varepsilon}$. This leads to the following uniform bound

$$\begin{aligned}&\sup_{j\in\mathbb{N}}\bigg[\int_0^T \|u_j(t)\|_{\mathrm{BV}(\Omega^*)}\,dt + \|\partial_t u_j\|_{L^2(\Omega_T^*)}\bigg]\\ &\qquad\le 4e^{\frac{T}{\varepsilon}}\big(1+T|\Omega_T^*|^{\frac12}\big)\big[\boldsymbol{F}(u_o)+1\big] + T\|u_o\|_{L^1(\Omega_T^*)}.\end{aligned}\tag{5.2}$$

At this stage we can apply a classical compactness result of J. Simon. More precisely, we apply [35, Theorem 1] with $p=1$ and $B=L^1(\Omega^*)$ to infer that $(u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1(\Omega_T^*)$; see Lemma 8.1 in the Appendix for the precise argument. Hence, we conclude the existence of a subsequence, again denoted by $(u_j)_{j\in\mathbb{N}}$ and a measurable function $u\colon\Omega_T^*\to\mathbb{R}$, such that

$$\begin{cases} u_j \to u & \text{strongly in } L^1(\Omega_T^*),\\ u_j\to u & \text{a.e. in } \Omega_T^*,\\ u_j \rightharpoonup u & \text{weakly in } L^2(\Omega_T^*).\\ \partial_t u_j \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T^*).\end{cases}$$

The sequence $(u_j)_{j\in\mathbb{N}}$ satisfies the hypotheses of Lemma 2.1, and therefore we obtain that $u\in L^1_w(0,T;\mathrm{BV}(\Omega^*))$. Next, we check that u fulfills the remaining requirements for being a function in the class $\mathcal{K}^*_{\varepsilon,u_o}$. By the pointwise a.e. convergence we find that

$$|u(x,t)-u_o(x)| \le |u(x,t)-u_j(x,t)| + |u_j(x,t)-u_o(x)| \to 0 \quad \text{as } j\to\infty$$

holds true for a.e. $(x,t)\in\Omega^*\setminus\Omega\times(0,T)$. Here we used the fact that the second term on the right hand side is identically zero. This implies that $u(x,t)=u_o(x)$ for a.e. $(x,t)\in(\Omega^*\setminus\Omega)\times(0,T)$. Next, we observe that for any $0\le s<t\le T$ there holds

$$\|u_j(t)-u_j(s)\|_{L^2(\Omega^*)} \le \sqrt{|t-s|}\,\|\partial_t u_j\|_{L^2(\Omega_T^*)} \le e^{\frac{T}{\varepsilon}}\big[\boldsymbol{F}(u_o)+1\big]\sqrt{|t-s|}.$$

Using the weak convergence $u_j\rightharpoonup u$ in $L^2(\Omega_T^*)$ and the last estimate with $s=0$, keeping in mind that $u_j(0)=u_o$, we find that

$$\tfrac1h\int_0^h \|u(t)-u_o\|^2_{L^2(\Omega^*)}\,dt \le \liminf_{j\to\infty}\tfrac1h\int_0^h\|u_j(t)-u_o\|^2_{L^2(\Omega^*)}\,dt \le e^{\frac{T}{\varepsilon}}\big[\boldsymbol{F}(u_o)+1\big]h.$$

But, this implies $\lim_{h\downarrow 0}\frac{1}{h}\int_0^h \|u(t)-u_o\|^2_{L^2(\Omega^*)}\,dt = 0$, so that $u(0)=u_o$ in the usual L^2-sense, i.e. the initial condition is preserved in the limit. Now we use the lower semi-continuity of certain quantities, i.e. of the total variation with respect to L^1-convergence, of the lower order term (cf. [23, Chapter VIII, Corollary 1.2]) and of the L^2-norm with respect to weak convergence. Using also Fatou's Lemma we obtain

$$\begin{aligned}
&\int_0^T e^{-\frac{t}{\varepsilon}}\bigg[\int_{\Omega^*}\tfrac12|\partial_t u|^2\,dx + \tfrac1\varepsilon \boldsymbol{F}(u(t))\bigg]\,dt\\
&\qquad\le \liminf_{j\to\infty}\int_0^T\int_{\Omega^*} e^{-\frac{t}{\varepsilon}}\tfrac12|\partial_t u_j|^2\,dxdt + \int_0^T e^{-\frac{t}{\varepsilon}}\tfrac1\varepsilon \liminf_{j\to\infty}\boldsymbol{F}(u_j(t))\,dt\\
&\qquad\le \liminf_{j\to\infty}\int_0^T\int_{\Omega^*} e^{-\frac{t}{\varepsilon}}\tfrac12|\partial_t u_j|^2\,dxdt + \liminf_{j\to\infty}\int_0^T e^{-\frac{t}{\varepsilon}}\tfrac1\varepsilon \boldsymbol{F}(u_j(t))\,dt\\
&\qquad\le \liminf_{j\to\infty}\int_0^T e^{-\frac{t}{\varepsilon}}\bigg[\int_{\Omega^*}\tfrac12|\partial_t u_j|^2\,dx + \tfrac1\varepsilon \boldsymbol{F}(u_j(t))\bigg]\,dt\\
&\qquad= \lim_{j\to\infty}\mathcal{F}_\varepsilon(u_j).
\end{aligned}$$

This proves that u is a minimizer of $\mathcal{F}_\varepsilon$ in the class $\mathcal{K}^*_{\varepsilon,u_o}$. The uniqueness follows, since the term containing the time derivative ensures the strict convexity of the functional $\mathcal{F}_\varepsilon$. □

5.2. **An equivalent formulation of the minimality.** We fix $\varepsilon\in(0,1]$ and consider $\varphi\in L^1_w(0,T;\mathrm{BV}_0(\Omega))$ with $\partial_t\varphi\in L^2(\Omega^*_T)$ and

$$\int_0^T \boldsymbol{F}\big((u_\varepsilon+\varphi)(t)\big)\,dt<\infty. \tag{5.3}$$

For $\delta\in(0,e^{-\frac{T}{\varepsilon}}]$ and $(x,t)\in\Omega^*_T$ we define

$$\widetilde\varphi_{\varepsilon,\delta}(x,t) := \delta e^{\frac{t}{\varepsilon}}\zeta(t)\varphi(x,t),$$

where $\zeta\in W^{1,\infty}(0,T;[0,1])$. We assume that either $\zeta(0)=0$ or $\varphi(0)=0$. Then, we set

$$v_{\varepsilon,\delta}(x,t) := u_\varepsilon(x,t)+\widetilde\varphi_{\varepsilon,\delta}(x,t) = u_\varepsilon(x,t)+\delta e^{\frac{t}{\varepsilon}}\zeta(t)\varphi(x,t).$$

We start with the observation $v_\varepsilon\in L^1_w(0,T;\mathrm{BV}(\Omega))$. This can easily be established by similar arguments as in §2.3. Next we need to establish that $\mathcal{F}_\varepsilon(v_{\varepsilon,\delta})<\infty$. Havin achieved this, it is straightforward to check the other properties guaranteeing that $v_{\varepsilon,\delta}\in\mathcal{K}^*_{\varepsilon,u_o}$, which means that $v_{\varepsilon,\delta}$ is a suitable comparison function in the minimality condition for u_ε. The finiteness of the the total $\mathcal{F}_\varepsilon$-energy of $v_{\varepsilon,\delta}$ is an easy consequence of the convexity of the functional $\boldsymbol{F}$, the fact that $v_{\varepsilon,\delta}$ is on fixed time slices $t\in[0,T]$ a convex combination of u_ε and $u_\varepsilon+\varphi$ and the finite energy assumption (5.3). More precisely, letting $\sigma(t):=\delta e^{\frac{t}{\varepsilon}}\zeta(t)$ we have $0\le\sigma(t)\le 1$ for δ as above, and by the convexity of $\boldsymbol{F}$ the contribution of the $\boldsymbol{F}$-part to the $\mathcal{F}_\varepsilon$-energy is estimated by

$$\begin{aligned}
\int_0^T e^{-\frac{t}{\varepsilon}}\boldsymbol{F}(v_{\varepsilon,\delta}(t))\,dt &\le \int_0^T e^{-\frac{t}{\varepsilon}}\Big[(1-\sigma(t))\boldsymbol{F}(u_\varepsilon(t))+\sigma(t)\boldsymbol{F}((u_\varepsilon+\varphi)(t))\Big]\,dt\\
&\le \int_0^T e^{-\frac{t}{\varepsilon}}\boldsymbol{F}(u_\varepsilon(t))\,dt+\int_0^T \boldsymbol{F}((u_\varepsilon+\varphi)(t))\,dt<\infty.
\end{aligned}$$

The remaining properties, such as $\partial_t v_{\varepsilon,\delta}\in L^2(\Omega^*_T)$ and the boundary and initial condition (note that either $\zeta(0)=0$ or $\varphi(0)=0$ by assumption), can be easily checked. Therefore, we have $v_{\varepsilon,\delta}\in\mathcal{K}^*_{\varepsilon,u_o}$. From the minimality of u_ε we conclude that

$$\mathcal{F}_\varepsilon(u_\varepsilon)\le\mathcal{F}_\varepsilon(v_{\varepsilon,\delta})<\infty.$$

The minimality condition can be re-written in the form

$$
\begin{aligned}
0 &\le \int_0^T e^{-\frac{t}{\varepsilon}} \int_{\Omega^*} \tfrac12\Big(|\partial_t u_\varepsilon + \delta\partial_t(e^{\frac{t}{\varepsilon}}\zeta\varphi)|^2 - |\partial_t u_\varepsilon|^2\Big)\,dxdt \\
&\qquad\qquad + \int_0^T e^{-\frac{t}{\varepsilon}}\tfrac{1}{\varepsilon}\Big(\boldsymbol{F}\big(u_\varepsilon(t) + \delta e^{\frac{t}{\varepsilon}}\zeta(t)\varphi(t)\big) - \boldsymbol{F}\big(u_\varepsilon(t)\big)\Big)\,dt \\
&\le \int_0^T e^{-\frac{t}{\varepsilon}} \int_{\Omega^*} \Big(\tfrac12\delta^2|\partial_t(e^{\frac{t}{\varepsilon}}\zeta\varphi)|^2 + \delta\partial_t u_\varepsilon\partial_t(e^{\frac{t}{\varepsilon}}\zeta\varphi)\Big)\,dx \\
&\qquad\qquad + \int_0^T \tfrac{\delta}{\varepsilon}\zeta(t)\Big(\boldsymbol{F}\big((u_\varepsilon+\varphi)(t)\big) - \boldsymbol{F}\big(u_\varepsilon(t)\big)\Big)\,dt.
\end{aligned}
$$

In the last line we used the convexity of the variational functional $\boldsymbol{F}$ exactly as before. We multiply the preceding inequality by ε/δ and then let $\delta\downarrow 0$. This yields

$$
\begin{aligned}
0 &\le \int_0^T e^{-\frac{t}{\varepsilon}} \int_{\Omega^*} \varepsilon\partial_t u_\varepsilon\partial_t(e^{\frac{t}{\varepsilon}}\zeta\varphi)\,dxdt + \int_0^T \zeta\Big(\boldsymbol{F}\big((u_\varepsilon+\varphi)(t)\big) - \boldsymbol{F}\big(u_\varepsilon(t)\big)\Big)\,dt \\
&= \int_0^T \zeta(t)\bigg[\int_{\Omega^*} \partial_t u_\varepsilon\varphi\,dx + \boldsymbol{F}\big((u_\varepsilon+\varphi)(t)\big) - \boldsymbol{F}\big(u_\varepsilon(t)\big)\bigg]\,dt \\
&\quad + \varepsilon\int_0^T\int_{\Omega^*}\Big[\zeta'\partial_t u_\varepsilon\varphi + \zeta\partial_t u_\varepsilon\partial_t\varphi\Big]\,dxdt.
\end{aligned}
$$

The preceding inequality can be re-written as follows:

$$
\begin{aligned}
\int_0^T \zeta(t)\boldsymbol{F}\big(u_\varepsilon(t)\big)\,dt &\le \int_0^T \zeta(t)\boldsymbol{F}\big((u_\varepsilon+\varphi)(t)\big)\,dt + \int_0^T\int_{\Omega^*}\zeta\partial_t u_\varepsilon\varphi\,dxdt \\
&\quad + \varepsilon\int_0^T\int_{\Omega^*}\Big[\zeta'\partial_t u_\varepsilon\varphi + \zeta\partial_t u_\varepsilon\partial_t\varphi\Big]\,dxdt.
\end{aligned}
\tag{5.4}
$$

The last inequality holds true for any $\zeta\in W^{1,\infty}(0,T)$ with $0\le\zeta\le 1$, and any testing function $\varphi\in L^1_w(0,T;\mathrm{BV}_0(\Omega))$ with $\partial_t\varphi\in L^2(\Omega^*_T)$ satisfying (5.3), and such that either $\zeta(0)=0$ or $\varphi(0)=0$.

5.3. **Energy bounds.** In this section we establish certain uniform energy bounds for $\mathcal{F}_\varepsilon$-minimizers $u_\varepsilon\in\mathcal{K}^*_{\varepsilon,u_o}$. Later on these bounds allow us to extract a converging subsequence in the limit procedure $\varepsilon\downarrow 0$. We let $\zeta\in W^{1,\infty}(0,T;[0,1])$, and define $[u_\varepsilon]_h$ according to (2.4) with u_o instead of v_o as initial values at $t=0$. Then, by Lemma 2.6 we have $[u_\varepsilon]_h\in L^1_w\big(0,T,\mathrm{BV}(\Omega^*)\big)$. Further, $[u_\varepsilon]_h(0)=u_o$ and therefore $\partial_t[u_\varepsilon]_h(0)=\frac1h(u_o-[u_\varepsilon]_h(0))=0$. From Corollary 2.7 we conclude that

$$
\int_0^T \zeta(t)\boldsymbol{F}\big((u_\varepsilon - h\partial_t[u_\varepsilon]_h)(t)\big)\,dt = \int_0^T \zeta(t)\boldsymbol{F}\big([u_\varepsilon]_h(t)\big)\,dt < \infty.
$$

Therefore, we are allowed to choose $\varphi=-h\partial_t[u_\varepsilon]_h$ in (5.4). Note that all the requirements on φ in (5.4) are fulfilled. The result is that

$$
\begin{aligned}
h\int_0^T\int_{\Omega^*}&\Big[(\zeta+\varepsilon\zeta')\partial_t u_\varepsilon\partial_t[u_\varepsilon]_h + \varepsilon\zeta\partial_t u_\varepsilon\partial_{tt}[u_\varepsilon]_h\Big]\,dxdt \\
&\le \int_0^T \zeta(t)\big[\boldsymbol{F}\big([u_\varepsilon]_h(t)\big) - \boldsymbol{F}(u_\varepsilon(t))\big]\,dt \\
&\le \int_0^T \zeta(t)\big[[\boldsymbol{F}(u_\varepsilon(t))]_h - \boldsymbol{F}(u_\varepsilon(t))\big]\,dt \\
&= -h\int_0^T \zeta(t)\partial_t[\boldsymbol{F}(u_\varepsilon(t))]_h\,dt,
\end{aligned}
$$

holds true. For the last inequality we used Corollary 2.7. The mollification $[\boldsymbol{F}(u_\varepsilon)]_h$ is defined according to (2.4) with v_o replaced by the initial value $\boldsymbol{F}(u_o)$. Now, we divide both sides by h and re-write and estimate the second terms integrand on the left-hand side as follows

$$\begin{aligned}\partial_t u_\varepsilon \partial_{tt}[u_\varepsilon]_h &= \partial_t[u_\varepsilon]_h\partial_{tt}[u_\varepsilon]_h + \big(\partial_t u_\varepsilon - \partial_t[u_\varepsilon]_h\big)\partial_{tt}[u_\varepsilon]_h \\ &= \tfrac12\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2 + \tfrac1h\big|\partial_t[u_\varepsilon]_h - \partial_t u_\varepsilon\big|^2 \\ &\geq \tfrac12\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2.\end{aligned}$$

Inserting this above, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega^*}\Big[(\zeta+\varepsilon\zeta')\partial_t u_\varepsilon\partial_t[u_\varepsilon]_h + \tfrac{\varepsilon}{2}\zeta\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2\Big]\,dxdt& \\ \leq -\int_0^T\zeta(t)\partial_t[\boldsymbol{F}(u_\varepsilon)(t)]_h\,dt.&\end{aligned}\tag{5.5}$$

We will use (5.5) in two different ways. First, we choose $\zeta\equiv 1$. We obtain

$$\begin{aligned}\int_0^T\int_{\Omega^*}\partial_t u_\varepsilon\partial_t[u_\varepsilon]_h\,dxdt &\leq -\int_0^T\partial_t[\boldsymbol{F}(u_\varepsilon(t))]_h\,dt - \tfrac12\varepsilon\int_0^T\int_{\Omega^*}\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2\,dxdt \\ &\leq \boldsymbol{F}(u_o).\end{aligned}$$

For the last inequality we used $[\boldsymbol{F}(u_\varepsilon)]_h(0)=\boldsymbol{F}(u_o)$, $\partial_t[u_\varepsilon]_h(0)=0$, $[\boldsymbol{F}(u_\varepsilon)]_h(T)\geq 0$ and $|\partial_t[u_\varepsilon]_h|^2(T)\geq 0$. In the preceding inequality we pass to the limit $h\downarrow 0$, and take into account $\partial_t[u_\varepsilon]_h\to\partial_t u_\varepsilon$ in $L^2(\Omega_T^*)$, since $\partial_t u_\varepsilon\in L^2(\Omega_T^*)$ (cf. Lemma 2.2). This yields the following uniform bound on the time derivative of u_ε:

$$\int_0^T\int_{\Omega^*}|\partial_t u_\varepsilon|^2\,dxdt\leq\boldsymbol{F}(u_o).\tag{5.6}$$

Similarly to the argument from §5.1 this implies the following L^2-bound for u_ε:

$$\|u_\varepsilon\|^2_{L^2(\Omega_T^*)}\leq T^2\|\partial_t u_\varepsilon\|^2_{L^2(\Omega_T^*)} + 2T\|u_o\|^2_{L^2(\Omega^*)}\leq T^2\boldsymbol{F}(u_o)+2T\|u_o\|^2_{L^2(\Omega^*)}.\tag{5.7}$$

Finally, using (5.6) we have for any $0\leq s<t\leq T$ that

$$\|u_\varepsilon(t)-u_\varepsilon(s)\|_{L^2(\Omega^*)}\leq\|\partial_t u_\varepsilon\|_{L^2(\Omega_T^*)}\sqrt{|t-s|}\leq\sqrt{\boldsymbol{F}(u_o)}\,\sqrt{|t-s|}\tag{5.8}$$

holds true. The estimates (5.7) and (5.8) imply that the family $(u_\varepsilon)_{\varepsilon>0}$ of $\mathcal{F}_\varepsilon$-minimizers is also uniformly bounded in $L^2(\Omega_T^*)$ and $C^{0,\frac12}([0,T];L^2(\Omega^*))$.

Now we take again (5.5) as starting point. For $0\leq t_1<t_2\leq T$ we choose $\zeta=\zeta_{t_1,t_2}$, where ζ_{t_1,t_2} is defined by

$$\zeta_{t_1,t_2}(t):=\begin{cases}1 & \text{if } t\in[0,t_1]\\ \frac{t_2-t}{t_2-t_1} & \text{if } t\in(t_1,t_2)\\ 0 & \text{if } t\in[t_2,T].\end{cases}$$

With this choice, we can re-write (5.5), with an integration by parts in the second term of the left-hand side and also in the right-hand side (note that in the left-hand side no boundary terms occur, while in the right-hand side the boundary term $\boldsymbol{F}(u_o)$ appears), as follows

$$\begin{aligned}&\int_0^T\int_{\Omega^*}\zeta_{t_1,t_2}(t)\partial_t u_\varepsilon\partial_t[u_\varepsilon]_h\,dxdt \\ &\quad\leq\boldsymbol{F}(u_o)+\int_0^T\zeta'_{t_1,t_2}(t)\bigg[[\boldsymbol{F}(u_\varepsilon(t))]_h+\int_{\Omega^*}\Big[\tfrac{\varepsilon}{2}\big|\partial_t[u_\varepsilon]_h\big|^2-\varepsilon\partial_t u_\varepsilon\partial_t[u_\varepsilon]_h\Big]dx\bigg]dt.\end{aligned}$$

In the preceding inequality we pass to the limit $h\downarrow 0$ and obtain that

$$\int_0^T\int_{\Omega^*}\zeta_{t_1,t_2}(t)|\partial_t u_\varepsilon|^2\,dxdt$$

$$\le \boldsymbol{F}(u_o) + \int_0^T \zeta'_{t_1,t_2}(t)\Big[\boldsymbol{F}(u_\varepsilon(t)) - \tfrac{\varepsilon}{2}\int_{\Omega^*}|\partial_t u_\varepsilon|^2 dx\Big]dt$$

holds true. Since $|\partial_t u_\varepsilon|^2 \ge 0$ this shows

$$\begin{aligned}\int_{t_1}^{t_2} \boldsymbol{F}(u_\varepsilon(t))\,dt &\le (t_2-t_1)\boldsymbol{F}(u_o) + \tfrac{\varepsilon}{2}\int_{t_1}^{t_2}\int_{\Omega^*}|\partial_t u_\varepsilon|^2\,dxdt\\ &\le \big(t_2-t_1+\tfrac{\varepsilon}{2}\big)\boldsymbol{F}(u_o),\end{aligned} \tag{5.9}$$

for any $0 \le t_1 < t_2 \le T$. For the last estimate we used the bound (5.6) for the time derivative. This preceding estimate implies in particular that

$$\int_{t_1}^{t_2} \mathbf{TV}(u(t))\,dt \le \big(t_2-t_1+\tfrac{\varepsilon}{2}\big)\boldsymbol{F}(u_o) \tag{5.10}$$

holds true.

5.4. **Passage to the limit.** In this Section we pass to the limit $\varepsilon \downarrow 0$ in the sequence of $\mathcal{F}_\varepsilon$-minimizers u_ε on Ω_T and thereby prove Theorem 1.2. By (5.6), (5.7) and (5.10) the family $(u_\varepsilon)_{\varepsilon>0}$ of $\mathcal{F}_\varepsilon$-minimizing functions is bounded in $L^2(\Omega_T^*)$, the corresponding time derivatives $\partial_t u_\varepsilon$ are bounded in $L^2(\Omega_T^*)$ and the total variations $t \mapsto \mathbf{TV}(u_\varepsilon(t))$ are bounded in $L^1(0,T)$; all assertions hold uniformly with respect to $\varepsilon \in (0,1]$. Therefore [35, Theorem 1] (see also Lemma 8.1 below) ensures the existence of a subsequence $\varepsilon_j \downarrow 0$, which we still denote by ε, and moreover, of a measurable function $u\colon \Omega_T^* \to \mathbb{R}$ such that there holds:

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } L^1(\Omega_T^*),\\ u_\varepsilon \to u & \text{a.e. on } \Omega_T^*,\\ u_\varepsilon \rightharpoonup u, \partial_t u_\varepsilon \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T^*).\end{cases} \tag{5.11}$$

From Lemma 2.1 we conclude that $u \in L^1_w(0,T;\mathrm{BV}(\Omega^*))$. By the lower-semicontinuity with respect to weak L^2-convergence and (5.6) we have

$$\int_0^\infty\int_{\Omega^*}|\partial_t u|^2\,dxdt \le \liminf_{\varepsilon\downarrow 0}\int_0^\infty\int_{\Omega^*}|\partial_t u_\varepsilon|^2\,dxdt \le \boldsymbol{F}(u_o). \tag{5.12}$$

Moreover, using the lower semicontinuity of the functional $\boldsymbol{F}$ on the time slices, Fatou's lemma and (5.9) we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\boldsymbol{F}(u(t))\,dt &\le \int_{t_1}^{t_2}\liminf_{\varepsilon\downarrow 0}\boldsymbol{F}(u_\varepsilon(t))\,dt\\ &\le \liminf_{\varepsilon\downarrow 0}\int_{t_1}^{t_2}\boldsymbol{F}(u_\varepsilon(t))\,dt\\ &\le (t_2-t_1)\boldsymbol{F}(u_o) < \infty.\end{aligned}$$

We apply this inequality with $t_1 = 0$ and $t_2 = T$. This yields

$$0 \le \int_0^T \boldsymbol{F}(u(t))\,dt < \infty, \tag{5.13}$$

proving that u has finite $\boldsymbol{F}$-energy, so that the left-hand side in (1.7) is finite. Now, from (5.8) and the fact that $u_\varepsilon(0) = u_o$, we conclude as in the proof of Lemma 5.1 that also $u(0) = u_o$ in the usual L^2-sense. Finally, we use the pointwise a.e. convergence from $(5.11)_2$, to deduce that

$$|u(x,t)-u_o(x)| \le |u(x,t)-u_\varepsilon(x,t)| + |u_\varepsilon(x,t)-u_o(x)| \to 0 \quad \text{as } \varepsilon\downarrow 0,$$

for a.e. $(x,t) \in (\Omega^*\setminus\Omega)\times(0,\infty)$. Hence, $u(x,t) = u_o(x)$ for a.e. $x \in \Omega^*\setminus\Omega$ and a.e. $t \in (0,T)$. At this stage it remains to show that the limit function u is a variational solution in the sense of Definition 1.1. In view of (5.13) it suffices to consider $v \in L^1_w(0,T;\mathrm{BV}_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega_T^*)$, having finite $\boldsymbol{F}$-energy, i.e.

$$\int_0^T \boldsymbol{F}(v(t))\,dt < \infty, \tag{5.14}$$

since otherwise (1.7) trivially holds. For $\theta \in (0, \frac{T}{2})$ we define the following cut-off function with respect to time:

$$\zeta_\theta(t) := \begin{cases} \frac{1}{\theta}t & \text{if } t \in [0,\theta) \\ 1 & \text{if } t \in [\theta, T-\theta] \\ \frac{1}{\theta}(T-t) & \text{if } t \in (T-\theta, T]. \end{cases}$$

For fixed $\varepsilon \in (0,1]$ we consider $\varphi := v - u_\varepsilon$. Since $\varphi \in L^1_w(0,T;\mathrm{BV}_0(\Omega))$ with $\partial_t\varphi \in L^2(\Omega_T^*)$ satisfies (5.3), we are allowed to use (5.4) with φ and $\zeta = \zeta_\theta$. The resulting inequality can be re-written as

$$\begin{aligned} &\int_0^T \boldsymbol{F}(u_\varepsilon(t))\,dt \\ &\quad\le \int_0^T (1-\zeta_\theta(t))\boldsymbol{F}(u_\varepsilon(t))\,dxdt + \int_0^T\int_{\Omega^*} \zeta_\theta \partial_t u_\varepsilon (v-u_\varepsilon)\,dxdt \\ &\qquad + \int_0^T \zeta_\theta(t)\boldsymbol{F}(v(t))\,dt + \varepsilon\int_0^T\int_{\Omega^*}\big[\zeta_\theta'\partial_t u_\varepsilon(v-u_\varepsilon) + \zeta_\theta\partial_t u_\varepsilon \partial_t(v-u_\varepsilon)\big]\,dxdt \\ &\quad =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon. \end{aligned}$$

The meaning of $\mathrm{I}_\varepsilon - \mathrm{IV}_\varepsilon$ is obvious in this context. If $\theta \ge \varepsilon$, the term I_ε can be bounded with the help of (5.9) as follows:

$$\mathrm{I}_\varepsilon \le \int_0^\theta \boldsymbol{F}(u_\varepsilon(t))\,dt + \int_{T-\theta}^T \boldsymbol{F}(u_\varepsilon(t))\,dt \le (2\theta+\varepsilon)\boldsymbol{F}(u_o) \le 3\theta\boldsymbol{F}(u_o).$$

The term II_ε can be re-written in the form

$$\mathrm{II}_\varepsilon = \int_0^T\int_{\Omega^*}\zeta_\theta\partial_t v(v-u_\varepsilon)\,dxdt - \tfrac12\int_0^T\int_{\Omega^*}\zeta_\theta\partial_t|v-u_\varepsilon|^2\,dxdt\,.$$

Since $\zeta_\theta(T) = 0 = \zeta_\theta(0)$ we obtain for the second term on the right-hand side by an integration by parts that

$$\begin{aligned} -\tfrac12\int_0^T\int_{\Omega^*}\zeta_\theta\partial_t|v-u_\varepsilon|^2\,dxdt &= \tfrac12\int_0^T\int_{\Omega^*}\zeta_\theta'|v-u_\varepsilon|^2\,dxdt \\ &= \tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_\varepsilon|^2\,dxdt - \tfrac{1}{2\theta}\int_{T-\theta}^T\int_{\Omega^*}|v-u_\varepsilon|^2\,dxdt. \end{aligned}$$

The first term on the right-hand of the preceding inequality is now treated as follows:

$$\begin{aligned} &\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_\varepsilon|^2\,dxdt \\ &\quad\le \bigg[\Big(\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_o|^2\,dxdt\Big)^{\frac12} + \Big(\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|u_\varepsilon-u_o|^2\,dxdt\Big)^{\frac12}\bigg]^2 \\ &\quad\le \bigg[\Big(\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_o|^2\,dxdt\Big)^{\frac12} + \Big(\tfrac{\theta}{2}\boldsymbol{F}(u_o)\Big)^{\frac12}\bigg]^2. \end{aligned}$$

Here we used (5.8) with $s=0$ to estimate the second term. Collecting terms and using the weak convergence $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega_T^*)$ we can pass to the limit $\varepsilon \downarrow 0$ in II_ε. We obtain that

$$\liminf_{\varepsilon\downarrow 0}\mathrm{II}_\varepsilon \le \int_0^T\int_{\Omega^*}\zeta_\theta\partial_t v(v-u)\,dxdt - \tfrac{1}{2\theta}\int_{T-\theta}^T\int_{\Omega^*}|v-u|^2\,dxdt$$

$$+\Bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_o|^2\,dxdt\bigg)^{\frac12}+\Big(\tfrac{\theta}{2}\boldsymbol{F}(u_o)\Big)^{\frac12}\Bigg]^2.$$

Finally, since $\partial_t u_\varepsilon$ and u_ε are uniformly bounded in $L^2(\Omega_T^*)$, we have that $\mathrm{IV}_\varepsilon\to 0$ in the limit $\varepsilon\downarrow 0$. Inserting the previous inequalities above and using the lower semicontinuity of the variational functional $\boldsymbol{F}$ with respect to L^1-convergence, we arrive at

$$\begin{aligned}\int_0^T\boldsymbol{F}(u(t))\,dt&\le\liminf_{\varepsilon\downarrow 0}\int_0^T\boldsymbol{F}(u_\varepsilon(t))\,dt\\&\le\int_0^T\zeta_\theta(t)\bigg[\int_{\Omega^*}\partial_t v(v-u)\,dx+\boldsymbol{F}(v(t))\bigg]\,dt+3\theta\boldsymbol{F}(u_o)\\&\quad+\Bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_{\Omega^*}|v-u_o|^2\,dxdt\bigg)^{\frac12}+\Big(\tfrac{\theta}{4}\boldsymbol{F}(u_o)\Big)^{\frac12}\Bigg]^2\\&\quad-\tfrac{1}{2\theta}\int_{T-\theta}^T\int_{\Omega^*}|v-u|^2\,dxdt.\end{aligned}$$

Note, that this last inequality holds true for any $\theta\in(0,\frac{T}{2})$, and therefore we can pass to the limit $\theta\downarrow 0$ in the right-hand side. We arrive at

$$\begin{aligned}\int_0^T\boldsymbol{F}(u(t))\,dt\le\int_0^T\bigg[\int_{\Omega^*}\big[\partial_t v(v-u)\,dx+\boldsymbol{F}(v(t))\bigg]\,dt\\+\tfrac12\|v(0)-u_o\|^2_{L^2(\Omega^*)}-\tfrac12\|(v-u)(T)\|^2_{L^2(\Omega^*)}.\end{aligned}$$

This proves the claim that u is a variational solution to (1.7) on Ω_T^*. By Lemma 4.2 u is the unique variational solution on Ω_T^*. Note, that $T>0$ was arbitrary. Therefore, given $0<T_1<T_2<\infty$, we denote by u_1 and u_2 the unique variational solutions on $\Omega^*_{T_1}$ and $\Omega^*_{T_2}$, respectively. Then, the argument in §2.3 ensures that u_2 is also a variational solution on the smaller cylinder $\Omega^*_{T_1}$, which coincides by the comparison principle with u_1 there, i.e. $u_1\equiv u_2$ on $\Omega^*_{T_1}$. This allows us to construct a unique global variational solution and finishes the proof of Theorem 1.2. □

6. Deblurring operators (Proof of Theorem 1.5)

Since the proof of Theorem 1.5 is quite similar to the one of Theorem 1.2 from the last section, we shall only indicate the main differences and refer to §5 for the details.

6.1. A sequence of minimizers to a variational functional on Ω_T. Let $T\in(0,\infty)$. For $\varepsilon\in(0,1]$ we consider variational integrals of the form

$$\mathcal{F}_\varepsilon(v):=\int_0^T e^{-\frac{t}{\varepsilon}}\bigg[\tfrac12\int_\Omega|\partial_t v|^2\,dx+\tfrac{1}{\varepsilon}\boldsymbol{F}(v(t))\bigg]\,dt,$$

where the dissipative part of the functional is given in terms of the functionals defined in (1.9), i.e. by

$$\boldsymbol{F}(v):=\mathbf{TV}(v)+\tfrac{\kappa}{2}\int_\Omega\big|\mathbf{K}[v]-u_o\big|^2\,dx.$$

Recall that $\mathbf{K}\colon L^1(\Omega)\to L^2(\Omega)$ is a bounded, linear operator, and $\kappa\ge 1$ is a large penalization parameter. We define the function space $\mathcal{K}_\varepsilon$ as in §5.1, but with Ω instead of Ω^*. The subclass $\mathcal{K}_{\varepsilon,u_o}$ consists of those mappings $v\in\mathcal{K}_\varepsilon$ satisfying the initial condition $v(0)=u_o$ in the usual L^2-sense. Note that – contrary to §5.1 – $\mathcal{F}_\varepsilon$ is finite on $\mathcal{K}_{\varepsilon,u_o}$, since $v\in L^2(\Omega_T)$ (actually we have $v\in C^{0,\frac12}\big([0,T],L^2(\Omega)\big)$) by the assumption $\partial_t v\in L^2(\Omega_T)$. We note that (5.1) continues to hold in the present setting (with Ω^* replaced by Ω).

Lemma 6.1. *For any $\varepsilon\in(0,1]$, the functional $\mathcal{F}_\varepsilon$ admits a unique minimizer $u_\varepsilon\in\mathcal{K}_{\varepsilon,u_o}$.*

Proof. We consider a minimizing sequence $u_j \in \mathcal{K}_{\varepsilon,u_o}$, i.e.

$$\lim_{j\to\infty} \mathcal{F}_\varepsilon(u_j) = \inf_{u\in\mathcal{K}_{\varepsilon,u_o}} \mathcal{F}_\varepsilon(u) \le \mathcal{F}_\varepsilon(u_o) = \boldsymbol{F}(u_o).$$

Without loss of generality we can assume that $\mathcal{F}_\varepsilon(u_j) \le \boldsymbol{F}(u_o) + 1$. Arguing exactly as in §5, i.e. following the argument leading to (5.2), we infer that u_j and $\partial_t u_j$ are uniformly bounded in $L^1(\Omega_T)$ and $(0,T) \ni t \mapsto \mathbf{TV}(v(t))$ is uniformly bounded in $L^1((0,T))$. This allows us to apply [35, Theorem 1] (see also Lemma 8.1 below) to deduce the existence of a subsequence, again denoted by $(u_j)_{j\in\mathbb{N}}$, and a measurable function $u\colon \Omega_T \to \mathbb{R}$, such that

$$\begin{cases} u_j \to u & \text{strongly in } L^1(\Omega_T), \\ u_j \to u & \text{a.e. in } \Omega_T, \\ u_j \rightharpoonup u,\ \partial_t u_j \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T). \end{cases}$$

Now, from Lemma 2.1 we conclude that $u \in L^1_w(0,T;\mathrm{BV}(\Omega))$. Next, for any $0 \le s < t \le T$ we get

$$\|u_j(t) - u_j(s)\|_{L^2(\Omega)} \le \sqrt{|t-s|}\, \|\partial_t u_j\|_{L^2(\Omega_T)} \le e^{\frac{T}{\varepsilon}} \big[\boldsymbol{F}(u_o) + 1\big] \sqrt{|t-s|},$$

so that $(u_j)_{j\in\mathbb{N}}$ is uniformly bounded in $C^{0,\frac12}([0,T];L^2(\Omega))$. Therefore we can choose the subsequence from above in such a way that it converges in $L^\infty(0,T;L^2(\Omega))$ to u. This implies in particular that the limit function u attains the initial datum $u(0) = u_o$ in the usual L^2-sense. Having arrived at this stage, we can use the boundedness of the operator $\mathbf{K}\colon L^1(\Omega) \to L^2(\Omega)$ to conclude that the lower order term is continuous with respect to $L^1(\Omega)$-convergence, i.e. for a.e. $t \in (0,T)$ we have

$$\lim_{j\to\infty} \int_\Omega \big|\mathbf{K}[u_j(t)] - u_o\big|^2 \,dx = \int_\Omega \big|\mathbf{K}[u(t)] - u_o\big|^2 \,dx.$$

In combination with the lower semi-continuity of the total variation with respect to $L^1(\Omega)$-convergence this implies the lower-semicontinuity of $\boldsymbol{F}$, i.e.

$$\boldsymbol{F}(u(t)) \le \liminf_{j\to\infty} \boldsymbol{F}(u_j(t))$$

holds true for a.e. $t \in (0,T)$. From this point the final part of the proof is exactly as in the proof of Lemma 5.1. Using the lower-semicontinuity of $\boldsymbol{F}$ and Fatou's Lemma we obtain

$$\begin{aligned} \int_0^T e^{-\frac{t}{\varepsilon}} &\Big[\int_\Omega \tfrac12 |\partial_t u|^2 \,dx + \tfrac{1}{\varepsilon}\boldsymbol{F}(u(t))\Big]\,dt \\ &\le \liminf_{j\to\infty} \int_0^T e^{-\frac{t}{\varepsilon}} \Big[\int_\Omega \tfrac12 |\partial_t u_j|^2 \,dx + \tfrac{1}{\varepsilon}\boldsymbol{F}(u_j(t))\Big]\,dt \\ &= \lim_{j\to\infty} \mathcal{F}_\varepsilon(u_j), \end{aligned}$$

proving that u is $\mathcal{F}_\varepsilon$-minimizing in $\mathcal{K}_{\varepsilon,u_o}$. The uniqueness easily follows, since the term containing the time derivative ensures the strict convexity of the functional $\mathcal{F}_\varepsilon$. □

6.2. **Proof of Theorem 1.5.** In principle one only has to replace in the proof of Theorem 1.2 the domain Ω^* by Ω and the functional $\boldsymbol{F}$ from (1.6) by the functional given in (1.9). The details are as follows: As in §5.2 we first establish (5.4). The argument is easier, since the maps $v_{\varepsilon,\delta}$ posses a finite $\mathcal{F}_\varepsilon$-energy. Also (5.3) is automatically fulfilled for functions $\varphi \in L^1_w(0,T;\mathrm{BV}(\Omega))$ with $\partial_t\varphi \in L^2(\Omega_T)$. The argument from §5.3 works exactly as before, again modulo the obvious replacements of Ω^* by Ω and of $\boldsymbol{F}$ from (1.6) by the functional defined in (1.9). The result is that (5.6), (5.9) and (5.10) hold true. Finally, the proof in §5.4 remains unchanged up to the uniqueness part, apart form the standard replacements (Ω^* by Ω and $\boldsymbol{F}$ from (1.6) by $\boldsymbol{F}$ from (1.9)). Finally, in the uniqueness argument we only have to apply Lemma 4.3 instead of Lemma 4.2.

7. Parabolic Minimizers To The Total Variation Flow

The notion of *variational solutions to the total variation flow* we are using throughout the paper has been introduced by Lichnewsky & Temam [30] in the context of the minimal surface problem. This notion is related to the concept of *parabolic minimizers*, introduced by Wieser [38] in the context of gradient flows to convex energy functionals. For the latter notion different regularity properties have been studied, such as the self-improving property of higher integrability, local boundedness and Hölder continuity in the scalar case or partial regularity again in the vectorial case. Moreover, the extension to the metric space setting came into the focus of research. In this Section we aim to establish that variational solutions to the total variation flow are parabolic minimizers. Before going into the details we first give the Definition of a parabolic minimizer, introduced by Wieser [38]. We only consider the case of functionals of the type (1.3), and assume that the initial datum u_o satisfies (1.5). For functionals from (1.9) we refer to Remark of the type (1.3), respectively (1.10) for functionals of the type (1.9) we refer to Remark 7.3.

Definition 7.1. A measurable map $u\colon \Omega^*_\infty \to \mathbb{R}$ is termed *parabolic minimizer of the total variation flow* if and only if for any $T>0$ we have

$$u \in L^1_w\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big) \quad \text{and} \quad \int_0^T \boldsymbol{F}(u(t))\,dt < \infty,$$

and moreover the following minimality condition

$$\int_0^T \bigg[\int_{\Omega^*} u\partial_t\varphi\,dx + \boldsymbol{F}(u(t))\bigg]\,dt \le \int_0^T \boldsymbol{F}\big((u+\varphi)(t)\big)\,dt \tag{7.1}$$

holds true, whenever $T>0$ and $\varphi\in C^\infty(\Omega^*_T)$ with $\operatorname{spt}\varphi\subset\Omega_T$. □

One could generalize Definition 7.1 in the sense that the right-hand side in the minimality condition (7.1) is replaced by

$$Q\int_0^T \boldsymbol{F}\big((u+\varphi)(t)\big)\,dt$$

for some fixed $Q\ge 1$. A map u satisfying instead of (7.1) the analogous inequality with the factor $Q>1$ in front of the right-hand side integral could be termed *parabolic Q-minimizer to the total variation flow*.

Now we are in the position to provide the argument establishing that variational solutions are in fact parabolic minimizers. The precise statement is as follows:

Proposition 7.2. *If u is a variational solution in the sense of Definition 1.1, then it is also a parabolic minimizer in the sense of Definition 7.1.*

Proof. We consider a fixed $T>0$ and a testing function $\varphi\in C^\infty(\Omega^*_T)$ with $\operatorname{spt}\varphi\subset\Omega_T$. By Theorem 1.3 we know that $\partial_t u\in L^2(\Omega^*_\infty)$. Our aim now, is to prove that (7.1) holds true. To establish this, we choose $v=u+s\varphi\in L^1_w\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big)$ with $s>0$ as comparison function in (1.7). Furthermore, without loss of generality we can assume that $\int_0^T \boldsymbol{F}((u+\varphi)(t))\,dt<\infty$, since otherwise (7.1) trivially holds. Taking into account that $\partial_t v\in L^2(\Omega^*_\infty)$, $v(0)=u_o$ and $v(T)=u(T)$ we obtain that

$$\int_0^T \boldsymbol{F}(u(t))\,dt \le \int_0^T \bigg[\int_{\Omega^*} \partial_t(u+s\varphi)s\varphi\,dx + \boldsymbol{F}\big((u+s\varphi)(t)\big)\bigg]\,dt$$

holds true. In the first term on the right-hand side we perform and integration by parts and note that no boundary terms occur, since $\varphi(0)=0=\varphi(T)$. In the second term on the right-hand side we use the convexity of the variational integral $\boldsymbol{F}$. Proceeding in this way we obtain

$$\int_0^T \bigg[\int_{\Omega^*} s(u+s\varphi)\partial_t\varphi\,dx + \boldsymbol{F}(u(t))\bigg]\,dt \le \int_0^T (1-s)\boldsymbol{F}(u(t)) + s\boldsymbol{F}\big((u+\varphi)(t)\big)\,dt.$$

Now, we subtract on both sides the integral $(1-s)\int_0^T \boldsymbol{F}(u(t))\,dt$, note that the integral is finite, and divide the resulting inequality by $s>0$. We obtain

$$\int_0^T \bigg[\int_{\Omega^*} (u+s\varphi)\partial_t\varphi\,dx + \boldsymbol{F}(u(t))\bigg]\,dt \le \int_0^T \boldsymbol{F}\big((u+\varphi)(t)\big)\,dt.$$

In the preceding inequality we pass to the limit $s\downarrow 0$ and end up with the inequality

$$\int_0^T \bigg[\int_{\Omega^*} u\partial_t\varphi\,dx + \boldsymbol{F}(u(t))\bigg]\,dt \le \int_0^T \boldsymbol{F}\big((u+\varphi)(t)\big)\,dt.$$

The previous inequality holds true for any $\varphi\in C^\infty(\Omega_T^*)$ with $\operatorname{spt}\varphi\subset\Omega_T$. But this means, that u is a parabolic minimizer in the sense of Definition 7.1 and completes the proof. □

Remark 7.3. The preceding considerations also apply to variational solutions in the sense of Definition 1.4 for functionals of the type (1.9). Since the definition and the argument from the proof of Proposition 7.2 are the same (modulo the obvious replacements), we skip the details. □

8. Appendix: Compactness

Throughout the paper we used several times a compactness argument for sequences of functions in $L^1_w-\mathrm{BV}$ to conclude sub-convergence in L^1. For the sake of completeness we will provide the argument (following [35, Theorem 1] closely) in this Section.

Lemma 8.1. *Suppose that the sequence $u_j\in L^1_w\big(0,T;\mathrm{BV}(\Omega_T)\big)$, $j\in\mathbb{N}$, satisfies*

$$\sup_{j\in\mathbb{N}}\bigg[\|u_j\|_{L^1(\Omega_T)} + \|\partial_t u_j\|_{L^1(\Omega_T)} + \int_0^T \mathbf{TV}(u_j(t))\,dt\bigg] < \infty. \tag{8.1}$$

Then, $(u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1(\Omega_T)$.

Proof. For $h\in(0,T)$ we define the Steklov mean $(u_j)_h$ of u_j by

$$(u_j)_h(t) := \tfrac{1}{h}\int_t^{t+h} u_j(s)\,ds \qquad \text{for } t\in[0,T-h].$$

Then, $(u_j)_h\in C^0\big([0,T-h];L^1(\Omega)\big)$ and for any $0\le t_1<t_2\le T-h$ there holds

$$\begin{aligned}\big\|(u_j)_h(t_2)-(u_j)_h(t_1)\big\|_{L^1(\Omega)} &= \bigg\|\int_{t_1}^{t_2}\partial_s(u_j)_h(s)\,ds\bigg\|_{L^1(\Omega)}\\ &= \bigg\|\frac{1}{h}\int_{t_1}^{t_2}\big[u_j(s+h)-u_j(s)\big]\,ds\bigg\|_{L^1(\Omega)}\\ &\le \tfrac{1}{h}\int_{t_1}^{t_2}\int_s^{s+h}\|\partial_\tau u_j(\tau)\|_{L^1(\Omega)}\,d\tau\,ds\\ &\le \tfrac{1}{h}(t_2-t_1)\|\partial_t u_j\|_{L^1(\Omega_T)}.\end{aligned}$$

Therefore, for any given fixed $h\in(0,T)$ the sequence of Steklov averages $\big((u_j)_h\big)_{j\in\mathbb{N}}$ is uniformly equicontinuous in $C^0\big([0,T-h];L^1(\Omega)\big)$. Now, for such a fixed $h\in(0,T)$ and $t\in[0,T-h]$ we consider on the time-slice the sequence $\big((u_j)_h(t)\big)_{j\in\mathbb{N}}$ of $L^1(\Omega)$-functions. For $\phi\in C^1_0(\Omega,\mathbb{R}^n)$ with $\|\phi\|_{L^\infty(\Omega)}\le 1$ we use Fubini's theorem and assumption (8.1) to conclude that

$$\begin{aligned}\int_\Omega (u_j)_h(t)\operatorname{div}\phi\,dx &= \tfrac{1}{h}\int_\Omega\int_t^{t+h} u_j(s)\,ds\operatorname{div}\phi\,dx = \tfrac{1}{h}\int_t^{t+h}\int_\Omega u_j(s)\operatorname{div}\phi\,dx\,ds\\ &\le \tfrac{1}{h}\int_t^{t+h}\mathbf{TV}(u_j(s))\,ds \le \tfrac{1}{h}\int_0^T\mathbf{TV}(u_j(s))\,ds.\end{aligned}$$

In the preceding inequality we first take the supremum over all $\phi \in C_0^1(\Omega,\mathbb{R}^n)$ with $\|\phi\|_{L^\infty(\Omega)} \le 1$, and then the supremum over $j\in\mathbb{N}$, to conclude that

$$\sup_{j\in\mathbb{N}} \mathbf{TV}\big((u_j)_h(t)\big) \le \tfrac{1}{h} \sup_{j\in\mathbb{N}} \int_0^T \mathbf{TV}(u_j(s))\,ds < \infty$$

holds true. Together with (8.1) this proves that for any fixed $h\in(0,T)$ the sequences $\big((u_j)_h(t)\big)_{j\in\mathbb{N}}$ with $t\in[0,T-h]$ are uniformly bounded in $\mathrm{BV}(\Omega)$, and therefore, by the BV-compactness theorem, relatively compact in $L^1(\Omega)$. Therefore, we are allowed to apply [35, Lemma 1], yielding that for any $h\in(0,T)$ the sequence $\big((u_j)_h\big)_{j\in\mathbb{N}}$ is relatively compact in $C^0\big([0,T-h];L^1(\Omega)\big)$ and hence also in $L^1\big(0,T-h;L^1(\Omega)\big)$.

We fix $T_1\in(0,T)$. Using again Fubini's theorem, we compute for $h\in(0,T-T_1)$ that

$$\begin{aligned}
\big\|(u_j)_h - u_j\big\|_{L^1(0,T_1;L^1(\Omega))} &= \int_0^{T_1}\int_\Omega \bigg| \frac{1}{h}\int_t^{t+h}\int_t^s \partial_\tau u_j(\tau)\,d\tau ds\bigg|\,dxdt \\
&\le \int_0^{T_1}\int_\Omega\int_t^{t+h} |\partial_\tau u_j(\tau)|\,d\tau dxdt \\
&= \int_0^{T_1+h}\int_{\max\{0,\tau-h\}}^{\min\{T_1,\tau\}}\int_\Omega \big|\partial_\tau u_j(\tau)\big|\,dtdxd\tau \\
&\le h\|\partial_t u_j\|_{L^1(\Omega_T)}
\end{aligned}$$

holds true. By (8.1) this proves that

$$\lim_{h\downarrow 0}\sup_{j\in\mathbb{N}} \big\|(u_j)_h - u_j\big\|_{L^1(0,T_1;L^1(\Omega))} = 0,$$

i.e. the sequence $(u_j)_{j\in\mathbb{N}}$ is the uniform limit in $L^1(0,T_1;L^1(\Omega))$ of $\big((u_j)_h\big)_{j\in\mathbb{N}}$ as $h\downarrow 0$; cf. [35, §2]. Since the sequence of Steklov averages $\big((u_j)_h\big)_{j\in\mathbb{N}}$ is relatively compact in $L^1\big(0,T_1;L^1(\Omega)\big)$ for any $h\in(0,T-T_1)$, we conclude that also the sequence $(u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1\big(0,T_1;L^1(\Omega)\big)$. In particular, choosing $T_1=\frac12 T$ we have shown that $(u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1\big(0,T/2;L^1(\Omega)\big)$.

Now, we apply the above argument to the sequence $(\tilde u_j)_{j\in\mathbb{N}}$, where $\tilde u_j\colon\Omega_T\to\mathbb{R}$ is defined by $\tilde u_j(x,t) := u_j(x,T-t)$; note that $\tilde u_j$ fulfills the assumption(8.1). We therefore conclude that $(\tilde u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1\big(0,T/2;L^1(\Omega)\big)$. But this is the same as the relative compactness of $(u_j)_{j\in\mathbb{N}}$ in $L^1\big(T/2,T;L^1(\Omega)\big)$. Combining this with the compactness property from above, yields the claim, i.e. that $(u_j)_{j\in\mathbb{N}}$ is relatively compact in $L^1\big(0,T;L^1(\Omega)\big)\cong L^1(\Omega_T)$. □

REFERENCES

[1] R. Acar and C.R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*10 (6):1217–1229, 1994.

[2] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures.* Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.

[3] F. Andreu, C. Ballester, V. Caselles and J.M. Mazón. The Dirichlet problem for the total variation flow. *J. Funct. Anal.* 180(2):347–403, 2001.

[4] F. Andreu, C. Ballester, V. Caselles and J.M. Mazón. Minimizing total variation flow. *Differential Integral Equations* 14(3):321–360, 2001.

[5] F. Andreu, V. Caselles, J.I. Díaz and J. I. J.M. Mazón. Some qualitative properties for the total variation flow. *J. Funct. Anal.* 188(2):516–547, 2002.

[6] F. Andreu, V. Caselles and J.M. Mazón. Parabolic quasilinear equations minimizing linear growth functionals. *Progress in Mathematics* 223, Birkhäuser Verlag, Basel, 2004.

[7] F. Andreu, J.M. Mazón and J.S. Moll. The total variation flow with nonlinear boundary conditions. *Asymptot. Anal.* 43(1-2): 9–46, 2005.

[8] F. Andreu, J.M. Mazón, J.S. Moll, and V. Caselles. The minimizing total variation flow with measure initial conditions. *Commun. Contemp. Math.* 6(3): 431–494, 2004.

[9] F. Andreu, V. Caselles and J.M. Mazón. A parabolic quasilinear problem for linear growth functionals. *Rev. Mat. Iberoamericana* 18: 135–185, 2002.
[10] F. Andreu, V. Caselles and J.M. Mazón. The Cauchy problem for a strongly degenerate quasilinear equation. *J. Eur. Math. Soc. (JEMS)* 7(3): 361–393, 2005.
[11] F. Andreu, V. Caselles and J.M. Mazón. A strongly degenerate quasilinear equation: the parabolic case. *Arch. Ration. Mech. Anal.*176 (3): 415–453, 2005.
[12] F. Andreu, V. Caselles, J.M. Mazón and J.S. Moll. Finite propagation speed for limited flux diffusion equations. *Arch. Ration. Mech. Anal.*182(2): 269–297, 2006.
[13] G. Bellettini, V. Caselles and M. Novaga. The total variation flow in R^N. *J. Differential Equations* 184 (2):475–525, 2002.
[14] V. Bögelein, F. Duzaar and P. Marcellini. Parabolic equations with p, q-growth. *J. Math. Pures Appl.*, 100(4):535–563, 2013.
[15] V. Bögelein, F. Duzaar and P. Marcellini. Existence of evolutionary variational solutions via the calculus of variations. *J. Differential Equations*, 256: 3912–3942, 2014.
[16] V. Bögelein, F. Duzaar and P. Marcellini. Parabolic systems with p, q-growth: a variational approach. *Arch. Ration. Mech. Anal.*, 210(1):219–267, 2013.
[17] K. Bredies and D. Lorenz. Mathematische Bildverarbeitung. *Viewig + Teubner Verlag 2011*.
[18] A. Chambolle and P.L. Lions. Image recovery via total variation minimization and related problems. *Numer. Math.* 76 (2):167–188, 1997.
[19] T.F. Chan and S. Esedoḡlu. Aspects of total variation regularized L^1 function approximation. *SIAM J. Appl. Math.*, 65(5):1817–1837, 2005.
[20] T.F. Chan and J. Shen. Image processing and analysis. *Society for Industrial and Applied Mathematics (SIAM)*, 2005.
[21] E. De Giorgi. Conjectures concerning some evolution problems. (Italian) A celebration of John F. Nash, Jr. *Duke Math. J.* 81(2):255–268, 1996.
[22] E. De Giorgi. *Selected papers.* Springer-Verlag, Berlin, 2006 (edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo).
[23] I. Ekeland and R. Teman. *Convex Analysis and Variational Problems*. North-Holland, 1976.
[24] S. Esedoḡlu and S. Osher. Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. *Comm. Pure Appl. Math.* 57(12) :1609–1626, 2004.
[25] E. Giusti. *Minimal surfaces and functions of bounded variation*. Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.
[26] H. Hakkarainen and J. Kinnunen. The BV-capacity in metric spaces. *Manuscripta Math.* 132:51–73, 2010.
[27] R. Hardt and X. Zhou. An evolution problem for linear growth functionals. *Commun. in Partial Differential Equations* 19(11&12):1879–1907, 1994.
[28] Z. Jin and X. Yang. Analysis of a new variational model for multiplicative noise removal, *J. Math. Anal. Appl.* 362:415–426, 2010.
[29] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)* 185(3):411–435, 2006.
[30] A. Lichnewsky and R. Temam. Pseudosolutions of the time-dependent minimal surface problem. *J. Differential Equations*, 30(3):340–364, 1978.
[31] S. Osher and O. Scherzer. G-norm properties of bounded variation regularization. *Commun. Math. Sci.* 2(2):237–254, 2004.
[32] L. Rudin, S. Osher and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D* 268:259–268, 1992.
[33] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen. *Variational methods in imaging.* Applied Mathematical Sciences, 167. Springer, New York, 2009.
[34] E. Serra and P. Tilli. Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. *Ann. of Math. (2)* 175(3):1551–1574, 2012.
[35] J. Simon. Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (IV) 146:65–96, 1987.
[36] E. Spadaro and U. Stefanelli. A variational view at the time-dependent minimal surface equation. J. Evol. Equ. 11(4) : 793–809, 2011.
[37] U. Stefanelli. The De Giorgi conjecture on elliptic regularization *Math. Models Methods Appl. Sci.* 21: 1377–1394, 2011.
[38] W. Wieser. Parabolic Q-minima and minimal solutions to variational flow. *Manuscripta Math.* 59(1):63–107, 1987.

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