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**Modica type gradient estimates for
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REPORT No. 43, 2013/2014, fall

ISSN 1103-467X

ISRN IML-R- -43-13/14- -SE+fall

Modica type gradient estimates for reaction-diffusion equations

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Abstract We continue the study of Modica type gradient estimates for inhomogeneous parabolic equations initiated in [BG]. First, we show that for the parabolic minimal surface equation with a semilinear force term if a certain gradient estimate is satisfied at $t = 0$, then it holds for all later times $t > 0$. We then establish analogous results for reaction-diffusion equations such as (5) below in $\Omega \times [0, T]$, where Ω is an epigraph such that the mean curvature of $\partial\Omega$ is nonnegative. We then turn our attention to settings where such gradient estimates are valid without any a priori information on whether the estimate holds at some earlier time. Quite remarkably (see Theorem 4.1, Theorem 4.4 and Theorem 5.1), this is true for $\mathbb{R}^n \times (-\infty, 0]$ and $\Omega \times (-\infty, 0]$, where Ω is an epigraph satisfying the geometric assumption mentioned above, and for $M \times (-\infty, 0]$, where M is a connected, compact Riemannian manifold with nonnegative Ricci tensor. As a consequence of the gradient estimate (7), we establish a rigidity result (see Theorem 6.1 below) for solutions to (5) which is the analogue of Theorem 5.1 in [CGS]. Finally, motivated by Theorem 6.1, we close the paper by proposing a parabolic version of the famous conjecture of De Giorgi also known as the ε -version of the Bernstein theorem.

1 introduction

In his pioneering paper [Mo] L. Modica proved that if u is a (smooth) bounded entire solution of the semilinear Poisson equation $\Delta u = F'(u)$ in \mathbb{R}^n , with nonlinearity $F \geq 0$, then u satisfies the a priori gradient bound

$$|Du|^2 \leq 2F(u). \tag{1}$$

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With a completely different approach from Modica's original one, this estimate was subsequently extended in [CGS] to nonlinear equations in which the leading operator is modeled either on the p -Laplacian $\operatorname{div}(|Du|^{p-2}Du)$, or on the minimal surface operator $\operatorname{div}((1+|Du|^2)^{-1/2}Du)$, and later to more general integrands of the calculus of variations in [DG]. More recently, in their very interesting paper [FV1] Farina and Valdinoci have extended the Modica estimate (1) to domains in \mathbb{R}^n which are epigraphs whose boundary has nonnegative mean curvature, and to compact manifolds having nonnegative Ricci tensor, see [FV3], and also the sequel paper with Sire [FSV].

It is by now well-known, see [Mo], [CGS], [AC], [DG], that, besides its independent interest, an estimate such as (1) implies Liouville type results, monotonicity properties of the relevant energy and it is also connected to a famous conjecture of De Giorgi (known as the ε -version of the Bernstein theorem) which we discuss at the end of this introduction and in Section 6 below, and which nowadays still constitutes a largely unsolved problem.

In the present paper we study Modica type gradient estimates for solutions of some nonlinear parabolic equations in \mathbb{R}^n and, more in general, in complete Riemannian manifolds with nonnegative Ricci tensor, and in unbounded domains satisfying the above mentioned geometric assumptions in [FV1]. In the first part of the paper we continue the study initiated in the recent work [BG], where we considered the following inhomogeneous variant of the normalized p -Laplacian evolution in $\mathbb{R}^n \times [0, T]$,

$$|Du|^{2-p} \{ \operatorname{div}(|Du|^{p-2}Du) - F'(u) \} = u_t, \quad 1 < p \leq 2. \quad (2)$$

In [BG] we proved that if a bounded solution u of (2) belonging to a certain class H (see [BG] for the relevant definition) satisfies the following gradient estimate at $t = 0$ for a.e. $x \in \mathbb{R}^n$,

$$|Du(x, t)|^p \leq \frac{p}{p-1} F(u(x, t)), \quad (3)$$

then such estimate continues to hold at any given time $t > 0$. On the function F we assumed that $F \in C_{loc}^{2, \beta}(\mathbb{R})$ and $F \geq 0$. These same assumptions will be assumed throughout this whole paper.

In Section 2 we show that a similar result is true for the following inhomogeneous variant of the minimal surface parabolic equation

$$(1 + |Du|^2)^{1/2} \left\{ \operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{1/2}} \right) - F'(u) \right\} = u_t, \quad (4)$$

see Theorem 2.1 below. The equation (4) encompasses two types of equations: when $F(u) = 0$ it represents the equation of motion by mean curvature studied in [EH], whereas when $u(x, t) = v(x)$, then (4) corresponds to the steady state which is prescribed mean curvature equation.

In Section 3 we establish similar results for the reaction diffusion equation in $\Omega \times [0, T]$

$$\Delta u = u_t + F'(u), \quad (5)$$

where now Ω is an epigraph and the mean curvature of $\partial\Omega$ is nonnegative. Theorem 3.1 below constitutes the parabolic counterpart of the cited result in [FV1] for the following problem

$$\begin{cases} \Delta u = F'(u), & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, & u \geq 0 \text{ on } \Omega. \end{cases} \quad (6)$$

In that paper the authors proved that a bounded solution u to (6) satisfies the Modica estimate (1), provided that the mean curvature of $\partial\Omega$ is nonnegative.

In Section 4 we turn our attention to settings where global versions of such estimates for solutions to (5) can be established, i.e., when there is no a priori information on whether such an estimate hold at some earlier time t_0 . In Theorems 4.1 and 4.4 we show that, quite remarkably, respectively in the case $\mathbb{R}^n \times (-\infty, 0]$ and $\Omega \times (-\infty, 0]$, where Ω is an epigraph that satisfies the geometric assumption mentioned above, the a priori gradient estimate

$$|Du(x,t)|^2 \leq 2F(u(x,t)) \quad (7)$$

holds globally on a bounded solution u of (5).

In Section 5 we establish a parabolic generalization of the result in [FV3], but in the vein of our global results in Section 4. In Theorem 5.1 we prove that if M is a compact Riemannian manifold with $\text{Ric} \geq 0$, with Laplace-Beltrami Δ , then any bounded entire solution u to (5) in $M \times (-\infty, 0]$ satisfies (7). It remains to be seen whether our result, or for that matter the elliptic result in [FV3], remain valid when M is only assumed to be complete, but not compact.

Finally in Section 6, as a consequence of the a priori estimate (7) in Section 4, we establish an analogue of Theorem 5.1 in [CGS] for solutions to (5) in $\mathbb{R}^n \times (-\infty, 0]$. More precisely, in Theorem 6.1 below we show that if the equality in (7) holds at some (x_0, t_0) , then there exists a function $g \in C^2(\mathbb{R})$, $a \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, such that

$$u(x,t) = g(\langle a, x \rangle + \alpha). \quad (8)$$

In particular, u is independent of time and the level sets of u are vertical hyperplanes in $\mathbb{R}^n \times (-\infty, 0]$. This result suggests a parabolic version of the famous conjecture of De Giorgi (also known as the ε -version of the Bernstein theorem for minimal graphs) which asserts that entire solutions to

$$\Delta u = u^3 - u, \quad (9)$$

such that $|u| \leq 1$ and $\frac{\partial u}{\partial x_n} > 0$, must be one-dimensional, i.e., must have level sets which are hyperplanes, at least in dimension $n \leq 8$, see [dG]. We recall that the conjecture of De Giorgi has been fully solved for $n = 2$ in [GG1] and $n = 3$ in [AC], and it is known to fail for $n \geq 9$, see [dPKW]. Remarkably, it is still an open question for $4 \leq n \leq 8$. Additional fundamental progress on De Giorgi's conjecture is contained in the papers [GG2], [Sa]. For results concerning the p -Laplacian version of De Giorgi's conjecture, we refer the reader to the interesting paper [SSV]. For

further results, the state of art and recent progress on De Giorgi's conjecture, we refer to [CNV], [FV2], [FSV1] and the references therein.

In Section 7 motivated by our Theorem 6.1 below, we close the paper by proposing a parabolic version of De Giorgi's conjecture. It is our hope that it will stimulate interesting further research.

Acknowledgment: First author was supported in part by the second author's NSF Grant DMS-1001317 and by a postdoctoral grant of the Institute Mittag-Leffler. Second author was supported in part by NSF Grant DMS-1001317 and by a grant of the University of Padova, "Progetti d'Ateneo 2013". The paper was finalized during the first author's stay at the Institut Mittag-Leffler during the semester-long program *Homogenization and Random Phenomenon*. The first author would like to thank the Institute and the organizers of the program for the kind hospitality and the excellent working conditions. We would like to thank Matteo Novaga for kindly bringing to our attention that Conjecture 1 at the end of this paper is violated by the traveling wave solutions in [CGHNR] and [G] and for suggesting the amended Conjecture 2.

2 Forward Modica type estimates in $\mathbb{R}^n \times [0, T]$ for the generalized motion by mean curvature equation

In [CGS] it was proved that if $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a solution to

$$\operatorname{div} \left(\frac{Du}{(1+|Du|^2)^{1/2}} \right) = F'(u), \quad (10)$$

such that $|Du| \leq C$, then the following Modica type gradient estimate holds

$$\frac{(1+|Du|^2)^{1/2} - 1}{(1+|Du|^2)^{1/2}} \leq F(u). \quad (11)$$

In Theorem 2.1 below we generalize this result to the parabolic minimal surface equation (4). Such result also provides the counterpart of the above cited main result (2) in [BG] for the normalized parabolic p -Laplacian (3). Henceforth, by $v \in C_{loc}^{2,1}$, we mean that v has continuous derivatives of up to order two in the x variable and up to order one in the t variable. We would also like to mention that unlike the case when $F = 0$, further requirements on F need to be imposed to ensure that a bounded solution to (4) has bounded gradient, see e.g. Theorem 4 in [LU1]. This is why an L^∞ gradient bound is assumed in the hypothesis of the next theorem.

We recall that throughout the whole paper we assume that $F \in C_{loc}^{2,\beta}(\mathbb{R})$ for some $\beta > 0$, and that $F \geq 0$.

Theorem 2.1 *For a given $\varepsilon > 0$, let $u \in C_{loc}^{2,1}(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times (-\varepsilon, T])$ be a classical solution to (4) in $\mathbb{R}^n \times [0, T]$ such that $|Du| \leq C$. If u satisfies the following gradient estimate*

$$\frac{(1 + |Du|^2)^{1/2} - 1}{(1 + |Du|^2)^{1/2}} \leq F(u) \quad (12)$$

at $t = 0$, then u satisfies (12) for all $t > 0$.

Proof. Since $|Du| \leq C$ and $F \in C_{loc}^{2,\beta}$, it follows from the Schauder regularity theory of uniformly parabolic non-divergence equations (see Chapters 4 and 5 in [Li]), that $u \in H_{3+\alpha}(\mathbb{R}^n \times [0, T])$ for some $\alpha > 0$ which depends on β and the bounds on u and Du (see Chapter 4 in [Li] for the relevant notion). Now we let

$$\phi(s) = (s^2 + 1)^{1/2}, \quad s \in \mathbb{R}. \quad (13)$$

With this notation we have that u is a classical solution to

$$\operatorname{div}(\phi'(|Du|^2)Du) = \phi'(|Du|^2)u_t + F'(u) \quad (14)$$

Now given that $u \in H_{3+\alpha}(\mathbb{R}^n \times [0, T])$, one can repeat the arguments as in the proof of Theorem 5.1 in [BG] with ϕ as in (13). We nevertheless provide the details for the sake of completeness and also because the corresponding growth of ϕ in s is quite different from the one in Theorem 5.1 in [BG]. Let

$$\xi(s) = 2s\phi'(s) - \phi(s), \quad (15)$$

and define $\Lambda = \xi'$. We also define P as follows

$$P(u, x, t) = \xi(|Du(x, t)|^2) - 2F(u(x, t)). \quad (16)$$

With ϕ as in (13) above, we have that

$$P = 2 \frac{(1 + |Du|^2)^{1/2} - 1}{(1 + |Du|^2)^{1/2}} - 2F(u). \quad (17)$$

We note that the hypothesis that (12) be valid at $t = 0$ can be reformulated by saying that $P(\cdot, 0) \leq 0$. We next write (14) in the following manner

$$a_{ij}(Du) u_{ij} = f(u) + \phi' u_t,$$

where for $\sigma \in \mathbb{R}^n$ we have let

$$a_{ij}(\sigma) = 2\phi'' \sigma_i \sigma_j + \phi' \delta_{ij}. \quad (18)$$

Therefore, u satisfies

$$d_{ij} u_{ij} = \frac{f}{\Lambda} + \frac{\phi'}{\Lambda} u_t, \quad (19)$$

where $d_{ij} = \frac{a_{ij}}{\Lambda}$. By differentiating (18) with respect to x_k , we obtain

$$(a_{ij} (u_k)_i)_j = f' u_k + \phi' u_{tk} + 2\phi'' u_{hk} u_h u_t. \quad (20)$$

From the definition of P in (16) we have,

$$P_i = 2\Lambda u_{ki} u_k - 2f u_i, \quad P_t = 2\Lambda u_{kt} u_k - 2f u_t. \quad (21)$$

We now consider the following auxiliary function

$$w = w_R = P - \frac{M}{R} \sqrt{|x|^2 + 1} - \frac{ct}{R^{1/2}},$$

where $R > 1$ and M, c are to be determined subsequently. Note that $P \geq w$ for $t \geq 0$. Consider the cylinder $Q_R = B(0, R) \times [0, T]$. One can see that if M is chosen large enough, depending on the L^∞ norm of u and its first derivatives, then $w < 0$ on the lateral boundary of Q_R . In this situation we see that if w has a strictly positive maximum at a point (x_0, t_0) , then such point cannot be on the parabolic boundary of Q_R . In fact, since $w < 0$ on the lateral boundary, the point cannot be on such set. But it cannot be on the bottom of the cylinder either since, in view of (12), at $t = 0$ we have $w(\cdot, 0) \leq P(u(\cdot, 0)) \leq 0$.

Our objective is to prove the following claim:

$$w \leq K \stackrel{def}{=} R^{-\frac{1}{2}}, \quad \text{in } Q_R, \quad (22)$$

provided that M and c are chosen appropriately. This claim will be established in (43) below. We first fix a point (y, s) in \mathbb{R}^n . Now for all R sufficiently large enough, we have that $(y, s) \in Q_R$. We would like to emphasize over here that finally we let $R \rightarrow \infty$. Therefore, once (22) is established, we obtain from it and the definition of w that

$$P(u, y, s) \leq \frac{K'}{R^{1/2}}, \quad (23)$$

where K' depends on $\varepsilon, (y, s)$ and the bounds of the derivatives of u of order three. By letting $R \rightarrow \infty$ in (23), we find that

$$P(u, y, s) \leq 0. \quad (24)$$

The sought for conclusion thus follows from the arbitrariness of the point (y, s) .

In order to prove the claim (22) we argue by contradiction and suppose that there exist $(x_0, t_0) \in \overline{Q_R}$ at which w attains its maximum and for which

$$w(x_0, t_0) > K.$$

It follows that at (x_0, t_0) we must have

$$(\varepsilon^2 + |Du(x_0, t_0)|^2)^{-1/2} |Du(x_0, t_0)|^2 \geq \frac{1}{2} P(x_0, t_0) \geq \frac{1}{2} w(x_0, t_0) > \frac{1}{2} K, \quad (25)$$

which implies, in particular, that $Du(x_0, t_0) \neq 0$. Therefore, we obtain from (25)

$$|Du(x_0, t_0)| \geq (1 + |Du(x_0, t_0)|^2)^{-1/2} |Du(x_0, t_0)|^2 \geq \frac{1}{2} P(x_0, t_0) > \frac{1}{2} K. \quad (26)$$

On the other hand, since (x_0, t_0) does not belong to the parabolic boundary, from the hypothesis that w has its maximum at such point, we conclude that $w_t(x_0, t_0) \geq 0$ and $Dw(x_0, t_0) = 0$. These conditions translate into

$$P_t \geq \frac{c}{R^{1/2}}, \quad (27)$$

and

$$P_i = \frac{M}{R} \frac{x_{0,i}}{(|x_0|^2 + 1)^{1/2}}. \quad (28)$$

Now

$$(d_{ij}w_i)_j = (d_{ij}P_i)_j - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j,$$

where

$$(d_{ij}P_i)_j = 2 \left(\frac{a_{ij}}{\Lambda} (\Lambda u_{ki} u_k - f u_i) \right)_j = 2(a_{ij} (u_k)_i u_k)_j - 2(f d_{ij} u_i)_j. \quad (29)$$

After a simplification, (29) equals

$$2a_{ij} (u_k)_i u_k + 2a_{ij} u_{ki} u_{kj} - 2f' d_{ij} u_i u_j - 2f d_{ij} u_{ij} - 2f (d_{ij})_j u_i.$$

We notice that

$$d_{ij}u_i u_j = \frac{2\phi'' u_i u_j u_i u_j + \phi' \delta_{ij} u_i u_j}{\Lambda} = |Du|^2.$$

Now by using (20) and by cancelling the term $2f'|Du|^2$, we get that the right-hand side in (29) equals

$$2\phi' u_{tk} u_k + 4\phi'' u_{hk} u_h u_k u_t + 2a_{ij} u_{ki} u_{kj} - 2f d_{ij} u_{ij} - 2f d_{ij,j} u_i.$$

Therefore by using the equation (19), we obtain

$$\begin{aligned} (d_{ij}P_i)_j &= 2a_{ij} u_{ki} u_{kj} + 2\phi' u_{tk} u_k + 4\phi'' u_{hk} u_h u_k u_t \\ &\quad - 2\frac{f^2}{\Lambda} - 2\frac{f \phi' u_t}{\Lambda} - 2f d_{ij,j} u_i. \end{aligned} \quad (30)$$

By using the extrema conditions (27), (28), we have the following two conditions at (x_0, t_0)

$$u_{kh} u_k u_h = \frac{f}{\Lambda} |Du|^2 + \frac{M}{2R\Lambda} \frac{x_h u_h}{(|x|^2 + 1)^{1/2}}, \quad (31)$$

$$2\Lambda u_{kt} u_k \geq 2f u_t + \frac{c}{R^{1/2}}. \quad (32)$$

Using the extrema conditions and by canceling $2\phi' u_{tk} u_k$ we obtain,

$$(d_{ij}w_i)_j \geq 2a_{ij} u_{ki} u_{kj} + \frac{4\phi'' f}{\Lambda} |Du|^2 u_t - \frac{2f^2}{\Lambda} - 2fd_{ij,j} u_i \quad (33)$$

$$+ \frac{2\phi'' M x_h u_h u_t}{R \Lambda (|x|^2 + 1)^{1/2}} + \frac{c \phi'}{R^{1/2} \Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j.$$

Now we have the following structure equation, whose proof is lengthy but straightforward,

$$d_{ij,j} u_i = \frac{2\phi''}{\Lambda} (|Du|^2 \Delta u - u_{hk} u_h u_k). \quad (34)$$

Using (32) in (34), we find

$$d_{ij,j} u_i = \frac{2\phi'' |Du|^2}{\Lambda} (\Delta u - \frac{f}{\Lambda} - \frac{M x_h u_h}{2R |Du|^2 \Lambda (|x|^2 + 1)^{1/2}}).$$

Using the equation (14), we have

$$2\phi'' u_{hk} u_h u_k + \phi' \Delta u = f + \phi' u_t.$$

Therefore,

$$\Delta u = \frac{f + \phi' u_t - 2\phi'' u_{hk} u_h u_k}{\phi'}. \quad (35)$$

Substituting the value for Δu in (35) and by using the extrema condition (32), we have the following equality at (x_0, t_0) ,

$$d_{ij,j} u_i = \frac{2\phi'' |Du|^2}{\Lambda \phi'} \left[f + u_t \phi' - 2\phi'' \frac{|Du|^2}{\Lambda} f - f \frac{\phi'}{\Lambda} \right. \quad (36)$$

$$\left. - \frac{\phi'' M x_h u_h}{R \Lambda (|x|^2 + 1)^{1/2}} - \frac{M x_h u_h \phi'}{2R |Du|^2 \Lambda (|x|^2 + 1)^{1/2}} \right].$$

Using the definition of Λ and cancelling terms in (36), we have that the right-hand side in (36) equals

$$2\phi'' \frac{|Du|^2 u_t}{\Lambda} - \frac{\phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}} - \frac{2(\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}}. \quad (37)$$

Therefore, by canceling the terms $4\phi'' f \frac{|Du|^2 u_t}{\Lambda}$ in (33), we obtain the following differential inequality at (x_0, t_0) ,

$$(d_{ij}w_i)_j \geq \frac{c \phi'}{R^{1/2} \Lambda} - \frac{2f^2}{\Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j + \frac{2\phi'' M x_h u_h u_t}{R \Lambda (|x|^2 + 1)^{1/2}} \quad (38)$$

$$+ \frac{2f \phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}} + \frac{4f (\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}} + 2a_{ij} u_{ki} u_{kj}.$$

Now by using the identity for DP in (21) above, we have

$$u_{ki} u_{kj} u_i u_j = \frac{(P_k + 2f u_k)^2}{4\Lambda^2}. \quad (39)$$

Also,

$$a_{ij} u_{kj} u_{ki} = \phi' u_{ik} u_{ik} + 2\phi'' u_{ik} u_i u_{jk} u_j.$$

Therefore, by Schwarz inequality, we have

$$a_{ij} u_{kj} u_{ki} \geq \phi' \frac{u_{ik} u_{jk} u_i u_j}{|Du|^2} + 2\phi'' u_{ik} u_i u_{jk} u_j = \frac{\Lambda u_{ik} u_i u_{jk} u_j}{|Du|^2}.$$

Then, by using (39) we find

$$a_{ij} u_{kj} u_{ki} \geq \frac{(P_k + 2f u_k)^2}{4\Lambda |Du|^2} = \frac{|DP|^2 + 4f^2 |Du|^2 + 2f \langle Du, DP \rangle}{4|Du|^2 \Lambda}. \quad (40)$$

At this point, using (40) in (38), we can cancel off $\frac{2f^2}{\Lambda}$ and consequently obtain the following inequality at (x_0, t_0) ,

$$\begin{aligned} (d_{ij} w_i)_j &\geq \frac{c\phi'}{R^{1/2}\Lambda} + \frac{f \langle Du, DP \rangle}{|Du|^2 \Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j + \frac{2\phi'' M x_h u_h u_i}{R \Lambda (|x|^2 + 1)^{1/2}} \\ &\quad + \frac{4f(\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}} + \frac{2f\phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}}. \end{aligned} \quad (41)$$

By assumption, since $w(x_0, t_0) \geq K$, we have that

$$|Du| \geq \frac{1}{2R^{1/2}}.$$

Moreover, since u has bounded derivatives up to order 3, for a fixed $\varepsilon > 0$, we have that ϕ' and Λ are bounded from below by a positive constant. Therefore by (28), the term $\frac{f \langle Du, DP \rangle}{|Du|^2 \Lambda}$ can be controlled from below by $-\frac{M''}{R^{1/2}}$ where M'' depends on ε and the bounds of the derivatives of u . Consequently, from (41), we have at (x_0, t_0) ,

$$(d_{ij} w_i)_j \geq \frac{C(c)}{R^{1/2}} - \frac{L(M)}{R} - \frac{M''}{R^{1/2}}. \quad (42)$$

Now in the very first place, if c is chosen large enough depending only on ε and the bounds of the derivatives of u up to order three, we would have the following inequality at (x_0, t_0) ,

$$(d_{ij} w_i)_j > 0.$$

This contradicts the fact that w has a maximum at (x_0, t_0) . Therefore, either $w(x_0, t_0) < K$, or the maximum of w is achieved on the parabolic boundary where $w < 0$. In either case, for an arbitrary point (y, s) such that $|y| \leq R$, we have that

$$w(y, s) \leq \frac{1}{R^{1/2}}. \quad (43)$$

3 Forward gradient bounds for the reaction-diffusion equation (5) in epigraphs

In this section we consider Modica type gradient bounds for solutions to the parabolic equation (5) in unbounded generalized cylinders of the type $\Omega \times [0, T]$. On the ground domain $\Omega \subset \mathbb{R}^n$ we assume that it is an epigraph, i.e., that

$$\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > h(x')\}. \quad (44)$$

Furthermore, we assume that $h \in C_{loc}^{2,\alpha}(\mathbb{R}^{n-1})$ and that

$$\|Dh\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} < \infty. \quad (45)$$

Before proving the main result of the section we establish a lemma which will be used throughout the rest of the paper.

Lemma 1. *Let u be a solution to (5), and assume that*

$$\inf_G |Du| > 0, \quad (46)$$

for some open set $G \in \mathbb{R}^n \times \mathbb{R}$. Define

$$P(x, t) \stackrel{def}{=} P(u, x, t) = |Du(x, t)|^2 - 2F(u(x, t)). \quad (47)$$

Then, we have in G that

$$(\Delta - \partial_t)P + \langle B, DP \rangle \geq \frac{|DP|^2}{2|Du|^2}, \quad (48)$$

where $B = \frac{2F'(u)Du}{|Du|^2}$.

Proof. The proof of the lemma follows from computations similar to that in the proof of Theorem 2.1, but we nevertheless provide details since this lemma will be crucially used in the rest of the paper. We first note that, since $F \in C_{loc}^{2,\beta}$, we have $u \in H_{3+\alpha,loc}$ for some α which also depends on β . By using (5), it follows from a simple computation that

$$(\Delta - \partial_t)P = 2\|D^2u\|^2 - 2F'(u)^2. \quad (49)$$

From the definition of P , it follows that

$$DP = 2D^2uDu - 2F'(u)Du.$$

This gives

$$4|D^2uDu|^2 = |DP + 2F'(u)Du|^2 = |DP|^2 + 4F'(u)^2|Du|^2 + 4F'(u) \langle DP, Du \rangle .$$

Therefore, from Cauchy-Schwartz inequality we obtain

$$4||D^2u|^2|Du|^2 \geq |DP|^2 + 4F'(u)^2|Du|^2 + 4F'(u) \langle DP, Du \rangle .$$

By dividing both sides of this inequality by $2|Du|^2$, and replacing in (49), the desired conclusion follows.

We now state the relevant result which is the parabolic analogue of Theorem 1 in [FV1].

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^n$ be as in (44), with h satisfying (45), and assume furthermore that the mean curvature of $\partial\Omega$ is nonnegative. Let u be a nonnegative bounded solution to the following problem*

$$\begin{cases} \Delta u = u_t + F'(u), \\ u = 0 \text{ on } \partial\Omega \times [0, T], \end{cases} \quad (50)$$

such that

$$|Du|^2(x, 0) \leq 2F(u)(x, 0). \quad (51)$$

Furthermore, assume that $\|u(\cdot, 0)\|_{C^{1,\alpha}(\bar{\Omega})} < \infty$. Then, the following gradient estimate holds for all $t > 0$ and all $x \in \Omega$,

$$|Du|^2(x, t) \leq 2F(u)(x, t). \quad (52)$$

Proof. Henceforth in this paper for a given function $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ we denote by

$$\Omega_g = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > g(x')\}$$

its epigraph. With α as in the hypothesis (45) above, we denote

$$\mathcal{F} = \left\{ g \in C^{2,\alpha}(\mathbb{R}^{n-1}) \mid \partial\Omega_g \text{ has nonnegative mean curvature and } \|Dg\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \leq \|Dh\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \right\}.$$

We now note that, given a bounded solution u to (50) above, then by Schauder regularity theory (see Chapters 4, 5 and 12 in [Li]) one has

$$\|u\|_{H_{1+\alpha}(\bar{\Omega} \times [0, T])} \leq C, \quad (53)$$

for some universal $C > 0$ which also depends on Ω and $\|u(\cdot, 0)\|_{C^{1,\alpha}(\bar{\Omega})}$, and for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|u\|_{H_{2+\alpha}(\overline{\Omega} \times [\varepsilon, T])} \leq C(\varepsilon). \quad (54)$$

Note that in (54), we cannot take $\varepsilon = 0$, since the compatibility conditions at the corner points need not hold. With C as in (53), we now define

$$\Sigma = \left\{ v \in C^{2,1}(\overline{\Omega}_g \times [0, T]) \mid \text{there exists } g \in \mathcal{F} \text{ for which } v \text{ solves (50) in } \Omega_g \times [0, T], \right. \\ \left. \text{with } 0 \leq v \leq \|u\|_{L^\infty}, \|v\|_{H_{1+\alpha}(\overline{\Omega}_g \times [0, T])} \leq C, P(v, x, 0) \leq 0 \right\}.$$

Note that in the definition of Σ we have that given any $v \in \Sigma$, there exists a corresponding $g^{(v)}$ in \mathcal{F} such that the assertions in the definition of the class Σ hold. Moreover Σ is non-empty since $u \in \Sigma$. From now on, with slight abuse of notation, we will denote the corresponding $\Omega_{g^{(v)}}$ by Ω_v .

We now set

$$P_0 = \sup_{v \in \Sigma, (x,t) \in \Omega_v \times [0, T]} P(v; x, t).$$

We note that P_0 is finite because by the definition of Σ , every $v \in \Sigma$ has $H_{1+\alpha}$ norm bounded from above by a constant C which is independent of v . Furthermore, by Schauder regularity theory we have that (54) holds uniformly for $v \in \Sigma$ in $\overline{\Omega}_v \times [0, T]$. Our objective is to establish that

$$P_0 \leq 0. \quad (55)$$

Assume on the contrary that $P_0 > 0$. For every $k \in \mathbb{N}$ there exist $v_k \in \Sigma$ and $(x_k, t_k) \in \Omega_{v_k} \times [0, T]$ such that

$$P_0 - \frac{1}{k} < P(v_k, x_k, t_k) \leq P_0.$$

By compactness, possibly passing to a subsequence, we know that there exists $t_0 \in [0, T]$ such that $t_k \rightarrow t_0$. We define

$$u_k(x, t) = v_k(x + x_k, t_k).$$

We then have that $u_k \in \Sigma$ and $0 \in \Omega_{u_k}$. Moreover,

$$P(u_k, 0, t_k) = P(v_k, x_k, t_k) \rightarrow P_0.$$

Now, if we denote by g_k the function corresponding to the graph of Ω_{u_k} , from the fact that $0 \in \Omega_{u_k}$ we infer that

$$g_k(0) \leq 0.$$

We now claim that $g_k(0)$ is bounded. If not, then there exists a subsequence such that

$$g_k(0) \rightarrow -\infty.$$

Moreover since $\|Dg_k\|_{C^{1,\alpha}}$ is bounded uniformly in k , we conclude that for every $x' \in \mathbb{R}^{n-1}$

$$g_k(x') \rightarrow -\infty, \quad (56)$$

and the same conclusion holds locally uniformly in x' . Since the u_k 's are uniformly bounded in $H_{1+\alpha}(\Omega_{u_k} \times [0, T])$, we have that $u_k \rightarrow w_0$ locally uniformly in $\mathbb{R}^n \times [0, T]$. Note that this can be justified by taking an extension of u_k to $\mathbb{R}^n \times [0, T]$ such that (53) hold in $\mathbb{R}^n \times [0, T]$, uniformly in k . Applying (54) to the u_k 's we see that the limit function w_0 solves (50) in $\mathbb{R}^n \times [0, T]$. Since by the definition of Σ we have $P(u_k, 0, 0) = P(v_k, x_k, 0) \leq 0$, we have that $t_0 > 0$, and therefore by (53) we conclude that $P(w_0, 0, t_0) = P_0 > 0$. Moreover, again by (53), we have $P(w_0, 0, 0) \leq 0$. This leads to a contradiction with the case $p = 2$ of Theorem 1.3 established in [BG]. Therefore, the sequence $\{g_k(0)\}$ must be bounded.

Now since g_k 's are such that Dg_k 's have uniformly bounded $C^{1,\alpha}$ norms, we conclude by Ascoli-Arzelà that there exists $g_0 \in \mathcal{F}$ such that $g_k \rightarrow g_0$ locally uniformly in \mathbb{R}^{n-1} . We denote

$$\Omega_0 = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > g_0(x')\}.$$

For each k , by taking an extension \tilde{u}_k of u_k to $\mathbb{R}^n \times [0, T]$ such that \tilde{u}_k has bounded $H_{1+\alpha}$ norm, we have that (possibly on a subsequence) $\tilde{u}_k \rightarrow u_0$ locally uniformly in \mathbb{R}^n . Moreover, because of (54) applied to u_k 's, the function u_0 solves the equation (5) in $\Omega_0 \times (0, T]$. We also note that since Dg_k 's have uniformly bounded $C^{1,\alpha}$ norms, $\partial\Omega_0$ has nonnegative mean curvature. Moreover, by arguing as in (33) and (34) in [FV1], we have that u_0 vanishes on $\partial\Omega_0 \times [0, T]$. Also, it follows that $P(u_0, x, 0) \leq 0$ for $x \in \Omega_0$, and therefore $u_0 \in \Sigma$. Arguing by compactness as previously in this proof, we infer that must be $t_0 > 0$, and since $u_0 \in \Sigma$, that

$$P_0 = P(u_0, 0, t_0) = \sup_{(x,t) \in \Omega_0 \times [0, T]} P(u_0, x, t) > 0.$$

Since $u_0 \geq 0$ and u_0 vanishes on $\partial\Omega_0 \times [0, T]$, indicating by ν the inward unit normal to $\partial\Omega_0$ at x , we have for each $(x, t) \in \partial\Omega_0 \times [0, T]$

$$\partial_\nu u_0(x, t) \geq 0. \quad (57)$$

Given (57) and from the fact that u_0 is bounded, by arguing as in (36)-(38) in [FV1], it follows that for all $t \in [0, T]$

$$\inf_{x \in \Omega_0} |Du_0(x, t)| = 0. \quad (58)$$

Next, we claim that if for a time level $t > 0$ we have $P(u_0, \bar{y}, t) = P_0$, then it must be $\bar{y} \in \partial\Omega_0$. To see this, suppose on the contrary that $\bar{y} \in \Omega_0$. Since $P_0 > 0$, this implies that $|Du_0(\bar{y}, t)| > 0$. Consider now the set

$$U = \{x \in \Omega_0 \mid P(u_0, x, t) = P_0\}.$$

Clearly, U is closed, and since $\bar{y} \in U$ by assumption, we also know that $U \neq \emptyset$. We now prove that U is open. Since $|Du_0(x, t)| > 0$ for every $x \in U$, by Lemma

1 and the strong maximum principle (we note that since $F \in C_{loc}^{2,\beta}$, we have that $u_0 \in H_{3+\alpha'}$ in the interior for some α' which also depends on β . Hence, $P(u_0, \cdot, \cdot)$ is a classical subsolution), we conclude that for every $x \in U$ there exists $\delta_x > 0$ such that $P(u_0, z, t) = P_0$ for $z \in B(x, \delta_x)$. This implies that U is open.

Since Ω_0 , being an epigraph, is connected, we conclude that $U = \Omega_0$. Now from (58) we have that for every fixed $t \in [0, T]$ there exists a sequence $x_j \in \Omega_0$ such that $Du_0(x_j, t) \rightarrow 0$ as $j \rightarrow \infty$. As a consequence, $\liminf_{j \rightarrow \infty} P(u_0, x_j, t) \leq 0$. This implies that for large enough j we must have $P(u_0, x_j, t) < P_0$, which contradicts the above conclusion that $U = \Omega_0$. Therefore this establishes the claim that if $P(u_0, \bar{y}, t) = P_0$, then $\bar{y} \in \partial\Omega_0$. Since $P(u_0, 0, t_0) = P_0$, and P_0 is assumed to be positive, this implies in particular that $(0, t_0) \in \partial\Omega_0 \times (0, T]$. Again, since $P_0 > 0$ by assumption, we must have that in (57) a strict inequality holds at $(0, t_0)$, i.e.

$$\partial_\nu u_0(0, t_0) > 0. \quad (59)$$

This is because if the normal derivative is zero at $(0, t_0)$, then it must also be $Du_0(0, t_0) = 0$ (since u_0 vanishes on the lateral boundary of $\Omega_0 \times [0, T]$), and this contradicts the fact that $P_0 > 0$.

From (59) we infer that $Du_0(0, t_0) \neq 0$, and therefore Lemma 1 implies that, near $(0, t_0)$, the function $P(u_0, \cdot, \cdot)$ is a subsolution to a uniformly parabolic equation. Now, by an application of the Hopf Lemma (see for instance Theorem 3' in [LN]) we have that

$$\partial_\nu P(u_0, 0, t_0) < 0. \quad (60)$$

Again by noting that u_0 vanishes on the lateral boundary, we have that $\partial_t u_0 = 0$ at $(0, t_0)$. Therefore, at $(0, t_0)$, the function u_0 satisfies the elliptic equation

$$\Delta u_0 = F'(u_0). \quad (61)$$

At this point, by using the fact that the mean curvature of $\partial\Omega_0$ is nonnegative and the equation (61) satisfied by u_0 at $(0, t_0)$, one can argue as in (50)-(55) in [FV1] to reach a contradiction with (60) above. Such contradiction being generated from having assumed that $P_0 > 0$, we conclude that (55) must hold, and this implies the sought for conclusion of the theorem.

Remark 3.2 Note that in the hypothesis of Theorem 3.1 instead of $u \geq 0$ we could have assumed that $\partial_\nu u \geq 0$ on $\partial\Omega \times [0, T]$.

Remark 3.3 We also note that the conclusion in Theorem 3.1 holds if Ω is of the form $\Omega = \Omega_0 \times \mathbb{R}^{n-n_0}$ for $1 \leq n_0 \leq n$, where Ω_0 is a bounded $C^{2,\alpha}$ domain with nonnegative mean curvature. The corresponding modifications in the proof would be as follows. Let D be the set of domains which are all translates of Ω . The classes H and Σ would be defined corresponding to D as in [FV1]. Then, by arguing as in the proof of Theorem 3.1, we can assume that sets $\Omega_{u_k} \in D$ are such that $\Omega_{u_k} = p_k + \Omega$ where $p_k = (p'_k, 0) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$. Since $0 \in \Omega_{u_k}$ and Ω_0 is bounded, this implies that p'_k is bounded independent of k . Therefore, up to a subsequence, $p_k \rightarrow p_0$ and

$u_k \rightarrow u_0$ such that u_0 solves (50) in $\Omega_1 = p_0 + \Omega$. The rest of the proof remains the same as that of Theorem 3.1.

Remark 3.4 *It remains an interesting open question whether Theorem 3.1 holds for the inhomogeneous variant of the normalized p -Laplacian evolution studied in [BG]. Note that unlike the case of \mathbb{R}^n , the Hopf lemma applied to P is a crucial step in the proof of Theorem 3.1 for which Lemma 1 is the key ingredient. As far as we are aware of, an appropriate analogue of Lemma 1 is not known to be valid for $p \neq 2$, even in the case when $F = 0$. Therefore, to be able to generalize Theorem 3.1 to the case of inhomogeneous normalized p -Laplacian evolution as studied in [BG], lack of an appropriate subsolution-type argument (i.e., Lemma 1), and a priori $H_{1+\alpha}$ estimates seem to be the two major obstructions at this point.*

4 Gradient estimates for the reaction-diffusion equation (5) in $\mathbb{R}^n \times (-\infty, 0]$ and $\Omega \times (-\infty, 0]$

In this section we turn our attention to the settings $\mathbb{R}^n \times (-\infty, 0]$ and $\Omega \times (-\infty, 0]$, where Ω is an epigraph satisfying the geometric assumptions as in the previous section. We investigate the validity of Modica type gradient estimates in a different situation with respect to that of Section 3, where such estimates were established under the crucial hypothesis that the initial datum satisfies a similar inequality. We first note that such unconstrained global estimates cannot be expected in $\mathbb{R}^n \times [0, T]$ without any assumption on the initial datum. This depends of the fact that, if at time $t = 0$ the initial datum is such that the function defined in (47) above satisfies $P(u, x, 0) > 0$ at some $x \in \mathbb{R}^n$, then by continuity $P(u, x, t) > 0$ for all $t \in [0, \varepsilon]$, for some $\varepsilon > 0$. This justifies our choice of the setting in this section. We now state our first main result.

Theorem 4.1 *Let u be a bounded solution to (5) in $\mathbb{R}^n \times (-\infty, 0]$. Then, with $P(u, \cdot, \cdot)$ as in (47) we have*

$$P(u, x, t) \leq 0, \quad \text{for all } (x, t) \in \mathbb{R}^n \times (-\infty, 0]. \quad (62)$$

Remark 4.2 *An explicit example of a bounded solution to (5) is an eternal traveling wave as in Section 7.*

Proof. The proof is inspired to that of Theorem 1.6 in [CGS]. We define the class Σ as follows.

$$\Sigma = \{v \mid v \text{ solves (5) in } \mathbb{R}^n \times (-\infty, 0], \ ||v||_{L^\infty} \leq ||u||_{L^\infty}\}. \quad (63)$$

Note that $u \in \Sigma$. Set

$$P_0 = \sup_{v \in \Sigma, (x,t) \in \mathbb{R}^n \times (-\infty, 0]} P(v, x, t). \quad (64)$$

Since $F \in C_{loc}^{2,\beta}(\mathbb{R})$ and the L^∞ norm of $v \in \Sigma$ is uniformly bounded by that of u , from the Schauder theory we infer all elements $v \in \Sigma$ have uniformly bounded $H_{3+\alpha}$ norms in $\mathbb{R}^n \times (-\infty, 0]$, for some α depending also on β . Therefore, P_0 is bounded.

We claim that $P_0 \leq 0$. Suppose, on the contrary, that $P_0 > 0$. Then, there exists $v_k \in \Sigma$ and corresponding points $(x_k, t_k) \in \mathbb{R}^n \times (-\infty, 0]$ such that $P(v_k, x_k, t_k) \rightarrow P_0$. Define now

$$u_k(x, t) = v_k(x + x_k, t + t_k). \quad (65)$$

Note that since $t_k \leq 0$, we have that $u_k \in \Sigma$ and $P(u_k, 0, 0) = P(v_k, x_k, t_k) \rightarrow P_0$. Moreover, since u_k 's have uniformly bounded $H_{3+\alpha}$ norms, for a subsequence, $u_k \rightarrow u_0$ which belongs to Σ . Moreover,

$$P(u_0, 0, 0) = \sup_{(x,t) \in \mathbb{R}^n \times (-\infty, 0]} P(u_0, x, t) = P_0 > 0.$$

As before, this implies that $Du_0(0, 0) \neq 0$. Now, by an application of Lemma 1, the strong maximum principle and the connectedness of \mathbb{R}^n , we have that $P(u_0, x, 0) = P_0$ for all $x \in \mathbb{R}^n$. On the other hand, since u_0 is bounded, it follows that

$$\inf_{x \in \mathbb{R}^n} |Du_0(x, 0)| = 0.$$

Then, there exists $x_j \in \mathbb{R}^n$ such that $|Du_0(x_j, 0)| \rightarrow 0$. However, we have that $P(u_0, x_j, 0) = P_0 > 0$ by assumption which is a contradiction for large enough j . Therefore, $P_0 \leq 0$ and the conclusion follows.

As an application of Theorem 4.1 one has the following result on the propagation of zeros whose proof is identical to that of Theorem 1.8 in [CGS] (see also Theorem 1.6 in [BG]).

Corollary 4.3 *Let u be a bounded solution to (5) in $\mathbb{R}^n \times (-\infty, 0]$. If $F(u(x_0, t_0)) = 0$ for some point $(x_0, t_0) \in \mathbb{R}^n \times (-\infty, 0]$, then $u(x, t) = u(x_0, t_0)$ for all $x \in \mathbb{R}^n$.*

We also have the following counterpart of Theorem 4.1 in an infinite cylinder of the type $\Omega \times (-\infty, 0]$ where Ω satisfies the hypothesis in Theorem 3.1.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^n$ be as in (44) above, with h satisfying (45). Furthermore, assume that the mean curvature of $\partial\Omega$ is nonnegative. Let u be a nonnegative bounded solution to the following problem*

$$\begin{cases} \Delta u = u_t + F'(u) \\ u = 0 \text{ on } \partial\Omega \times (-\infty, 0]. \end{cases} \quad (66)$$

Then, we have that $P(u, x, t) \leq 0$ for all $(x, t) \in \Omega \times (-\infty, 0]$.

Proof. By Schauder theory we have that

$$\|u\|_{H_{3+\alpha}(\Omega \times (-\infty, 0])} \leq C, \quad (67)$$

for some C which also depends on β . We let \mathcal{F} be as in the proof of Theorem 3.1 and define

$$\Sigma = \{v \in C^{2,1}(\overline{\Omega}_g \times [0, T]) \mid \text{there exists } g \in \mathcal{F} \text{ for which } v \text{ solves (5) in } \Omega_g \times [0, T], \\ \text{with } 0 \leq v \leq \|u\|_{L^\infty}, v = 0 \text{ on } \partial\Omega_g \times (-\infty, 0]\}.$$

As before, note that in the definition of Σ we have that, given any $v \in \Sigma$, there exists a corresponding $g^{(v)}$ in \mathcal{F} such that the assertions in the definition of the class Σ hold. With slight abuse of notation, we will denote the corresponding $\Omega_{g^{(v)}}$ by Ω_v . Again by Schauder theory, we have that any $v \in \Sigma$ satisfies (67) in $\Omega_v \times (-\infty, 0]$, where the constant C is independent of v . We now set,

$$P_0 = \sup_{v \in \Sigma, (x,t) \in \Omega_v \times (-\infty, 0]} P(v, x, t).$$

As before, we claim that $P_0 \leq 0$. This claim would of course imply the sought for conclusion. From the definition of \mathcal{H} , we note that P_0 is bounded. Suppose, on the contrary, that $P_0 > 0$. Then, there exists v_k 's and corresponding points (x_k, t_k) such that $P(v_k, x_k, t_k) \rightarrow P_0$. We define,

$$u_k(x, t) = v_k(x + x_k, t + t_k). \quad (68)$$

Since $t_k \leq 0$, we note that $u_k \in \Sigma$. Now, by an application of Theorem 4.1 and a compactness type argument as in the proof of Theorem 3.1, we conclude that if g_k is function corresponding to $\Omega_{g_k} = \Omega_{u_k}$ for each k , then $g_k(0)$'s are bounded and since Dg_k 's have uniformly bounded $C^{1,\alpha}$ norms, then g_k 's are bounded locally uniformly in \mathbb{R}^{n-1} . From this point on the proof follows step by step the lines of that of Theorem 3.1 and we thus skip pointless repetitions. There exists a $g_0 \in \mathcal{F}$ for which $g_k \rightarrow g_0$ locally uniformly in \mathbb{R}^{n-1} as in that proof and we call Ω_0 the epigraph of g_0 . From the uniform Schauder type estimates, possibly passing to a subsequence, we conclude the existence of a solution $u_0 \geq 0$ of (5) in $\Omega_0 \times (-\infty, 0]$ such that $\Omega_{u_k} \rightarrow \Omega_0$, and $u_k \rightarrow u_0$ which solves such that $\partial\Omega_0$ has nonnegative mean curvature. Moreover, u_0 vanishes on the lateral boundary and $P(u_0, 0, 0) = \sup P(u_0, 0, 0) = P_0$. The rest of the proof remains the same as that of Theorem 3.1, but with $(0, 0)$ in place of $(0, t_0)$.

Remark 4.5 *As indicated in Remark 3.4, the conclusion of Theorem 4.4 remains valid with minor modifications in the proof when $\Omega = \Omega_0 \times \mathbb{R}^{n-n_0}$, $1 \leq n_0 \leq n$, where Ω_0 is a bounded smooth domain with boundary having nonnegative mean curvature.*

5 Modica type estimates for reaction-diffusion equations on compact manifolds with nonnegative Ricci tensor

Let (M, g) be a connected, compact Riemannian manifold with Laplace-Beltrami Δ_g , and suppose that the Ricci tensor be nonnegative. In the paper [FV3] the authors established a Modica type estimate for bounded solutions in M of the semilinear Poisson equation

$$\Delta_g u = F'(u), \quad (69)$$

under the assumption that $F \in C^2(\mathbb{R})$, and $F \geq 0$. Precisely, they proved that following inequality holds

$$|\nabla_g u(x)|^2 \leq 2F(u), \quad (70)$$

where ∇_g is the Riemannian gradient on M .

In this section, we prove a parabolic analogue of (70). Our main result can be stated as follows.

Theorem 5.1 *Let M be a connected compact Riemannian manifold with $\text{Ric} \geq 0$, and let u be a bounded solution to*

$$\Delta_g u = u_t + F'(u) \quad (71)$$

on $M \times (-\infty, 0]$ where $F \in C^{2,\beta}(\mathbb{R})$ and $F \geq 0$. Then, the following estimate holds in $M \times (-\infty, 0]$

$$|\nabla_g u(x, t)|^2 \leq 2F(u(x, t)). \quad (72)$$

Proof. By Schauder theory, we have that $u \in H_{3+\alpha}(M \times (-\infty, 0])$ for some α which additionally depends on β . This follows from writing the equation in local coordinates and by using the compactness of M . We next recall the Bochner-Weitzenböck formula, which holds for any $\phi \in C^3(M)$

$$\frac{1}{2} \Delta_g |\nabla_g \phi|^2 = |H_\phi|^2 + \langle \nabla_g \phi, \nabla_g \Delta_g \phi \rangle + \text{Ric}_g \langle \nabla_g \phi, \nabla_g \phi \rangle. \quad (73)$$

Here, H_ϕ is the Hessian of ϕ and the square of the Hilbert-Schmidt norm of H_ϕ is given by

$$|H_\phi|^2 = \sum_i \langle \nabla_{X_i} \nabla_g, \nabla_{X_i} \nabla_g \rangle,$$

where $\{X_i\}$ is a local orthonormal frame. Moreover, Cauchy-Schwarz inequality gives

$$|H_\phi|^2 \geq |\nabla_g |\nabla_g \phi||^2. \quad (74)$$

See for instance [FSV] for a proof of this fact. Now we define the class

$$\mathcal{F} = \{v \mid v \text{ is a classical solution to (71) in } M \times (-\infty, 0], \|v\|_{L^\infty(M)} \leq \|u\|_{L^\infty(M)}\}.$$

By the Schauder theory we see as before that for every $v \in \mathcal{F}$ the norm of v in $H_{3+\alpha}(M \times (-\infty, 0])$ is bounded independent of v for some α which additionally depends on β . In particular, without loss of generality, one may assume that the

choice of the exponent α is the same as for u . Now, given any $v \in \mathcal{F}$, we let

$$P(v, x, t) = |\nabla_g v(x, t)|^2 - 2F(v(x, t)). \quad (75)$$

Applying (73) we find

$$\begin{aligned} (\Delta_g - \partial_t)P(v, x, t) &= 2|H_v|^2 + 2(\langle \nabla_g v, \nabla_g \Delta_g v \rangle + \text{Ric}_g \langle \nabla_g v, \nabla_g v \rangle) - \\ &= 2 \langle \nabla_g v, \nabla_g v_t \rangle - 2F'(v)(\Delta_g v - v_t) - 2 \langle \nabla_g v, \nabla_g F'(v) \rangle. \end{aligned}$$

Using the fact that v solves (71), we obtain

$$\begin{aligned} (\Delta_g - \partial_t)P(v, x, t) &= 2|H_v|^2 + 2 \langle \nabla_g v, \nabla_g F'(v) \rangle \\ &+ 2\text{Ric}_g \langle \nabla_g v, \nabla_g v \rangle - 2F'(v)^2 - 2 \langle \nabla_g v, \nabla_g F'(v) \rangle. \end{aligned}$$

After cancelling off the term $2 \langle \nabla_g v, \nabla_g F'(v) \rangle$, and by using (74) and the fact that the Ricci tensor is nonnegative, we find

$$(\Delta_g - \partial_t)P(v, x, t) = 2|H_v|^2 + 2\text{Ric}_g \langle \nabla_g v, \nabla_g v \rangle - 2F'(v)^2 \geq 2|\nabla_g(|\nabla_g v|)|^2 - 2F'(v)^2 \quad (76)$$

Now from the definition of P ,

$$\nabla_g P - 2F'(v)\nabla_g v = \nabla_g(|\nabla_g v|^2).$$

Therefore,

$$|\nabla_g P|^2 + 4F'(v)^2|\nabla_g v|^2 - 4F'(v) \langle \nabla_g v, \nabla_g P \rangle = |\nabla_g(|\nabla_g v|^2)|^2 = 4|\nabla_g v|^2|\nabla_g(|\nabla_g v|)|^2.$$

By dividing by $2|\nabla_g v|^2$ in the latter equation, we find

$$\frac{|\nabla_g P|^2}{2|\nabla_g v|^2} = 2|\nabla_g(|\nabla_g v|)|^2 - 2F'(v)^2 + 2\frac{F'(v)}{|\nabla_g v|^2} \langle \nabla_g v, \nabla_g P \rangle. \quad (77)$$

Combining (76) and (77), we finally obtain

$$(\Delta_g - \partial_t)P + 2\frac{F'(v)}{|\nabla_g v|^2} \langle \nabla_g v, \nabla_g P \rangle \geq \frac{|\nabla_g P|^2}{2|\nabla_g v|^2}. \quad (78)$$

The inequality (78) shows that $P(v, x, t)$ is a subsolution to a uniformly parabolic equation in any open set where $|\nabla_g v| > 0$. Now we define

$$P_0 = \sup_{v \in \mathcal{F}, (x, t) \in M \times (-\infty, 0]} P(v, x, t).$$

Our goal as before is to show that $P_0 \leq 0$, from the which the conclusion of the theorem would follow. Suppose on the contrary that $P_0 > 0$. Then, there exists $v_k \in \mathcal{F}$ and $(x_k, t_k) \in M \times (-\infty, 0]$ such that $P(v_k, x_k, t_k) \rightarrow P_0$. We define

$$u_k(x, t) = v_k(x_k, t + t_k).$$

Since $t_k \leq 0$ we have that $u_k \in \mathcal{F}$, and since M is compact, $x_k \rightarrow x_0$ after possibly passing to a subsequence. Moreover, $P(u_k, x_0, 0) \rightarrow P_0$. By compactness, we have that $u_k \rightarrow u_0$ in $H_{3+\alpha}$, where u_0 is a solution to (71), and $P(u_0, x_0, 0) = P_0 > 0$. Since since $F \geq 0$ this implies that $\nabla_g u_0(x_0, 0) \neq 0$. By continuity, we see that $\nabla_g u_0 \neq 0$ in a parabolic neighborhood of $(x_0, 0)$. By (78) and by the strong maximum principle we infer that $P(u_0, x, 0) = P_0$ in a neighborhood of x_0 , and since M is connected, we conclude that for all $x \in M$

$$P(u_0, x, 0) = P_0 > 0. \quad (79)$$

Since $u_0(\cdot, 0) \in C^1(M)$ and M is compact, there exists $y_0 \in M$ at which $u_0(\cdot, 0)$ attains its absolute minimum. At such point one has

$$\nabla_g u_0(y_0, 0) = 0.$$

Since $F \geq 0$, this implies that

$$P(u_0, y_0, 0) \leq 0, \quad (80)$$

which is a contradiction to (79). Therefore $P_0 \leq 0$ and the theorem is proved.

Remark 5.2 *It remains an interesting question whether the conclusion of Theorem 5.1 (and for that matter even the corresponding elliptic result in [FV3]) continue to hold when M is only assumed to be complete and not compact. In such a case, one would need to bypass the compactness argument which uses translation in a crucial way (see for instance (65) as in the proof of Theorem 4.1). We intend to come back to this question in a future study.*

6 On a conjecture of De Giorgi and level sets of solutions to (5)

In 1978 Ennio De Giorgi formulated the following conjecture, also known as ε -version of the Bernstein theorem: *let u be an entire solution to*

$$\Delta u = u^3 - u, \quad (81)$$

such that $|u| \leq 1$ and $\frac{\partial u}{\partial x_n} > 0$. Then, u must be one-dimensional, i.e., must have level sets which are hyperplanes, at least in dimension $n \leq 8$.

As mentioned in the introduction, the conjecture of De Giorgi has been fully solved for $n = 2$ in [GG1] and $n = 3$ in [AC], and it is known to fail for $n \geq 9$, see [dPKW]. For $4 \leq n \leq 8$ it is still an open question. Additional fundamental progress on De Giorgi's conjecture is contained in the papers [GG2], [Sa]. Besides these developments, in [CGS] it was established that for entire bounded solutions to

$$\operatorname{div}(|Du|^{p-2}Du) = F'(u), \quad (82)$$

if the equality holds at some point $x_0 \in \mathbb{R}^n$ for the corresponding gradient estimate

$$|Du|^p \leq \frac{p}{p-1} F(u), \quad (83)$$

then u must be one dimensional. The result in [CGS] actually regarded a more general class of equations than (82), and in [DG] some further generalizations were presented. We now establish a parabolic analogue of that result in the case $p = 2$.

Theorem 6.1 *Let u be a bounded solution to (5) in $\mathbb{R}^n \times (-\infty, 0]$. Furthermore, assume that the zero set of F is discrete. With P as in (47) above, if $P(u, x_0, t_0) = 0$ for some point $(x_0, t_0) \in \mathbb{R}^n \times (-\infty, 0]$, then there exists $g \in C^2(\mathbb{R})$ such that $u(x, t) = g(\langle a, x \rangle + \alpha)$ for some $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. In particular, u is independent of time, and the level sets of u are vertical hyperplanes in $\mathbb{R}^n \times (-\infty, 0]$.*

Proof. We begin by observing that it suffices to prove the theorem under the hypothesis that $t_0 = 0$. In fact, once that is done, then if $t_0 < 0$ we consider the function $v(x, t) = u(x, t + t_0)$. For such function we have $P(v, x, 0) = P(u, x, t_0)$ and therefore v satisfies the same hypothesis as u , except that $P(v, x_0, 0) = 0$. But then we conclude that $v(x, t) = u(x, t + t_0) = g(\langle a, x \rangle + \alpha)$, which implies the desired conclusion for u as well.

We thus assume without restriction that $P(u, x_0, 0) = 0$, and consider the set

$$A = \{x \in \mathbb{R}^n \mid P(u, x, 0) = 0\}.$$

By the continuity of P we have that A is closed, and since $(x_0, 0) \in A$, this set is also non-empty. We distinguish two cases:

- Case 1: There exists $x_1 \in A$ such that $Du(x_1, 0) = 0$;
- Case 2: $Du(x, 0) \neq 0$ for every $x \in A$.

If Case 1 occurs, then from the fact that $P(u, x_1, 0) = 0$ we obtain that $F(u(x_1, 0)) = 0$. By Corollary 4.3 we thus conclude that must be $u(\cdot, 0) \equiv u_0 = u(x_1, 0)$. At this point we observe that, since by assumption $F \geq 0$, and $F(u_0) = 0$, we must also have $F'(u_0) = 0$. Therefore, if we set $v = u - u_0$, then by the continuity of F'' and the fact that $u \in L^\infty(\mathbb{R}^n)$, we have

$$|F'(u)| = |F'(v + u_0)| = |F'(v + u_0) - F'(u_0)| \leq \int_{u_0}^{v+u_0} |F''(s)| ds \leq C|v|.$$

Since by (5) we have $\Delta v - \partial_t v = \Delta u - \partial_t u = F'(u)$, we see that v is thus a solution of the following inequality

$$|\Delta v - \partial_t v| \leq C|v|.$$

Since $v(\cdot, 0) = 0$, by the backward uniqueness result in Theorem 2.2 in [C], we have that $u \equiv u_0$ in $\mathbb{R}^n \times (-\infty, 0]$, from which the desired conclusion follows in this case.

If instead Case 2 occurs, we prove that A is also open. But then, by connectedness, we conclude in such case that $A = \mathbb{R}^n$. To see that A is open fix $x_1 \in A$. Since $Du(x_1, 0) \neq 0$, by the continuity of Du we conclude the existence of $r > 0$ such that $Du(x, t) \neq 0$ for every $(x, t) \in G = B(x_1, r) \times (-r^2, 0]$. By Lemma 1 above we conclude that $P(u, \cdot, \cdot)$ is a sub-caloric function in G . Since by Theorem 4.1 we know that $P(u, \cdot, \cdot) \leq 0$, by the strong maximum principle we conclude that $P(u, \cdot, \cdot) \equiv 0$ in G . In particular, $P(u, x, 0) = 0$ for every $x \in B(x_1, r)$, which implies that A is open.

Since as we have seen the desired conclusion of the theorem does hold in Case 1, we can without loss of generality assume that we are in Case 2, and therefore $Du(x, 0) \neq 0$ for every $x \in A = \mathbb{R}^n$. Furthermore, since for $x \in A$ we have $P(u, x, 0) = 0$, we also have

$$|Du(x, 0)|^2 = 2F(u(x, 0)), \quad \text{for every } x \in \mathbb{R}^n. \quad (84)$$

Next, we consider the set

$$K = \{(x, t) \in \mathbb{R}^n \times (-\infty, 0] \mid P(u, x, t) = 0\}.$$

We note that K is closed and non-empty since by assumption we know that $(x_0, 0) \in A$ (in fact, by (84) we now know that $\mathbb{R}^n \times \{0\} \subset K$). Let $(y_1, t_1) \in K$. If $Du(y_1, t_1) = 0$, we can argue as above (i.e., as if it were $t_1 = 0$) and conclude by backward uniqueness that $u \equiv u(y_1, t_1)$ in $\mathbb{R}^n \times (-\infty, t_1]$. Then, by the forward uniqueness of bounded solutions, see Theorem 2.5 in [LU2], we can infer that $u \equiv u(y_1, t_1)$ in $\mathbb{R}^n \times (t_1, 0]$. All together, we would have proved that $u \equiv u(y_1, t_1)$ in $\mathbb{R}^n \times (-\infty, 0]$ and therefore the conclusion of the theorem would follow.

Therefore from now on, without loss of generality, we may assume that Du never vanishes in K . With this assumption in place, if $(y_1, t_1) \in K$, then since $Du(y_1, t_1) \neq 0$, by continuity there exists $r > 0$ such that Du does not vanish in $G = B_r(y_1) \times (t_1 - r^2, t_1)$. But then, again by Lemma 1, the function $P(u, \cdot, \cdot)$ is sub-caloric in G . Since $P(u, \cdot, \cdot) \leq 0$ in G (Theorem 4.4) and $P(u, y_1, t_1) = 0$ ($(y_1, t_1) \in K$), we can apply the strong maximum principle to conclude that $P \equiv 0$ in G . Then, again by connectedness, as in the case when $t_1 = 0$, we conclude that $\mathbb{R}^n \times \{t_1\} \subset K$. In particular, we have that $P(u, y_1, t) = 0$ when $t \in (t_1 - r^2, t_1]$. Therefore, we can now repeat the arguments above with (y_1, t) in place of (y_1, t_1) for each such t and conclude that $P \equiv 0$ in $\mathbb{R}^n \times (t_1 - r^2, t_1]$.

We now claim that:

$$K = \mathbb{R}^n \times (-\infty, 0], \quad \text{or equivalently } P(u, x, t) = 0, \quad \text{for every } (x, t) \in \mathbb{R}^n \times (-\infty, 0]. \quad (85)$$

Suppose the claim not true, hence $P \not\equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$. From the above arguments it follows that if for $t_2 < 0$ there exists $y_2 \in \mathbb{R}^n$ such that $P(u, y_2, t_2) \neq 0$, then it must be $P(u, x, t_2) \neq 0$ for all $x \in \mathbb{R}^n$. We define

$$T_0 = \sup\{t < 0 \mid P(u, \cdot, t) \neq 0\}.$$

Since we are assuming the claim not true, we must have $\{t < 0 \mid P(u, \cdot, t) \neq 0\} \neq \emptyset$, hence $T_0 \leq 0$ is well-defined. We first observe that $T_0 < 0$. In fact, since by the hypothesis $(x_0, 0) \in K$ and we are assuming that we are in Case 2, we have already proved above the existence of $r > 0$ such that $\mathbb{R}^n \times (-r^2, 0] \subset K$. This fact shows that $T_0 \leq -r^2 < 0$. Next, we see that it must be $P(u, \cdot, T_0) = 0$. In fact, if this were not the case there would exist $y_2 \in \mathbb{R}^n$ such that $P(u, y_2, T_0) < 0$. Since $T_0 < 0$, by continuity we would have that $P(u, y_2, t) < 0$, for all $t \in [T_0, T_0 + \delta_1)$ for some $\delta_1 > 0$. By the arguments above, this would imply that P never vanishes in $\mathbb{R}^n \times [T_0, T_0 + \delta_1)$, in contradiction with the definition of T_0 . Since, as we have just seen, $P(u, \cdot, T_0) = 0$, arguing again as above we conclude that $P \equiv 0$ in $\mathbb{R}^n \times (T_0 - r^2, T_0]$ for some $r > 0$. But this contradicts the definition of T_0 .

This contradiction shows that $\{t < 0 \mid P(u, \cdot, t) \neq 0\} = \emptyset$, hence the claim (85) must be true. We also recall that we are assuming that Du never vanishes in $K = \mathbb{R}^n \times (-\infty, 0]$.

In conclusion, we have that

$$|Du|^2 = 2F(u) \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \quad \text{and } Du \neq 0. \quad (86)$$

At this point we argue as in the proof of Theorem 5.1 in [CGS], and we let $v = H(u)$, where H is a function to be suitably chosen subsequently. Then, we have that

$$\Delta v - v_t = H''(u)|Du|^2 + H'(u)\Delta u - H'(u)u_t.$$

By using (5) and (86), we conclude that

$$\Delta v - v_t = 2H''(u)F(u) + H'(u)F'(u). \quad (87)$$

Let $u_0 = u(0, 0)$ and define

$$H(u) = \int_{u_0}^u (2F(s))^{-1/2} ds.$$

Since $|Du|(x, t) > 0$ for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$, we have from (86) that $F(u(x, t)) > 0$. Therefore, if the zero set of F is ordered in the following manner, $a_0 < a_1 < a_2 < a_3 < a_4 < \dots$, then by connectedness, we have that $F(u(\mathbb{R}^n \times (-\infty, 0])) \subset (a_i, a_{i+1})$ for some i . We infer that H is well defined and is $C^{2,\beta}$, and with this H it is easy to check that the right-hand side in (87) is zero, i.e., v is a solution to the heat equation in $\mathbb{R}^n \times (-\infty, 0]$. Moreover, by the definition of H and (86),

$$|Dv|^2 = H'(u)^2 |Du|^2 = 1,$$

i.e., Dv is bounded in $\mathbb{R}^n \times (-\infty, 0]$. Since $v_i = D_{x_i} v$ is a solution to the heat equation for each $i \in 1, \dots, n$, by Liouville's theorem in $\mathbb{R}^n \times (-\infty, 0]$ applied to v_i , we conclude that Dv is constant, hence $\Delta v = 0$. This implies $v_t = 0$, hence v is time-independent. Hence, there exist $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $v = \langle a, x \rangle + \alpha$. The desired conclusion now follows by taking $g = H^{-1}$. This completes the proof of the theorem.

7 A parabolic version of the conjecture of De Giorgi

Motivated by the result in Theorem 6.1, the fact that $\mathbb{R}^n \times (-\infty, 0]$ is the appropriate setting for the parabolic Liouville type theorems, and the crucial role played by them in the proof of the original conjecture of De Giorgi, at least for $n \leq 3$ (see [GG1], [AC], [GG2]), it is tempting to propose the following parabolic version of De Giorgi's conjecture:

CONJECTURE 1: *Let u be a solution in $\mathbb{R}^n \times (-\infty, 0]$ to*

$$\Delta u - u_t = u^3 - u,$$

such that $|u| \leq 1$, and $\partial_{x_n} u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$. Then, u must be one dimensional and independent of t , at least for $n \leq 8$. In other words, for $n \leq 8$ the level sets of u must be vertical hyperplanes, parallel to the t axis.

However, Matteo Novaga has kindly brought to our attention that, stated this way, the conjecture is not true. There exist in fact eternal traveling wave solutions of the form

$$v(x', x_n, t) = u(x', x_n - ct), \quad c \geq 0, \quad (88)$$

for which $\partial_{x_n} u(x) > 0$. This suggests that one should amend the above in the following way.

CONJECTURE 2: *Let u be a solution in $\mathbb{R}^n \times (-\infty, 0]$ to*

$$\Delta u - u_t = u^3 - u,$$

such that $|u| \leq 1$, and $\partial_{x_n} u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$. Then, u must be an eternal traveling wave, i.e. after a change of coordinates, u must be of the form (88).

It also remains to be seen what additional assumption needs to be imposed in the hypothesis of Conjecture 1 so that the corresponding conclusion holds. We hope that these questions will stimulate interesting further research.

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