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**A characterisation of the generic rigidity of
2-dimensional point-line frameworks**

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A characterisation of the generic rigidity of 2-dimensional point-line frameworks*

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Abstract

A 2-dimensional point-line framework is a collection of points and lines in the plane which are linked by pairwise constraints that fix some angles between pairs of lines and also some point-line and point-point distances. It is rigid if every continuous motion of the points and lines which preserves the constraints results in a point-line framework which can be obtained from the initial framework by a translation or a rotation. We characterise when a generic point-line framework is rigid. Our characterisation gives rise to a polynomial algorithm for solving this decision problem.

Keywords: point-line framework, combinatorial rigidity, count matroid, submodular function, matroid union, Dilworth truncation, polynomial algorithm.

1 Introduction

A point-line framework is a collection of points and lines in d -dimensional Euclidean space which are linked by pairwise constraints that fix the angles between some pairs of lines, the distances between some pairs of points and the distances between some pairs of points and lines. The placing of the

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pairwise constraints is represented by a point-line graph where the vertices in the graph correspond to the points and lines, and an edge in the graph corresponds to the existence of a pairwise constraint. A point-line framework is obtained from a point-line graph by assigning coordinates to the points and lines, see Figure 1.

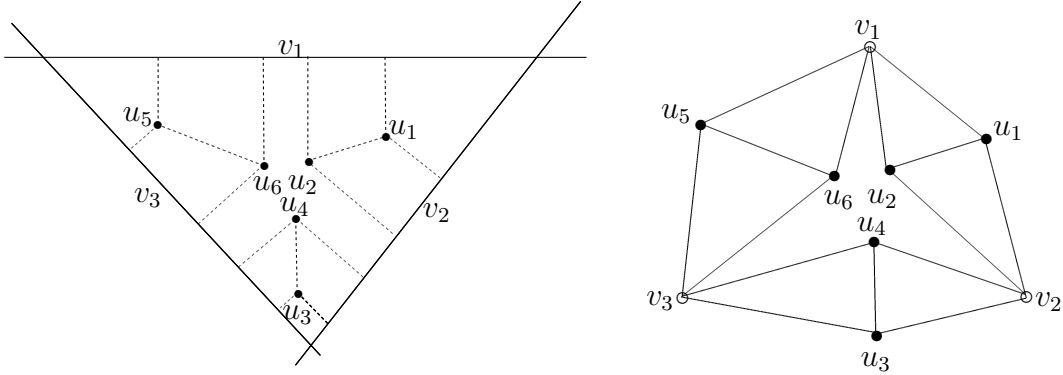


Figure 1: A point-line framework in \mathbb{R}^2 and its associated point-line graph.

A point-line framework is rigid if every continuous motion of the points and lines which preserves the constraints results in a point-line framework which can be obtained from the initial framework by an isometry of the whole space.

The constraints of a point-line framework determine its rigidity matrix, and the linearly independent sets of rows of the rigidity matrix give rise to the independent sets in its rigidity matroid. For generic frameworks the linear independence of any set of rows of the rigidity matrix does not depend on the coordinates of the points and lines. Hence the rigidity matroid is completely determined by the dimension and the point-line graph. We will denote the 2-dimensional point-line rigidity matroid of a point-line graph G by $\mathcal{M}_{PL}(G)$ (or simply \mathcal{M}_{PL} where the graph is implied).

Point-line frameworks with no lines correspond to the much studied bar-joint frameworks. Such frameworks provide a model for a variety of physical systems such as bar and joint structures [24] (where points correspond to universal joints and bars correspond to distance constraints) or molecular structures [13] (where points correspond to atoms and distance constraints correspond to bonds).

Laman [17] obtained the following characterisation of independence in the generic 2-dimensional bar-joint rigidity matroid. Given a graph $G = (V, E)$, let $\nu : 2^E \rightarrow \mathbb{Z}$ by taking $\nu(S)$ to be the number of vertices incident to S for all $S \subseteq E$. Then S is independent in the generic 2-dimensional bar-joint rigidity matroid of G if and only if $|S'| \leq 2\nu(S') - 3$ for all $\emptyset \neq S' \subseteq S$. The analogous condition that $|S'| \leq d\nu(S') - d(d+1)/2$ is a necessary condition for independence in the d -dimensional bar-joint rigidity matroid but it is not sufficient when $d \geq 3$. Characterising independence in the d -dimensional bar-joint rigidity matroid is an important open problem. We refer the reader to the survey article of Whiteley [24] for more information on bar-joint frameworks.

The point-line graph on the right of Figure 1 can be used to illustrate the difference between point-line and bar-joint frameworks. We can use Laman's theorem to deduce that every generic realisation of this graph as a 2-dimensional bar-joint framework (with all vertices as points and all edges as distance constraints) is rigid. In contrast we will see below that no generic realisation as a 2-dimensional point-line framework is rigid.

Point-line graphs have been used extensively in computer aided design [21, 22, 2] and computer aided geometry [25, 12]. The rigidity of the corresponding point-line framework determines when a geometric design is well-dimensioned and is useful in determining the decomposition of a geometric design into rigid components.

In this paper we give the first characterisation of independence in the 2-dimensional point-line rigidity matroid \mathcal{M}_{PL} . Our characterisation uses two count matroids¹ $\mathcal{M}(\nu_L)$ and $\mathcal{M}(2\nu_P + \nu_L - 2)$ which are defined as follows. Given a point-line graph $G = (V, E)$, let $\nu_P, \nu_L : 2^E \rightarrow \mathbb{Z}$ by taking $\nu_P(S)$ and $\nu_L(S)$ to be the number of point-vertices, respectively line-vertices, incident to S for all $S \subseteq E$. Then S is independent in $\mathcal{M}(\nu_L)$, respectively $\mathcal{M}(2\nu_P + \nu_L - 2)$, if and only if $|S'| \leq \nu_L(S')$, respectively $|S'| \leq 2\nu_P(S') + \nu_L(S') - 2$, for all $\emptyset \neq S' \subseteq S$. We will show that \mathcal{M}_{PL} is a Dilworth truncation of the matroid union of $\mathcal{M}(\nu_L)$ and $\mathcal{M}(2\nu_P + \nu_L - 2)$. Our characterisation of \mathcal{M}_{PL} leads immediately to a polynomial time algorithm to determine a maximal set of independent edges in \mathcal{M}_{PL} .

We can consider a line as a one dimensional affine subspace of the usual

¹We use the term *count matroid* to mean a matroid on the edge set of a graph $G = (V, E)$, in which the independence of a set $S \subseteq E$ is determined by counting the number of vertices incident to each subset of S .

two dimensional Euclidean space. An alternative approach, which we use here, is to consider a line as an oriented hyperplane with codimension one. When $d = 2$, the essential difference in the second approach is that the distance from a point to a line becomes a signed quantity which passes linearly through zero as the point crosses the line. This corresponds to the coordinatisation of a line which is commonly used in computer aided design [22] and is used by Yang [26] in his extension of Cayley-Menger determinants and distance geometry to include lines when $d = 2$. It is straightforward to show that these two formulations lead to the same generic rigidity matroid when $d = 2$. However for $d \geq 3$ a line and an oriented hyperplane have different dimensions and hence give rise to different rigidity matrices and matroids. It remains an open problem to characterise the generic rigidity matroid in either case when $d \geq 3$.

We have so far considered only the infinitesimal rigidity of point-line frameworks. It may also be of interest to consider global rigidity. (Generic global rigidity for the special case of 2-dimensional bar-joint frameworks was characterised in [5, 15].) Note that it may be significant whether we view a line as a one dimensional subspace or as an oriented hyperplane for global rigidity even when $d = 2$.

The remainder of this paper is organised as follows. In Section 2 we give formal definitions for a point-line graph G , a 2-dimensional point-line framework (G, p) , its rigidity matrix $R(G, p)$ and the generic rigidity matroid $\mathcal{M}_{PL}(G)$. We also obtain two necessary conditions for a rigidity matrix to have linearly independent rows by considering its null space.

In Section 3 we use these two necessary conditions to define a matroid \mathcal{M}_{\sharp} on the edge set of a point-line graph with the property that every independent set of \mathcal{M}_{\sharp} is independent in \mathcal{M}_{PL} . The derivation of this necessary condition for independence in \mathcal{M}_{PL} is purely combinatorial (and should extend to $d > 2$). We investigate the properties of \mathcal{M}_{\sharp} and show that it can be described as a Dilworth truncation of the matroid union of $\mathcal{M}(\nu_L)$ and $\mathcal{M}(2\nu_P + \nu_L - 2)$.

We complete our characterisation of \mathcal{M}_{PL} in Section 4 by showing that every independent set of edges in \mathcal{M}_{\sharp} is also independent in \mathcal{M}_{PL} . Our result generalises Laman's Theorem (which corresponds to the case when there are no line-vertices). Our proof technique differs from that of Laman, however. Laman's proof is based on a recursive construction of the family of graphs whose edge set is independent in the generic bar-joint rigidity matroid. We have not been able to obtain a similar recursive construction for point-line graphs. Instead we adapt an alternative proof technique for

Laman’s Theorem due to Whiteley [23], and give a direct construction for a point-line framework (G, p) with the property that E is independent in $\mathcal{M}_{PL}(G, p)$ whenever E is independent in $\mathcal{M}_\#$.

In order to make this construction as simple as possible, we restrict our attention to ‘naturally bipartite’ point-line graphs, i.e. point-line graphs in which every edge is incident with both a point-vertex and a line-vertex. We then complete the proof by using the fact that any point-line graph can be made naturally bipartite by replacing every edge between a pair of points or a pair of lines with a copy of the naturally bipartite graph $K_{3,3}$.

We close Section 4 by deriving a formula for the rank function of \mathcal{M}_{PL} . When there are no points, this reduces to the formula for the rank function of the cycle matroid of a graph. When there are no lines it reduces to the formula for the rank function of the 2-dimensional bar-joint rigidity matroid given by Lovász and Yemini in [19].

In Section 5 we consider the algorithmic implications of our characterisation of \mathcal{M}_{PL} . We give a brief description of Edmond’s algorithm [8] for constructing a maximum independent set in the union of two matroids. We then adapt the network flow approach used by Berg and Jordán [2] to find circuits in $\mathcal{M}(\nu_L)$ and in $\mathcal{M}(2\nu_P + \nu_L - 2)$. Finally we describe how these two algorithms can be combined to determine the rank of any generic set of point-line constraints. We give a more detailed description of our implementation in the appendix.

2 Definitions and preliminary results

A *point-line graph* is a graph $G = (V, E)$ without loops together with an ordered pair (V_P, V_L) of, possibly empty, disjoint sets whose union is V . We refer to vertices in V_P and V_L as *point-vertices* and *line-vertices*, respectively. We label the vertices as $V_P = \{u_1, \dots, u_s\}$ and $V_L = \{v_1, \dots, v_t\}$, and the edges as $E = \{e_1, e_2, \dots, e_m\}$. We use E_{PP} , E_{PL} and E_{LL} to denote the sets of edges incident to two point-vertices, to a point-vertex and a line-vertex and to two line-vertices respectively. For $e \in E$, we write $e = xy$ to mean that the end-vertices of e are x and y . We will assume that a point-line graph is simple (without parallel edges) unless it is explicitly described as a point-line multigraph. We supplement the above notation when it is not obvious which graph we are referring to by using $V(G)$, $E(G)$, etc. We say that the point-line graph G is *naturally bipartite* if G is a bipartite graph with bipartition

$\{V_P, V_L\}$. In this case $E_{PP} = E_{LL} = \emptyset$ and $E = E_{PL}$.

A *point-line framework* is a pair (G, p) where G is a point-line graph and $p : V \rightarrow \mathbb{R}^{2|V|}$. We put $p(u_i) = (x_i, y_i)$ for each $u_i \in V_P$ and $p(v_i) = (a_i, b_i)$ for each $v_i \in V_L$. This gives rise to a geometric representation of (G, p) by taking the point corresponding to u_i to have cartesian coordinates (x_i, y_i) , and the equation of the line corresponding to v_i to be $x = a_i y + b_i$. We say that (G, p) is *degenerate* if either $V_L = \emptyset$ and $p(u_i) = p(u_j)$ for all $u_i, u_j \in V_P$, or $V_P = \emptyset$ and $a_i = a_j$ for all $v_i, v_j \in V_L$.

Given a point-line graph $G = (V, E)$, the *rigidity map* $f_G : \mathbb{R}^{2|V|} \rightarrow \mathbb{R}^{|E|}$ is a rational polynomial map defined as follows. For each $p \in \mathbb{R}^{2|V|}$ we consider the point-line framework (G, p) and take $f_G(p) = (f_1(p), f_2(p), \dots, f_m(p))$ where

$$f_i(p) = \begin{cases} (x_j - x_k)^2 + (y_j - y_k)^2 & \text{if } e_i = u_j u_k \in E_{PP} \\ (x_j - y_j a_k - b_k)(1 + a_k^2)^{-1/2} & \text{if } e_i = u_j v_k \in E_{PL} \\ \tan^{-1} a_j - \tan^{-1} a_k & \text{if } e_i = v_j v_k \in E_{LL} \text{ and } i < j \end{cases}$$

The expressions for $f_i(p)$ are: the squared distance between the points represented by u_j, u_k when $e_i = u_j u_k \in E_{PP}$; the signed distance between the point represented by u_j and the line represented by v_k when $e_i = u_j v_k \in E_{PL}$; the angle between the lines represented by v_j, v_k when $e_i = v_j v_k \in E_{LL}$.

A point-line framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that every point-line framework (G, q) which satisfies $f_G(q) = f_G(p)$ and $\|p(w) - q(w)\| < \epsilon$ for all $w \in V$ can be obtained from (G, p) by a rotation or translation of \mathbb{R}^2 .

Let $J(G, p)$ be the Jacobean matrix of f_G evaluated at some point $p \in \mathbb{R}^{2|V|}$. The point p is said to be a *regular point* of f_G if $\text{rank } J(G, p) = \max_{q \in \mathbb{R}^{2|V|}} \{\text{rank } J(G, q)\}$. Asimow and Roth [1] used methods from differential geometry to show that the rigidity of a bar-joint framework (G, p) is determined by the rank of $J(G, p)$ when p is a regular point of f_G . Similar arguments, based on the facts that the rotations and translations of \mathbb{R}^2 generate a 3-dimensional subspace of the null space of $J(G, p)$ when (G, p) is non-degenerate, and that there exists an open neighbourhood U of p such that $\{q \in U : f_G(q) = f_G(p)\}$ is a manifold of dimension $2|V| - \text{rank } J(G, p)$ when p is a regular point of f_G , can be used to show:

Lemma 2.1 *Let (G, p) be a non-degenerate point-line framework. Then*

- (a) $\text{rank } J(G, p) \leq 2|V| - 3$.
- (b) *If $\text{rank } J(G, p) = 2|V| - 3$ then (G, p) is rigid.*

(c) If p is a regular point of f_G and $\text{rank } J(G, p) < 2|V| - 3$ then (G, p) is not rigid.

It is straightforward to determine when a degenerate point-line framework is rigid. We will determine when a given point-line graph G can be realised as a non-degenerate point-line framework (G, p) with $\text{rank } J(G, p) = 2|V| - 3$. To this end it will be helpful to apply row and column operations to $J(G, p)$ to obtain the following simpler $|E| \times 2|V|$ -matrix $R(G, p)$. The rows of $R(G, p)$ are indexed by E and pairs of columns by V . We label the two columns indexed by a vertex $u_i \in V_P$ as $u_{i,x}$ and $u_{i,y}$, respectively, and the two columns indexed by a vertex $v_i \in V_L$ as $v_{i,a}$ and $v_{i,b}$, respectively.

- A row in $R(G, p)$ indexed by an edge $e = u_i u_j \in E_{PP}$ has entries

$$x_i - x_j, y_i - y_j, x_j - x_i, y_j - y_i$$

in the columns indexed by $u_{i,x}, u_{i,y}, u_{j,x}$ and $u_{j,y}$, respectively, and zeros elsewhere.

- A row in $R(G, q)$ indexed by an edge $e = u_i v_j \in E_{PL}$ has entries

$$1, -a_j, -x_i a_j - y_i, -1$$

in the columns indexed by $u_{i,x}, u_{i,y}, v_{j,a}$ and $v_{j,b}$, respectively, and zeros elsewhere.

- A row in $R(G, q)$ indexed by an edge $e = v_i v_j \in E_{LL}$ with $i < j$ has entries

$$1, -1$$

in the columns indexed by $v_{i,a}$ and $v_{j,a}$, respectively, and zeros elsewhere.

The Jacobean matrix $J(G, p)$ can be constructed from $R(G, p)$ using the following operations. For each $v_i \in V_L$, we multiply the column of $R(G, p)$ indexed by $v_{i,b}$ by $a_i b_i$ and add it to the column indexed by $v_{i,a}$, then divide the resulting column by $(1 + a_i^2)$. For each $e \in E_{PP}$, we multiply the row indexed by e by 2. For each $e = u_i v_j \in E_{PL}$, we divide the row indexed by e by $(1 + a_j^2)^{1/2}$. This construction immediately implies

Lemma 2.2 *Let (G, p) be a point-line framework. Then $\text{rank } J(G, p) = \text{rank } R(G, p)$.*

We say that a point-line framework (G, p) is: *independent* if $\text{rank } R(G, p) = |E|$; *infinitesimally rigid* if it is non-degenerate and $\text{rank } R(G, p) = 2|V| - 3$; *isostatic* if it is both independent and infinitesimally rigid; *generic* if the set of coordinates $\{x_i, y_i, a_j : u_i \in V_P, v_j \in V_L\}$ are algebraically independent over \mathbb{Q} .

Note that the infinitesimal rigidity of (G, p) is equivalent to the condition that $\text{rank } J(G, p) = 2|V| - 3$ whenever (G, p) is non-degenerate. Since the entries of $R(G, p)$ are rational functions of the coordinates x_i, y_i, a_j , the rank of $R(G, p)$ will be maximised whenever (G, p) is generic. We say that the point-line graph G is *rigid* if some, or equivalently every, generic realisation (G, p) of G is infinitesimally rigid. Lemmas 2.1 and 2.2 tell us that G is rigid if and only if some, or equivalently every, generic realisation (G, p) of G is rigid.

The *rigidity matroid* $\mathcal{M}(G, p)$ of a point-line framework (G, p) is the row matroid of its rigidity matrix $R(G, p)$. Its ground set is E and a set $S \subseteq E$ is independent if the rows of $R(G, p)$ indexed by S are linearly independent. The matroid $\mathcal{M}(G, p)$ will be the same for all generic (G, p) . In this case we refer to $\mathcal{M}(G, p)$ as the *rigidity matroid of G* and denote it by $\mathcal{M}_{PL}(G)$, or simply \mathcal{M}_{PL} when it is obvious which graph we are referring to. We denote the rank of $\mathcal{M}_{PL}(G)$ by $r_{PL}(G)$. Thus G is independent if and only if $r_{PL}(G) = |E|$ and is rigid if and only if $r_{PL}(G) = 2|V| - 3$.

When (G, p) is a point-line framework with no point-vertices, $R(G, p)$ is the $\{0, 1, -1\}$ edge/vertex incidence matrix of an orientation of G . This immediately implies

Lemma 2.3 *Let (G, p) be a point-line framework with $V_P = \emptyset$. Then $\mathcal{M}_{PL}(G, p)$ is the cycle matroid of G . In particular (G, p) is independent if and only if G is a forest.*

We can use this result to obtain the following necessary conditions for (G, p) to be independent.

Lemma 2.4 *Suppose (G, p) is an independent point-line framework and H is a subgraph of G with $|E(H)| > 0$. Then $|E(H)| \leq 2|V(H)| - 3$. In addition, if $V_P(H) = \emptyset$, then $|E(H)| \leq |V(H)| - 1$.*

Proof. Since the rank of the rigidity matrix is maximised for generic realisations of G , we may assume that (G, p) is generic. The hypothesis that

(G, p) is independent implies that $(H, p|_H)$ is independent. We can now use Lemmas 2.1(a) and 2.2(b) to deduce that $|E(H)| = \text{rank } R(H, p|_H) = \text{rank } J(H, p|_H) \leq 2|V(H)| - 3$. When $V_P(H) = \emptyset$, Lemma 2.3 gives $|E(H)| = \text{rank } R(H, p|_H) \leq |V(H)| - 1$. \bullet

3 A count matroid for point-line graphs

Given a point-line graph $G = (V, E)$, the necessary conditions for independence in $\mathcal{M}_{PL}(G)$ given by Lemma 2.4 are not sufficient - we shall see that the point-line graph in Figure 1 is a counterexample. Indeed the family \mathcal{F} of sets satisfying these conditions need not even define a matroid on E . On the other hand, we will show that $\mathcal{M}_{PL}(G)$ is equal to the matroid $\mathcal{M}_\#(G)$ on E whose family of independent sets is the maximum subset of \mathcal{F} which satisfies the matroid axioms.

We construct $\mathcal{M}_\#(G)$ from two simpler count matroids using the operations of matroid union and Dilworth truncation. We then use Lemma 2.4 to prove the partial result that every independent set in $\mathcal{M}_{PL}(G)$ is an independent set in $\mathcal{M}_\#(G)$. (More precisely we will show that if I is an independent set in some matroid on E whose independent sets satisfy Lemma 2.4, then I is independent in $\mathcal{M}_\#(G)$.) The reverse implication will be proved in the next section.

We first recall some results from matroid theory. We refer a reader unfamiliar with submodular functions and matroids to [9].

Let E be a set. A *subpartition* of E is a (possibly empty) collection of pairwise disjoint nonempty subsets of E . A function $f : 2^E \rightarrow \mathbb{R}$ is *nondecreasing* if $f(A) \leq f(B)$ for all $A \subseteq B \subseteq E$, *submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq E$, and *intersecting submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq E$ with $A \cap B \neq \emptyset$. We will need the following result of Dunstan [6], see [9, Theorem 12.1.1].

Theorem 3.1 *Suppose E is a set and $f : 2^E \rightarrow \mathbb{Z}$ is intersecting submodular. Let $g : 2^E \rightarrow \mathbb{Z}$ be defined by*

$$g(A) = \min \left\{ \sum_{i=1}^s f(A_i) \right\}$$

where the minimum is taken over all partitions $\{A_1, \dots, A_s\}$ of A . Then g is submodular.

The following result of Edmonds [8], see [9, Theorem 13.4.2], tells us how an intersecting submodular function can be used to define a matroid on E .

Theorem 3.2 *Suppose E is a set and $f : 2^E \rightarrow \mathbb{Z}$ is nondecreasing, intersecting submodular and nonnegative on $2^E \setminus \{\emptyset\}$. Let*

$$\mathcal{I} = \{I \subseteq E : |J| \leq f(J) \text{ for all } \emptyset \neq J \subseteq I\}.$$

Then \mathcal{I} is the family of independent sets of a matroid $\mathcal{M}(f) = (E, \mathcal{I})$. The rank of any $A \subseteq E$ in $\mathcal{M}(f)$ is given by

$$r(A) = \min \left\{ \left| A \setminus \bigcup_{i=1}^s A_i \right| + \sum_{i=1}^s f(A_i) \right\}$$

where the minimum is taken over all subpartitions $\{A_1, \dots, A_s\}$ of A . In addition we have:

- (a) if $f(e) \leq 1$ for all $e \in E$, then $r(A) = \min \{\sum_{i=1}^s f(A_i)\}$ where the minimum is taken over all partitions $\{A_1, \dots, A_s\}$ of A ;*
- (b) if f is submodular and $f(\emptyset) = 0$, then $r(A) = \min_{B \subseteq A} \{|A \setminus B| + f(B)\}$;*
- (c) if f is submodular, $f(\emptyset) = 0$ and $f(e) \leq 1$ for all $e \in E$, then $r(A) = f(A)$.*

The matroid $\mathcal{M}(f)$ given in Theorem 3.2 is referred to as the matroid *induced by f* . The function $f - 1$ will also satisfy the hypotheses of Theorem 3.2 whenever f has $f(e) \geq 1$ for all $e \in E$. In this case we will refer to $\mathcal{M}(f - 1)$ as the *Dilworth truncation* of $\mathcal{M}(f)$.

The next result describes a method to combine two matroids on the same ground set to obtain a new matroid.

Theorem 3.3 *Suppose that $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ are two matroids with the same ground set E . Let*

$$\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}.$$

Then \mathcal{I} is the family of independent sets of a matroid $\mathcal{M}_1 \vee \mathcal{M}_2$. The rank of any $A \subseteq E$ in $\mathcal{M}_1 \vee \mathcal{M}_2$ is given by

$$r(A) = \min_{B \subseteq A} \{r_1(B) + r_2(B) + |A \setminus B|\}$$

where r_1, r_2 are the rank functions of $\mathcal{M}_1, \mathcal{M}_2$, respectively.

The matroid $\mathcal{M}_1 \vee \mathcal{M}_2$ is the *matroid union* of \mathcal{M}_1 and \mathcal{M}_2 .

The expressions for the rank functions in Theorems 3.2(b) and 3.3 immediately give the following relationship between matroids induced by submodular functions and their matroid union.

Lemma 3.4 [20] *Suppose E is a set and $f, g : 2^E \rightarrow \mathbb{Z}$ are nondecreasing nonnegative submodular functions. Then $f+g$ is a nondecreasing nonnegative submodular function and $\mathcal{M}(f+g) = \mathcal{M}(f) \vee \mathcal{M}(g)$.*

Note that the conclusion of Lemma 3.4 may not hold if the functions f, g are allowed to take negative values on the empty set.

Let $G = (V, E)$ be a point-line graph. For each $A \subseteq E$ let $\nu_P(A)$ and $\nu_L(A)$ be the numbers of point-vertices and line-vertices, respectively which are incident to edges in A . It is easy to see that $\nu_P(A)$ and $\nu_L(A)$ are both nondecreasing nonnegative submodular functions on 2^E .

Lemma 3.5 *Let $G = (V, E)$ be a point-line graph. Let $\rho : 2^E \rightarrow \mathbb{Z}$ be defined by putting*

$$\rho(A) = \min \left\{ \sum_{i=1}^s (2\nu_P(A_i) + \nu_L(A_i) - 2) \right\}$$

for all $A \subseteq E$, where the minimum is taken over all partitions $\{A_1, \dots, A_s\}$ of A . Then ρ and $\rho + \nu_L$ are nondecreasing, submodular and nonnegative, and $\rho + \nu_L - 1$ is nondecreasing, submodular and nonnegative on $2^E \setminus \{\emptyset\}$.

Proof. The function $2\nu_P + \nu_L - 2$ is nondecreasing and submodular because ν_P and ν_L are both nondecreasing and submodular. This, and Theorem 3.1, imply that ρ is nondecreasing and submodular. The facts that $2\nu_P + \nu_L - 2$ is nonnegative on $2^E \setminus \{\emptyset\}$ and $\rho(\emptyset) = 0$ imply that ρ is nonnegative. The assertion that $\rho + \nu_L$ is nondecreasing, submodular and nonnegative now follows since ν_L is nondecreasing, submodular and nonnegative. This immediately implies that $\rho + \nu_L - 1$ is nondecreasing and submodular. The assertion that $\rho + \nu_L - 1$ is nonnegative on $2^E \setminus \{\emptyset\}$ follows since it is nondecreasing and $\rho(e) + \nu_L(e) - 1 = 2\nu_P(e) + 2\nu_L(e) - 3 = 1$ for all $e \in E$. \bullet

We can now define our promised matroid $\mathcal{M}_{\#}(G)$ by putting $\mathcal{M}_{\#}(G) = \mathcal{M}(\rho + \nu_L - 1)$. Since $\mathcal{M}_{\#}(G)$ is the Dilworth truncation of $\mathcal{M}(\rho + \nu_L)$, it will aid our understanding of $\mathcal{M}_{\#}(G)$ to express $\mathcal{M}(\rho + \nu_L)$ as a matroid union of two simpler matroids.

Lemma 3.6 *Suppose $G = (V, E)$ is a point-line graph. Then*

$$\mathcal{M}(\rho + \nu_L) = \mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L).$$

Proof. Since ρ and ν_L are both nondecreasing and submodular with $\rho(\emptyset) = 0 = \nu_L(\emptyset)$, we have $\mathcal{M}(\rho + \nu_L) = \mathcal{M}(\rho) \vee \mathcal{M}(\nu_L)$ by Lemma 3.4. It remains to show that $\mathcal{M}(\rho) = \mathcal{M}(2\nu_P + \nu_L - 2)$. Let r_1, r_2 be the rank functions of $\mathcal{M}(\rho)$ and $\mathcal{M}(2\nu_P + \nu_L - 2)$, respectively. We can use Theorem 3.2(b) and the definition of ρ to deduce that

$$\begin{aligned} r_1(A) &= \min_{B \subseteq A} \{|A \setminus B| + \rho(B)\} \\ &= \min_{B \subseteq A} \min_{\mathcal{P}_B} \left\{ |A \setminus B| + \sum_{B_i \in \mathcal{P}_B} (2\nu_P(B_i) + \nu_L(B_i) - 2) \right\} \end{aligned}$$

for all $A \subseteq E$, where the second minimum runs over all partitions \mathcal{P}_B of B . On the other hand Theorem 3.2 gives

$$r_2(A) = \min_{\mathcal{Q}_A} \left\{ \left| A \setminus \bigcup_{B_i \in \mathcal{Q}_A} B_i \right| + \sum_{B_i \in \mathcal{Q}_A} (2\nu_P(B_i) + \nu_L(B_i) - 2) \right\}$$

where the minimum runs over all subpartitions \mathcal{Q}_A of A . We can now deduce that $r_1(A) = r_2(A)$ by putting $B = \bigcup_{B_i \in \mathcal{Q}_A} B_i$. •

Given a graph $G = (V, E)$ and $w_1, w_2 \in V$ we denote the graph obtained by adding a new edge w_1w_2 to G by $G + w_1w_2$. Note that if $e = w_1w_2$ already exists as an edge in G then we add a new edge e' parallel to e in $G + w_1w_2$.

Lemma 3.7 *Suppose $G = (V, E)$ is a point-line graph. Then the following statements are equivalent.*

- (a) G is $\mathcal{M}(\rho + \nu_L - 1)$ -independent.
- (b) $G + w_1w_2$ is $\mathcal{M}(\rho + \nu_L)$ -independent for all $w_1, w_2 \in V$.
- (c) $G + w_1w_2$ is $\mathcal{M}(\rho + \nu_L)$ -independent for all $w_1w_2 \in E$.

Proof. (a) \Rightarrow (b). Suppose G is $\mathcal{M}(\rho + \nu_L - 1)$ -independent and let $w_1, w_2 \in V$. Choose $A \subseteq E$. Then $|A| \leq \rho(A) + \nu_L(A) - 1$ and $|A + w_1w_2| = |A| + 1 \leq \rho(A) + \nu_L(A) \leq \rho(A + w_1w_2) + \nu_L(A + w_1w_2)$. Hence $G + w_1w_2$ is $\mathcal{M}(\rho + \nu_L)$ -independent.

(b) \Rightarrow (c) is immediate.

(c) \Rightarrow (a). We prove the contrapositive. Suppose G is $\mathcal{M}(\rho + \nu_L - 1)$ -dependent. Then there exists a $\emptyset \neq B \subseteq E$ such that $|B| > \rho(B) + \nu_L(B) - 1$. Choose a partition $\{B_1, B_2, \dots, B_t\}$ of B such that $\rho(B) = \sum_{i=1}^t (2\nu_P(B_i) + \nu_L(B_i) - 2)$. Choose $e = w_1w_2 \in B_1$ and put $B'_1 = B_1 + w_1w_2$ and $B'_i = B_i$ for all $2 \leq i \leq t$. Then $\{B'_1, B'_2, \dots, B'_t\}$ is a partition of $B + w_1w_2$, so

$$\rho(B + w_1w_2) \leq \sum_{i=1}^t (2\nu_P(B'_i) + \nu_L(B'_i) - 2) = \sum_{i=1}^t (2\nu_P(B_i) + \nu_L(B_i) - 2) = \rho(B).$$

We also have $\nu_L(B + w_1w_2) = \nu_L(B)$. Hence $|B + w_1w_2| = |B| + 1 > \rho(B) + \nu_L(B) \geq \rho(B + w_1w_1) + \nu_L(B + w_1w_2)$ so $G + w_1w_2$ is $\mathcal{M}(\rho + \nu_L)$ -dependent. \bullet

The remainder of this section is devoted to showing that every independent set in $\mathcal{M}_{PL}(G)$ is independent in $\mathcal{M}_{\#}(G)$. The converse statement will be proved in the next section.

Given a point-line graph G , let $r_{\#}(G)$ and $r_{PL}(G)$ denote the ranks of $\mathcal{M}_{\#}(G)$ and $\mathcal{M}_{PL}(G)$, respectively. Our next result gives an upper bound on $r_{PL}(G)$.

Lemma 3.8 *Let $G = (V, E)$ be a point-line graph. Then*

$$r_{PL}(G) \leq \rho(E) + |V_L| - 1.$$

Proof. By the definition of ρ , it will suffice to show that every partition $\mathcal{F} = \{E_1, E_2, \dots, E_t\}$ of E satisfies

$$r_{PL}(G) \leq \sum_{i=1}^t (2\nu_P(E_i) + \nu_L(E_i) - 2) + |V_L| - 1. \quad (1)$$

We will prove the stronger result that

$$r_{PL}(G) \leq \sum_{i=1}^t (2\nu_P(E_i) + \nu_L(E_i) - 2) + |V_L| - c_L(\mathcal{F}) \quad (2)$$

where $c_L(\mathcal{F})$ is the number of components in the graph with vertex set \mathcal{F} , in which two vertices E_i, E_j are adjacent if $V_L(E_i) \cap V_L(E_j) \neq \emptyset$.

Let H_i be the subgraph of G induced by E_i . We may assume that each H_i is a complete graph since adding an edge between two vertices of H_i to G

will not change the right hand side of (2) and cannot decrease the left hand side of (2). Let I be a maximal set of edges in E_{LL} which are independent in $\mathcal{M}_{PL}(G)$, B be a base of $\mathcal{M}_{PL}(G)$ which contains I , $B_i = B \cap E_i$ and $I_i = B_i \cap I$. Then

$$r_{PL}(G) = |B| = \sum_{i=1}^t |B_i| = \sum_{i=1}^t |B_i \setminus I_i| + |I|. \quad (3)$$

We first consider the case when $V_L = \emptyset$. Then $|B_i| \leq 2\nu_P(E_i) - 3$ for all $1 \leq i \leq t$ by Lemma 2.4 and (2) follows from (3).

Hence we may suppose that $V_L \neq \emptyset$. Then $|I| \leq |V_L| - c_L(\mathcal{F})$ by Lemma 2.4, and inequality (2) will follow from (3) if we can show that $|B_i \setminus I_i| \leq 2\nu_P(E_i) + \nu_L(E_i) - 2$ for all $1 \leq i \leq t$. This last inequality follows immediately from Lemma 2.4 (with strict inequality) when $\nu_L(E_i) = 0$. Hence we may suppose that $\nu_L(E_i) \neq 0$.

Let I_i^* be a maximal set of edges in $E_i \cap E_{LL}$ which are independent in $\mathcal{M}_{PL}(H_i)$. Since H_i is complete, I_i^* will induce a spanning tree on the line-vertices of H_i , by Lemma 2.3, and hence $|I_i^*| = \nu_L(E_i) - 1$. The maximality of I implies that $r_{PL}(I + e) = r_{PL}(I)$ for all edges $e \in I_i^*$. This in turn implies that $(B_i \setminus I_i) \cup I_i^*$ is independent in $\mathcal{M}_{PL}(G)$, since if $(B_i \setminus I_i) \cup I_i^*$ were dependent then $(B_i \setminus I) \cup I = B_i \cup I$ would also be dependent in $\mathcal{M}_{PL}(G)$, which would contradict the fact that $B_i \cup I \subseteq B$. Lemma 2.4 now gives $|(B_i \setminus I_i) \cup I_i^*| \leq 2\nu_P(E_i) + 2\nu_L(E_i) - 3$. Since $|I_i^*| = \nu_L(E_i) - 1$ we have $|B_i \setminus I_i| \leq 2\nu_P(E_i) + \nu_L(E_i) - 2$, as required. \bullet

Lemma 3.9 *Let $G = (V, E)$ be a point-line graph and $F \subset E$ be independent in $\mathcal{M}_{PL}(G)$. Then F is independent in $\mathcal{M}_{\#}(G)$.*

Proof. Since $\mathcal{M}_{\#}(G) = \mathcal{M}(\rho + \nu_L - 1)$, it will suffice to show that $|F'| \leq \rho(F') + \nu_L(F') - 1$ for all $F' \subseteq F$. Let $H = G[F']$. Since $F' \subseteq F$, F' is independent in $\mathcal{M}_{PL}(G)$ and hence $r_{PL}(H) = |F'|$. We can now apply Lemma 3.8 to H to deduce that $|F'| = r_{PL}(H) \leq \rho(F') + \nu_L(F') - 1$. \bullet

4 A characterisation of the generic point-line rigidity matroid

We now complete the proof that $\mathcal{M}_{PL}(G) = \mathcal{M}_{\sharp}(G)$. Our approach is to first construct a linear representation for $\mathcal{M}_{\sharp}(G)$ when G is *naturally bipartite* i.e. G is a bipartite graph with bipartition (V_P, V_L) . We use this to construct a point-line framework (G, p) such that the rows of the rigidity matrix $R(G, p)$ are linearly independent whenever G is \mathcal{M}_{\sharp} -independent and naturally bipartite. Together with Lemma 3.9, this will imply $\mathcal{M}_{PL}(G) = \mathcal{M}_{\sharp}(G)$ when G is naturally bipartite. We then deduce that this equality holds for an arbitrary point-line graph by reducing to the bipartite case.

4.1 Linear representations of point-line frames

We will need the following general result on linear representations of matroid unions.

Lemma 4.1 [4, Lemma 7.6.14(1)] *Suppose that \mathcal{M}_1 and \mathcal{M}_2 are two matroids with the same ground set and that \mathcal{M}_i is the row matroid of an $m \times n_i$ real matrix M_i for $i = 1, 2$. Let X be the $m \times m$ diagonal matrix $\text{diag}(x_1, x_2, \dots, x_m)$ where x_1, x_2, \dots, x_m are algebraically independent over $\mathbb{Q}[M_1]$. Then $\mathcal{M}_1 \vee \mathcal{M}_2$ is the row matroid of $M = (M_1, XM_2)$.*

A *point-line frame* is a triple (G, t, c) where $G = (V, E)$ is a bipartite multigraph with bipartition $V = V_P \cup V_L$, $t : V_L \rightarrow \mathbb{R}$ and $c : E \rightarrow \mathbb{R}$. We label the vertices in V_P and V_L as u_1, u_2, \dots, u_m and v_1, v_2, \dots, v_n , respectively. We denote $t(v_j)$ by t_j and $c(e)$ by c_e . We associate the following three matrices with (G, t, c) .

- The *A-matrix*, $A(G, t, c)$, is the $|E| \times 2|V|$ matrix in which the entries in the row indexed by an edge $e = u_i v_j \in E$ are: $(1, t_j)$ in the columns indexed by u_i ; $(c_e, -1)$ in the columns indexed by v_j ; and zeros elsewhere.
- the *B-matrix*, $B(G, c)$, is the $|E| \times (2|V_P| + |V_L|)$ matrix in which the entries in the row indexed by an edge $e = u_i v_j \in E$ are: $(1, c_e)$ in the columns indexed by u_i ; -1 in the column indexed by v_j ; zeros elsewhere.

- the C -matrix, $C(G, t)$, is the $|E| \times (2|V_P| + |V_L|)$ matrix in which the entries in the row indexed by an edge $e = u_i v_j \in E$ are: $(1, t_j)$ in the columns indexed by u_i ; -1 in the column indexed by v_j ; zeros elsewhere.

The point-line frame (G, t, c) is *generic* if the set $\{t_j, c_e : v_j \in V_L, e \in E\}$ is algebraically independent over \mathbb{Q} . We shall show that the matrices $B(G, c)$, $C(G, t)$ and $A(G, t, c)$ provide linear representations for the matroids $\mathcal{M}(2\nu_P + \nu_L - 1)$, $\mathcal{M}(2\nu_P + \nu_L - 2)$ and $\mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L)$, respectively, when (G, t, c) is generic. We first need to express $\mathcal{M}(2\nu_P + \nu_L - 1)$ as the matroid union $\mathcal{M}(\nu - 1) \vee \mathcal{M}(\nu_P)$, where $\nu = \nu_P + \nu_L$ (and hence $\mathcal{M}(\nu - 1)$ is the well known *cycle matroid* of G). Note that this does not follow from Lemma 3.4 because $\nu(\emptyset) - 1 = -1$.

Lemma 4.2 *Suppose $G = (V, E)$ is a naturally bipartite multigraph. Then*

$$\mathcal{M}_G(2\nu_P + \nu_L - 1) = \mathcal{M}_G(\nu - 1) \vee \mathcal{M}_G(\nu_P).$$

Proof. Let r_1 and r_2 be the rank functions of the matroids $\mathcal{M}_G(\nu - 1)$ and $\mathcal{M}_G(\nu_P)$, respectively. For $K \subset E$, let $\Pi(K)$ be the partition of K induced by the connected components of $G[K]$. It is well known that $r_1(K) = \sum_{K_i \in \Pi(K)} (\nu(K_i) - 1)$ for all $F \subseteq E$, and it is not difficult to check that we also have $r_2(K) = \sum_{K_i \in \Pi(K)} \nu_P(K_i)$. We can now apply [16, Lemma 2.2] to deduce that $\mathcal{M}_G(\nu - 1) \vee \mathcal{M}_G(\nu_P) = \mathcal{M}_G(\nu + \nu_P - 1) = \mathcal{M}_G(2\nu_P + \nu_L - 1)$. •

Lemma 4.3 *Suppose (G, t, c) is a generic point-line frame. Then:*

- the row matroid of $B(G, c)$ is $\mathcal{M}_G(2\nu_P + \nu_L - 1)$;*
- the row matroid of $C(G, t)$ is $\mathcal{M}_G(2\nu_P + \nu_L - 2)$;*
- the row matroid of $A(G, t, c)$ is $\mathcal{M}_G(2\nu_P + \nu_L - 2) \vee \mathcal{M}_G(\nu_L)$.*

Proof. (a) Let \vec{G} be the directed graph obtained by directing all edges of G from V_L to V_P . Then the matroid $\mathcal{M}_G(\nu - 1)$ is the row matroid of the $|E| \times |V|$ matrix M_1 which is the $\{0, 1, -1\}$, edge/vertex incidence matrix for \vec{G} . Similarly, the matroid $\mathcal{M}_G(\nu_P)$ is the row matroid of the $|E| \times |V_P|$ matrix M_2 , which is the $\{0, 1\}$, edge/point-vertex incidence matrix for G . Lemma 4.1 now implies that $\mathcal{M}_G(\nu - 1) \vee \mathcal{M}_G(\nu_P)$ is the row matroid of $B(G, c)$, after a suitable reordering of its columns, and (a) follows from Lemma 4.2.

(b) It will suffice to show that the rows of $C(G, t)$ are independent if and only if E is independent in $\mathcal{M}_G(2\nu_P + \nu_L - 2)$.

Suppose that the rows of $C(G, t)$ are independent. Choose $F \subseteq E$ and let H be the subgraph of G induced by the edges in F . Then the rows of $C(H, t)$ are independent, since the entries in the rows of $C(G, t)$ indexed by F and columns indexed by vertices not incident to F are all zero. We can express the vectors in $\text{Null } C(H, t)$ in the form (q, b) where $q : V_P(H) \rightarrow \mathbb{R}^2$ and $b : V_L(H) \rightarrow \mathbb{R}$. Then the two vectors (q, b) and (q', b') defined by $q_i = (1, 0)$ and $q'_i = (0, 1)$ for all $u_i \in V_P(H)$, and $b_j = 1$ and $b'_j = t_j$ for all $v_j \in V_L(H)$, are linearly independent and belong to $\text{Null } C(H, t)$. This implies that

$$|F| = \text{rank } C(H, t) \leq |V_L(H)| + 2|V_P(H)| - 2 = \nu_L(F) + 2\nu_P(F) - 2.$$

Since this holds for all $F \subseteq E$, E is independent in $\mathcal{M}_G(2\nu_P + \nu_L - 2)$.

We next suppose that E is independent in $\mathcal{M}_G(2\nu_P + \nu_L - 2)$. We may assume further that each vertex in V_P is incident with an edge of G since, if this is not the case, then we may add an edge incident to an isolated vertex of V_P in G and preserve independence in $\mathcal{M}_G(2\nu_P + \nu_L - 2)$. Choose $e = u_i v_j \in E$ and let $G + e' = (V, E + e')$ be the bipartite multigraph obtained by adding a new edge e' parallel to e in G . Let $(G + e', \tilde{t}, \tilde{c})$ be a generic frame for $G + e'$ and $B(G + e', \tilde{c})$ be its B -matrix. The fact that E is independent in $\mathcal{M}_G(2\nu_P + \nu_L - 2)$ implies that $E + e'$ is independent in $\mathcal{M}_G(2\nu_P + \nu_L - 1)$ and hence, by (a), the rows of $B(G + e', \tilde{c})$ are linearly independent. We can represent each vector in the null space of $B(G + e', \tilde{c})$ as (q, b) where $q : V_P \rightarrow \mathbb{R}^2$ and $b : V_L \rightarrow \mathbb{R}$. Let $q(u_k) = (q_{k,1}, q_{k,2})$ for all $u_k \in V_P$, and $b(v_k) = b_k$ for all $v_k \in V_L$. Since $e = u_i v_j$ and $\text{rank } B(G, \tilde{c}) = \text{rank } B(G + e', \tilde{c}) - 1$, we can find a $(q, b) \in \text{Null } B(G, \tilde{c})$ such that $q_{i,2} \neq 0$.

Repeating the above argument for each $e \in E$ and taking a suitable linear combination of the vectors we obtain, we can construct a $(q, b) \in \text{Null } B(G, \tilde{c})$ such that $q_{i,2} \neq 0$ for all $u_i \in V_P$. The fact that $(q, b) \in \text{Null } B(G, \tilde{c})$ gives

$$-b_j + q_{i,1} + \tilde{c}_e q_{i,2} = 0 \quad \text{for all } e = u_i v_j \in E. \quad (4)$$

Equation (4) enables us to transform the matrix $C(G, b)$ to the matrix $B(G, \tilde{c})$ by subtracting $q_{i,1}$ times column $u_{i,1}$ from column $u_{i,2}$ and then dividing column $u_{i,2}$ by $q_{i,2}$, for all $u_i \in V_P$. This implies that $\text{rank } C(G, b) = \text{rank } B(G, \tilde{c}) = |E|$. Since t is generic we have $\text{rank } C(G, t) \geq \text{rank } C(G, b)$. Hence the rows of $C(G, t)$ are linearly independent.

(c) This follows from (b), Lemma 4.1 and the fact that the matroid $\mathcal{M}_G(\nu_L)$

is the row matroid of the $\{0, 1\}$, edge/line-vertex incidence matrix for G . •

Lemma 4.4 *Let $G = (V, E)$ be a naturally bipartite point-line graph. Suppose that E is independent in $\mathcal{M}_\#(G)$. Then E is independent in $\mathcal{M}_{PL}(G)$.*

Proof. Recall that $\mathcal{M}_\#(G) = \mathcal{M}(\rho + \nu_L - 1)$ where ρ is as defined in Lemma 3.5. We may assume that each vertex in V_L is incident with an edge of G since, if this is not the case, then we may add an edge incident to an isolated vertex of V_L in G and preserve independence in $\mathcal{M}_G(\rho + \nu_L - 1)$. Choose $e = u_i v_j \in E$ and let $G + e' = (V, E + e')$ be the bipartite multigraph obtained by adding a new edge e' parallel to e in G . Let $(G + e', t, c)$ be a generic frame for $G + e'$. Since E is independent in $\mathcal{M}_G(\rho + \nu_L - 1)$, Lemma 3.7 implies that $E + e'$ is independent in $\mathcal{M}_G(\rho + \nu_L)$ and hence, by Lemmas 3.6 and 4.3(c), the rows of $A(G + e', t, c)$ are linearly independent. We can represent each vector in the null space of $A(G + e', t, c)$ as (q, h) where $q : V_P \rightarrow \mathbb{R}^2$ and $h : V_L \rightarrow \mathbb{R}^2$. Let $q(u_k) = (q_{k,1}, q_{k,2})$ for all $u_k \in V_P$ and $h(v_k) = (h_{k,1}, h_{k,2})$ for all $v_k \in V_L$. Since $e = u_i v_j$ and $\text{rank } A(G, t, c) = \text{rank } A(G + e', t, c) - 1$, we can find a $(q, h) \in \text{Null } A(G, t, c)$ such that $h_{j,1} \neq 0$.

Repeating this argument for each $e \in E$ and taking a suitable linear combination of the vectors we obtain, we can construct a $(q, h) \in \text{Null } A(G, t, c)$ such that $h_{j,1} \neq 0$ for all $v_j \in V_L$. The fact that $(q, h) \in \text{Null } A(G, t, c)$ gives

$$q_{i,1} + t_j q_{i,2} + c_e h_{j,1} - h_{j,2} = 0 \quad \text{for all } e = u_i v_j \in E. \quad (5)$$

Construct a point-line framework (G, p) by putting $p(u_i) = (-q_{i,2}, q_{i,1})$ for all $u_i \in V_P$ and $p(v_j) = (-t_j, 0)$ for all $v_j \in V_L$. Equation (5) enables us to transform the rigidity matrix $R(G, p)$ to the A -matrix $A(G, t, c)$ by subtracting $h_{j,2}$ times column $v_{j,2}$ from column $v_{j,1}$ for all $v_j \in V_L$, and then dividing column $v_{j,1}$ by $h_{j,1}$. This implies that $\text{rank } R(G, p) = \text{rank } A(G, t, c) = |E|$. It follows that the rows of the rigidity matrix of any generic realisation of G as a point-line framework will be linearly independent. Hence E is independent in $\mathcal{M}_{PL}(G)$. •

4.2 The non-bipartite case

We reduce the general case to the naturally bipartite case by replacing each ‘non-bipartite edge’ by a copy of a naturally bipartite $K_{3,3}$. We need the following lemma to show that this operation preserves independence in $\mathcal{M}_\#$.

Lemma 4.5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be point-line graphs. Suppose that G_1 and G_2 are $\mathcal{M}_\#$ -independent, $e = z_1 z_2 \in E_1$, and $V(G_1) \cap V(G_2) = \{z_1, z_2\} \in \{V_P(G_1) \cap V_P(G_2), V_L(G_1) \cap V_L(G_2)\}$. Then $G = (G_1 - e) \cup G_2$ is $\mathcal{M}_\#$ -independent.*

Proof. Suppose $G = (V, E)$ is not $\mathcal{M}(\rho + \nu_L - 1)$ -independent. Then there exists a nonempty $A \subseteq E$ such that $\rho(A) + \nu_L(A) - 1 \leq |A| - 1$. Since G_1, G_2 are $\mathcal{M}(\rho + \nu_L - 1)$ -independent, we have $A \cap E_1 \neq \emptyset \neq A \cap E_2$. The definition of ρ implies that there exists a partition $\mathcal{F} = \{A_1, A_2, \dots, A_t\}$ of A such that

$$\sum_{i=1}^t (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A) - 1 \leq |A| - 1. \quad (6)$$

We consider two cases.

Case 1: $z_1, z_2 \in V_L$. We may assume that \mathcal{F} has been chosen amongst all partitions satisfying (6) to make $|\mathcal{F}|$ as large as possible. This choice ensures that $A_i \subset E_j$ for each $1 \leq i \leq t$ and some $1 \leq j \leq 2$, since if $A_i \cap E_1 \neq \emptyset \neq A_i \cap E_2$ then the partition $\mathcal{F}' = \mathcal{F} \setminus \{A_i\} \cup \{A_i \cap E_1, A_i \cap E_2\}$ would still satisfy (6).

For $1 \leq j \leq 2$, let $\mathcal{F}_j = \{A_i \in \mathcal{F} : A_i \subseteq E_j\}$. Then \mathcal{F}_j is a partition of $A'_j = A \cap E_j$. Since $V(A'_1) \cap V(A'_2) \subseteq \{z_1, z_2\} \subseteq V_L$, we have

$$\begin{aligned} |A'_1| + |A'_2| = |A| &\geq \sum_{i=1}^t (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A) \\ &\geq \sum_{A_i \in \mathcal{F}_1} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A'_1) - 1 + \\ &\quad \sum_{A_i \in \mathcal{F}_2} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A'_2) - 1. \end{aligned}$$

Since G_j is $\mathcal{M}(\rho + \nu_L - 1)$ -independent we must have $|A'_j| = \sum_{A_i \in \mathcal{F}_j} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A'_j) - 1$ and $z_1, z_2 \in V_L(A'_j)$ for both $j = 1, 2$. We can now put $\mathcal{F}'_1 = \mathcal{F}_1 + \{e\}$. Then \mathcal{F}'_1 is a partition of $A''_1 = A'_1 + e$ which satisfies

$$\begin{aligned} \sum_{B_i \in \mathcal{F}'_1} (2\nu_P(B_i) + \nu_L(B_i) - 2) + \nu_L(A''_1) - 1 &= \\ \sum_{A_i \in \mathcal{F}_1} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A'_1) - 1 &= |A'_1| = |A''_1| - 1. \end{aligned}$$

This contradicts the hypothesis that G_1 is $\mathcal{M}(\rho + \nu_L - 1)$ -independent, since $A_1'' \subseteq E_1$.

Case 2: $z_1, z_2 \in V_P$. We may assume that \mathcal{F} has been chosen amongst all partitions satisfying (6) to make $|\mathcal{F}|$ as small as possible. This choice ensures that $A_i \cap A_j \cap V_P = \emptyset$ for all $1 \leq i < j \leq t$, since if $A_i \cap A_j \cap V_P \neq \emptyset$ then the partition $\mathcal{F}' = \mathcal{F} \setminus \{A_i, A_j\} \cup \{A_i \cup A_j\}$ would still satisfy (6). It follows, in particular that z_1, z_2 each belong to at most one set $A_i \in \mathcal{F}$.

For $1 \leq j \leq 2$, let $\mathcal{F}_j = \{A_i \cap E_j : A_i \in \mathcal{F} \text{ and } A_i \cap E_j \neq \emptyset\}$. Then \mathcal{F}_j is a partition of $A'_j = A \cap E_j$ for $j = 1, 2$. Since $V(A'_1) \cap V(A'_2) \subseteq \{z_1, z_2\} \subseteq V_P$, we have

$$\begin{aligned} |A'_1| + |A'_2| = |A| &\geq \sum_{i=1}^t (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A) \\ &\geq \sum_{B_i \in \mathcal{F}_1} (2\nu_P(B_i) + \nu_L(B_i) - 2) + \nu_L(A'_1) + \\ &\quad \sum_{B_i \in \mathcal{F}_2} (2\nu_P(B_i) + \nu_L(B_i) - 2) + \nu_L(A'_2) - 2s \end{aligned}$$

where $s = 1$ if there exists $A_i \in \mathcal{F}$ with $z_1, z_2 \in A_i$ and $A_i \cap E_1 \neq \emptyset \neq A_i \cap E_2$, and $s = 0$ otherwise. Since G_j is $\mathcal{M}(\rho + \nu_L - 1)$ -independent we must have $|A'_j| = \sum_{B_i \in \mathcal{F}_j} (2\nu_P(B_i) + \nu_L(B_i) - 2) + \nu_L(A'_j) - 1$ for both $j = 1, 2$, and $s = 1$. The construction of \mathcal{F}_1 now implies that $z_1, z_2 \in V_P(B_k)$ for some $B_k \in \mathcal{F}_1$. We can now put $\mathcal{F}'_1 = \mathcal{F}_1 \setminus \{B_k\} \cup \{B_k + e\}$. Then \mathcal{F}'_1 is a partition of $A''_1 = A'_1 + e$ which satisfies

$$\begin{aligned} \sum_{C_i \in \mathcal{F}'_1} (2\nu_P(C_i) + \nu_L(C_i) - 2) + \nu_L(A''_1) - 1 = \\ \sum_{B_i \in \mathcal{F}_1} (2\nu_P(B_i) + \nu_L(B_i) - 2) + \nu_L(A'_1) - 1 = |A'_1| = |A''_1| - 1. \end{aligned}$$

This contradicts the hypothesis that G_1 is $\mathcal{M}(\rho + \nu_L - 1)$ -independent, since $A''_1 \subseteq E_1$. •

Lemma 4.6 *Let $G = (V, E)$ be a naturally bipartite point-line graph which is isomorphic to $K_{3,3}$. Then G is $\mathcal{M}_\#$ -independent.*

Proof. Choose $e = uv \in E$. It is straightforward to check that $E(G + uv) = A \cup B$ where A is $\mathcal{M}(2\nu_P + \nu_L - 2)$ -independent and B is $\mathcal{M}(\nu_L)$ -independent. Symmetry and Lemmas 3.6 and 3.7 now imply that G is $\mathcal{M}_\#$ -independent. •

Lemma 4.7 *Let $G = (V, E)$ be a point-line graph which is $\mathcal{M}_\#$ -independent. Then G is \mathcal{M}_{PL} -independent.*

Proof. We use induction on $|E_{PP} \cup E_{LL}|$. The lemma follows from Lemma 4.4 when $E_{PP} \cup E_{LL} = \emptyset$. Hence we may suppose that we have an edge $e = w_1w_2 \in E_{PP} \cup E_{LL}$. Let H be a naturally bipartite point-line graph which is isomorphic to $K_{3,3}$ and label its vertices such that $\{w_1, w_2\} = V_P(G) \cap V_P(H)$ or $\{w_1, w_2\} = V_L(G) \cap V_L(H)$. Let $G^+ = (G - e) \cup H$. Then Lemmas 4.5 and 4.6 imply that G^+ is $\mathcal{M}_\#$ -independent. We can now use induction to deduce that G^+ is \mathcal{M}_{PL} -independent. This implies that G is \mathcal{M}_{PL} -independent (since if G were \mathcal{M}_{PL} -dependent then the matroid circuit axiom and the fact that $H + e$ is \mathcal{M}_{PL} -dependent would imply that $G^+ = (G \cup (H + e)) - e$ contains an \mathcal{M}_{PL} -circuit). •

Theorem 4.8 *Let $G = (V, E)$ be a point-line graph. Then $\mathcal{M}_{PL}(G) = \mathcal{M}_\#(G)$.*

Proof. Choose $F \subset E$. If F is independent in $\mathcal{M}_{PL}(G)$ then F is independent in $\mathcal{M}_\#(G)$ by Lemma 3.9. On the other hand, if F is independent in $\mathcal{M}_\#(G)$ then we may apply Lemma 4.7 to $G[F]$ to deduce that F is independent in $\mathcal{M}_{PL}(G)$. •

4.3 The rank function

Theorem 4.8 tells us that $\mathcal{M}_{PL}(G)$ is the matroid induced by $\rho + \nu_L - 1$ and we can now use Theorem 3.2(a) to deduce that its rank function is given by

$$r_{PL}(A) = \min_{\mathcal{F}} \left\{ \sum_{A_i \in \mathcal{F}} (\rho(A_i) + \nu_L(A_i) - 1) \right\} \quad (7)$$

for all $A \subseteq E$, where the minimum is taken over all partitions \mathcal{F} of A . We close this section by simplifying this expression for $r_{PL}(A)$. We first need to

introduce some notation. Given a partition \mathcal{F} of A let $c_L(\mathcal{F})$ be the number of connected components in the graph with vertex set \mathcal{F} in which two vertices $A_i, A_j \in \mathcal{F}$ are adjacent if $V_L(A_i) \cap V_L(A_j) \neq \emptyset$.

Theorem 4.9 *Let $G = (V, E)$ be a point-line graph and $A \subseteq E$. Then*

$$r_{PL}(A) = \nu_L(A) + \min_{\mathcal{F}} \left\{ \sum_{A_i \in \mathcal{F}} (2\nu_P(A_i) + \nu_L(A_i) - 2) - c_L(\mathcal{F}) \right\} \quad (8)$$

where the minimum is taken over all partitions \mathcal{F} of A .

Proof. The fact that the right hand side gives an upper bound on $r_{PL}(A)$ follows from inequality (2) in the proof of Lemma 3.8. Hence it will suffice to show that there exists a partition of A which gives equality in (8).

By (7), we may choose a partition \mathcal{P} of A such that

$$r_{PL}(A) = \sum_{B_i \in \mathcal{P}} (\rho(B_i) + \nu_L(B_i) - 1) \quad (9)$$

and, subject to this condition, $|\mathcal{P}|$ is as small as possible. We claim that $V_L(B_i) \cap V_L(B_j) = \emptyset$ for all distinct $B_i, B_j \in \mathcal{P}$.

Suppose to the contrary that $V_L(B_i) \cap V_L(B_j) \neq \emptyset$. Let $\mathcal{Q} = \mathcal{P} \setminus \{B_i, B_j\} \cup \{B_i \cup B_j\}$. Then

$$\begin{aligned} \sum_{B_k \in \mathcal{P}} (\rho(B_k) + \nu_L(B_k) - 1) - \sum_{C_k \in \mathcal{Q}} (\rho(C_k) + \nu_L(C_k) - 1) = \\ \rho(B_i) + \rho(B_j) - \rho(B_i \cup B_j) + \nu_L(B_i) + \nu_L(B_j) - \nu_L(B_i \cup B_j) - 1 \geq 0 \end{aligned}$$

since ρ, ν_L are submodular and nonnegative, and $\nu_L(B_i \cap B_j) \geq 1$. Equations (7) and (9) now imply that $r_{PL}(A) = \sum_{C_k \in \mathcal{Q}} (\rho(C_k) + \nu_L(C_k) - 1)$ and hence \mathcal{Q} contradicts the choice of \mathcal{P} .

The fact that the sets in \mathcal{P} are line-vertex disjoint now gives

$$r_{PL}(A) = \sum_{B_i \in \mathcal{P}} (\rho(B_i) + \nu_L(B_i) - 1) = \nu_L(A) + \left(\sum_{B_i \in \mathcal{P}} \rho(B_i) \right) - |\mathcal{P}|. \quad (10)$$

For each $B_i \in \mathcal{P}$ we have

$$\rho(B_i) = \min_{\mathcal{P}_i} \left\{ \sum_{A_{i,j} \in \mathcal{P}_i} (2\nu_P(A_{i,j}) + \nu_L(A_{i,j}) - 2) \right\},$$

where the minimum is taken over all partitions \mathcal{P}_i of B_i . Choose a partition \mathcal{F}_i of B_i such that $\rho(B_i) = \sum_{A_{i,j} \in \mathcal{F}_i} (2\nu_P(A_{i,j}) + \nu_L(A_{i,j}) - 2)$ and, subject to this condition, such that $|\mathcal{F}_i|$ is as small as possible. We may use the argument from the first paragraph of Case 2 of the proof of Theorem 4.5 to deduce that the sets in \mathcal{F}_i are point-vertex disjoint. Let $\mathcal{F} = \bigcup_{B_i \in \mathcal{P}} \mathcal{F}_i$. Then \mathcal{F} is a partition of A and we may use (10) to obtain

$$r_{PL}(A) = \nu_L(A) + \sum_{A_{i,j} \in \mathcal{F}} (2\nu_P(A_{i,j}) + \nu_L(A_{i,j}) - 2) - |\mathcal{P}|.$$

It remains to show that $|\mathcal{P}| = c_L(\mathcal{F})$, or equivalently, that $c_L(\mathcal{F}_i) = 1$ for all $B_i \in \mathcal{P}$.

Suppose to the contrary that $c_L(\mathcal{F}_i) \geq 2$ for some $B_i \in \mathcal{P}$. Then there exists a partition of \mathcal{F}_i into two sets $\mathcal{F}'_i, \mathcal{F}''_i$ such that $V_L(A_{i,j}) \cap V_L(A_{i,k}) = \emptyset$ for all $A_{i,j} \in \mathcal{F}'_i$ and $A_{i,k} \in \mathcal{F}''_i$. Let $B'_i = \bigcup_{A_{i,j} \in \mathcal{F}'_i} A_{i,j}$ and $B''_i = \bigcup_{A_{i,k} \in \mathcal{F}''_i} A_{i,k}$. Then

$$\begin{aligned} \rho(B_i) &= \sum_{A_{i,j} \in \mathcal{F}_i} (2\nu_P(A_{i,j}) + \nu_L(A_{i,j}) - 2) \\ &= \sum_{A_{i,j} \in \mathcal{F}'_i} (2\nu_P(A_{i,j}) + \nu_L(A_{i,j}) - 2) + \sum_{A_{i,k} \in \mathcal{F}''_i} (2\nu_P(A_{i,k}) + \nu_L(A_{i,k}) - 2) \\ &\geq \rho(B'_i) + \rho(B''_i). \end{aligned} \tag{11}$$

Let $\mathcal{R} = (\mathcal{P} \setminus \{B_i\}) \cup \{B'_i, B''_i\}$. Then \mathcal{R} is a partition of A into line-vertex disjoint sets and

$$\begin{aligned} &\sum_{B_j \in \mathcal{P}} (\rho(B_j) + \nu_L(B_j) - 1) - \sum_{C_j \in \mathcal{R}} (\rho(C_j) + \nu_L(C_j) - 1) \\ &= \nu_L(A) + \left(\sum_{B_j \in \mathcal{P}} \rho(B_j) \right) - |\mathcal{P}| - \nu_L(A) - \left(\sum_{C_j \in \mathcal{R}} \rho(C_j) \right) + |\mathcal{R}| \\ &= \rho(B_i) - \rho(B'_i) - \rho(B''_i) + 1 \geq 1 \end{aligned}$$

by (11). This contradicts the minimality of $\sum_{B_j \in \mathcal{P}} (\rho(B_j) + \nu_L(B_j) - 1)$. Hence $c_L(\mathcal{F}_i) = 1$. \bullet

Example Consider the point-line graph $G = (V, E)$ on the right hand side of Figure 1. Let A_1, A_2, A_3 be the sets of edges in the three copies of $K_4 - e$.

Then $\mathcal{F} = \{A_1, A_2, A_3\}$ is a partition of E . We have $\nu_L(E) = 3$, $c_L(\mathcal{F}) = 1$ and $2\nu_P(A_i) + \nu_L(A_i) - 2 = 4$ for all $1 \leq i \leq 3$. Theorem 4.9 now gives $r_{PL}(G) \leq 3 + 3 \times 4 - 1 = 14 < 2|V| - 3$ so G is not rigid.

5 Algorithmic Implications

Lemma 3.6 tells us that $\mathcal{M}(\rho + \nu_L) = \mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L)$. This implies that we may use Edmond's algorithm for matroid union [7], see also Gabow and Westermann [11], to efficiently test for independence in $\mathcal{M}(\rho + \nu_L)$, as long as we can efficiently test for independence in both $\mathcal{M}(2\nu_P + \nu_L - 2)$ and $\mathcal{M}(\nu_L)$. The latter can be accomplished by adapting existing algorithms for count matroids based on graph orientations [2] or the pebble game [14, 18]. In addition, Theorem 4.8 gives $\mathcal{M}_{PL} = \mathcal{M}(2\nu_P + \nu_L - 1)$, and Lemma 5.3 implies that the problem of testing for independence in $\mathcal{M}(2\nu_P + \nu_L - 1)$ can be reduced to that of testing for independence in $\mathcal{M}(\rho + \nu_L)$. This gives us the following recursive procedure for constructing a maximum independent set in $\mathcal{M}_{PL}(G)$.

Suppose we have already constructed an \mathcal{M}_{PL} -independent set I in G . For each $e \in E \setminus I$ we determine whether or not $I + e$ is \mathcal{M}_{PL} -independent. We do this by adding *two* copies of e to I and testing whether $I + e + \bar{e}$ is $\mathcal{M}(\rho + \nu_L)$ -independent. If it is then $I + e$ is \mathcal{M}_{PL} -independent. If it isn't, then we delete e and choose another edge in $E \setminus I$. We stop when $I + e + \bar{e}$ is not $\mathcal{M}(\rho + \nu_L)$ -independent for all $e \in E \setminus I$. Then I is a maximum independent in $\mathcal{M}_{PL}(G)$.

5.1 Matroid union and Augmenting Paths

We now give a brief description of Edmond's algorithm for matroid union. We refer the reader to [11] for more details. We are given matroids $\mathcal{M}_1, \mathcal{M}_2$ with the same groundset E , an independent set I of $\mathcal{M}_1 \vee \mathcal{M}_2$, a partition (I_1, I_2) of I with I_q independent in \mathcal{M}_q for $q \in \{1, 2\}$. For any $e \in E \setminus I_q$ such that $I_q + e$ is not independent in \mathcal{M}_q let $C(I_q, e, \mathcal{M}_q)$ be the unique circuit of \mathcal{M}_q contained in $I_q + e$. We determine whether $I + e$ is independent in $\mathcal{M}_1 \vee \mathcal{M}_2$ by searching for an *augmenting path*. This is a sequence of elements $e = e_0, e_1, \dots, e_s$ of $I + e$ with the following properties (where subscripts j on I_j and \mathcal{M}_j are to be read modulo two).

For some $q \in \{1, 2\}$

- $e_{i+1} \in I_{q+i}$ for all $0 \leq i \leq s-1$.
- For all $0 \leq i \leq s-1$, $I_{q+i} + e_i$ is dependent in \mathcal{M}_{q+i} and $e_{i+1} \in C(I_{q+i}, e_i, \mathcal{M}_{q+i}) - e_i$.
- For all $1 \leq i+1 < j \leq s$, $e_j \notin C(I_{q+i}, e_i, \mathcal{M}_{q+i})$.
- $I_{q+s} + e_s$ is independent in \mathcal{M}_{q+s} .

If we find an augmenting path then we conclude that $I + e$ is independent in $\mathcal{M}_1 \vee \mathcal{M}_2$. We output $I + e$ together with the partition (I'_q, I'_{q+1}) of $I + e$ where $I'_q = I_q \Delta \{e_0, e_1, \dots, e_s\}$ is independent in \mathcal{M}_q , $I'_{q+1} = I_{q+1} \Delta \{e_1, \dots, e_s\}$ is independent in \mathcal{M}_{q+1} , and Δ denotes symmetric difference. The requirement that the augmenting path has no short cuts (third bullet point above) ensures that I'_q is independent in \mathcal{M}_q for $q \in \{1, 2\}$. If no augmenting path exists then we conclude that $I + e$ is dependent in $\mathcal{M}_1 \vee \mathcal{M}_2$.

To implement this algorithm we need subroutines which determine whether $I_q + e$ is independent in \mathcal{M}_q , and determine the unique circuit of \mathcal{M}_q contained in $I_q + e$ when it is dependent. We next describe these subroutines when $\mathcal{M}_1 = \mathcal{M}(2\nu_P + \nu_L - 2)$ and $\mathcal{M}_2 = \mathcal{M}(\nu_L)$.

5.2 Graph orientations

We will adapt the algorithms for count matroids given by Berg and Jordán [2] to test for independence in $\mathcal{M}(2\nu_P + \nu_L - 2)$ and $\mathcal{M}(\nu_L)$.

Let $G = (V, E)$ be a graph. An *orientation* of G is a directed graph D obtained by replacing each edge $wz \in E$ by a directed edge (directed from w to z or from z to w). For $w \in V$, let $d_D^-(w)$ be the number of edges entering w in D . Given a map $g : V \rightarrow \mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$, an orientation D of G is said to be a *g -orientation* if $d_D^-(w) \leq g(w)$ for all $w \in V$. For $X \subseteq V$, let $g(X) = \sum_{v \in X} g(v)$ and let $i_G(X)$ be the number of edges induced by X . The following result of Frank and Gyárfás [10] characterises when a graph has a g -orientation.

Theorem 5.1 *Let $G = (V, E)$ be a graph and $g : V \rightarrow \mathbb{Z}^+$. Then G has a g -orientation if and only if $i_G(X) \leq g(X)$ for all $X \subseteq V$.*

Given a point-line graph $G = (V, E)$ and $i, j \in \mathbb{Z}^+$, let $g_{i,j} : V \rightarrow \mathbb{Z}^+$ be defined by $g_{i,j}(x) = i$ if $x \in V_P$ and $g_{i,j}(x) = j$ if $x \in V_L$. We will be

interested in (special kinds of) $g_{2,1}$ - and $g_{0,1}$ -orientations but it will be more efficient to consider general $g_{i,j}$ -orientations. Given $w, z \in V$ and $k \in \mathbb{Z}^+$, a $g_{i,j,k}^{wz}$ -orientation of G is a $g_{i,j}$ -orientation D such that

$$d_D^-(w) + d_D^-(z) \leq g_{i,j}(w) + g_{i,j}(z) - k.$$

We will assume henceforth that $k \leq \min\{2i, 2j\}$. In this case, the nondecreasing submodular function $i\nu_P + j\nu_L - k$ is nonnegative on $2^E \setminus \{\emptyset\}$ and hence induces a matroid on E by Theorem 3.2. Let $\mathcal{M}(i, j, k) = \mathcal{M}(i\nu_P + j\nu_L - k)$. The following results give relationships between $\mathcal{M}(i, j, k)$ and $g_{i,j}$ -orientations.

Lemma 5.2 *Suppose $G = (V, E)$ is a point-line graph. Then G is $\mathcal{M}(i, j, k)$ -independent if and only if G has a $g_{i,j,k}^{wz}$ -orientation for all $w, z \in V$.*

Proof. Suppose that G has no $g_{i,j,k}^{wz}$ -orientation for some $w, z \in V$. Let $h : V \rightarrow \mathbb{Z}^+$ be such that $h(w) + h(z) = g_{i,j}(w) + g_{i,j}(z) - k$, $h(x) \leq g_{i,j}(x)$ for all $x \in \{w, z\}$, and $h(x) = g_{i,j}(x)$ for all $x \in V \setminus \{w, z\}$. Then G has no h -orientation because any h -orientation would be a $g_{i,j,k}^{wz}$ -orientation. By Lemma 5.1 there is a set $X \subseteq V$ with $i_G(X) > h(X)$. Let $A \subseteq E$ be the set of edges of G induced by X . We have $|A| = i_G(X) > h(X) \geq g_{i,j}(X) - k = i\nu_P(A) + j\nu_L(A) - k$. Hence G is not $\mathcal{M}(i, j, k)$ -independent.

Conversely suppose that G is not $\mathcal{M}(i, j, k)$ -independent. Then $|A| > i\nu_P(A) + j\nu_L(A) - k$ for some nonempty $A \subseteq E$. Let $X = V(A)$. Then $i_G(X) \geq |A| > i\nu_P(A) + j\nu_L(A) - k$ and Lemma 5.1 implies that G has no $g_{i,j,k}^{wz}$ -orientation for any distinct $w, z \in V(A)$. \bullet

For $I \subset E$ let $G(I)$ denote the spanning subgraph of G with edge set I .

Lemma 5.3 *Let $G = (V, E)$ be a point-line graph, $I \subset E$ be independent in $\mathcal{M}(i, j, k)$ and $e = wz \in E \setminus I$. Then $I + e$ is independent in $\mathcal{M}(i, j, k)$ if and only if $G(I)$ has a $g_{i,j,k+1}^{wz}$ -orientation.*

Proof. Suppose that $I + e$ is not independent in $\mathcal{M}(i, j, k)$. Then there exists an $A \subseteq I + e$ such that $|A| > i\nu_P(A) + j\nu_L(A) - k$. Since I is independent in $\mathcal{M}(i, j, k)$, we must have $e \in A$, and $|A - e| = i\nu_P(A) + j\nu_L(A) - k$. Let $X = V(A)$. Then $w, z \in X$ and $i_{G(A-e)}(X) \geq |A - e| = i\nu_P(A) + j\nu_L(A) - k$. Lemma 5.1 now implies that $G(I)$ has no $g_{i,j,k+1}^{wz}$ -orientation.

Conversely, suppose that $G(I)$ has no $g_{i,j,k+1}^{wz}$ -orientation. Then $G(I+e)$ has no $g_{i,j,k}^{wz}$ -orientation. Lemma 5.2 now implies that $I+e$ is dependent in $\mathcal{M}(i,j,k)$. \bullet

Suppose \mathcal{M} is a matroid, I is an independent set in \mathcal{M} and e is an element of \mathcal{M} such that $I+e$ is dependent. $I+e$. Constructing the circuit $C(I,e,\mathcal{M})$ is an important step in the algorithm for matroid union outlined in Section 5.1. Our next lemma tells us how to do this when $\mathcal{M} = \mathcal{M}(i,j,k)$.

Lemma 5.4 *Let $G = (V,E)$ be a point-line graph, $I \subset E$ be independent in $\mathcal{M}(i,j,k)$ and $I+e$ be dependent for some $e = wz \in E \setminus I$. Then $G(I)$ has a $g_{i,j,k}^{wz}$ -orientation D . Furthermore, if Y is the set of all vertices which are connected to $\{w,z\}$ by directed paths in D , and F is the set of all edges of I which are induced by Y , then $C(I,e) = F+e$.*

Proof. The fact that $G(I)$ has a $g_{i,j,k}^{wz}$ -orientation follows from Lemma 5.2. By definition, $C = C(I,e)$ is the minimal $\mathcal{M}(i,j,k)$ -dependent subset of $I+e$. Hence $|C| = i\nu_P(C) + j\nu_L(C) - k + 1$ and $|C-e| = i\nu_P(C-e) + j\nu_L(C-e) - k$. Let $Y' = V(C)$. Since I is independent, $e \in C$ and hence $w, z \in Y'$. Let $G' = (Y', C-e)$ and D' be the restriction of D to G' . The facts that D' is a $g_{i,j,k}^{wz}$ -orientation of G' and $|C-e| = i\nu_P(C-e) + j\nu_L(C-e) - k$ imply that $d_{D'}^-(y) = g_{i,j}(y)$ for all $y \in Y' \setminus \{w, z\}$ and $d_{D'}^-(w) + d_{D'}^-(z) = g_{i,j}(w) + g_{i,j}(z) - k$. Since D is a $g_{i,j,k}^{wz}$ -orientation of G , this gives $d_{D'}^-(y) = d_D^-(y)$ for all $y \in Y'$. Thus there are no directed edges in D from $V \setminus Y'$ to Y' and hence $Y \subseteq Y'$.

On the other hand, the definition of Y implies that there are no directed edges in D from $V \setminus Y$ to Y . Thus, if F is the set of edges of I induced by Y and D'' is the restriction of D to (Y, F) , we will have $d_{D''}^-(y) = d_D^-(y)$ for all $y \in Y$. This gives

$$|F| = \sum_{y \in Y} d_{D''}^-(y) = \sum_{y \in Y} d_D^-(y) = i\nu_P(F) + j\nu_L(F) - k.$$

This implies that $F+e$ is dependent in $\mathcal{M}(i,j,k)$ and the minimality of C now gives $C = F+e$ (and $Y' = Y$). \bullet

Lemmas 5.2, 5.3 and 5.4 give rise to the following linear time algorithm which either increases the size of an independent set I in $\mathcal{M}(i,j,k)$ by adding a new element e to it, or finds the fundamental circuit $C(I,e)$ when $I+e$ is dependent.

Suppose that we are given I together with a $g_{i,j}$ -orientation D of $G(I)$ and an edge $e = wz \in E \setminus I$. If D is a $g_{i,j,k+1}^{wz}$ -orientation of $G(I)$ then we conclude that $I + e$ is independent. We orient e so that $D + e$ is a $g_{i,j,k}^{wz}$ -orientation of $G(I + e)$ and output $(I + e, D + e)$. Otherwise we construct the set Y of all vertices which are connected to $\{w, z\}$ by directed paths in D . If some vertex $y \in Y \setminus \{w, z\}$ has $d_D^-(y) < g_{i,j}(y)$ then we construct a new orientation by reversing the direction of all edges on a directed path from y to $\{w, z\}$ in D and then iterate. (Note that the reorientation will reduce $d_D^-(w) + d_D^-(z)$ by one.) After at most k iterations, we will arrive at either a $g_{i,j,k+1}^{wz}$ -orientation of $G(I)$, or a $g_{i,j,k}^{wz}$ -orientation with $d_D^-(y) = g_{i,j}(y)$ for all $y \in Y \setminus \{w, z\}$. In the latter case we conclude that $I + e$ is dependent and output $C(I, e) = F + e$, where F is the set of all edges of I induced by Y .

We may combine this algorithm for $\mathcal{M}(2, 1, 2)$ and $\mathcal{M}(0, 1, 0)$ with the augmenting path algorithm of Section 5.1 to give an $O(|V|^2)$ -algorithm which determines whether $I + e$ is independent in \mathcal{M}_{PL} and hence obtain an $O(|V|^2|E|)$ -algorithm for constructing a maximum independent set in \mathcal{M}_{PL} . A more detailed description of this algorithm as a pebble game is given in the Appendix.

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6 Appendix

We will use the language of a pebble game [14, 18] to describe an implementation of our algorithm to determine the rank of set of edges in the matroid $\mathcal{M}_{PL}(G)$ for any point-line graph $G = (E, V)$. We will assume that we have already determined a set of edges $I \subset E$ which are independent in \mathcal{M}_{PL} and for any edge $e \in E \setminus I$ we will determine whether $I + e$ is independent.

We will assume that we have a partition of I into two sets R and T such that R is independent in $\mathcal{M}(\nu_L)$ and T is independent in $\mathcal{M}(2\nu_P + \nu_L - 2)$. We represent these partitions and orientations by the matching of edges with pebbles on a point-line graph $G = (V, E)$ as follows.

Every $u \in V_P$ is given two t -pebbles and every $v \in V_L$ is given one r -pebble and one t -pebble. Every edge in I is matched with a pebble from one or other of its incident vertices. An edge in E_{LL} must be matched with an r -pebble and an edge in E_{PL} may be matched with either a t -pebble or an r -pebble. Since an edge in E_{PP} is incident to vertices which have only t -pebbles it can only be matched with a t -pebble. If an edge is matched with a pebble we also say that the pebble is assigned to the edge or the pebble is consumed by the edge. A pebble may be assigned to at most one of its incident edges and every edge must be assigned a pebble.

An assignment of pebbles to edges which satisfies the rules in the preceding paragraph is called valid. An edge is in R if and only if it is assigned an r -pebble and an edge is in T if and only if it is assigned a t -pebble. If an edge wz is assigned a pebble from w it is directed into w otherwise it is assigned a pebble from z and it is directed into z . Any pebble on a vertex which has not been assigned to one of its incident edges is called a free pebble.

Every valid assignment of pebbles gives a partition of edges into sets R and T . The assignments imply an orientation of $G(R)$ and $G(T)$ and these are a $g_{0,1}$ -orientation and a $g_{2,1}$ -orientation respectively. Further if two vertices $w, z \in V(I)$ have a total of r free r -pebbles and t free t -pebbles then the orientations are a $g_{0,1,r}^{wz}$ -orientation and a $g_{2,1,t}^{wz}$ -orientation respectively.

Conversely if a graph $G = (E, V)$ has a partition of its edges into sets R and T such that $G(R)$ and $G(T)$ have a $g_{0,1}$ -orientation and a $g_{2,1}$ -orientation respectively then G has a valid assignment of pebbles. This implies that every set I which is independent in $\mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L)$ has a valid assignment of pebbles on its graph $G(I)$ and every valid assignment has at least two free t -pebbles. Furthermore, if I is independent in $\mathcal{M}_{PL}(G)$, then every valid assignment has at least three free pebbles.

We allow the assignment of pebbles on a graph to be changed in two ways.

The first way does not involve changing edge partitions. If a vertex $v \in V_L$ has a free r -pebble then every vertex $w \in V_L$ which can be reached from v by a directed path of edges which have been assigned an r -pebble can be given an additional free r -pebble by reversing the direction of all edges on the directed path from w to z . Note that these edges must all be in E_{LL} because an edge in E_{PL} which is assigned an r -pebble cannot be redirected and still be assigned an r -pebble. We describe this as the collection of a free r -pebble onto w . Similarly we collect an additional free t -pebble onto w by following directed paths of edges which have been assigned a t -pebble in the reverse direction starting from w . In this case the edges which are assigned

a t -pebble may be in either E_{PP} or in E_{PL} . If a search for a free pebble fails then we can record all the edges which are visited during the search. If we are searching for an r pebble then all visited edges are in the R partition and if we are searching for a t -pebble then all visited edges are in the T partition. By following directed paths of edges in the reverse direction we can collect any available free t -pebbles or r -pebbles onto any vertex in $O(|I|)$ steps.

Since pebble collection occurs without changing the sets R and T it is straightforward to show that if I is independent in $\mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L)$ then for every valid assignment of pebbles at least two free t -pebbles can be collected onto any pair of vertices $w, z \in V(I)$. Furthermore if I is independent in $\mathcal{M}_{PL}(G)$ then three free pebbles can be collected onto w and z unless $w, z \in V_L$. (The exception occurs when the r -pebbles on w and z are both assigned to edges in E_{PL} .)

This type of free pebble collection is encapsulated in the following procedure which attempts to collect three free pebbles onto a pair of vertices $x, y \in V(I)$ which are specified as an edge $xy \in E$. It is not necessary to specify the type of pebbles which are collected. We assume that I is independent in $\mathcal{M}(2\nu_P + \nu_L - 2) \vee \mathcal{M}(\nu_L)$ with a corresponding partition $I = T \cup R$. Two free t -pebbles can always be collected onto x and y . The remaining pebble type is a t -pebble for $xy \in E_{PP}$ and an r -pebble for $xy \in E_{LL}$. If $xy \in E_{PL} \cap I$ then if xy has been assigned an r -pebble we try to collect a third t -pebble and if xy has been assigned a t -pebble we attempt to collect an r -pebble. If $xy \in E_{PL} \setminus I$ then we may assume I is independent in $\mathcal{M}_{PL}(G)$ and the collection of a third pebble (either a t -pebble or an r -pebble) onto x and y is always possible.

```

procedure collect-three-pebbles( $C, xy$ )
{
  Input: edge  $xy \in E$ 
  Output: success or visited edges in  $C$ 
  collect two free  $t$ -pebbles onto  $x$  and  $y$ 
  attempt to collect a third free pebble onto  $x$  and  $y$ 
  if this fails return failure and the visited edges
  otherwise return success
}

```

The following procedure searches for an augmenting path in I which starts

on the edge $wz \in E \setminus I$. It uses a queue Q to implement a breadth first search of all potential augmenting paths. This ensures that the augmenting paths which are found have no short cuts. All edges in I are initially unmarked. If an unmarked edge $g \in E_{PL}$ is found to be in a circuit with an edge f from a failed call to `collect-three-pebbles(f)` it is put into Q , marked and has a pointer set to f which is the preceding edge in a potential augmenting path which starts with wz and includes f . Marked edges and edges in $E_{PP} \cup E_{LL}$ are ignored. This ensures that any edge is placed only once in Q .

```

procedure find-augmenting-path( $wz$ )
{
  Input: edge  $wz \in E \setminus I$ 
  Output: an augmenting path if one exists
  unmark all edges in  $I$ 
  define a queue  $Q$  and add  $wz$  to  $Q$ 
  while  $Q$  is not empty
  {
    select next edge  $f$  from  $Q$ 
    if collect-three-pebbles(C, f) succeeds
    return success and the final edge  $f$  in the augmenting path
    else
    for all unmarked circuit edges  $g \in E_{PL} \cup C$ 
    {
      mark  $g$ 
      set preceding edge of  $g$  to  $f$ 
      add  $g$  to  $Q$ .
    }
  }
  return failure
}

```

The following procedure will change the partition for an edge $wz \in E_{PL} \cap I$ in an augmenting path and maintain a valid pebble assignment for $G(I)$. This procedure must be called in reverse order along the augmenting path.

```

procedure change-partition( $wz$ )
{
  Input:  $wz \in E_{PL} \cap I$ 

```

```

Output: changed pebble type assigned to  $wz$ 
collect-three-pebbles( $wz$ )
change the type of pebble assigned to  $wz$ 
}

```

We add an edge wz to I with the following procedure. One copy of wz can always be added. If the addition of a second copy is successful $I + wz$ is independent in \mathcal{M}_{PL} and a valid pebble assignment is returned for $I + wz$. Otherwise $I + wz$ is dependent in \mathcal{M}_{PL} and e is not added to I . In this case all of the edges which were marked during the failed search for an augmenting path are in a fundamental circuit with wz .

```

procedure add-edge-to-I( $wz$ )
{
Input: edge  $wz \in E \setminus I$ 
find-augmenting-path( $e = wz$ )
for all edges  $f \neq e$  on the augmenting path
    change-partition( $f$ )
collect-three-pebbles( $e$ )
assign a free pebble to  $e$  from  $w$  or  $z$  leaving two free  $t$ -pebbles
if(find-augmenting-path( $\bar{e} = wz$ ) succeeds)
{
for all edges  $f \neq \bar{e}$  on augmenting path
change-partition( $f$ )
return success
}
else
unassign pebble from  $e$  and return failure
}

```