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An exponential-type upper bound for Folkman numbers

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Abstract

For given integers k and r, the Folkman number $f(k;r)$ is the smallest number of vertices in a graph G which contains no clique on $k+1$ vertices, yet for every partition of its edges into r parts, some part contains a clique of order k. The existence (finiteness) of Folkman numbers was established by Folkman (1970) for $r=2$ and by Nešetřil and Rödl (1976) for arbitrary r, but these proofs led to very weak upper bounds on $f(k;r)$.

We give an upper bound on $f(k;r)$ which is only exponential in a polynomial of k and r, which is comparable to the known lower bound $2^{\Omega(kr)}$. Our proof relies on a recent result of Saxton and Thomason (or, alternatively, on a recent result of Balogh, Morris, and Samotij) from which we deduce a quantitative version of the threshold for Ramsey's theorem in random graphs. We also provide another, self-contained proof of that theorem which yields a double exponential upper bound on $f(k;2)$.

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1 Introduction

For two graphs, G and F, and an integer $r \geq 2$ we write $G \to (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. By a copy we mean here a subgraph of G isomorphic to F. Let K_k stand for the complete graph on k vertices and let $R(k;r)$ be the r-color Ramsey number, that is, the smallest integer n such that $K_n \to (K_k)_r$. As it is customary, we suppress $r = 2$ and write $R(k) := R(k;2)$ as well as $G \to F$ for $G \to (F)_2$.

In 1967 Erdős and Hajnal [8] asked if for some l, $k+1 \leq l \leq R(k)$, there exists a graph G such that $G \to K_k$ and $G \not\supseteq K_l$. Graham [11] answered this question in positive for $k = 3$ and $l = 6$ (with a graph on eight vertices), and Pósa (unpublished) for $k = 3$ and $l = 5$. Folkman [10] proved, by an explicit construction, that such a graph exists for every $k \geq 3$ and $l = k+1$.

For integers k and r, a graph G is called $(k;r)$-*Folkman* if $G \to (K_k)_r$ and $G \not\supseteq K_{k+1}$. We define the r-*color Folkman number* for K_k by

$$f(k;r) = \min\{n \in \mathbb{N} \colon \exists \ G \text{ such that } |V(G)| = n \text{ and } G \text{ is } (k;r)\text{-Folkman}\}\,.$$

For $r = 2$ we set $f(k) := f(k;2)$. It follows from [10] that $f(k)$ is well defined for every integer k, i.e., $f(k) < \infty$. This was extended by Nešetřil and Rödl [14], who showed that $f(k;r) < \infty$ for an arbitrary number of colors r.

Already the determination of $f(3)$ is a difficult, open problem. In 1975, Erdős [7] offered max(100 dollars, 300 Swiss francs) for a proof or disproof of $f(3) < 10^{10}$. For the history of improvements of this bound see [5], where a computer assisted construction is given yielding $f(3) < 1000$. For general k, the only previously known upper bounds on $f(k)$ come from the constructive proofs in [10] and [14]. However, these bounds are tower functions of height polynomial in k. On the other hand, since $f(k) \geq R(k)$, it follows by the well known lower bound on the Ramsey number that $f(k) \geq 2^{k/2}$.

We prove an upper bound on $f(k;r)$ which is exponential in a polynomial of k and r. Set $R := R(k;r)$ for the r-color Ramsey number for K_k and recall that

$$r^{k/2} < R < r^{rk}. \tag{1}$$

Theorem 1. *For all integers $r \geq 2$ and sufficiently large k,*

$$f(k;r) \leq k^{400k^4} R^{40k^2} = 2^{O(k^4 \log k + k^3 r \log r)}\,.$$

To prove Theorem 1, we consider a random graph $G(n,p)$, $p = Cn^{-\frac{2}{k+1}}$, and carefully estimate from below the probabilities $\mathbb{P}(G(n,p) \to (K_k)_r)$ and $\mathbb{P}(G(n,p) \not\supseteq K_{k+1})$, so that their sum is strictly greater than 1. The latter probability is easily bounded by the FKG inequality. However, to prove a bound on $\mathbb{P}(G(n,p) \to (K_k)_r)$ we rely on a recent general result of Saxton and Thomason [19] and also use some ideas of Nenadov and Steger [20]. Let us remark that instead of the Saxton and Thomason result from [19] we could have used a concurrent result of Balogh, Morris, and Samotij [1], which, by using our method, yields only a slightly worse upper bound on the Folkman numbers $f(k;r)$ than Theorem 1 (roughly, k^4 in the exponent gives way to k^6).

In the second part of the paper, we combine ideas from the proofs appearing in [9, 16, 18] and obtain, for $r = 2$, another proof of the Ramsey threshold theorem that provides a self-contained derivation of a double-exponential bound for the two-color Folkman numbers $f(k)$. Independently, a similar double exponential bound for $f(k;r)$ (with r arbitrary) was obtained by Conlon and Gowers [2] by a different method.

Theorem 2. *There exists an absolute constant $c > 0$ such that for every $k \geq 3$*

$$f(k) \leq 2^{k^{ck^2}}.$$

While Theorem 1 supersedes Theorem 2, we still include it here, since the proof of Theorem 2 is self-contained.

Motivated by the original question of Erdős and Hajnal, for $r = 2$, all $k \geq 3$, and $k+1 \leq l \leq R(k)$, one can also define *relaxed Folkman numbers* as

$$f(k,l) = \min\{n : \exists\ G \text{ such that } |V(G)| = n, \quad G \to K_k \text{ and } G \not\supset K_l\}.$$

Note that $f(k, k+1) = f(k)$. As mentioned above, Graham [11] found out that $f(3,6) = 8$, while Nenov [13] and Piwakowski, Radziszowski and Urbański [15] determined that $f(3,5) = 15$ (see also [22]). Of course, the problem is easier when the difference $l-k$ is bigger. Our final result provides an exponential bound of the form $f(k,l) \leq \exp\{-ck\}$, when l is close to but bigger than $4k$ (the constant c is proportional to the reciprocal of the difference between l/k and 4).

Theorem 3. *For every $0 < \alpha < \frac{1}{4}$ there exists k_0 such that for k and l satisfying $k \geq k_0$ and $k \leq \alpha l$,*

$$f(k,l) \leq 2^{4k/(1-4\alpha)}.$$

It would be interesting to decide if the true order of the logarithm of $f(k, k+1) = f(k)$ is also linear in k.

The paper is organized as follows. In the next section we prove our main result, Theorem 1. Section 3.1 contains a proof of Theorem 2, while Theorem 3 is proved in Section 4. Finally, a short Section 5 offers a brief discussion of the analogous problem for hypergraphs.

Most logarithms in this paper are binary and are denoted by log. Only occasionally, when citing a result from [19] (Theorem 6 in Section 2 below), we will use the natural logarithms, denoted by ln.

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2 Proof of Theorem 1

Let $G(n,p)$ be the binomial random graph, where each of $\binom{n}{2}$ possible edges is present, independently, with probability p. We will prove Theorem 1 by a version of the probabilistic method. Namely, we will show that for every $n \ge k^{40k^4}R^{10k^2}$ and a suitable function $p = p(n)$, with positive probability, $G(n,p)$ will have simultaneously two properties: $G(n,p) \to (K_k)_r$ and $G(n,p) \not\supset K_{k+1}$. We begin with a simple bound on $\mathbb{P}(G(n,p) \not\supset K_{k+1})$ from below.

Lemma 4. *For all $k, n \ge 3$ and $C > 0$, if $p = Cn^{-2/(k+1)} \le \frac{1}{2}$ then*

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp(-C^{\binom{k+1}{2}}n)\,.$$

Proof. By applying the FKG inequality (see, e.g., [12, Theorem 2.12 and Corollary 2.13], we obtain the bound

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) \ge \left(1 - p^{\binom{k+1}{2}}\right)^{\binom{n}{k+1}} \ge \exp\left(-2C^{\binom{k+1}{2}}n^{-k}\binom{n}{k+1}\right) > \exp\left(-C^{\binom{k+1}{2}}n\right)\,,$$

where we used the inequalities $\binom{n}{k+1} < n^{k+1}/2$ and $1 - x \ge e^{-2x}$ for $0 < x < \frac{1}{2}$. □

The main ingredient of the proof of Theorem 1 traces back to a theorem from [16] (c.f. Theorem 13 in Section 3.1). A special case of that result states that for all integers $k \ge 3$ and $r \ge 2$ there exists a constant C such that if $p = p(n) \ge Cn^{-\frac{2}{k+1}}$ then $\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (K_k)_r) = 1$. Adapting an idea of Nenadov and Steger [20], and based on a result of Saxton and Thomason [19], we obtain the following quantitative version of the above result. Recall our notation $R = R(k;r)$ for the r-color Ramsey number.

Lemma 5. *For all integers $r \ge 2$, $k \ge 3$, and*

$$n \ge k^{400k^4}R^{40k^2} \tag{2}$$

the following holds. Set

$$b = \frac{1}{2R^2}, \quad C = 2^{5\sqrt{\log n \log k}}R^{16}\,, \quad \text{and} \quad p = Cn^{-\frac{2}{k+1}}. \tag{3}$$

Then

$$\mathbb{P}(G(n,p) \to (K_k)_r) \ge 1 - \exp\left(-bp\binom{n}{2}\right)\,.$$

We devote the next two subsections to the proof of Lemma 5. Now, we deduce Theorem 1 from Lemmas 4 and 5.

Proof of Theorem 1. For given r and k, let n be as in (2), and let b, C, and p be as in (3). Below we will show that these parameters satisfy not only the assumptions of Lemma 5, but also the assumption $p \le \frac{1}{2}$ of Lemma 4, as well as an additional inequality

$$n \ge (5/b)^{\frac{k+1}{k-1}}\,C^{\binom{k+2}{2}}. \tag{4}$$

With these two inequalities at hand, we may quickly finish the proof of Theorem 1. Indeed, (4) implies that

$$bp\binom{n}{2} \geq \frac{2}{5}bpn^2 \geq 2(b/5)Cn^{1+\frac{k-1}{k+1}} \overset{(4)}{\geq} 2C^{\binom{k+1}{2}}n \tag{5}$$

which, by Lemma 4, implies in turn that

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left(-bp\tbinom{n}{2}\right).$$

Since, by Lemma 5,

$$\mathbb{P}(G(n,p) \to (K_k)_r) \geq 1 - \exp\left(-bp\tbinom{n}{2}\right),$$

we conclude that

$$\mathbb{P}(G(n,p) \to (K_k)_r \text{ and } G(n,p) \not\supset K_{k+1}) > 0.$$

Thus, there exists a $(k;r)$-Folkman graph on n vertices, and thus, $f(k) \leq k^{400k^4}R^{40k^2}$.

It remains to show that $p \leq \frac{1}{2}$ and that (4) holds. The first inequality is equivalent to

$$n \geq (2C)^{\frac{k+1}{2}}. \tag{6}$$

We will now show that this inequality is a consequence of (4) and then establish (4) itself. Since $C > 1$ and $5/b \overset{(3)}{=} 10R^2 \overset{(1)}{\geq} 10 \cdot 2^k$, we infer that

$$(5/b)^{\frac{k+1}{k-1}}C^{\binom{k+2}{2}} \geq (10 \cdot 2^k)^{\frac{k+1}{k-1}}C^{\binom{k+2}{2}} \geq (2C)^{\frac{k+1}{2}},$$

and hence, (6) indeed follows from (4).

Finally, we establish (4). In doing so we will use again the identity $5/b \overset{(3)}{=} 10R^2$, as well as the inequalities $100 \leq C$, which follows from (2) and (3), $\binom{k+2}{2} \leq 2k^2 - 1$, and $\frac{k+1}{k-1} \leq 2$, valid for all $k \geq 3$. The R-H-S of (4) can be bounded from above by

$$(10R^2)^{\frac{k+1}{k-1}}C^{\binom{k+2}{2}} \leq 100R^4C^{\binom{k+2}{2}} \leq R^4C^{2k^2} = 2^{10k^2\sqrt{\log n \log k}}R^{20k^2}.$$

Hence, it suffices to show that

$$n \geq 2^{10k^2\sqrt{\log n \log k}}R^{20k^2}. \tag{7}$$

Observe that, by (2), $\frac{1}{2}\log n \geq 20k^2 \log R$, and thus, it remains to check that

$$\frac{1}{2}\log n \geq 10k^2\sqrt{\log n \log k},$$

or equivalently that

$$\log n \geq 400k^4 \log k.$$

This, however, follows trivially from (2). $\square$

2.1 The proof of Lemma 5 – preparations

In this and the next subsection we present a proof of Lemma 5, which was inspired by the work of Nenadov and Steger [20] and is based on a recent general result of Saxton and Thomason [19] on the distribution of independent sets in hypergraphs. For a hypergraph H, a subset $I \subseteq V(H)$ is *independent* if the subhypergraph $H[I]$ induced by I in H has no edges.

For an h-graph H, the degree $d(J)$ of a set $J \subset V(H)$ is the number of edges of H containing J. (Since in our paper letter r is reserved for the number of colors, we will use h for hypergraph uniformity.) We will write $d(v)$ for $d(\{v\})$, the ordinary vertex degree. We further define, for a vertex $v \in V(H)$ and $j = 2, \dots, h$, the maximum j-degree of v as

$$d_j(v) = \max\left\{d(J) : v \in J \subset \binom{V(H)}{j}\right\}.$$

Finally, the co-degree function of H with a formal variable τ is defined in [19] as

$$\delta(H, \tau) = \frac{2^{\binom{h}{2}-1}}{nd} \sum_{j=2}^{h} \frac{\sum_v d_j(v)}{2^{\binom{j-1}{2}} \tau^{j-1}}, \tag{8}$$

where the inner sum is taken over all vertices $v \in V(H)$ and d is the average vertex degree in H, that is, $d = \frac{1}{n}\sum_v d(v)$.

Theorem 6 below is an abridged version of [19, Corollary 2.7, page 10], where we suppress part of conclusion (a) (about the sets T_i), as well as the "Moreover" part therein, since we do not use this additional information here. Most importantly, we provide an explicit value of the constant $c = c(h)$ as $864h!^3h$. Indeed, it follows from the calculations in [19, page 22] that

$$c \le \frac{288h!^2h}{\ln 1/\varepsilon}\left(1 + \frac{\ln \varepsilon}{\ln(1 - 1/2h!)}\right) \le \frac{288h!^2h}{\ln 1/\varepsilon}\left(1 + 2h! \ln 1/\varepsilon\right) \le 864h!^3h.$$

In part (c) of the theorem below, for convenience, we switch from ln to log, but only on the R-H-S of the upper bound on $\ln |\mathcal{C}|$.

Theorem 6 (Saxton & Thomason, [19])**.** *Let H be an h-graph on vertex set $[n]$ and let ε and τ be two real numbers such that $0 < \varepsilon < 1/2$,*

$$\tau \le 1/(144h!^2h) \qquad \text{and} \qquad \delta(H, \tau) \le \varepsilon/12h!.$$

Then there exists a collection $\mathcal{C}$ of subsets of $[n]$ such that the following three properties hold.

(a) For every independent set I in H there exists a set $C \in \mathcal{C}$ such that $I \subset C$.

(b) For all $C \in \mathcal{C}$, we have $e(H[C]) \le \varepsilon e(H)$.

(c) $\ln |\mathcal{C}| \le c \log(1/\varepsilon)\tau \log(1/\tau)n$, where $c = 864h!^3h$.

We will now tailor the above result to our application. The hypergraphs we consider have a very symmetric structure. Given k and n, let $H(n,k)$ be the hypergraph with vertex set $\binom{[n]}{2}$, the edges of which correspond to all copies of K_k in the K_n with vertex set $[n]$. Thus, $H(n,k)$ has $\binom{n}{2}$ vertices, $\binom{n}{k}$ edges, and is $\binom{k}{2}$-uniform. Every vertex of $H(n,k)$ has degree $d = \binom{n-2}{k-2}$, while for each $j = 2, \ldots, \binom{k}{2}$, $d_j(v) = \binom{n-l_j}{k-l_j}$, where ℓ_j is the smallest integer such that $j \le \binom{\ell_j}{2}$. Let

$$\delta(n,k,\tau) := \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{\tau^{j-1} n^{\ell_j - 2}}.$$

The co-degree function of $H(n,k)$ can be bounded by $\delta(n,k,\tau)$.

Claim 7.

$$\delta(H(n,k),\tau) \le \delta(n,k,\tau).$$

Proof. By the definition of $\delta(H,\tau)$ in (8) with h replaced by $\binom{k}{2}$, n by $\binom{n}{2}$, d by $\binom{n-2}{k-2}$, $d_j(v)$ by $\binom{n-l_j}{k-l_j}$, and with $2^{\binom{j-1}{2}}$ dropped out from the denominator, we have

$$\delta(H(n,k),\tau) \le 2^{k^4} \sum_{j=2}^{\binom{k}{2}} \frac{\binom{n-l_j}{k-l_j}}{\tau^{j-1}\binom{n-2}{k-2}}.$$

Now, observe that $\frac{\binom{n-l_j}{k-l_j}}{\binom{n-2}{k-2}} \le (k/n)^{l_j - 2}$ and $l_j \le k$. □

The most important property of hypergraph $H(n,k)$ is that a subset S of the vertices of H corresponds to a graph G with vertex set $[n]$ and edge set S, and S is an independent set in $H(n,k)$ if and only if the corresponding graph G is K_k-free. We now apply Theorem 6 to $H(n,k)$.

Corollary 8. *Let $k \ge 3$, $n \ge 3$, and let ϵ and τ be two real numbers such that $0 < \varepsilon < 1/2$,*

$$\tau \le \left(k^2!\right)^{-2} \qquad \text{and} \qquad \delta(n,k,\tau) \le \frac{\varepsilon}{k^2!}\,. \tag{9}$$

Then there exists a collection $\mathcal{C}$ of subgraphs of K_n such that the following three properties hold.

(*a*) *For every K_k-free graph $G \subseteq K_n$ there exists a set $C \in \mathcal{C}$ such that $G \subset C$.*

(*b*) *For all $C \in \mathcal{C}$, C contains at most $\varepsilon\binom{n}{k}$ copies of K_k.*

(*c*) $\ln|\mathcal{C}| \le (2k^2)!\log(1/\varepsilon)\tau\log(1/\tau)\binom{n}{2}$.

Proof. Note that for $k \geq 3$,

$$k^2! > 12\tbinom{k}{2}! \quad \text{and, consequently,} \quad k^2!^2 > 144\tbinom{k}{2}!\tbinom{k}{2},$$

and that, by Claim 7, $\delta(H(n,k),\tau) \leq \delta(n,k,\tau)$. Thus, the assumptions of Theorem 6 hold for $H := H(n,k)$ with $h = \binom{k}{2}$, and its conclusions (a-c) translate into the corresponding properties (a-c) of Corollary 8. Finally, notice that

$$(2k^2)! > c = 864\tbinom{k}{2}!^3\tbinom{k}{2}.$$

□

In the next subsection we deduce Lemma 5 from Corollary 8. First, however, we make a simple observation about the number of monochromatic copies of K_k in every coloring of K_n. Recall that $R = R(k;r)$ is the r-color Ramsey number for K_k and set

$$\alpha = \binom{R}{k}^{-1}. \tag{10}$$

Proposition 9. *Let $n \geq R$. For every $(r+1)$-coloring of the edges of K_n either there are at least $\frac{\alpha}{2}\binom{n}{k}$ monochromatic copies of K_k colored by the first r colors, or at least $\frac{1}{R^2}\binom{n}{2}$ edges receive color $r+1$.*

Proof. Consider an $(r+1)$-coloring of the edges of K_n. Let $x\binom{n}{R}$ be the number of the R-element subsets of the vertices of K_n with no edge colored by color $r+1$. By the definition of R, each of these subsets induces in K_n a monochromatic copy of K_k. Thus, counting repetitions, there are at least

$$x\frac{\binom{n}{R}}{\binom{n-k}{R-k}} = x\frac{\binom{n}{k}}{\binom{R}{k}} = x\alpha\binom{n}{k}$$

monochromatic copies of K_k colored by one of the first r colors. Suppose that their number is smaller than

$$\frac{\alpha}{2}\binom{n}{k}.$$

Then $x \leq \frac{1}{2}$, that is, at least a half of the R-element subsets of $V(K_n)$ contain at least one edge colored by $r+1$. Hence, color $r+1$ appears on at least

$$\frac{\frac{1}{2}\binom{n}{R}}{\binom{n-2}{R-2}} = \frac{\frac{1}{2}\binom{n}{2}}{\binom{R}{2}} > \frac{1}{R^2}\binom{n}{2}$$

edges of K_n. This completes the proof. □

2.2 Proof of Lemma 5 – details

Let $r \geq 2$, $k \geq 3$, and let n, b, C, and p be as in Lemma 5, see (3) and (2). We have to show that

$$\mathbb{P}(G(n,p) \to (K_k)_r) \geq 1 - \exp(-bp\tbinom{n}{2})\,.$$

First we set up a few auxiliary constants required for the application of Corollary 8. Recalling that α is defined in (10), let

$$\varepsilon = \frac{\alpha}{2r}, \tag{11}$$

$$C_0 = 2^{4\sqrt{\log n}} R^{10/k}, \quad \text{and} \quad \tau = C_0 n^{-\frac{2}{k+1}}\,. \tag{12}$$

We will now prove that the above defined constants ε and τ satisfy the assumptions of Corollary 8.

Claim 10. *Condition (9) holds true for every $k \geq 3$.*

Proof. In order to verify the first condition, note that by the definitions of τ and C_0 in (12) and the obvious bound $x! < x^x$,

$$(k^2!)^2\tau \leq k^{4k^2} 2^{4\sqrt{\log n}} R^{10/k} n^{-\frac{2}{k+1}}. \tag{13}$$

It remains to show that the R-H-S of (13) is smaller than one, or, by taking logarithms, that

$$4k^2 \log k + 4\sqrt{\log n} + \frac{10}{k}\log R < \frac{2}{k+1}\log n.$$

This, however, follows from

$$4\sqrt{\log n} < \frac{1}{k+1}\log n,$$

or equivalently,

$$16(k+1)^2 < \log n,$$

and from

$$4k^2(k+1)\log k + \frac{10}{k}(k+1)\log R < \log n,$$

both of which are true by the lower bound on n in (2).

To prove the second inequality in (9), note that since $\tau \leq 1$ and $j \leq \binom{\ell_j}{2}$, the quantity $\tau^{j-1} n^{\ell_j - 2}$ is minimized when $j = \binom{\ell_j}{2}$. Thus, we have

$$\tau^{j-1} \cdot n^{\ell_j-2} \geq \tau^{\binom{\ell_j}{2}-1} \cdot n^{\ell_j-2} = C_0^{\binom{\ell_j}{2}-1} n^{-\frac{(\ell_j-2)(\ell_j+1)}{k+1}+\ell_j-2} = C_0^{\binom{\ell_j}{2}-1} n^{\frac{(\ell_j-2)(k-\ell_j)}{k+1}}\,. \tag{14}$$

Observe that for $j \geq 2$ we have $\ell_j \geq 3$. In what follows we obtain a lower bound on the right-hand side of (14) by distinguishing two cases: $\ell_j < k$ and $\ell_j = k$. If $\ell_j \leq k$, then $(\ell_j - 2)(k - \ell_j)$ is minimized for $\ell_j = 3$ and $\ell_j = k-1$ and owing to $C_0 > 1$ we infer

$$\tau^{j-1} \cdot n^{\ell_j-2} \overset{(14)}{\geq} C_0^{\binom{\ell_j}{2}-1} n^{\frac{(\ell_j-2)(k-\ell_j)}{k+1}} > n^{\frac{k-3}{k+1}} \overset{(2)}{\geq} k^{80k^4} R^{8k^2}\,,$$

where we also used the bound $\frac{k+1}{k-3} \leq 5$ for all $k \geq 4$, which holds due to $3 \leq \ell_j < k$. If, on the other hand, $\ell_j = k$, then observe that by the definition of C_0 in (12) and the bound on n (2),

$$C_0 \geq 2^{80k^2} R^{10/k}. \tag{15}$$

Hence, in view of (15), and the fact that $\binom{k}{2} - 1 \geq \frac{1}{5}k^2$ for $k \geq 3$, we infer that

$$\tau^{j-1} \cdot n^{\ell_j - 2} \overset{(14)}{\geq} C_0^{\binom{k}{2}-1} \geq \left(2^{80k^2} R^{10/k}\right)^{k^2/5} = 2^{16k^4} R^{2k}.$$

Consequently, using the trivial bounds $k^k \cdot k^2! < 2^{k^4}$, $\binom{R}{k} < R^k$, and $r < R$ (see (1)), we infer that

$$\sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{\tau^{j-1} n^{\ell_j - 2}} \leq \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{2^{16k^4} R^{2k}} \leq \frac{k^k}{2^{15k^4} R^{2k}} \leq \frac{1}{2r\binom{R}{k} \cdot k^2!} \overset{(10),(11)}{=} \frac{\varepsilon}{k^2!}. \quad \square$$

In view of Claim 10, the conclusions of Corollary 8 hold true with ε and τ defined in, resp., (11) and (12). That is, there exists a collection $\mathcal{C}$ of subgraphs of K_n such that Properties (a), (b), and (c) of Corollary 8 are satisfied for these specific values of ε and τ.

To continue with the proof of Lemma 5 consider a random graph $G(n,p)$ and let $\mathcal{E}$ be the event that $G(n,p) \not\to (K_k)_r$. By the definition of $\mathcal{E}$, for each $G \in \mathcal{E}$, there exists an r-coloring $\varphi\colon E(G) \to [r]$ yielding no monochromatic copy of K_k. (Further on we will call such a coloring *proper*.) In other words, there are K_k-free graphs $G_1, \ldots, G_r$ (defined by $G_i = \varphi^{-1}(i)$) such that $G_1 \cup \ldots \cup G_r = G$. According to Property (a) of Corollary 8, for every $i \in [r]$ there exists a graph $C_i \in \mathcal{C}$ such that $G_i \subseteq C_i$. Consequently, the graph

$$X_{G,\varphi} := K_n \setminus \bigcup_{i=1}^{r} C_i$$

is edge-disjoint from G, i.e., $G \cap X_{G,\varphi} = \emptyset$. (Recall that we identify graphs with their edge sets.)

Notice that since each graph $X_{G,\varphi}$ is completely determined by an r-tuple of graphs $C_1, \ldots, C_r$ belonging to the family $\mathcal{C}$, there are only at most $|\mathcal{C}|^r$ distinct candidate graphs for becoming an $X_{G,\varphi}$ for some G and φ. We next show that the graphs $X_{G,\varphi}$ are dense (see Claim 11). As it is unlikely for a random graph $G(n,p)$ to miss completely one of only few dense graphs, it will follow that $\mathbb{P}(\mathcal{E}) = o(1)$.

Claim 11. *For every $G \in \mathcal{E}$ and a proper r-coloring φ of $E(G)$, the above defined graph $X_{G,\varphi}$ has at least $\binom{n}{2}/R^2$ edges.*

Proof. For a graph G with $G \not\to (K_k)_r$ and a proper coloring φ of G, consider the graphs C_i, $i \in [r]$, and $X_{G,\varphi}$ as above. These graphs form an $(r+1)$-coloring of K_n, more precisely, an $(r+1)$-coloring where, for each $i = 1, \ldots, r$, the edges of color i are contained in C_i, while all edges of $X_{G,\varphi}$ are colored with color $r+1$. (Note that this coloring may not be unique, as the graphs T_i are not necessarily mutually disjoint.) By Proposition 9, this $(r+1)$-coloring yields either at least $(\alpha/2)\binom{n}{k}$ monochromatic copies of K_k in the first r colors or at least $\binom{n}{2}/R^2$ edges in the last color. Since for each $i \in [r]$, the i-th color class is contained in C_i, it follows from Property (b) that there at most

$$r \cdot \varepsilon \binom{n}{k} \overset{(11)}{=} \frac{\alpha}{2}\binom{n}{k}$$

monochromatic copies of K_k in the first r colors. Consequently, we must have

$$e(X_{G,\varphi}) \geq \frac{1}{R^2}\binom{n}{2}. \quad \square \tag{16}$$

Based on Claim 11 we can bound from above $\mathbb{P}(\mathcal{E}) = \mathbb{P}(G(n,p) \not\to (K_s)_r)$.

Claim 12.

$$\mathbb{P}(G(n,p) \not\to (K_s)_r) \leq |\mathcal{C}|^r \exp\left\{-\frac{p\binom{n}{2}}{R^2}\right\}$$

Proof. Set

$$X(C_1, \ldots, C_r) = K_n \setminus \bigcup_{i=1}^{r} C_i,$$

where $C_i \in \mathcal{C}$, $i = 1, \ldots, r$. Let $\mathcal{F}$ be the event that $G(n,p) \cap X(C_1, \ldots, C_r) = \emptyset$ for at least one r-tuple of graphs $C_i \in \mathcal{C}$, $i = 1, \ldots, r$, such that

$$e(X(C_1, \ldots, C_r)) \geq \frac{1}{R^2}\binom{n}{2}. \tag{17}$$

We have $\mathcal{E} \subseteq \mathcal{F}$. Indeed, if $G \in \mathcal{E}$ then there is a proper coloring φ of G and graphs $C_1, \ldots, C_r \in \mathcal{C}$ such that $G \subseteq \bigcup_{i=1}^{r} C_i$ and, by Claim 11, the graph $X_{G,\varphi} = X(C_1, \ldots, C_r)$ has at least $\frac{1}{R^2}\binom{n}{2}$ edges and is disjoint from G. Thus, $G \in \mathcal{F}$. Consequently,

$$\mathbb{P}(G(n,p) \not\to (K_k)_r) \leq \mathbb{P}(\mathcal{F}).$$

To estimate $\mathbb{P}(\mathcal{F})$ we write $\mathcal{F} = \bigcup \mathcal{F}(C_1, \ldots, C_r)$, where the summation runs over all collections $(C_1, \ldots, C_r)$ with $C_i \in \mathcal{C}^r$, $i = 1, \ldots, r$, satisfying (17) and the event $\mathcal{F}(C_1, \ldots, C_r)$ means that $G(n,p) \cap X(C_1, \ldots, C_r) = \emptyset$. Clearly,

$$\mathbb{P}(\mathcal{F}(C_1, \ldots, C_r)) = (1-p)^{e(X(C_1, \ldots, C_r))} \leq (1-p)^{\binom{n}{2}/R^2}.$$

Finally, applying the union bound, we have

$$\mathbb{P}(G(n,p) \not\to (K_s)_r) \leq \mathbb{P}(\mathcal{F}) \leq |\mathcal{C}|^r (1-p)^{\binom{n}{2}/R^2} \leq |\mathcal{C}|^r \exp\left\{-\frac{p\binom{n}{2}}{R^2}\right\}. \quad \square$$

Observe that by Property (c) of Corollary 8,

$$|\mathcal{C}|^r \le \exp\left\{ r(2k^2)! \log(1/\varepsilon)\tau \log(1/\tau)\binom{n}{2} \right\}. \tag{18}$$

In view of Claim 12 and inequality (18), to complete the proof of Lemma 5, it suffices to show that

$$r(2k^2)! \log(1/\varepsilon)\tau \log(1/\tau)\binom{n}{2} \le \frac{p\binom{n}{2}}{2R^2},$$

or, equivalently, after applying the definitions of p and τ ((3) and (12), resp.) and dividing sidewise by $n^{-\frac{2}{k+1}}\binom{n}{2}$, that

$$r(2k^2)! \log(1/\varepsilon) C_0 \log(1/\tau) \le C/(2R^2). \tag{19}$$

To this end, observe that, since $C_0 \ge 1$ and $2r < R$ (see (1)),

$$\log(1/\tau) \overset{(12)}{\le} \tfrac{2}{k+1} \log n$$

and

$$\log(1/\varepsilon) \overset{(11)}{=} \log(2r\tbinom{R}{k}) \le (k+1)\log R\,.$$

Hence, the L-H-S of (19) can be upper bounded by $2r(2k^2)! C_0 \log R \log n$. Consequently, using also the bounds $(2k^2)! < (2k)^{4k^2}$ and $4r < R$, we realize that (19) will follow from

$$R^3 (2k)^{4k^2} \log n \le C/C_0. \tag{20}$$

On the other hand,

$$C/C_0 \overset{(3),(12)}{=} 2^{5\sqrt{\log n \log k} - 4\sqrt{\log n}} R^{16-10/k} \ge 2^{\sqrt{\log n \log k} + 4\sqrt{\log n}(\sqrt{\log k} - 1)} R^{12}.$$

Thus, (20) is an immediate consequence of the following two inequalities, which are themselves easy consequences of (2):

$$2^{\sqrt{\log n \log k}} \overset{(2)}{\ge} 2^{20k^2 \log k} \ge (2k)^{4k^2}$$

and

$$2^{4\sqrt{\log n}(\sqrt{\log k} - 1)} > 2^{\sqrt{\log n}} \ge \log n.$$

For the latter inequality we first used $k \ge 3$ and $\sqrt{\log 3} > \frac{5}{4}$, and then the fact that $2^{\sqrt{x}} \ge x$ for all $x \ge 16$, which can be easily verified by checking the first derivative (note that by (2), $\log n \ge 16$). This completes the proof of Lemma 5.

3 Proof of Theorem 2

In [16] the first two authors established a threshold edge probability for the Ramsey property $G(n,p) \to (F)_r$, where, recall, $G(n,p)$ is a random graph obtained by including each edge of the complete graph on n vertices, independently, with probability p.

For a graph F let v_F and e_F stand for, respectively, the number of vertices and edges of F. For a graph F with $e_F \geq 1$ define

$$d_F = \begin{cases} \frac{e_F-1}{v_F-2} & \text{if} \quad e_F > 1 \\ \frac{1}{2} & \text{if} \quad e_F = 1 \end{cases}, \tag{21}$$

and

$$m_F = \max\{d_H \colon\ H \subseteq F \text{ and } e_H \geq 1\}\,. \tag{22}$$

Observe that $m_F = \frac{1}{2}$ for every F with $\Delta(F) = 1$, while for every F with $\Delta(F) \geq 2$ we have $m_F \geq 1$. Moreover, for every k-vertex graph we have

$$m_F \leq m_{K_k} = \frac{k+1}{2}\,.$$

We now state the main result of [16] in a slightly abridged form.

Theorem 13 ([16]). *For every integer $r \geq 2$ and a graph F with $\Delta(F) \geq 2$ there exists a constant $C_{F,r}$ such that if $p = p(n) \geq C_{F,r} n^{-1/m_F}$ then*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)_r) = 1.$$

The original proof of Theorem 13 was based on the regularity lemma of Szemerédi [21] and this led to tower-type dependencies of the involved parameters. In [18] it was noticed that for *two* colors the usage of the regularity lemma could be replaced by a simple Ramsey-type argument. Here we follow that thread and for $r = 2$ prove a quantitative version of Theorem 13 with only doubly exponential dependencies between the constants.

In order to state the result, we first define inductively four parameters indexed by the number of edges of a k-vertex graph F. For fixed $k \geq 3$ we set

$$a_1 = \frac{1}{2}, \qquad b_1 = \frac{1}{8}, \qquad C_1 = 1, \qquad \text{and} \qquad n_1 = 1 \tag{23}$$

and for each $i = 1, \ldots, \binom{k}{2} - 1$, define

$$a_{i+1} = \frac{a_i^{19k^4}}{2^{55k^6}}, \quad b_{i+1} = \frac{a_i^{37k^2}}{2^{118k^4}} b_i^4, \quad C_{i+1} = \frac{2^{122k^4}}{b_i^4 a_i^{37k^2}} C_i, \quad \text{and} \quad n_{i+1} = \frac{2^{14k^3}}{a_i^{4k}} n_i\,. \tag{24}$$

Note that a_i and b_i decrease with i, while C_i and n_i increase. Finally, for a graph F on k vertices, denote by

$$\mu_F = \binom{n}{k} \frac{k!}{\mathrm{aut}(F)} p^{e_F}$$

the expected number of copies F in $G(n,p)$ and note that

$$\binom{n}{k} p^{e_F} \le \mu_F \le n^k p^{e_F} = n^{v_F} p^{e_F}. \tag{25}$$

For a real number $\lambda > 0$ we write $G \xrightarrow{\lambda} F$ if every 2-coloring of the edges of G produces at least λ monochromatic copies of F. We call a graph F *k-admissible* if $v_F = k$ and either $e_F = 1$ or $\Delta(F) \ge 2$. Now, we are ready to state our first quantitative version of Theorem 13.

Lemma 14. *For every $k \ge 3$, every k-admissible graph F, and for all $n \ge n_{e_F}$ and $p \ge C_{e_F} n^{-1/m_F}$,*

$$\mathbb{P}\left(G(n,p) \xrightarrow{a_{e_F}\mu_F} F\right) \ge 1 - \exp(-b_{e_F} p \tbinom{n}{2}).$$

Note, that owing to the monotonicity of the Ramsey property, for $r = 2$ Theorem 13 is an immediate corollary of Lemma 14.

3.1 The proof of Lemma 14 – preparations

Before we start with the proof of Lemma 14, we need to recall abridged versions of two useful facts from [12, Lemmas 2.52 and 2.51] (see also [16, 18]), which we formulate as Propositions 15 and 16 below.

Given a set Γ and a real number p, $0 \le p \le 1$, let Γ_p be the random binomial subset of Γ, that is, a subset obtained by independently including each element of Γ with probability p. Further, given an increasing family $\mathcal{Q}$ of subsets of a set Γ and an integer h, we denote by $\mathcal{Q}_h$ the subfamily of $\mathcal{Q}$ consisting of the sets $A \in \mathcal{Q}$ having the property that all subsets of A with at least $|A| - h$ elements still belong to $\mathcal{Q}$.

Proposition 15. *Let $0 < c < 1$, $\delta = c^2/9$, $Np \ge 72/\delta^2 = 2^3 3^6/c^4$, and $h = \delta Np/2$. Then for every increasing family $\mathcal{Q}$ of subsets of an N-element set Γ the following holds. If*

$$\mathbb{P}\left(\Gamma_{(1-\delta)p} \notin \mathcal{Q}\right) \le \exp(-cNp)$$

then

$$\mathbb{P}\left(\Gamma_p \notin \mathcal{Q}_h\right) \le \exp(-\delta^2 Np/9).$$

Proof. We want to apply [12, Lemma 2.52], which is very similar to Proposition 15. Lemma 2.52 from [12] states that if c and $\delta > 0$ satify

$$\delta(3 + \log(1/\delta)) \le c \tag{26}$$

and

$$\mathbb{P}(\Gamma_{(1-\delta)p} \notin \mathcal{Q}) \le \exp(-cNp)$$

then

$$\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) \le 3\sqrt{Np}\exp(-cNp/2) + \exp(-\delta^2 Np/8). \tag{27}$$

To this end we first note that by assumption of Proposition 15 we have $\delta < 1/9$. Since $\sqrt{x}(\log(1/x)$ is increasing for $x \in (0, 1/\mathrm{e}^2]$ it follows for every $\delta \leq 1/9$ that

$$\sqrt{\delta}\log(1/\delta) \leq \frac{\log(9)}{3} \leq 2\,.$$

Consequently, $\sqrt{\delta}(3 + \log(1/\delta)) \leq 3$ and owing to the assumption $\delta = c^2/9$ this is equivalent to (26). Moreover, since $Np \geq 2^3 3^6/c^4 > (12/c)^2$ we have

$$3\sqrt{Np} \leq \exp(3\sqrt{Np}) \leq \exp(cNp/4)\,.$$

Hence, (27) yields

$$\begin{aligned}\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) &\leq \exp(-cNp/4) + \exp(-\delta^2 Np/8) \leq 2\exp(-\delta^2 Np/8)\\ &\leq \exp(-\delta^2 Np/8 + 1) \leq \exp(-\delta^2 Np/9),\end{aligned}$$

where the last inequality follows by our assumption $Np \geq 72/\delta^2$. □

The following proposition will be also needed in the proof of Lemma 14. We state the version from [12] (with $t = 2$).

Proposition 16 ([12, Lemma 2.51])**.** *Let $\mathcal{S} \subseteq \binom{\Gamma}{s}$, $0 \leq p \leq 1$, and $\lambda = |\mathcal{S}|p^s$. Then for every nonnegative integer h, with probability at least $1 - \exp(-\frac{h}{2s})$, there exists a subset $E_0 \subseteq \Gamma_p$ of size h such that $\Gamma_p \setminus E_0$ contains at most 2λ sets from $\mathcal{S}$.*

In the proof of Lemma 14 we will also use an elementary fact about (ϱ, d)-dense graphs (see Proposition 17(b) below). For constants ϱ and d with $0 < d, \varrho \leq 1$ we call an n-vertex graph Γ (ϱ, d)-dense if every induced subgraph on $m \geq \varrho n$ vertices contains at least $d(m^2/2)$ edges. It follows by an easy averaging argument that it suffices to check the above inequality only for $m = \lceil \varrho n \rceil$. Note also that every induced subgraph of a (ϱ, d)-dense n-vertex graph on at least cn vertices is $(\frac{\varrho}{c}, d)$-dense.

It turns out that for a suitable choice of the parameters, (ϱ, d)-dense graphs enjoy a Ramsey-like property. For a two-coloring of (the edges of) Γ we call a sequence of vertices $(v_1, \ldots, v_\ell)$ *canonical* if for each $i = 1, \ldots, \ell - 1$ all the edges $\{v_i, v_j\}$, for $j > i$ are of the same color. Part (a) of Proposition 17, or, in fact, part (a) of its consequence, Corollary 18 below, will only be needed in the proof of Proposition 3 in Section 5.

Proposition 17. *For every $\ell \geq 2$ and $d \in (0, 1)$ the following two statements hold.*

(a) If $n \geq (2/d)^{\ell-2}$ and $0 < \varrho \leq (d/2)^{\ell-2}$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains a canonical sequence of length ℓ.

(b) If $n \geq 2(4/d)^{\ell-2}$ and $0 < \varrho \leq (d/4)^{\ell-2}/2$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains at least

$$f_n(\ell) := \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} n^\ell$$

canonical sequences of length ℓ.

Proof. The proof of part (a) is by a standard Ramsey-type inductive argument, based on the fact that every graph on n vertices and with at least $dn^2/2$ edges contains a vertex of degree at least dn; moreover, at least half of the edges incident to that vertex have the same color. We omit the details as they are similar but simpler than those given below for the proof of part (b).

For part (b), note that as long as $\varrho \le 1/2$ every (ϱ, d)-dense graph contains at least $n/2$ vertices with degrees at least $dn/2$. Indeed, otherwise a set of $m = \lceil (n+1)/2 \rceil$ vertices of degrees smaller than $dn/2$ would induce less than $mdn/4 \le d(m^2/2)$ edges, a contradiction.

We prove Proposition 17 by induction on ℓ. For $\ell = 2$, every ordered pair of adjacent vertices is a canonical sequence and there are at least $2d\binom{n}{2} > f_n(2)$ such pairs if $n \ge 2$. Assume that the proposition is true for some $\ell \ge 2$ and consider an n-vertex (ϱ, d)-dense graph Γ, where $\varrho \le (d/4)^{\ell-1}/2$ and $n \ge 2(4/d)^{\ell-1}$. As observed above, there is a set U of at least $n/2$ vertices with degrees at least $dn/2$. Fix one vertex $u \in U$ and let M_u be a set of at least $dn/4$ neighbors of u connected to u by edges of the same color. Let $\Gamma_u = \Gamma[M_u]$ be the subgraph of Γ induced by the set M_u. Note that Γ_u has $n_u \ge dn/4 \ge 2(4/d)^{\ell-2}$ vertices and is (ϱ_u, d)-dense with $\varrho_u \le (d/4)^{\ell-2}/2$. Hence, by the induction assumption, there are at least

$$f_{n_u}(\ell) \ge \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} \left(\frac{dn}{4}\right)^{\ell} = \left(\frac{1}{4}\right)^{\binom{\ell}{2}+\ell} d^{\binom{\ell+1}{2}} n^{\ell}$$

canonical sequences of length ℓ in Γ_u. Each of these sequences preceded by the vertex u makes a canonical sequence of length $\ell+1$ in Γ. As there are at least $n/2$ vertices in U, there are at least

$$\frac{n}{2} f_{n_u}(\ell) \ge \left(\frac{1}{4}\right)^{\binom{\ell+2}{2}} d^{\binom{\ell+1}{2}} n^{\ell+1}$$

canonical sequences of length $\ell+1$ in Γ. This completes the inductive proof of Proposition 17. □

Corollary 18. *For every $k \ge 2$, every graph F on k vertices, and every $d \in (0,1)$ the following two statements hold.*

(a) If $n \ge (2/d)^{2k}$ and $0 < \varrho \le (d/2)^4$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains a monochromatic copy of F.

(b) If $n \ge (4/d)^{2k}$ and $0 < \varrho \le (d/4)^{2k}$ then every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least γn^k monochromatic copies of F, where $\gamma = d^{2k^2} 2^{-5k^2}$.

Proof. Every canonical sequence $(v_1, \dots, v_{2k-2})$ contains a monochromatic copy of K_k. Indeed, among the vertices $v_1, \dots, v_{2k-3}$, some $k-1$ have the same color on all the "forward" edges. Therefore, these vertices together with vertex v_{2k-2} form a monochromatic copy of K_k. This proves part (a). To prove part (b), notice that every such copy is contained in no more than $k!\binom{2k-2}{k} n^{k-2} = (2k-2)_k n^{k-2}$ canonical sequences of length $2k-2$. Finally, every copy of K_k contains at least one copy of F, and different copies of K_k contain different copies of F. Consequently, by Proposition 17, every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least

$$\frac{f_n(2k-2)}{(2k-2)_k n^{k-2}} = \frac{1}{(2k-2)_k}\left(\frac{1}{4}\right)^{\binom{2k-1}{2}} d^{\binom{2k-2}{2}} n^k > \frac{d^{2k^2}}{2^{5k^2}} n^k$$

monochromatic copies of F. □

3.2 Proof of Lemma 14

For given $n \in \mathbb{N}$, $p \in (0,1)$, and a k-vertex graph F we denote by X_F the random variable counting the number of copies of F in $G(n,p)$. We also recall that $\mu_F = \mathbb{E}X_F$.

For fixed $k \geq 3$ we prove Lemma 14 by induction on e_F. We may assume $n \geq k$, as for $n < k$ we have $\mu_F = 0$ and there is nothing to prove.

Induction start. Let F_1 be a graph consisting of one edge and $k-2$ isolated vertices. Note that $m_{F_1} = 1/2$ (see (22)) and for every two-coloring of the edges of $G(n,p)$ every copy of F_1 in $G(n,p)$ is monochromatic. Clearly,

$$X_{F_1} = \binom{n-2}{k-2} X_{K_2} \quad \text{and} \quad \mu_{F_1} = \binom{n-2}{k-2}\mu_{K_2} = \binom{n-2}{k-2}\binom{n}{2}p.$$

Thus, by Chernoff's bound (see, e.g., [12, ineq. (2.6)]) we have

$$\mathbb{P}\left(X_{F_1} \leq \frac{1}{2}\mu_{F_1}\right) = \mathbb{P}\left(X_{K_2} \leq \frac{1}{2}\mu_{K_2}\right) \leq \exp\left(-\frac{1}{8}\binom{n}{2}p\right),$$

which holds for any values of p and n. Hence, Lemma 14 follows for $F = F_1$ and for the constants $a_1 = 1/2$, $b_1 = 1/8$, and $C_1 = n_1 = 1$ as given in (23) and it remains to prove the inductive step.

Inductive step. Let F_{i+1} be a graph with $i+1 \geq 2$ edges and maximum degree $\Delta(F_{i+1}) \geq 2$. If $i+1 \geq 3$, then we can remove one edge from F_{i+1} in such a way that the resulting graph F_i still contains at least one vertex of degree at least two, i.e., $\Delta(F_i) \geq 2$. If $i+1 = 2$, the graph $F_{i+1} = F_2$ consists of a path of length two and $k-3$ isolated vertices and removing any of the two edges results in the graph $F_i = F_1$. In either case, we may fix an edge $f \in E(F_{i+1})$ such that the graph $F_i = F_{i+1} - f$ is k-admissible. Hence, we can assume that Lemma 14 holds for F_i and for the constants a_i, b_i, C_i, and n_i inductively defined by (23) and (24).

We have to show that Lemma 14 holds for F_{i+1} and constants a_{i+1}, b_{i+1}, C_{i+1}, and n_{i+1} given in (24). For that let $n \geq n_{i+1}$ and $p \geq C_{i+1}n^{-1/m_{F_{i+1}}}$. We will expose the random graph $G(n,p)$ in two independent rounds $G(n,p_{\mathrm{I}})$ and $G(n,p_{\mathrm{II}})$ and have $G(n,p) = G(n,p_{\mathrm{I}}) \cup G(n,p_{\mathrm{II}})$. For that we will fix p_{I} and p_{II} as follows. First we fix auxiliary constants[1]

$$d = \frac{a_i^2}{64^{k^2}}\,, \qquad \varrho = \left(\frac{d}{4}\right)^{2k}\,, \qquad \gamma = \frac{d^{2k^2}}{2^{5k^2}}\,, \qquad \delta_{\mathrm{II}} = \frac{\gamma^4}{9\cdot 16^{k^2}}\,, \quad \text{and} \quad \alpha = \frac{\delta_{\mathrm{II}}^2\gamma}{36}\,. \tag{28}$$

Then p_{I} and $p_{\mathrm{II}} \in (0,1)$ are defined by the equations

$$p = p_{\mathrm{I}} + p_{\mathrm{II}} - p_{\mathrm{I}}p_{\mathrm{II}} \qquad \text{and} \qquad p_{\mathrm{I}} = \alpha p_{\mathrm{II}}\,. \tag{29}$$

Clearly, we have

$$p \geq p_{\mathrm{II}} \geq \frac{p}{2} \geq \alpha p \geq \alpha p_{\mathrm{II}} = p_{\mathrm{I}} \geq \alpha\frac{p}{2}\,. \tag{30}$$

We continue with a short outline of the main ideas of the forthcoming proof.

Outline. First we consider a two-coloring χ with colors red and blue of the edges of $G(n,p_{\mathrm{I}})$. Owing to the induction assumption (Lemma 14 for F_i) we note that with high probability the coloring χ yields many monochromatic copies of F_i. We will say that an unordered pair of vertices $e = \{u,v\}$ is *χ-rich* if $G(n,p_{\mathrm{I}}) + e$ possesses "many" (to be defined later) copies of F_{i+1}, in which e plays the role of the edge f and the rest is a monochromatic copy of F_i. Let Γ_χ be an auxiliary graph of all χ-rich pairs. We will show that with "high" (to be defined later) probability Γ_χ is, in fact, (ϱ,d)-dense for d and ϱ defined in (28).

The auxiliary graph Γ_χ is naturally two-colored (by azure and pink), since every χ-rich pair closes either many blue or many red copies of F_i (or both and then we pick the color for that edge, azure or blue, arbitrarily). Consequently, Corollary 18 yields many monochromatic copies of F_{i+1} in Γ_χ and at least half of them are colored, say, pink. That is, there are many copies of F_{i+1} in Γ_χ such that each of their edges closes many red copies of F_i in $G(n,p_{\mathrm{I}})$ under the coloring χ. By Janson's inequality combined with Proposition 15, with high probability, many pink copies will be still present in $\Gamma_\chi \cap G(n,p_{\mathrm{II}})$ even after a fraction of edges is deleted. Thus, we are facing a "win-win" scenario. Namely, if an extension of χ colors only few pink edges of $\Gamma_\chi \cap G(n,p_{\mathrm{II}})$ red then, by the above, many copies of F_{i+1} in $\Gamma_\chi \cap G(n,p_{\mathrm{II}})$ have to be colored completely blue. Otherwise, many pink edges of $\Gamma_\chi \cap G(n,p_{\mathrm{II}})$ are red, which, by the definition of a pink edge, results in many red copies of F_{i+1} in $G(n,p)$. We now give the technical details of this proof.

Useful estimates. For the verification of several inequalities in the proof, it will be useful to appeal to the following lower bounds for γ, α, and ϱ in terms of powers of a_i and 2. From the definitions in (28), for sufficiently large k, one obtains the following

[1] The proof requires several auxiliary constants, which at first may appear a bit unmotivated. For example, we now define δ_{II}, while δ_{I} is be defined only later. Both δ's will be used in the applications of Proposition 15, and δ_{II} will appear in the second one.

bounds.

$$\begin{aligned}
\gamma =&\frac{a_i^{4k^2}}{2^{12k^4+5k^2}} \geq \frac{a_i^{4k^2}}{2^{13k^4}}\,,\\
\alpha =&\frac{a_i^{36k^2}}{3^6\cdot 2^{108k^4+53k^2+2}} \geq \frac{a_i^{36k^2}}{2^{109k^4}}\,,\\
\varrho =&\frac{a_i^{4k}}{2^{12k^3+4k}} \geq \frac{a_i^{4k}}{2^{13k^3}}\,.
\end{aligned} \tag{31}$$

We will also make use of the inequalities

$$np \geq C_{i+1}, \tag{32}$$

valid because $m_{F_{i+1}} \geq 1$, and

$$n^{v_H-2}p^{e_H-1} \geq C_{i+1}^{e_H-1}, \tag{33}$$

valid for every subgraph H of F_{i+1} with $v_H \geq 3$, because

$$m_{F_{i+1}} \geq d_H = \frac{e_H-1}{v_H-2}.$$

Of course, (32) follows from (33).

Details. As outlined above, in the first round we want to show that with high probability the random graph $G(n,p_{\mathrm{I}})$ has the property that for every two-coloring χ the auxiliary graph Γ_χ is (ϱ,d)-dense. For that we set

$$\delta_{\mathrm{I}} = \frac{b_i^2}{36} \tag{34}$$

and for a two-coloring χ we define a pair $\{u,v\}$ of vertices to be *χ-rich* if it closes at least

$$\ell = \frac{a_i}{4^{k^2}}(\varrho n)^{k-2}p_{\mathrm{I}}^i \tag{35}$$

monochromatic copies of F_i in $G(n,p_{\mathrm{I}})$ to a copy of F_{i+1}. Then Γ_χ is an auxiliary n-vertex graph with the edge set being the set of *χ-rich* pairs.

Let $\mathcal{E}$ be the event (defined on $G(n,p_{\mathrm{I}})$) that for every two-coloring χ of $G(n,p_{\mathrm{I}})$ the graph Γ_χ is (ϱ,d)-dense.

Claim 19.

$$\mathbb{P}(\mathcal{E}) \geq 1-\exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}}+n+2k^2\right)$$

Proof. Let χ be a two-coloring of $G(n,p_{\mathrm{I}})$. Fix a set $U\subseteq[n]$ with $|U|=\varrho n$ (throughout we assume that ϱn is an integer) and consider the random graph $G(n,p_{\mathrm{I}})$ induced on U

$$G(U,p_{\mathrm{I}}) := G(n,p_{\mathrm{I}})[U]\,.$$

By the induction assumption, if $\varrho n \geq n_i$ and $p_i \geq C_i(\varrho n)^{-1/m_{F_i}}$ then, with high probability, there are many monochromatic copies of F_i in $G(U, p_{\mathrm{I}})$. For technical reasons that will become clear only later, we want to strengthen the above Ramsey property so that it is resilient to deletion of a small fraction of edges. For that we apply the induction assumption to the random graph $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$, followed by an application of Propositions 15. We begin by verifying the assumptions of Lemma 14 with respect to F_i and $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. First, note that

$$|U| = \varrho n \geq \varrho n_{i+1} \overset{(31)}{\geq} \frac{a_i^{4k}}{2^{13k^3}} n_{i+1} \overset{(24)}{=} \frac{a_i^{4k}}{2^{13k^3}} \cdot \frac{2^{14k^3}}{a_i^{4k}} n_i = 2^{k^3} n_i \geq n_i \,. \tag{36}$$

It remains to check that

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \geq C_i(\varrho n)^{-1/m_{F_i}} \,. \tag{37}$$

To this end, we simply note that using $\delta_{\mathrm{I}} \leq 1/2$, $\varrho \leq 1$, and $m_{F_{i+1}} \geq \max(1, m_{F_i})$ we have

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \overset{(30)}{\geq} \frac{\alpha p}{4} \geq \frac{\alpha}{4} C_{i+1} \varrho^{1/m_{F_{i+1}}} (\varrho n)^{-1/m_{F_{i+1}}} \geq \frac{\alpha}{4} C_{i+1} \varrho (\varrho n) n^{-1/m_{F_i}}.$$

Furthermore, we have

$$\frac{\alpha\varrho}{4} C_{i+1} \overset{(31)}{\geq} \frac{a_i^{36k^2+4k}}{2^{109k^4+13k^3+2}} \cdot C_{i+1} \overset{(24)}{=} \frac{a_i^{37k^2}}{2^{110k^4}} \cdot \frac{2^{122k^4} C_i}{b_i^4 a_i^{37k^2}} = \frac{2^{12k^2} C_i}{b_i^4} \geq C_i \,. \tag{38}$$

and (37) follows. Thus, we are in position to apply the induction assumption to $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$ and F_i. Let

$$\mu := \mu_{F_i}^{\varrho, \delta_{\mathrm{I}}} := \binom{\varrho n}{k} \frac{k!}{\mathrm{aut}(F_i)} ((1-\delta_{\mathrm{I}})p_{\mathrm{I}})^i \geq \frac{1}{4^{k^2}} (\varrho n)^k p_{\mathrm{I}}^i \tag{39}$$

denote the expected number of copies of F_i in $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. By Lemma 14 we infer that

$$\begin{aligned} \mathbb{P}\left(G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}}) \xrightarrow{a_i \mu} F_i \right) &\geq 1 - \exp\left(-b_i (1-\delta_{\mathrm{I}}) p_{\mathrm{I}} \tbinom{\varrho n}{2}\right) \\ &\geq 1 - \exp\left(-\tfrac{b_i}{2} p_{\mathrm{I}} \tbinom{\varrho n}{2}\right) \,. \end{aligned} \tag{40}$$

Next we head for an application of Proposition 15 with $c = b_i/2$, $\delta = \delta_{\mathrm{I}}$, $N = \binom{\varrho n}{2}$, and p_{I}. Note that, indeed, $\delta_{\mathrm{I}} = b_i^2/36 = c^2/9$ (see (34)). Moreover, using $\varrho n \geq 3$ (see (36)) and (32) we see that

$$p_{\mathrm{I}} \binom{\varrho n}{2} \overset{(30)}{\geq} \frac{\alpha p}{2} \cdot \varrho n \geq \frac{\alpha \varrho}{2} \cdot C_{i+1} \overset{(38)}{\geq} \frac{2^{12k^3+1}}{b_i^4} \geq \frac{72}{\delta_{\mathrm{I}}^2}$$

and the assumptions of Proposition 15 are verified. From (40) we infer by Proposition 15 that with probability at least

$$1 - \exp\left(-\frac{\delta_{\mathrm{I}}^2}{9} \binom{\varrho n}{2} p_{\mathrm{I}}\right) \tag{41}$$

$G(U,p_{\rm I})$ has the property that for every subgraph $G' \subseteq G(U,p_{\rm I})$ with

$$\big|E(G(U,p_{\rm I}))\setminus E(G')\big| \le \frac{\delta_{\rm I}}{2}\binom{\varrho n}{2}p_{\rm I}$$

we have

$$G' \xrightarrow{a_i\mu} F_i\,. \tag{42}$$

Recall our goal which is to show that, with high probability, any two-coloring χ of $G(U,p_{\rm I})$ yields at least $d(|U|^2/2)$ χ-rich edges, and ultimately, by repeating this argument for every set $U \subseteq [n]$ with ϱn vertices, that Γ_χ is (ϱ,d)-dense.

The Ramsey statement above asserts that with high probability there are many monochromatic copies of F_i contained in $G(U,p_{\rm I})$. If these copies were clustered at relatively few pairs, then we might fall short of the required lower bound on the number of χ-rich pairs. However, we will show that in the random graph $G(n,p_{\rm I})$ it is unlikely that many copies of F_i share the same χ-rich pair. For that we will consider the distribution of the graphs T consisting of two copies of F_i which share the vertices of a missing edge f (and possibly other vertices). We will require that the number of those copies is of the same order of magnitude as its expectation and that this holds with probability $1-\exp(-\Omega(p_{\rm I}\binom{|U|}{2}))$. Such a sharp concentration result is known to be false, but Proposition 16 asserts that it can be obtained on the cost of removing a few edges from $G(U,p_{\rm I})$. The above "robust" Ramsey property (42) means that after applying Proposition 16 to $G(U,p_{\rm I})$ the resulting subgraph of $G(U,p_{\rm I})$ will still have the Ramsey property with high probability.

In order to apply Proposition 16 we need first to define the graphs T and estimate their expectation in $G(U,p_{\rm I})$. Let $\{T_1,T_2,\dots,T_t\}$ be the family of all pairwise non-isomorphic graphs which are unions of two copies of F_i, say $F_i'\cup F_i''$, with the property that adding a single edge completes both, F_i' and F_i'' to a copy of F_{i+1}. We will refer to these graphs as *double creatures* (of F_i). Clearly,

$$t \le 2^{\binom{2k-2}{2}}\,. \tag{43}$$

Let X_j be the number of copies of T_j in $G(U,p_{\rm I})$, $j=1,\dots,t$.

Fact 20. *For every $j=1,\dots,t$*

$$\mathbb{E}X_j \le (\varrho n)^{2k-2}p_{\rm I}^{2i}\,.$$

Proof. Let $T := T_j = F_i'\cup F_i''$ be a double creature and set $S = F_i'\cap F_i''$. Then the expected number of copies of T is bounded from above by

$$\mathbb{E}X_T \overset{(25)}{\le} (\varrho n)^{v_T}p_{\rm I}^{e_T} = \frac{(\varrho n)^{2k}p_{\rm I}^{2i}}{(\varrho n)^{v_S}p_{\rm I}^{e_S}}\,,$$

and it remains to show that

$$(\varrho n)^{v_S}p_{\rm I}^{e_S} \ge (\varrho n)^2.$$

There is nothing to prove when $v_S = 2$ (and thus $e_S = 0$). Otherwise, pick a pair of vertices f in T such that both, $F_i' + f$ and $F_i'' + f$, are isomorphic to F_{i+1}. Then $J := S + f \subseteq F_{i+1}$. Note that $e_J = e_S + 1$ and $3 \le v_J = v_S \le k$. Since $C_{i+1} \ge 2/\alpha$,

$$(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S} \overset{(30)}{\ge} (\varrho n)^{v_J} \left(\frac{\alpha}{2}\right)^{e_S} p^{e_J-1} \overset{(33)}{\ge} \varrho^{v_S-2} \left(\frac{\alpha}{2}\right)^{e_S} C_{i+1}^{e_S} (\varrho n)^2$$
$$\ge \varrho^k \frac{\alpha}{2} C_{i+1} (\varrho n)^2 \overset{(31)}{\ge} \frac{1}{2} \frac{a_i^{4k^2}}{2^{13k^4}} \frac{a_i^{36k^2}}{2^{109k^4}} C_{i+1} (\varrho n)^2 \overset{(24)}{\ge} \frac{2^{13k^4-1}}{b_i^4} C_i (\varrho n)^2 \ge (\varrho n)^2 . \quad \square \quad (44)$$

Let Y be the random variable counting the number of double creatures in $G(U, p_{\mathrm{I}})$. It follows from Fact 20 that

$$\mathbb{E}Y \le t(\varrho n)^{2k-2} p_{\mathrm{I}}^{2i} . \tag{45}$$

Hence, by Proposition 16, applied for every $j = 1, \dots, t$ to the families $\mathcal{S}_j$ of all copies of T_j in $G(U, p_{\mathrm{I}})$ with

$$h_{\mathrm{I}} = \frac{\delta_{\mathrm{I}}}{2t} \binom{\varrho n}{2} p_{\mathrm{I}} \tag{46}$$

we conclude that with probability at least

$$1 - \sum_{j=1}^{t} \exp\left(-\frac{h_{\mathrm{I}}}{2e(T_j)}\right) \ge 1 - t \exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right) \tag{47}$$

there exists a subgraph $G_0 \subseteq G(U, p_{\mathrm{I}})$ with $|E(G(U, p_{\mathrm{I}}) \setminus E(G_0)| \le th_{\mathrm{I}}$ such that G_0 contains at most $2\mathbb{E}Y$ double creatures. Since

$$th_{\mathrm{I}} \overset{(46)}{=} \frac{\delta_{\mathrm{I}}}{2} \binom{\varrho n}{2} p_{\mathrm{I}},$$

the robust Ramsey property (42) holds with $G' = G_0$.

Recall that a two-coloring χ of $G(n, p_{\mathrm{I}})$ is fixed. For $\{u, v\} \subset U$, let x_{uv} be the number of monochromatic copies of F_i in G_0 which together with the pair $\{u, v\}$ form a copy of F_{i+1}. Owing to (42), we have

$$\sum_{\{u,v\} \in \binom{U}{2}} x_{uv} \ge a_i \mu. \tag{48}$$

By the above application of Proposition 16 we infer that

$$\sum_{\{u,v\} \in \binom{U}{2}} x_{uv}^2 \le 2 \cdot \binom{k}{2} \cdot |DC(G_0)| \le 4 \binom{k}{2} \mathbb{E}Y \overset{(45)}{\le} 4^{k^2-1} (\varrho n)^{2k-2} p_{\mathrm{I}}^{2i} , \tag{49}$$

where $DC(G_0)$ is the set of all double creatures in G_0. Recall that $\{u, v\} \in E(\Gamma_\chi)$ if it is χ-rich, which is implied by $x_{uv} \ge \ell$, where ℓ is defined in (35). We want to show that

$e(\Gamma_\chi[U]) \geq d(\varrho n)^2/2$. Since $\ell \leq a_i\mu/(\varrho n)^2$ (compare (35) and (39)), it follows from (48) that

$$\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq\ell}} x_{uv} \geq \frac{a_i\mu}{2} \overset{(39)}{\geq} \frac{1}{2}\cdot\frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\,.$$

Squaring the last inequality and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left(\frac{1}{2}\cdot\frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\right)^2 \leq \left(\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq\ell}} x_{uv}\right)^2 &\leq\; e(\Gamma_\chi[U]) \sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq\ell}} x_{uv}^2\\ &\overset{(49)}{\leq}\; e(\Gamma_\chi[U])\cdot 4^{k^2-1}(\varrho n)^{2k-2}p_{\mathrm{I}}^{2i}\,.\end{aligned}$$

Consequently,

$$e(\Gamma_\chi[U]) \geq \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \geq \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \overset{(28)}{=} d(\varrho n)^2/2\,.$$

Summarizing the above, we have shown that if $G(U,p_{\mathrm{I}})$ has the robust Ramsey property for F_i and if the conclusion of Proposition 16 holds for all $j = 1,\dots,t$, then $e(\Gamma_\chi[U]) \geq d(\varrho n)^2/2$. The probability that at least one of these events fails is at most (see (41) and (47))

$$\exp\left(-\frac{\delta_{\mathrm{I}}^2}{9}\binom{\varrho n}{2}p_{\mathrm{I}}\right) + t\exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right)\,.$$

Recalling that $t \leq 4^{k^2}$ (see (43)) and the definition of h_{I} in (46), Claim 19 now follows by summing up these probabilities over all choices of $U \subseteq [n]$ with $|U| = \varrho n$. More precisely, using the union bound and the estimate $\binom{n}{\varrho n} \leq 2^n$, we conclude that the probability that there is a coloring χ for which the graph Γ_χ is not (ϱ, d)-dense is

$$\begin{aligned}\mathbb{P}(\neg\mathcal{E}) \leq 2^n\exp\left(-\frac{1}{9}\delta_{\mathrm{I}}^2\binom{\varrho n}{2}p_{\mathrm{I}}\right) + 2^n 4^{k^2}\exp\left(-\frac{1}{k^2 4^{k^2}}\delta_{\mathrm{I}}\binom{\varrho n}{2}p_{\mathrm{I}}\right)&\\ \leq \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2\right)&\end{aligned}$$

□

This ends the analysis of the first round.

Let $\mathcal{B}$ be the conjunction of $\mathcal{E}$ and the event that $|G(n,p_{\mathrm{I}})| \leq n^2 p_{\mathrm{I}}$. In the second round we will condition on the event $\mathcal{B}$ and sum over all two-colorings χ of $G(n,p_{\mathrm{I}})$. Formally, let $\mathcal{A}$ be the (bad) event that there is a two-coloring of the edges of $G(n,p)$ with fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}. (That is, $\neg\mathcal{A}$ is the Ramsey property $G(n,p) \xrightarrow{a_{i+1}\mu_{F_{i+1}}} F_{i+1}$.) Further, given a two-coloring χ of $G(n,p_{\mathrm{I}})$, let $\mathcal{A}_\chi$ be the event that there exists an extension of χ to a coloring $\bar{\chi}$ of $G(n,p)$ yielding altogether fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}.

The following pair of inequalities exhibit the skeleton of our proof of Lemma 14:

$$\mathbb{P}(\mathcal{A}) \le \mathbb{P}(\neg\mathcal{B}) + \sum_{G\in\mathcal{B}} \mathbb{P}\Big(\mathcal{A}|G(n,p_{\mathrm{I}})=G\Big)\mathbb{P}\Big(G(n,p_{\mathrm{I}})=G\Big) \tag{50}$$

and

$$\mathbb{P}(\mathcal{A}|G(n,p_{\mathrm{I}})=G) = \mathbb{P}\left(\bigcup_{\chi}\mathcal{A}_\chi\middle|G(n,p_{\mathrm{I}})=G\right) \le 2^{n^2p_{\mathrm{I}}}\max_{\chi}\mathbb{P}(\mathcal{A}_\chi|G(n,p_{\mathrm{I}})=G). \tag{51}$$

By Claim 19 and Chernoff's inequality (see, e.g., [12, ineq. (2.5)])

$$\begin{aligned}\mathbb{P}(\neg\mathcal{B}) &\le \mathbb{P}(\neg\mathcal{E}) + \mathbb{P}\Big(|G(n,p_{\mathrm{I}})| > n^2p_{\mathrm{I}}\Big)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2\right) + \exp\left(-\frac{1}{3}\binom{n}{2}p_{\mathrm{I}}\right)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2 + 1\right) =: q_{\mathrm{I}}\,. \end{aligned}\tag{52}$$

To complete the proof of Lemma 14 it is crucial to find an upper bound on $\mathbb{P}(\mathcal{A}_\chi|G(n,p_{\mathrm{I}})=G)$ which substantially beats the factor $2^{n^2p_{\mathrm{I}}}$.

Claim 21. *For every $G\in\mathcal{B}$ and every two-coloring χ of G,*

$$\mathbb{P}(\mathcal{A}_\chi|G(n,p_{\mathrm{I}})=G) \le \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}}\right).$$

Proof. For a given two-coloring χ of $G(n,p_{\mathrm{I}})$, in the second round $G(n,p_{\mathrm{II}})$ we only focus on some of the χ-rich edges, namely, on a suitably selected subgraph of $(\Gamma_\chi)_{p_{\mathrm{II}}}$.

The edges of Γ_χ are naturally two-colored according to the majority color among the monochromatic copies of F_i attached to them. We color an edge of Γ_χ *pink* if it closes at least $\ell/2$ red copies of F_i and we color it *azure* otherwise. Subsequently, we apply Corollary 18 to Γ_χ for F_{i+1} and d (chosen in (28)). Note that in (28) we chose ϱ to facilitate such an application. Moreover, the required lower bound on n is equivalent to $\varrho n \ge 1$ and this follows from (36). Hence, by Corollary 18 and the choice of γ in (28), we may assume without loss of generality, that there are at least $\gamma n^k/2$ pink copies of F_{i+1} in Γ_χ. In particular, all these copies of F_{i+1} consist entirely of edges closing each at least $\ell/2$ red copies of F_i (from the first round). Let us denote by $\mathcal{F}_\chi$ the family of these copies of F_{i+1}, and let $\Gamma_\chi^{\text{pink}}$ be the subgraph of Γ_χ containing the pink edges. Since every edge may belong to at most n^{k-2} copies of F_{i+1}, we have

$$e(\Gamma_\chi^{\text{pink}}) \ge \frac{(i+1)\cdot|\mathcal{F}_\chi|}{n^{k-2}} \ge \frac{(i+1)\cdot\gamma n^k/2}{n^{k-2}} \ge \gamma n^2\,. \tag{53}$$

In the analysis of the second round we intend to use again Proposition 15, this time with $\Gamma = \Gamma_\chi^{\text{pink}}$ and $\mathcal{Q}$ – the property of containing at least

$$\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1} \tag{54}$$

copies of F_{i+1} belonging to $\mathcal{F}_\chi$.

Fact 22. *With δ_{II} chosen in (28),*

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \le \exp\left(-\frac{\gamma^2}{4^{k^2}} e(\Gamma_\chi^{\text{pink}}) p_{\text{II}}\right).$$

Proof. Consider a random variable Z counting the number of copies F_{i+1} belonging to $\mathcal{F}_\chi$ which are subgraphs of $G(n,(1-\delta_{\text{II}})p_{\text{II}})$. We have

$$\mathbb{E}Z = |\mathcal{F}_\chi|((1-\delta_{\text{II}})p_{\text{II}})^{i+1} \ge \frac{1}{2}\gamma n^k ((1-\delta_{\text{II}})p_{\text{II}})^{i+1} \ge \frac{1}{2} \cdot \frac{1}{2^{\binom{k}{2}}} \gamma n^k p_{\text{II}}^{i+1}, \tag{55}$$

where we used the bound $\delta_{\text{II}} \le 1/2$.

By Janson's inequality (see, e.g., [12, Theorem 2.14]),

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \le \mathbb{P}\left(Z \le \frac{1}{2}\mathbb{E}Z\right) \le \exp\left(-\frac{(\mathbb{E}Z)^2}{8\bar{\Delta}}\right),$$

where

$$\bar{\Delta} = \sum_{F' \in \mathcal{F}_\chi} \sum_{F'' \in \mathcal{F}_\chi} \mathbb{P}(F' \cup F'' \subseteq G(n,(1-\delta_{\text{II}})p_{\text{II}})),$$

with the double sum ranging over all ordered pairs $(F', F'') \in \mathcal{F}_\chi \times \mathcal{F}_\chi$ with $E(F') \cap E(F'') \neq \emptyset$. The quantity $\bar{\Delta}$ can be bounded from above by

$$\bar{\Delta} \le \sum_{\widetilde{F} \subseteq F_{i+1}} n^{2k - v(\widetilde{F})} p_{\text{II}}^{2(i+1)-e(\widetilde{F})}, \tag{56}$$

where the sum is taken over all subgraphs $\widetilde{F}$ of F_{i+1} with at least one edge. If $e(\widetilde{F}) = 1$ then

$$n^{v(\widetilde{F})} p_{\text{II}}^{e(\widetilde{F})} = n^{v(\widetilde{F})} p_{\text{II}} \ge n^2 p_{\text{II}}. \tag{57}$$

Otherwise,

$$n^{v(\widetilde{F})} p_{\text{II}}^{e(\widetilde{F})} \ge \frac{n^{v(\widetilde{F})} p^{e(\widetilde{F})}}{2^{e(\widetilde{F})}} \overset{(33)}{\ge} \frac{n^2 p C_{i+1}^{e(\widetilde{F})-1}}{2^{e(\widetilde{F})}} \ge n^2 p \overset{(30)}{\ge} n^2 p_{\text{II}}, \tag{58}$$

where we also used the fact that $C_{i+1} \ge 4$ (see (24)). Combining (56) with the bounds (57) and (58) yields

$$\bar{\Delta} \le 2^{i+1} n^{2k-2} p_{\text{II}}^{2i+1} \le 2^{\binom{k}{2}} n^{2k-2} p_{\text{II}}^{2i+1}.$$

Finally, plugging this estimate for $\bar{\Delta}$ and (55) into Janson's inequality we obtain

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \leq \exp\left(-\frac{\gamma^2 n^2 p_{\text{II}}}{32 \cdot 2^{2\binom{k}{2}} \cdot 2^{\binom{k}{2}}}\right) \leq \exp\left(-\frac{\gamma^2}{4^{k^2}} e(\Gamma_\chi^{\text{pink}}) p_{\text{II}}\right). \quad \square$$

We plan to apply Proposition 15 with $c = \gamma^2/4^{k^2}$, $\delta_{\text{II}} = \gamma^4/(9 \cdot 16^{k^2})$ (see (28)), $N = e(\Gamma_\chi^{\text{pink}})$, and p_{II}. Therefore, first we have to verify that $e(\Gamma_\chi^{\text{pink}})p_{\text{II}} \geq 72/\delta_{\text{II}}^2$. Indeed,

$$e(\Gamma_\chi^{\text{pink}}) \cdot p_{\text{II}} \overset{(30,53)}{\geq} \gamma n^2 \cdot \frac{p}{2} \overset{(32)}{\geq} \frac{\gamma}{2} n C_{i+1} \overset{(24)}{\geq} \frac{\gamma}{2} \cdot \frac{2^{122k^4}}{a_i^{37k^3}} \overset{(31)}{\geq} \frac{72 \cdot 81 \cdot 16^{2k^2}}{\gamma^8} = \frac{72}{\delta_{\text{II}}^2}.$$

Consequently, by Proposition 15, we conclude that with probability at least

$$1 - \exp\left(-\frac{\delta_{\text{II}}^2}{9} e(\Gamma_\chi^{\text{pink}}) p_{\text{II}}\right) \overset{(53)}{\geq} 1 - \exp\left(-\frac{\delta_{\text{II}}^2 \gamma}{9} n^2 p_{\text{II}}\right), \tag{59}$$

the random graph $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ has the property that for every subgraph $\Gamma' \subseteq (\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ with

$$\big|E((\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}) \setminus E(\Gamma')\big| \leq \frac{\delta_{\text{II}}\gamma}{2} n^2 p_{\text{II}} =: h_{\text{II}} \tag{60}$$

we have $\Gamma' \in \mathcal{Q}$, that is, Γ' contains at least $\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1}$ copies of F_{i+1} belonging to $\mathcal{F}_\chi$ (see (54)).

Consider now an extension $\bar{\chi}$ of the coloring χ from $G(n, p_{\text{I}})$ to $G(n,p)$. If in the coloring $\bar{\chi}$ fewer than h_{II} edges of $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ are colored red, then, by the above consequence of Proposition 15, the blue part of $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ contains at least

$$\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1} \overset{(30)}{\geq} \frac{\gamma}{4^{k^2}} n^k p^{i+1}$$

copies of F_{i+1}. If, on the other hand, more than h_{II} edges of $(\Gamma_\chi^{pink})_{p_{\text{II}}}$ are colored red, then, by the definition of a pink edge, noting that $i \leq k^2/2$, at least

$$\begin{aligned} h_{\text{II}} \times \frac{\ell}{2} \times \frac{1}{i+1} &\overset{(35,60)}{\geq} \frac{\delta_{\text{II}}\gamma}{2} n^2 p_{\text{II}} \times \frac{a_i}{4^{k^2} k^2} (\varrho n)^{k-2} p_{\text{I}}^i \\ &\overset{(30)}{\geq} \frac{\delta_{\text{II}}\gamma}{4} n^2 p \times \frac{a_i \varrho^k}{4^{k^2} k^2} \left(\frac{\alpha}{2}\right)^i n^{k-2} p^i \\ &\geq \frac{\delta_{\text{II}}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} n^k p^{i+1} \end{aligned}$$

red copies of F_{i+1} arise. Owing to (28), (31), and the choice of a_{i+1} in (24) we have

$$\frac{\gamma}{4^{k^2}} \overset{(31)}{\geq} \frac{a_i^{4k^2}}{2^{13k^4+2k^2}} \overset{(24)}{\geq} a_{i+1}$$

and

$$\frac{\delta_{\rm II}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} \overset{(28)}{=} \frac{\gamma^5 \varrho^k \alpha^{k^2/2}}{9\cdot 2^{8k^2}} a_i \overset{(31)}{\geq} \frac{a_i^{18k^4+24k^2}}{2^{55k^6}} \overset{(24)}{\geq} a_{i+1}\,.$$

Therefore, we have shown that with probability as in (59), indeed any extension $\bar{\chi}$ of χ yields at least

$$\min\left(\frac{\gamma}{4^{k^2}}\,,\ \frac{\delta_{\rm II}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} n^k p^{i+1}\right) \geq a_{i+1} n^k p^{i+1} \overset{(25)}{\geq} a_{i+1}\mu_{F_{i+1}}$$

monochromatic copies of F_{i+1}. This finishes the proof of Claim 21. $\square$

To finish the proof of Lemma 14 it is left to verify that indeed $\mathbb{P}(\mathcal{A}) \leq \exp(-b_{i+1}\binom{n}{2}p)$. The error probability of the first round is (see (52))

$$\mathbb{P}(\neg\mathcal{B}) \leq q_{\rm I}.$$

Turning to the second round, by Claim 21 and (51), for any $G \in \mathcal{B}$,

$$\mathbb{P}(\mathcal{A}|G(n,p_{\rm I}) = G) \leq 2^{n^2 p_{\rm I}} \cdot \exp\left(-\frac{\delta_{\rm II}^2\gamma}{9} n^2 p_{\rm II}\right) \leq \exp\left(-\frac{\delta_{\rm II}^2\gamma}{9} n^2 p_{\rm II} + n^2 p_{\rm I}\right) =: q_{\rm II}\,, \quad (61)$$

and, consequently, by (50),

$$\mathbb{P}(\mathcal{A}) \leq q_{\rm I} + q_{\rm II}.$$

Below we show (see Fact 23) that $q_{\rm I}$ and $q_{\rm II}$ are each upper bounded by $\exp(-b_{i+1}n^2p)$. Consequently,

$$\mathbb{P}(\mathcal{A}) \leq 2\exp(-b_{i+1}n^2p) \leq \exp(1 - b_{i+1}n^2p) \leq \exp(-\tfrac{b_{i+1}}{2}n^2p) \leq \exp(-b_{i+1}\tbinom{n}{2}p)$$

because

$$\frac{b_{i+1}}{2} n^2 p \overset{(32)}{\geq} \frac{b_{i+1}}{2} C_{i+1} n_{i+1} \overset{(24)}{\geq} C_i n_{i+1} \geq 1\,.$$

Fact 23.

$$\max(q_{\rm I}, q_{\rm II}) \leq \exp(-b_{i+1}n^2p)$$

Proof. We first bound $q_{\rm I}$. Since $\varrho n \geq 3$ (see (36)),

$$\frac{\delta_{\rm I}^2}{16^{k^2}}\binom{\varrho n}{2} p_{\rm I} \overset{(30)}{\geq} \frac{\delta_{\rm I}^2 \varrho^2 \alpha}{16^{k^2}\cdot 6} n^2 p \overset{(31,34)}{\geq} \frac{b_i^4 a_i^{36k^2+8k}}{6^5\cdot 2^{109k^4+26k^3+4k^2}} n p^2 \overset{(24)}{\geq} 2b_{i+1}n^2p$$

while, since $i+1 \geq 2$,

$$n + 2k^2 + 1 \leq n + n_{i+1} \leq 2n \overset{(24)}{\leq} b_{i+1}C_{i+1}n \overset{(32)}{\leq} b_{i+1}n^2p\,.$$

Consequently,

$$q_{\rm I} \leq \exp(-2b_{i+1}n^2p + b_{i+1}n^2p) = \exp(-b_{i+1}n^2p)\,.$$

Now we derive the same upper bound for q_{II}. Since

$$p_{\mathrm{I}} \overset{(30)}{\leq} \alpha p \overset{(28)}{=} \frac{\delta_{\mathrm{II}}^2 \gamma}{36} p$$

while $p_{\mathrm{II}} \overset{(30)}{\geq} p/2$,

$$q_{\mathrm{II}} = \exp\left(-\frac{\delta_{\mathrm{II}}^2 \gamma}{9} n^2 p_{\mathrm{II}} + n^2 p_{\mathrm{I}}\right) \leq \exp\left(-\frac{\delta_{\mathrm{II}}^2 \gamma}{36} n^2 p\right).$$

Therefore, the required bound follows from

$$\frac{\delta_{\mathrm{II}}^2 \gamma}{36} \overset{(28)}{=} \frac{\gamma^9}{36 \cdot 81 \cdot 16^{2k^2}} \overset{(31)}{\geq} \frac{a_i^{36k^2}}{2^{118k^4}} \overset{(24)}{\geq} b_{i+1}. \quad \square$$

This concludes the proof of the induction step, i.e., the proof of Lemma 14 for F_{i+1}. The proof of Lemma 14 is thus completed.

3.3 Proof of Theorem 2

In order to deduce Theorem 2 from Lemma 14, we first need to estimate the parameters a_i, b_i, C_i, n_i, $i = 1, \ldots, \binom{k}{2}$, defined recursively in (24).

Proposition 24. *There exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that for every $k \geq 3$*

$$a_{K_k} \geq 2^{-k^{(c_1 \cdot k^2)}} \qquad b_{K_k} \geq 2^{-k^{(c_2 \cdot k^2)}} \qquad C_{K_k} \leq 2^{k^{(c_3 \cdot k^2)}} \qquad n_{K_k} \leq 2^{k^{(c_4 \cdot k^2)}}.$$

Proof. Throughout the proof we assume that $k \geq k_0$ for some sufficiently large constant k_0. Let $x = 19k^4$, $y = 55k^6$, and set $\alpha_i = \log a_i$, $i = 1, \ldots, \binom{k}{2}$. Recall that $a_1 = \frac{1}{2}$. The recurrence relation (24) becomes now

$$\alpha_i = x\alpha_{i-1} - y,$$

whose solution can be easily found as

$$\alpha_i = -x^{i-1} - y\frac{x^{i-1} - 1}{x - 1}$$

(note that $\alpha_1 = -1$). Hence, for all $i = 1, \ldots, \binom{k}{2}$, and some constant $c_1 > 0$,

$$-\alpha_i = x^{i-1} + y\frac{x^{i-1} - 1}{x - 1} \leq k^{c_1 \cdot i}. \tag{62}$$

In particular,

$$a_{\binom{k}{2}} \geq 2^{-k^{c_1 \cdot \binom{k}{2}}} \geq 2^{-k^{(c_1 \cdot k^2)}}.$$

The recurrence relation for the b_i's is more complex. With $u = 37k^2$ and $v = 118k^4$, it reads as

$$b_i = b_{i-1}^4 a_{i-1}^u 2^{-v}.$$

Thus, recalling that $b_1 = \frac{1}{8}$,

$$b_i 8^{4^{i-1}} = \prod_{j=2}^{i} \left(\frac{b_j}{b_{j-1}^4} \right)^{4^{i-j}} = \prod_{j=2}^{i} \left(a_j^u 2^{-v} \right)^{4^{i-j}}.$$

Setting, $\beta_i = \log b_i$, and taking logarithms of both sides and using (62) we obtain, for some constant $c_2 > 0$,

$$\begin{aligned} -\beta_i = 3 \cdot 4^{i-1} + \sum_{j=2}^{i} 4^{i-j} \left(u(-\alpha_j) + v \right) &\leq 4^i + (i-1)4^{i-2} \left(u(-\alpha_i) + v \right) \\ &\leq 4^i \left[1 + i \left(uk^{(c_1 \cdot i)} + v \right) \right] \leq k^{c_2 \cdot i}, \end{aligned} \tag{63}$$

where in the last step above we used estimates $4^i \leq k^{2i}$ and $i \leq k^2$. In particular,

$$b_{\binom{k}{2}} \geq 2^{-k^{(c_2 \cdot k^2)}}.$$

The recurrence relation for C_i involves not only C_{i-1} and a_{i-1} but also b_{i-1}. Nevertheless, its solution follows the steps of that for b_i. Indeed, we have

$$\frac{C_i}{C_{i-1}} = \frac{2^z}{b_{i-1}^4 a_{i-1}^w},$$

where $z = 122k^4$ and $w = 37k^2$. Recalling that $C_1 = 1$,

$$C_i = \prod_{j=2}^{i} \frac{C_j}{C_{j-1}} = \prod_{j=2}^{i} \frac{2^z}{b_{j-1}^4 a_{j-1}^w}$$

and, consequently, by (62) and (63), for some constant $c_3 > 0$,

$$\begin{aligned} \log C_i \leq (i-1)z + \sum_{j=2}^{i} \left(4(-\beta_j) + w(-\alpha_j) \right) &\leq (i-1) \left(z + 4(-\beta_i) + w(-\alpha_i) \right) \\ &\leq k^2 \left(z + 4k^{(c_2 \cdot i)} + wk^{(c_1 \cdot i)} \right) \leq k^{c_3 \cdot i}. \end{aligned}$$

In particular,

$$C_{\binom{k}{2}} \leq 2^{k^{(c_3 \cdot k^2)}}.$$

Similarly, for some constant $c_4 > 0$,

$$n_i = \prod_{j=2}^{i} \frac{n_j}{n_{j-1}} = \prod_{j=2}^{i} \frac{2^{14k^3}}{a_{j-1}^{4k}} \leq 2^{k^{(c_4 \cdot i)}}$$

and, consequently,

$$n_{\binom{k}{2}} \le 2^{k^{(c_4 \cdot k^2)}}. \quad \square$$

Now, we are ready to complete the proof of Theorem 2 which, in its final steps, reminds the proof of Theorem 1. For convenience, set $\bar{b} = b_{\binom{k}{2}}$, $\bar{C} = C_{\binom{k}{2}}$, and $\bar{n} = n_{\binom{k}{2}}$. Let $n \ge \bar{n}$ and $p = \bar{C} n^{-2/(k+1)}$. By Lemma 14

$$\mathbb{P}(G(n,p) \to K_k) \ge 1 - \exp\left\{-\bar{b}p\binom{n}{2}\right\}.$$

Let, in addition, $n \ge (2\bar{C})^{(k+1)/2}$ Then, by Lemma 4,

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left\{-\bar{b}p\binom{n}{2}\right\}$$

and, in turn,

$$\mathbb{P}(G(n,p) \to K_k \text{ and } G(n,p) \not\supset K_{k+1}) > 0.$$

Consequently, for every

$$n \ge n_0 := \max(\bar{n}, (2\bar{C})^{(k+1)/2})$$

there exists a $(k;2)$-Folkman graph with n vertices. Finally, by Proposition 24, there exists $c > 0$ such that $n_0 \le 2^{k^{c \cdot k^2}}$. This way we have proved that $f(k) \le n_0 \le 2^{k^{c \cdot k^2}}$.

4 Relaxed Folkman numbers

In this section we prove Theorem 3. Let $n = 2^{4k/(1-4\alpha)}$. Consider a random graph $G(n,p)$ where

$$p = 2n^{-\frac{7+4\alpha}{16k}} = 2^{-\frac{20\alpha+3}{4(1-4\alpha)}}.$$

By elementary estimates one can bound the expected number of l-cliques in $G(n,p)$ by

$$\left(\frac{en}{l} p^{\frac{l-1}{2}}\right)^l.$$

Thus, if

$$\frac{l-1}{2} \ge \frac{\log n}{\log(1/p)} = \frac{16k}{20\alpha+3}$$

then, as $k \to \infty$, a.a.s. there are no l-cliques in $G(n,p)$. By assumption,

$$\frac{l-1}{2} \ge \frac{k-\alpha}{2\alpha} \ge \frac{16k}{20\alpha+3},$$

where the last inequality, equivalent to $(3-12\alpha)k \ge 20\alpha^2 + 3\alpha$, holds if $k \ge \frac{2}{3(1-4\alpha)}$ (we used here the assumption that $\alpha < \frac{1}{4}$).

Further, by a straightforward application of Chernoff's bound (see, e.g., [12, ineq. (2.6)]), a.a.s. $G(n,p)$ is $(\varrho, p-o(p))$-dense, where $\varrho = \frac{\log^2 n}{n}$, say. Indeed, setting $t = \varrho n = \log^2 n$, $\epsilon = \epsilon(n) = (\log n)^{-1/3}$, and $d = (1-\epsilon)p$, the probability that a fixed set T of t vertices spans in $G(n,p)$ fewer than $dt^2/2$ edges is at most

$$\mathbb{P}(e(T) \le (1-\epsilon)pt^2/2) \le \mathbb{P}\left(e(T) \le (1-\epsilon/2)p\binom{t}{2}\right) \le \exp\left\{-\frac{\epsilon^2}{8}p\binom{t}{2}\right\} \le \exp\left\{-\frac{\epsilon^2}{24}pt^2\right\}.$$

Finally, note that the above bound, even multiplied by $\binom{n}{t}$, the number of all t-element subsets of vertices in $G(n,p)$, still converges to zero (recall that p is a constant).

Using that $\epsilon k = O(\log^{2/3} n)$ one can easily verify that both assumptions of Corollary 18(a), that is, $n \ge (2/d)^{2k}$ and $\varrho \le (d/2)^{2k}$, hold true. Indeed,

$$(d/2)^{2k} = (1-\epsilon)^{2k} n^{-1+\delta} \ge \varrho \ge \frac{1}{n},$$

for n large enough, that is, for k large enough.

In conclusion, a.a.s. $G(n,p)$ is such that

- it contains no K_l, and
- by Corollary 18(a), for every two-coloring of its edges, there is a monochromatic copy of K_k.

Hence, there exists an n-vertex graph with the above two properties and, consequently, $f(k,l) \le n = 2^{4k/(1-4\alpha)}$.

5 Hypergraph Folkman numbers

Hypergraph Folkman numbers are defined in an analogous way to their graph counterparts. Given three integers h, k, and r, the h-uniform Folkman number $f_h(k;r)$ is the minimum number of vertices in an h-uniform hypergraph H such that $H \to (K_k^{(h)})_r$ but $H \not\supset K_{k+1}^{(h)}$. Here $K_k^{(h)}$ stands for the complete h-uniform hypergraph on k vertices, that is, one with $\binom{k}{h}$ edges. The finiteness of hypergraph Folkman numbers was proved by Nešetřil and Rödl in [14, Colloary 6, page 206] and besides the gigantic upper bound stemming from their construction, no reasonable bounds have been proven so far. Much better understood are the vertex-Folkman numbers (where instead of edges, the vertices are colored), which for both, graphs and hypergraphs, are bounded from above by an almost quadratic function of k, while from below the bound is only linear in k (see [6, 4].

The study of Ramsey properties of random hypergraphs began in [17] where a threshold was found for $K_4^{(3)}$, the 3-uniform clique on 4 vertices. Also there a general conjecture was stated that a theorem analogous to Theorem 13 holds for hypergraphs too. This was confirmed for h-uniform hypergraphs in [18], and, finally, for all h-partite hypergraphs in [9] and, independently, in [3].

We believe that our approach based on the Saxton-Thomasson (or the Balogh-Morris-Samotij) theorem should also yield a much simpler proof of the hypergraph Ramsey threshold theorem from [9, 3] and, in effect, provide an upper bound on $f_h(k;r)$ exponential in a polynomial function of k and r.

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