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On Some Open Questions for Ramsey and Folkman Numbers

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Abstract

We discuss some of our favorite open questions about Ramsey numbers and a related problem on edge Folkman numbers. For the classical two-color Ramsey numbers we first focus on constructive bounds for the difference between consecutive Ramsey numbers. We present the history of progress on the Ramsey number $R(5,5)$ and discuss the conjecture that it is equal to 43. For the multicolor Ramsey numbers we focus on the growth of $R_r(k)$, in particular for $k = 3$. Two concrete conjectured cases, $R(3,3,3,3) = 51$ and $R(3,3,4) = 30$, are discussed in some detail. For Folkman numbers, we present the history, recent developments and potential future progress on $F_e(3,3;4)$, defined as the smallest number of vertices in any K_4-free graph which is not a union of two triangle-free graphs. Although several problems discussed in this paper are concerned with concrete cases, and some involve significant computational approaches, there are interesting and important theoretical questions behind each of them.

Keywords: Ramsey numbers, Folkman numbers
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1 Introduction and Notation

In 2005, Arnold [4] wrote: *From the deductive mathematics point of view most of these results are not theorems, being only descriptions of several millions of particular observations. However, I hope that they are even more important than the formal deductions from the formal axioms, providing new points of view on difficult problems where no other approaches are that efficient.* The paper appeared in the *Journal of Mathematical Fluid Mechanics*, and it has not much to do with Ramsey theory. Yet, the motivation of our paper is somewhat similar in that we may seem to focus much on concrete cases. Although several problems in this paper are concerned with concrete cases, and some involve significant computational approaches, there are interesting and important theoretical questions behind each of them.

We obviously also try to look further for general results, but we don't want to skip observing what is happening with the basic small open cases. Understanding them better may lead to surprising general conclusions. For example, our work on an old construction for $R_4(6)$ a decade ago, recently led to interesting general connections between our methods and the Shannon capacity [78] discussed in Section 3.1.

The standard reference for Ramsey theory is a great book by Graham, Rothschild and Spencer [39], *"Ramsey Theory"*. The subject first concerned mathematical logic, but over the years found its way into several areas of mathematics, computing, and other fields. For the discussion of numerous applications see the survey paper by Rosta [66], and a very useful website by Gasarch [31]. There is also a colorful book by Soifer [72] on the history and results in Ramsey theory, followed by a collection of essays and technical papers based on presentations from the 2009 Ramsey theory workshop at DIMACS [73]. A regularly updated survey of the most recent results on the best known bounds on various types of Ramsey numbers is maintained by the second author [63].

The most important operation involved in the concept of Ramsey and Folkman numbers is that of arrowing, which is defined as follows.

Definition 1.1 Arrowing
Graph F arrows graphs $G_1, ..., G_r$, written $F \rightarrow (G_1, ..., G_r)$, if and only if every r-coloring of the edges of F contains a monochromatic copy of G_{s_i} in color i, for some $1 \le i \le r$.

The definition of the classical two-color *Ramsey numbers* can be stated in terms of the arrowing relation as $R(s,t) = \min\{n \mid K_n \rightarrow (K_s, K_t)\}$, with a straightforward generalization for more colors and noncomplete graphs. If all graphs G_i are the same G, we will use notation $R_r(G)$ for $R(G_1, ..., G_r)$, and if the graphs G_i are complete we will write $s_i = |V(G_i)|$ instead of G_i. So for example, $R(5,5) = R(K_5, K_5)$, and $R_r(3)$ is the smallest n such that the r-color arrowing $K_n \rightarrow (K_3, ..., K_3)$ holds. The latter two cases are discussed in detail in Sections 2.2 and 3, respectively.

Ramsey proved a theorem which implies the following.

Theorem 1.2 Ramsey 1930 [65]
For $r \ge 1$ and all graphs $G_1, ..., G_r$, the Ramsey number $R(G_1, ..., G_r)$ exists.

Any edge r-coloring witnessing $K_n \not\rightarrow (G_1, ..., G_r)$ will be called a $(G_1, ..., G_r)$- or $(G_1, ..., G_r; n)$-coloring. Clearly, constructing any $(G_1, ..., G_r; n)$-coloring implies a lower bound $n < R(G_1, ..., G_r)$.

In the case of two colors, we will talk about (G, H)- and $(G, H; n)$-graphs, which are simply $(G, H; n)$-colorings of K_n, where the first color is interpreted as the graph, while the second as its complement. Let $\alpha(F)$ denote the independence number of graph F. Note that $R(s,t)$ can be defined equivalently as the smallest integer n such that every graph on n vertices contains K_s or has independence $\alpha(F) \ge t$. An (s,t)-graph G will be called *Ramsey-critical* (for (s,t)) if it has $R(s,t) - 1$ vertices, i.e. it is an $(s,t; R(s,t) - 1)$-graph. $\delta(G)$ and $\Delta(G)$ will denote the minimum and maximum degree in G, respectively, and $K_n - e$ is the complete graph on n vertices with one edge removed. We will sometimes write $n(G) = |V(G)|$ for the number of vertices in G.

Next, we define the set of *edge Folkman graphs* by

$$\mathcal{F}_e(s,t;k) = \{F \mid F \rightarrow (s,t) \text{ and } K_k \not\subseteq F\}.$$

Then, the corresponding *edge Folkman number* $F_e(s,t;k)$ is the smallest order $n(F)$ of any graph F in $\mathcal{F}_e(s,t;k)$. Folkman proved that these graphs exist, as follows.

Theorem 1.3 Folkman 1970 [28]
If $k > \max(s,t)$, then $F_e(s,t;k)$ exist.

Edge Folkman numbers have obvious generalizations to arrowing graphs other than complete, and to more colors as in $\mathcal{F}_e(s_1, ..., s_r; k)$ and $F_e(s_1, ..., s_r; k)$ [58]. One can also color vertices instead of edges, which leads to the so-called *vertex Folkman numbers.* In general, much less is known about edge Folkman numbers than for more studied vertex Folkman numbers [17]. Here, however, we will discuss only the case of $F_e(3, 3; k)$, in Section 4.

The problem of deciding whether a graph F arrows triangles, that is whether $G \rightarrow (3, 3)$, is of particular interest in Ramsey theory. This is **coNP**-complete, and it appeared in the classical complexity text by Garey and Johnson in 1979 [32]. Some related Ramsey graph coloring problems are **NP**-hard or lie even higher in the polynomial hierarchy. For example, Burr [8, 9] showed that arrowing $(3, 3)$ is **coNP**-complete together with other results about arrowing, and Schaefer [68] showed that for general graphs F, G, and H, $F \rightarrow (G, H)$ is $\mathbf{\Pi_2^P}$-complete.

2 Two-color Ramsey Numbers

2.1 Difference and Connectivity

The estimates of the difference between consecutive (in various meanings) Ramsey numbers are difficult. What we know, most of the time gives weaker bounds from what we seem to observe.

Problem 2.1.1 Erdős-Sós 1980 [19, 13]
Let $\Delta_k = R(3, k) - R(3, k-1)$. Is it true that

$$\Delta_k \xrightarrow{k} \infty \ ? \qquad \Delta_k/k \xrightarrow{k} 0 \ ?$$

Only easy bounds $3 \leq \Delta_k \leq k$ are known. The upper bound k is obvious since the maximum degree of $(3, k)$-graphs is at most $k - 1$. The lower bound of 3 looks misleadingly simple, it is not trivial (Theorems 2.1.2 and 2.1.3 imply it). It was argued in [35] that better understanding of Δ_k may come from the study of $R(K_3, K_k - e)$ relative to $R(K_3, K_k) = R(3, k)$, since

$$\Delta_k = \big(R(K_3, K_k) - R(K_3, K_k - e)\big) + \big(R(K_3, K_k - e) - R(K_3, K_{k-1})\big).$$

Recent progress on what we know for small cases is significant [34, 35], however still some very simple-looking questions remain open. For example, we do not even know for certain whether $R(K_3, K_k - e) - R(K_3, K_{k-1})$ is positive for all large k.

The following three theorems were proved by constructive methods as parts of Theorems 2 and 3 in [81], and Theorem 9 in [80].

Theorem 2.1.2 [81] *Given a (k,s)-graph G and a (k,t)-graph H, for some $k \geq 3$ and $s,t \geq 2$, if both G and H contain an induced subgraph isomorphic to some K_{k-1}-free graph M, then $R(k,s+t-1) \geq n(G)+n(H)+n(M)+1$.*

Theorem 2.1.3 [81] *If $2 \leq s \leq t$ and $k \geq 3$, then*

$$R(k,s+t-1) \geq R(k,s)+R(k,t)+\begin{cases} k-3, & if\ s=2; \\ k-2, & if\ s \geq 3. \end{cases}$$

The first inequality of Theorem 2.1.3 for $s=2$, $R(k,t+1) \geq R(k,t)+2k-3$, was proved by Burr et al. in 1989 [10].

Theorem 2.1.4 [80] *If $k \geq 2, s \geq 5$, then $R(2k-1,s) \geq 4R(k,s-1)-3$.*

We think that the progress on constructive lower bounds illustrated in Theorems 2.1.2–2.1.4 is quite representative for the area, but it seems slow. Much slower than it was once anticipated by Erdős, Faudree, Schelp and Rousseau. In 1980, Paul Erdős wrote in [19], page 11 (using r for our R): *Faudree, Schelp, Rousseau and I needed recently a lemma stating*

$$\lim_{n\to\infty} \frac{r(n+1,n)-r(n,n)}{n} = \infty \qquad (a)$$

We could prove (a) without much difficulty, but could not prove that $r(n+1,n)-r(n,n)$ increases faster than any polynomial of n. We of course expect

$$\lim_{n\to\infty} \frac{r(n+1,n)}{r(n,n)} = C^{\frac{1}{2}}, \qquad (b)$$

where $C = \lim_{n\to\infty} r(n,n)^{1/n}$.

Based on the above theorems and considerations in [79], the best known lower bound estimate for the difference in (a) seems to be barely $\Omega(n)$. Asking others, including collaborators of Erdős, did not lead us to any proof of this result, leaving however some possibility that Erdős knew it. In summary, we think that it is reasonable to consider (a) to be only a conjecture.

Beveridge and Pikhurko in [5], using Theorem 2.1.3, proved that the connectivity of any $(k,s;R(k,s)-1)$-graph, i.e. Ramsey-critical (k,s)-graph, is at least $k-1$ for all $k,s \geq 3$. In Theorem 8 of [79], we increased this bound on connectivity to k for $k \geq 5$, and then we obtained further results about which Ramsey-critical graphs must be Hamiltonian.

Theorem 2.1.5 [79] *If $k \geq 5$ and $s \geq 3$, then the connectivity of any $(k,s;R(k,s)-1)$-graph is at least k. Furthermore, if $k \geq s-1 \geq 1$ and $k \geq 3$, except $(k,s)=(3,2)$, then any $(k,s;R(k,s)-1)$-graph is Hamiltonian.*

In particular, all diagonal Ramsey-critical (k,k)-graphs are Hamiltonian for every $k \geq 3$. It remains an open question for which k and s, when $3 \leq k < s-1$, Ramsey-critical (k,s)-graphs are still Hamiltonian. We think that the answer is positive at least in the cases when $s-k$ is small.

Conjecture 2.1.6 *For all $k \geq 2$, there exists a Ramsey-critical $(k+1,k)$-graph with maximum degree at least $R(k+1,k)/2-1$.*

This conjecture seems weak, but we still have no idea how to prove or disprove it. Many would even readily agree with an intuition that any Ramsey-critical $(k+1,k)$-graph G satisfies the bound $\Delta(G) \geq |V(G)|/2$. On the other hand, we clearly have $\Delta(G) < R(k,k)$. Putting it together with the classical bound $R(k+1,k+1) \leq 2R(k+1,k)$, we propose the next conjecture.

Conjecture 2.1.7 *$R(k+1,k) \leq 2R(k,k)$ and $R(k+1,k+1) \leq 4R(k,k)$.*

By the comments above, a yes answer to Conjecture 2.1.6 implies a yes for Conjecture 2.1.7. We note that a very similar inequality, $R(k+1,k+1) \leq 4R(k+1,k-1)+2$, was proved by Walker [75] in 1968. There are straightforward generalizations of these thoughts to other close-to-diagonal cases and to more than two colors, but we stop short of proposing them as conjectures.

2.2 On the Ramsey number $R(5,5)$

What is the largest number of vertices in any K_5-free graph with independence number less than 5? The answer is $R(5,5)-1$. The values of $R(s,t)$ are known for all s and t with $s+t < 10$ [63], so in this sense $R(5,5)$ is the smallest open case in Ramsey theory.

The progress of knowledge about lower and upper bounds on $R(5,5)$ first spanned more than three decades, then it apparently stopped in 1997. What we know now is almost the same as 17 years ago, while a significant gap between bounds remains unchanged. The effort required to lower the upper bound on $R(5,5)$ from 50 down to 49 was very significant, but still 49 is quite far from the best known lower bound of 43, which was obtained by Exoo in 1989 [22].

Theorem 2.2.1 [22] $43 \leq R(5,5) \leq 49$ [57].

The history of bounds on $R(5,5)$ is presented in Table 1. None of the results in references listed until 1973 depended in a significant way on computer algorithms. All of the later items involved at least some computational components to the degree that their full verification by hand seems infeasible. Note that Table 1 stops the listing in 1997. It is not the case that people did not try since then. We are aware of several such attempts, but it seems that none of them was finally published. The constructions allegedly improving on the lower bound of 43, which we have seen, each contained an error. A few attempts to improve the upper bound tried to derive some properties of, say, $(5,5;45)$-graphs, however we are not aware of any recognized and significant results in this direction.

year	reference	lower	upper	comments
1965	Abbott [1]	38		quadratic residues in $\mathcal{Z}_{37}$
1965	Kalbfleisch [43]		59	pointer to a future paper
1967	Giraud [33]		58	combinatorics, LP
1968	Walker [75]		57	combinatorics, LP
1971	Walker [76]		55	combinatorics, LP
1973	Irving [42]	42		sum-free sets
1989	Exoo [22]	43		simulated annealing
1992	McKay-Radziszowski [54]		53	$(4,4)$-graph enumeration, LP
1994	McKay-Radziszowski [55]		52	LP, computation
1995	McKay-Radziszowski [56]		50	implication of $R(4,5)=25$
1997	McKay-Radziszowski [57]		49	

Table 1. The history of bounds on $R(5,5)$, based on [57]
(LP refers to linear programming techniques)

In 1997, McKay and the second author [57] posed the following conjecture.

Conjecture 2.2.2 $R(5,5)=43$, *and the number of* $(5,5;42)$*-graphs is precisely* 656.

The authors of [57] provided some strong evidence for its correctness. Of particular strength seems to be the fact that a few distinct methods to generate $(5,5;42)$-graphs ended up in the same final set of 656 graphs. 328 of these graphs, with the number of edges ranging from 423 to 430, are posted at a website by McKay [52], the other 328 on at least 431 edges are their complements. All of the 656 graphs have the minimum degree 19 and maximum degree 22. The automorphism groups of these graphs are surprisingly small; none has order larger than 2, or more precisely 232 are involutions without fixed points, and the remaining 424 groups are trivial. This is somewhat against an intuition that complete sets of extreme graphs for typical Ramsey cases should contain some graph with a larger automorphism group. We note, however, that graphs with more symmetries in general are easier to find, and thus we think that any such $(5,5;42)$-graph would have been already found if it existed.

In 2014, McKay and Lieby [53] provided the following new evidence for Conjecture 2.2.2, which required a computational effort of about 9 CPU years. Define the distance between two graphs on n vertices to be k if their largest common induced subgraph has $n-k$ vertices. McKay and Lieby report that any new $(5,5;42)$-graph H would have to be in distance at least 6 from every graph in the set of 656 known $(5,5;42)$-graphs.

Some improvement of the upper bound in Theorem 2.2.1 might be possible, but we consider that lowering it even just by 1 would be a great accomplishment.

Although the authors of this work share their ideas on most problems presented herein, there is an exception in our positions on the so-called almost regular Ramsey graphs, and in consequence on Conjecture 2.2.2. A graph G is *almost regular* if $\Delta(G) - \delta(G) \leq 1$. The following Conjecture 2.2.3 on almost regular Ramsey graphs was proposed by the first author, who explored it with Zehui Shao and Linqiang Pan in 2008. Shao's computational work in this direction appears in his thesis [70], but otherwise was not published.

Conjecture 2.2.3 *For all positive s and t, and every $1 \leq n < R(s,t)$, there exists an almost regular $(s,t;n)$-graph.*

Needless to say, no counterexample to Conjecture 2.2.3 is known. However, since none of the 656 known $(5,5;42)$-graphs is almost regular, hence if Conjecture 2.2.2 holds then Conjecture 2.2.3 is false. The first author supports Conjecture 2.2.3, but not Conjecture 2.2.2, while the second author supports Conjecture 2.2.2 and thus not Conjecture 2.2.3, unless the latter is restated only for sufficiently large s and t.

2.3 Constructive lower bounds for $R(3,k)$

In 1995, Kim [46] obtained a breakthrough result establishing the asymptotics of $R(3,k)$ up to a multiplicative constant, when he raised the lower bound to match the upper bound.

Theorem 2.3.1 [46] $R(3,k) = \Theta(k^2/\log k)$.

Recently, in independent work by Bohman and Keevash [7] and by Fiz, Griffiths and Morris [26], an impressive further progress has been obtained in closing on the actual constants of Theorem 2.3.1.

Theorem 2.3.2 [7, 26] $\left(\frac{1}{4} + o(1)\right)k^2/\log k \leq R(3,k) \leq (1 + o(1))k^2/\log k$ [71].

The progress on asymptotic lower bounds for $R(3,k)$ was obtained by the probabilistic method [74, 46, 7, 26], which often yields very weak bounds for concrete small cases. The upper bound of Theorem 2.3.2 is implicit in [71]. The best specific constructions are usually obtained by insight, computations and ad hoc means. We lack general constructions which give both clear structure of the graphs and good Ramsey lower bound. One of the most known and elegant constructions is a recursive method by Chung, Cleve and Dagum from 1993 [12]. We present an instance of it in Figure 1 below.

Let G be a triangle-free graph on n vertices with independence $\alpha(G) = k$, i.e. G is a $(3, k+1; n)$-graph. Consider graph H, called a *fibration* of G, formed by 6 disjoint copies of G with two types of edges joining them (see Figure 1), as described in [12]. Chung et al. proved that their construction produces a $(3, 4k+1; 6n)$-graph H, which easily gives $R(3, 4k+1) \geq 6R(3, k+1) - 5$. By solving the recurrence one obtains the asymptotic lower bound $R(3,k) = \Omega(k^{\log 6/\log 4}) \approx \Omega(k^{1.29})$.

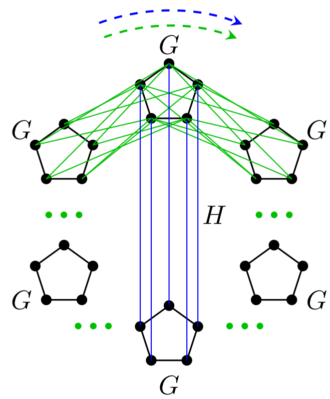

Figure 1: Construction of a $(3, 9; 30)$-graph H using C_5 as a $(3, 3; 5)$-graph G, for $k = 2$.

Other explicit constructions for $R(3, k)$ leading to a better lower bound $\Omega(k^{3/2})$ were presented by Alon in 1994 [2], and Codenotti, Pudlák and Resta in 2000 [15]. In 2010, Kostochka, Pudlák and Rödl [47] improved further known constructive lower bounds for $R(k, n)$ for fixed $4 \le k \le 6$, but their results still lagged behind those obtained by the probabilistic method. For example with $k = 4$, the probabilistic K_4-free process used by Bohman yields $R(4, n) = \Omega(n^{5/2}/\log^2 n)$ [6], while the constructive approach of [47] gives only $R(4, n) = \Omega(n^{8/5})$.

Challenge 2.3.3 Design a recursive lower bound construction of $(3, k; n)$-graphs for $R(3, k)$, with the number of vertices n larger than $\Omega(k^{3/2})$.

3 Multicolor Ramsey Numbers

Using elementary methods in 1955, Greenwod and Gleason [40] established that for the multicolor Ramsey numbers, for all $k_i \ge 2$ and $r \ge 2$, we have

$$R(k_1, \ldots, k_r) \le 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r), \tag{1}$$

with strict inequality if the right hand side of (1) is even and the sum has an even term. The bound (1) reduces to the classical two-color upper bound for $r = 2$. There are only two known multicolor cases ($r \ge 3$), for the parameters (3,3,3,3) and (3,3,4), where this bound was improved. On the other hand, very likely the bound (1) is never tight for $r \ge 3$, except for (3,3,3). In subsections 3.2 and 3.3 we will discuss more in detail the special cases of $R_4(3) = R(3, 3, 3, 3)$ and $R(3, 3, 4)$, respectively.

3.1 Constructions and limits

In 1973, Chung [11] proved constructively that $R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$, and in 1983 Chung and Grinstead [14] showed that the limit

$$L = \lim_{r\to\infty} R_r(3)^{\frac{1}{r}} \tag{2}$$

exists, though it may be infinite.

One of the most successful techniques for deriving lower bounds on $R_r(3)$ are constructions based on Schur partitions, and closely related cyclic and linear colorings. A Schur partition of the integers from 1 to n, $[1, n]$, is a partition into sum-free sets. The *Schur number* $s(r)$ is the maximum n for which there exists a Schur partition of $[1, n]$ into r sets. A simple argument gives $s(r) + 2 \leq R_r(3)$.

In an early work, Abbott [1] showed that $s(r) > 89^{r/4-c\log r} > 3.07^r$. After much more effort the exact values of $s(r)$ have been found for $1 \leq r \leq 4$. What we know now about Schur numbers $s(r)$ provides the best known lower bound of 3.199 for L, which is implied by the lower bound of 536 on $s(6)$. This was obtained by Fredricksen and Sweet in 2000 [27]. In Table 2, which summarizes the best known bounds on $R_r(3)$, three lower bounds for $5 \leq r \leq 7$ are implied by constructions of partitions for Schur numbers. For additional results and comments on constructive lower bounds on $R_r(3)$ and general $R(k_1, \cdots, k_r)$ see [80].

Recently, we improved a construction from [80] to one which permits to double the number of colors in a special way, as stated in the next theorem.

Theorem 3.1.1 [78] *For integers $k, n, m, s \geq 2$, let G be a $(k, ..., k; s)$-coloring with n colors containing an induced subcoloring of K_m using less than n colors. Then*

$$R_{2n}(k) \geq s^2 + m(R_n(k-1, k, \ldots, k) - 1) + 1.$$

The *Shannon capacity* $c(G)$ of a noisy channel modelled by graph G is equal to $\lim_{n\to\infty} \alpha(G^n)^{1/n}$, where $\alpha(G^n)$ is the independence number of the n-th power of graph G using the strong product of graphs. We proved in [78] that the construction in the proof of Theorem 3.1.1 with $k = 3$ implies the following:

Theorem 3.1.2 [78] *The supremum of the Shannon capacity over all graphs with independence number 2 cannot be achieved by any finite graph power.*

We also generalized Theorem 3.1.2 to graphs with bounded independence number. The link between Shannon capacity and multicolor Ramsey numbers was first studied by Erdős, McEliece and Taylor [21] in 1971, but it was not much exploited afterwards. As we showed in [78], the limits involved in the definition of $c(G)$ and L can be linked via constructions as in Theorem 3.1.1. We note that at least three different graph products are used in the work in this area: strong product in the definition of $c(G)$ [69], simple product [1], and the so-called composition used by us in [80, 78]. Each of these products is useful in a different way. We now propose two conjectures related to Theorem 3.1.2.

Conjecture 3.1.3 *For each $k \geq 3$, there does not exist any finite graph G with independence number equal to $k-1$ such that $c(G) = \lim_{n\to\infty} R_n(k)^{1/n}$.*

Conjecture 3.1.4 *There exists a positive integer k such that $\lim_{n\to\infty} R_n(k)^{1/n} = \infty$.*

The limit $\lim_{n\to\infty} R_n(k)^{1/n}$ exists for each $k \geq 3$ by an argument similar to that in the proof for $k = 3$ [14]. What remains open is for which k this limit is infinite. The second of these conjectures seems a little easier, if it is true. If Conjecture 3.1.3 is false, then $L_k = \lim_{n\to\infty} R_n(k)^{\frac{1}{n}}$ is finite, and actually we have $L_k \leq |V(G)|$ where G is a counterexample graph. Hence, a proof of Conjecture 3.1.4 would imply a proof for Conjecture 3.1.3. We note that Conjecture 3.1.3 is not true for infinite graphs. This was not considered in [21], but one could prove it using the same methods as in [21].

The known values and bounds on $R_r(3)$ for small r are listed in Table 2 below. The first open case for $r = 4$ is perhaps the most studied specific multicolor Ramsey number, and we give more details about it in the next subsection. The lower bounds in Table 2 for $5 \leq r \leq 7$ were obtained by constructions of Schur colorings.

r	value or bounds	references
2	6	folklore
3	17	Greenwood-Gleason 1955 [40]
4	51–62	Chung 1973 [11] – Fettes-Kramer-R 2004 [25]
5	162–307	Exoo 1994 [24] – bound (1)
6	538–1838	Fredricksen-Sweet 2000 [27] – bound (1)
7	1682–12861	Fredricksen-Sweet 2000 [27] – bound (1)

Table 2. Bounds and values of $R_r(3)$

3.2 On the Ramsey number $R(3, 3, 3, 3)$

The best known bounds on $R_4(3) = R(3, 3, 3, 3)$ are given in the next theorem, after which we conjecture that the actual value is 51.

Theorem 3.2.1 [11] $51 \leq R(3, 3, 3, 3) \leq 62$ [25].

Conjecture 3.2.2 $R(3, 3, 3, 3) = 51$.

year	references	lower	upper
1955	Greenwood-Gleason [40]	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert [37]	46	
1973	Whitehead [77]	50	65
1973	Chung [11], Porter cf. [11]	51	
1974	Folkman [29]		65
1995	Sánchez-Flores [67]		64
1995	Kramer (no computer) [48]		62
2004	Fettes-Kramer-R (computer) [25]		62

Table 3. History of bounds on $R_4(3)$, based on [25]

We first overview the history of the upper bounds. The bound of 66 follows from (1) and $R_3(3) = 17$ [40]. The result $R(3,3,3,3) \leq 65$ appeared first in a 1973 paper by Whitehead [77], although he gives credit for part of the proof to Folkman. Notes by Folkman were printed posthumously in 1974 [29]. No progress was made on lowering further the upper bound until Sánchez-Flores [67] gave a computer-free proof that $R(3,3,3,3) \leq 64$. In his 1995 article, Sánchez-Flores proved some properties of 3- and 4-colorings of K_n without monochromatic triangles, and then used them to derive the new upper bound. In 1994, Kramer [48] gave a series of talks at a graph theory seminar at Iowa State University to show that $R(3,3,3,3) \leq 62$. These talks led to a 116 pages long unpublished manuscript [48], which provided the spark to develop the algorithms for the computational verification of this result in [25]. We consider it feasible to decide whether $R(3,3,3,3) \leq 61$ with the techniques similar to those in [25], however we also consider that going down to 60 or less would necessarily require a significantly new insight.

Between 1955 and 1973 the best known lower bound was moving from 42 to 51 as listed in Table 3. In her 1973 article, Chung took an incidence matrix for one of the two proper 3-colorings of K_{16} and constructed from it the incidence matrix corresponding to a good 4-coloring of K_{50}, thereby establishing $R(3,3,3,3) > 50$. Actually, this is a special case of the general construction by Chung for any number of colors, mentioned at the opening of subsection 3.1. To date it gives the best known lower bound for 4 colors. Many other nonisomorphic proper 4-colorings of K_{50} were obtained by the second author, though all of them had the structure very similar to the one constructed by Chung, in that all of them have significantly less edges in one of the colors. We summarize all these developments in Table 3.

We are aware of several attempts to use heuristic algorithms for the lower bound, which had a hard time to produce correct constructions for the number of vertices well below 50. Actually, we consider that designing a general heuristic method which can come close to, match, or perhaps even beat the Chung's bound is an interesting challenge for the computationally oriented approach. There exists a very large number of 4-colorings of K_n without monochromatic triangles for n equal to 49 or slightly less, yet the standard heuristic search techniques somehow fail to find them. Understanding why this is happening could give new insights on how to design better general search techniques.

3.3 On the Ramsey number $R(3,3,4)$

In the multicolor case, when only complete graphs are avoided, the only known nontrivial value of such type of Ramsey number is $R(3,3,3) = 17$ [40]. The only other case whose evaluation does not look hopeless is $R(3,3,4)$, which currently is known to be equal to 30 or 31. The lower bound $30 \le R(3,3,4)$ was obtained by Kalbfleisch in 1966 [44], while the best known upper bound $R(3,3,4) \le 31$ by Piwakowski and the second author [60] in 1998. The same authors obtained some further constraints on the final outcome in 2001 [61]. We are not aware of any further progress on this case since then. Perhaps it is time to attack it again.

Conjecture 3.3.1 [60, 61] $R(3,3,4) = 30$.

It is known that if $R(3,3,4) = 31$, then any witness $(3,3,4;30)$-coloring must be very special. The known results of [43, 44, 60, 61], all obtained with the help of computer algorithms, are summarized in the next three theorems. For edge coloring C of K_n, the set $C[k]$ consists of the edges in color k.

Theorem 3.3.2 [43, 60] $30 \le R(3,3,4) \le 31$, *and* $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$*-coloring* C *such that every triangle* $T \subset C[3]$ *has a vertex* $x \in T$ *with* $\deg_{C[3]}(x) = 13$. *Furthermore,* C *has at least* 14 *vertices* v *such that* $\deg_{C[1]}(v) = \deg_{C[2]}(v) = 8$ *and* $\deg_{C[3]}(v) = 13$.

Theorem 3.3.3 [61] $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$*-coloring* C *such that every triangle* $T \subseteq C[3]$ *has at least two vertices* $x, y \in T$ *with* $\deg_{C[3]}(x) = \deg_{C[3]}(y) = 13$.

Theorem 3.3.4 [61] $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$*-coloring* C *such that every edge in the third color has at least one endpoint* x *with* $\deg_{C[3]}(x) = 13$. *Furthermore,* C *has at least* 25 *vertices* v *such that* $\deg_{C[1]}(v) = \deg_{C[2]}(v) = 8$ *and* $\deg_{C[3]}(v) = 13$.

Further elimination of all vertices of degree at least 14 in the third color, on triangles in the third color, is perhaps within the reach of feasible computations. Unfortunately, we don't know of any approach which likely could be efficient enough to proceed similarly as in [60, 61] for the remaining cases (including a 13-regular graph in the third color).

If you like this type of problems and wish to attack $R(3,3,4)$, we would recommend to try first a somewhat similar case of $R_3(K_4 - e)$. This is almost certainly easier than $R(3,3,4)$, but still difficult enough to pose a serious computational challenge. The best known bounds are [23] $28 \leq R_3(K_4 - e) = R(K_4 - e, K_4 - e, K_4 - e) \leq 30$ [59]. With some new approach, and a lot of good luck, it might be even possible to solve this case without the help of intensive computations.

Finally, we note that the Ramsey numbers of the form $R(3, 3, k)$ are special since their asymptotics is known up to a poly-log factor. A surprising result by Alon and Rödl from 2005 [3] implies that $R(3, 3, k) = \Theta(k^3 \text{poly-log}\, k)$. They actually prove a more general result that for every fixed number of colors $r \geq 2$, when we avoid triangles in the first $r - 1$ colors and K_k in color r, we have $R(3, \ldots, 3, k) = \Theta(k^r \text{poly-log}\, k)$.

4 Edge Folkman Numbers

In 1967, Erdős and Hajnal [20] posed a problem asking for a construction of a K_6-free graph G whose every coloring of the edges with two colors contains a monochromatic triangle. The proposers also expected (but didn't prove it) that for every number of colors r there is a K_4-free graph G whose every coloring of the edges with r colors contains a monochromatic triangle. The latter for $r = 2$ reduces to the question: Does there exist a K_4-free graph that is not a union of two triangle-free graphs? In 1970, Folkman [28] proved a general result implying that such graphs exist, but far from providing their effective construction. We recommend chapter 27 in a book by Soifer [72] for an earlier, alternate and complementary perspective on problems discussed in this section.

Using notation from Section 1, we wish to understand the structure of the graphs in the set $\mathcal{F}_e(s, t; k)$, and in particular those with the smallest number of vertices which define the value of the corresponding Folkman number $F_e(s, t; k)$. Much work has been done for the general cases, but here we concentrate mainly on the simplest looking, but already difficult case of arrowing triangles, namely for $s = t = 3$.

The state of knowledge about the cases $F_e(3, 3; k)$ is summarized in Table 4 below. It is easy to see that $k > R(s, t)$ implies $F_e(s, t; k) = R(s, t)$, which gives the first row. Graham [38] found that $C_5 + K_3 \rightarrow (3, 3)$, which solved the first question by Erdős and Hajnal, and it gives the second row with $k = 6$. The next entry for $k = 5$, after numerous papers on this case, was finally completed in 1999 by Piwakowski, Urbański and the second author [62] who used significant help of computer algorithms. The case of $k = 4$ is the hardest and still open. The known bounds are stated in Theorem 4.1 below. We expect that any further improvements to these bounds can be very hard to obtain. We discuss $F_e(3, 3; 4)$ more in detail in the remainder of this section.

Theorem 4.1 [64] $19 \leq F_e(3, 3; 4) \leq 786$ [49].

k	$F_e(3,3;k)$	graphs	references
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham 1968 [38]
5	15	659 graphs	P-R-U 1999 [62]
4	19–786	see Table 5	2007 [64], 2014 [49]

Table 4. Edge Folkman numbers $F_e(3,3;k)$

The history of events and progress on $F_e(3,3;4)$ is summarized in Table 5 starting with Erdős and Hajnal's [20] original question. The positive answer follows from a theorem by Folkman [28] proved in 1970, which when instantiated to 2 colors produces a very large upper bound for $F_e(3,3;4)$. In 1975, Erdős [18] offered \$100 (or 300 Swiss francs) for deciding if $F_e(3,3;4) < 10^{10}$, which later resulted to be remarkably close to what can be obtained by using probabilistic methods. This question remained open for over 10 years. Frankl and Rödl [30] nearly met Erdős' request in 1986 when they showed that $F_e(3,3;4) < 7.02 \times 10^{11}$. In 1988, Spencer [74], using probabilistic techniques, proved the existence of a Folkman graph of order 3×10^9 (after an erratum by Hovey), without explicitly constructing it. The main idea of these probabilistic proofs [30][74] is quite simple. Any K_4-free graph G such that $G \to (3,3)$ proves the bound $F_e(3,3;4) \leq |V(G)|$. How to find such a G? First, take randomly a graph F from the set $G(n,p)$ of all graphs on n vertices with edge probability p, and then remove one edge from every K_4 in F. The resulting graph G is clearly K_4-free and so has some probability of being the graph we need. The difficult part is showing that this probability is positive for certain values of n and p.

In 2008, Lu [51] showed that $F_e(3,3;4) \leq 9697$ by constructing a family of K_4-free circulant graphs and showing that some such graphs arrow $(3,3)$ using spectral analysis. Dudek and Rödl [16] developed a strategy to construct new Folkman graphs by approximating the maximum cut of a related graph, and used it to improve the upper bound to 941. Lange and the authors [49] improved this bound first to 860, and then further to 786 with the MAX-CUT semidefinite programming relaxation as in the Goemans-Williamson algorithm. The results of [49] were obtained by 2012, though its publication year is 2014. During the 2012 SIAM Conference on Discrete Mathematics in Halifax, Nova Scotia, Ronald Graham announced a \$100 award for determining if $F_e(3,3;4) < 100$.

Conjecture 4.2 $50 \leq F_e(3,3;4) \leq 94$.

At the end of chapter 27 of *The Mathematical Coloring Book* by Soifer [72], it is stated that a double prize of \$500 was offered by the second author of this paper for proving the bounds $50 \leq F_e(3,3;4) \leq 127$. These bounds are much stronger than the best known bounds in Theorem 4.1, but note that we are lowering further the upper bound in Conjecture 4.2 because of Conjecture 4.4 and comments after it.

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal [20]
1970	exist	Folkman [28]
1972	10 –	Lin [50]
1975	– 10^{10}?	Erdős offers $100 for proof [18]
1986	– 8×10^{11}	Frankl-Rödl [30]
1988	– 3×10^{9}	Spencer [74]
1999	16 –	Piwakowski-R-Urbański, implicit in [62]
2007	19 –	R-X [64]
2008	– 9697	Lu [51]
2008	– 941	Dudek-Rödl [16]
2012	– 786	Lange-R-X [49]
2012	– 100?	Graham offers $100 for proof

Table 5. History of the edge Folkman number $F_e(3, 3; 4)$

Next, we give more details on the upper bounds obtained in recent years. Building off other methods, Dudek and Rödl [16] showed how to construct a graph H_G from graph G, such that the maximum cut size of H_G determines whether or not $G \to (3, 3)$. The vertices of H_G are the edges of G, so $|V(H_G)| = |E(G)|$. For $e_1, e_2 \in V(H_G)$, if edges $\{e_1, e_2, e_3\}$ form a triangle in G, then $\{e_1, e_2\}$ is an edge in H_G. Let $t(G)$ denote the number of triangles in G, so $|E(H_G)| = 3t(G)$. Let $MC(H)$ denote the MAX-CUT size of graph H.

Theorem 4.3 Dudek-Rödl 2008 [16]
$G \to (3, 3)$ *if and only if* $MC(H_G) < 2t(G)$.

The intuition behind Theorem 4.3 is as follows. Any coloring of the edges G can be seen as a partition of the vertices in H_G, with two colors giving a bipartition of $V(H_G)$. If a triangle in G is not monochromatic, then its edges are in both parts. If we treat this bipartition as a cut, then the size of the cut counts each triangle twice for the two edges that cross it. Since there is only one triangle in a graph that contains two given edges, this counts the number of non-monochromatic triangles. Therefore, if there exists a cut of size $2t(G)$, then it defines an edge 2-coloring of G without monochromatic triangles. However, if $MC(H_G) < 2t(G)$, then in each coloring all three edges of some triangle are in one part, and thus $G \to (3, 3)$.

A benefit of converting the problem of arrowing $(3, 3)$ to MAX-CUT is that the latter is well-known and has been studied extensively in computer science and mathematics. The related decision problem MAX-CUT(H, k) asks whether $MC(H) \geq k$. MAX-CUT is **NP**-hard and its decision problem was one of Karp's 21 **NP**-complete problems [45].

The Goemans-Williamson MAX-CUT approximation algorithm [36] is a polynomial-time algorithm that relaxes the problem to a semidefinite program (SDP). It involves the first use of SDP in combinatorial approximation and has since inspired a variety of other successful algorithms. This randomized algorithm returns a cut with expected size at least 0.878 of the optimal value. However, in our case, all that is needed is the solution to the SDP, as it gives an upper bound on $MC(H)$. Another often effective method approximates MAX-CUT using minimum eigenvalue, or one can combine partial exhaustive search with one of the approximation methods [16, 49].

Define graphs $G_{n,r}$ on vertices $\mathcal{Z}_n$ with an edge connecting x and y if and only if $x - y = \alpha^r$ for some nonzero $\alpha \in \mathcal{Z}_n$. If the graph $G_{n,r}$ is K_4-free, then it may be a good candidate for a witness to the upper bound of n. Using the minimum eigenvalue method, Dudek and Rödl [16] found that the graph $G_{941,5}$ is a witness of $F_e(3,3;4) \leq 941$. A reduction of the same graph led to a better bound 860 [49], and some modifications of graphs considered by Lu [51] produced the best to date bound of 786 [49].

A puzzling question about triangle arrowing is however for a much smaller graph, namely for $G_{127,3}$. This graph was used by Hill and Irving [41] in 1982 to establish the bound $128 \leq R(4,4,4)$. About 10 years ago Exoo proposed to consider this graph for the triangle arrowing. Since then, Exoo, us, and many others tried to decide whether $G_{127,3}$ forces a monochromatic triangle if its edges are colored with two colors. As far as we are aware, all to no avail. Nevertheless, all failed attempts build up more evidence for the positive answer to the following.

Conjecture 4.4 Exoo, $G_{127,3} \rightarrow (3,3)$.

Exoo suggested that even a 94-vertex induced subgraph of $G_{127,3}$, obtained by removing from it 3 disjoint independent sets of order 11, may still work. If true, this would imply $F_e(3,3;4) \leq 94$.

One of the approaches for verifying the conjecture is by reducing $\{G \mid G \not\rightarrow (3,3)\}$ to the problem 3-SAT. We map the edges $E(G)$ to the variables of $\phi_G \in$ 3-SAT, and for each (edge)-triangle xyz in $E(G)$ we add to ϕ_G two clauses $(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z})$. One can easily see that $G \not\rightarrow (3,3)$ if and only if ϕ_G is satisfiable. Conjecture 4.4 above is equivalent to the unsatisfiability of ϕ_G for $G = G_{127,3}$. In this case the formula ϕ_G has 2667 variables and 19558 3-clauses, two for each of the 9779 triangles. In all, this is considered of only moderate size for the state of art SAT-solvers. Still, all of several attempts to decide this ϕ_G by us and others failed.

The lower bound on $F_e(3,3;4)$ is a challenge as well, as it is quite surprising that only 19 is the best known. Even an improvement to $20 \leq F_e(3,3;4)$ would be a good progress. Lin [50] obtained a lower bound of 10 in 1972 without the help of a computer. All 659 graphs on 15 vertices witnessing $F_e(3,3;5) = 15$ [62] contain K_4, thus giving the bound $16 \leq F_e(3,3;4)$. In 2007, the authors gave a computer-free proof of $18 \leq F_e(3,3;4)$ and improved the lower bound further to 19 with the help of computations [64]. Any proof or computational technique improving further the lower bound of 19 very likely will be of significant interest.

We wish to mention an interesting open problem about a related Folkman number, namely $F_e(K_4-e, K_4-e; 4)$. Note that clearly we have $F_e(3,3;4) \leq F_e(K_4-e, K_4-e; 4)$. As commented by Lu [51] in his work on $F_e(3,3;4)$, he also obtained as a side result the bound $F_e(K_4-e, K_4-e; 4) \leq 30193$. The gap here between the known lower and upper bounds is much larger than that for $F_e(3,3;4)$.

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