THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Hadwigers Conjecture for inflations of 3-chromatic graphs

C. J. Casselgren and A. S. Pedersen

REPORT No. 28, 2013/2014, spring

ISSN 1103-467X
ISRN IML-R- -28-13/14- -SE+spring

Hadwiger's Conjecture for inflations of 3-chromatic graphs

Carl Johan Casselgren[*] *Anders Sune Pedersen*[†‡]

September 23, 2014

Abstract. Hadwiger's Conjecture states that every k-chromatic graph has a complete minor of order k. A graph G' is an inflation of a graph G if G' is obtained from G by replacing each vertex v of G by a clique C_v and joining two vertices of distinct cliques by an edge if and only if the corresponding vertices of G are adjacent. We present an algorithm for computing an upper bound on the chromatic number $\chi(G')$ of any inflation G' of any 3-chromatic graph G. As a consequence, we deduce that Hadwiger's Conjecture holds for any inflation of any 3-colorable graph.

Keywords: Hadwiger's Conjecture, graph coloring, inflation, 3-chromatic graph, complete minor

1 Introduction

A *proper k-coloring* of a graph G is a function $f : V(G) \to \{1, \dots, k\}$ such that $f(v) \neq f(u)$ whenever u and v are adjacent. The *chromatic number* $\chi(G)$ of G is the smallest k such that there is a proper k-coloring of G. A graph G is *k-chromatic* if $\chi(G) = k$.

Hadwiger's Conjecture is one of the fundamental open questions in graph coloring. It dates back to 1943, when Hadwiger [7] suggested that every k-chromatic graph G contains a complete minor of order k, i.e. a complete graph of order k can be obtained from G by deleting and/or contracting edges.

The conjecture is a far-reaching generalization of the well-known Four Color Problem, which asks if every planar graph has chromatic number at most 4, and it remains open for all k greater than 6. (See [15] for a survey on Hadwiger's Conjecture.) The case $k \leq 4$ was proved by Hadwiger in his original paper [7]. Wagner [16] proved that the case $k = 5$ is equivalent to the Four Color

[*]Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden.
E-mail address: carl.johan.casselgren@liu.se
Part of the work done while the author was a postdoc at University of Southern Denmark and at Mittag-Leffler Institute. Research supported by SVeFUM and Mittag-Leffler Institute.
[†]Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense, Denmark.
E-mail address: asp@imada.sdu.dk
[‡]Research Clinic on Gambling Disorders, Center for Functionally Integrative Neuroscience, Aarhus University Hospital, Trøjborgvej 72, Bygning 30, 8200 Aarhus N, Denmark.

Problem. The latter problem was solved in the affirmative by Appel and Haken [1, 2] in 1977, and in 1993 Robertson et al. [12] proved Hadwiger's Conjecture for $k = 6$.

Hadwiger's Conjecture has also been proved to hold for some special families of graphs, e.g. line graphs [11] and quasi-line graphs [13]. Bollobás et al. [5] proved that Hadwiger's Conjecture is true for almost every graph.

In this paper we study Hadwiger's Conjecture for inflations of graphs: given a graph G with vertex set $V(G) = \{v_1, \dots, v_n\}$ and non-negative integers $k_1, \dots, k_n$, we define the *inflation* $G' = G(k_1, \dots, k_n)$ of G to be the graph obtained from G by replacing vertices $v_1, \dots, v_n$ by disjoint cliques $A_1, \dots, A_n$ of size $k_1, \dots, k_n$, respectively, such that vertices x and y, where $x \in V(A_s)$ and $y \in V(A_t)$, $s \neq t$, are adjacent if and only if v_s and v_t are adjacent in G. The cliques $A_1, \dots, A_n$ are referred to as the *inflated vertices*, and the numbers $k_1, \dots, k_n$ are referred to as *inflation sizes* of G'. If $k_1 = \cdots = k_n$, then G' is a *uniform* inflation. We also say that G' is obtained *by inflating* G.

One motivation for studying Hadwiger's Conjecture for inflations of graphs stems from Hajós' Conjecture which states that every k-chromatic graph contains a subdivision of the complete graph on k vertices. In 1979, Catlin [6] showed that this latter conjecture is false for all values of k greater than 6. Catlin's counterexamples are surprisingly simple: they are just uniform inflations of the 5-cycle. Catlin's counterexamples to Hajós' Conjecture are not counterexamples to Hadwiger's Conjecture, but perhaps a similar construction might yield a counterexample to Hadwiger's Conjecture. Thomassen [14] proved that a graph G is perfect if and only if every inflation of G satisfies Hajós' Conjecture. In particular, this means that any non-perfect graph can be inflated to a counterexample to Hajós' Conjecture. We prove that no counterexample to Hadwiger's Conjecture can be constructed by inflating a 3-colorable graph.

There are some other results on Hadwiger's Conjecture for inflations of graphs in the literature: Plummer et al. [10] proved that no counterexample to Hadwiger's Conjecture can be obtained by inflating a graph with independence number at most 2 (complements of triangle-free graphs) and order at most 11. Kawarabayashi conjectured that Hadwiger's Conjecture holds for any inflation of a outerplanar graph [private communication to Pedersen, 2012]. Since every outerplanar graph is 3-colorable, the main result of this paper settles that conjecture in the affirmative. Pedersen [9] proved that Hadwiger's Conjecture holds for any inflation of the Petersen graph. Here we prove the following stronger proposition.

Theorem 1. *Hadwiger's Conjecture is true for any inflation of any* 3*-colorable graph.*

2 Proof of Theorem 1

Let $\eta(G)$ denote the Hadwiger number of G, i.e., the order of the largest complete minor of G. Hadwiger's Conjecture then states that $\eta(G) \geq \chi(G)$ for every graph G. In this section we will prove that for any inflation G' of any 3-colorable graph G, we have $\eta(G') \geq \chi(G')$.

Inflations of graphs are studied in e.g. [3, 4]. Therein the authors were, among other things, interested in determining the chromatic number of (uniform) inflations. Here we do not attempt to calculate the chromatic number of such graphs explicitly; rather we obtain an upper bound on the chromatic number of any (possibly non-uniform) inflation G' of any 3-colorable graph G and give a lower bound on the Hadwiger number of G'.

Suppose that G' is an inflation of G with inflation sizes $k_1, k_2, \dots, k_s$. We denote by $G_{k_1,k_2,\dots,k_t}$ the subgraph of G induced by the vertices which are replaced by cliques with sizes in the set $\{k_1, k_2, \dots, k_t\}$ in G'. Similarly, $G'_{k_1,k_2,\dots,k_t}$ denotes the subgraph of G' induced by all cliques with sizes in $\{k_1, k_2, \dots, k_t\}$ that correspond to vertices of G.

Given two graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) \neq \emptyset$, we define the *intersection of* G_1 *and* G_2, denoted by $G_1 \cap G_2$, as the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. Similarly, we define the *union of* G_1 *and* G_2, denoted by $G_1 \cup G_2$, as the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

In the following we will present an algorithm for computing an upper bound on the chromatic number $\chi(G')$ of any inflation G' of any 3-chromatic graph G. By analyzing this algorithm we will then be able to prove that Hadwiger's Conjecture is true for any inflation of any 3-colorable graph. We shall need some preliminary results. The following was noted by Albertson et al. [3].

Lemma 1. *Let G be a graph, and G' the inflation obtained from G by replacing each vertex by a clique of size k. Then, $\chi(G') \leq k\chi(G)$.*

If G is a graph and G' an inflation of G, then an edge $e = uv$ of G is called an *$\alpha\beta$-edge* if in G' u and v are replaced by cliques of size α and β, respectively. Similarly, a vertex in G which is replaced by a clique of size α in G' is called an *α-vertex.* We will use the following observation, which easily follows from the well-known fact that the chromatic number of a graph equals the maximum of the chromatic numbers of its blocks, and so we leave the proof to the reader.

Lemma 2. *Let G be a graph and let E_c denote a set of cut-edges in G. Suppose that G' is some inflation of G. Denote by H the graph $G - E_c$, and let H' denote the subgraph of G' obtained by removing all edges corresponding to edges of E_c. Then*

$$\chi(G') \leq \max\big(\{\chi(H')\} \cup \{\alpha + \beta \mid e \in E_c \text{ is an } \alpha\beta\text{-edge}\}\big)$$

We shall repeatedly apply the following consequence of Lovasz' Perfect Graph Theorem [8].

Theorem 2. *Every inflation of a perfect graph is perfect.*

Proof of Theorem 1. Suppose the result is false. Let G be a vertex-minimal graph with chromatic number at most 3 such that there is an inflation G' of G that is a counterexample to Hadwiger's Conjecture. Moreover, let G' be vertex-minimal with respect to the property of being an inflation of G that is a counterexample to Hadwiger's Conjecture. It is straightforward to see that G must be 2-connected. Suppose that G is 2-colorable. By Theorem 2, any inflation of a perfect graph is perfect and so $\chi(G') = \omega(G') \leq \eta(G')$, a contradiction to the assumption that G' is a counterexample to Hadwiger's Conjecture. Hence, we may assume that G is 3-chromatic.

Let a_1 be the largest inflation size of G'. If $\chi(G_{a_1}) = 3$, then it follows from Lemma 1 that $\chi(G') \leq 3a_1$. Furthermore, $\eta(G') \geq 3a_1$, because G_{a_1} contains a cycle. Hence, $\eta(G') \geq \chi(G')$ which contradicts that G' is a counterexample to Hadwiger's Conjecture. Thus, we conclude that $\chi(G_{a_1}) \leq 2$. Since $\chi(G) = 3$, this means that a_1 is not the only inflation size of G'.

Let $a_1, \dots, a_m, b_1, \dots, b_n$ denote the inflation sizes in G', where

$$a_1 > \cdots > a_m > b_1 > \cdots > b_n,$$

and $\chi(G_{a_1,\dots,a_m}) \leq 2$ while $\chi(G_{a_1,\dots,a_m,b_1}) = 3$.

Let $\mathcal{A}$ denote the set $\{a_1, \dots, a_m\}$, and let $\mathcal{S}$ be the set of all ordered pairs (a_i, a_j) of $\mathcal{A}$ with $a_i \geq a_j$ for which there is an $a_i a_j$-edge in G. Since $\chi(G_{a_1,\dots,a_m}) \leq 2$, Theorem 2 yields that

$$\chi(G'_{a_1,\dots,a_m}) = \max(\{a_i + a_j \mid (a_i, a_j) \in \mathcal{S}\} \cup \{a_1, \dots, a_m\}). \tag{1}$$

We define the graph $G''_{a_1,\dots,a_m}$ to be the graph obtained from $G'_{a_1,\dots,a_m}$ by removing b_1 vertices from each of the inflated vertices of $G_{a_1,\dots,a_m}$. Similarly, we set $b_{n+1} = 0$, and, for each $i \in [n]$, we let $G''_{a_1,\dots,a_m,b_1,\dots,b_i}$ denote the graph obtained from $G'_{a_1,\dots,a_m,b_1,\dots,b_i}$ by removing b_{i+1} vertices from each of the inflated vertices of $G'_{a_1,\dots,a_m,b_1,\dots,b_i}$ in such a way that $G''_{a_1,\dots,a_m,b_1,\dots,b_i}$ is a subgraph of $G''_{a_1,\dots,a_m,b_1,\dots,b_j}$ whenever $i < j$. (This is possible since $G'_{a_1,\dots,a_m,b_1,\dots,b_i} \subseteq G'_{a_1,\dots,a_m,b_1,\dots,b_j}$ and $b_i > b_j$ if $i < j$.) As a shorthand, we will often write

$$G_{\geq b_i}, G'_{\geq b_i}, \text{ and } G''_{\geq b_i}$$

for the graphs

$$G_{a_1,\dots,a_m,b_1,\dots,b_i}, G'_{a_1,\dots,a_m,b_1,\dots,b_i} \text{ and } G''_{a_1,\dots,a_m,b_1,\dots,b_i},$$

respectively. The analogue of (1) for $G''_{\geq a_m}$ then reads

$$\chi(G''_{\geq a_m}) = \max(\{a_i + a_j - 2b_1 \mid (a_i, a_j) \in \mathcal{S}\} \cup \{a_1 - b_1, \dots, a_m - b_1\}). \tag{2}$$

Below we shall give our algorithm for computing a useful upper bound on the chromatic number of G'. First we discuss it informally:

The algorithm proceeds by steps and at Step i of the algorithm ($1 \leq i \leq n$) it considers the graph $G_{\geq b_i}$, and defines the sets $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$, $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$, the set $\mathcal{T}_{i+1}$ from $\mathcal{T}_i$, and the auxiliary sets $\mathcal{S}'_i$, $\mathcal{A}'_i$, $\mathcal{A}''_i$ and $\mathcal{T}'_i$. Each step consists of the three parts (a), (b) and (c), and at each such part certain sets are defined.

At the beginning of Step 1 we have $\mathcal{A}_1 := \mathcal{A}$, $\mathcal{S}_1 := \mathcal{S}$, and $\mathcal{T}_1 := \varnothing$. Then at Step i ($1 \leq i \leq n$) the set $\mathcal{S}_{i+1}$ is constructed from $\mathcal{S}_i$ by adding a new element (α, b_i) if

- $\alpha \in \mathcal{A}_i$,
- there is no α-vertex in a cycle of $G_{\geq b_i}$, and
- there is an αb_i-edge in $G_{\geq b_i}$,

and removing any element (α, β) such that there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_i}$.

The set $\mathcal{A}_{i+1}$ is constructed from $\mathcal{A}_i$ at Step i by removing any element α such that there is an α-vertex on a cycle in $G_{\geq b_i}$.

Finally, the set $\mathcal{T}_{i+1}$ is constructed from $\mathcal{T}_i$ at Step i by adding any element (α, β, b_i) such that there is an $\alpha\beta$-edge in a cycle of $G_{\geq b_i}$, and adding every element (α, b_i, b_i) such that there is an α-vertex in a cycle of $G_{\geq b_i}$, *and* there is no $\beta > b_i$, such that there is an $\alpha\beta$-edge in a cycle of $G_{\geq b_i}$.

Note that if $(\alpha, \beta) \in \mathcal{S}_j \setminus \mathcal{S}_{j+1}$, then j is the minimum integer q such that there is an $\alpha\beta$-edge in a cycle of $G_{\geq b_q}$, and one might think of the set $\mathcal{S}_{i+1}$ as "the set of pairs (α, β) such that $\alpha \geq \beta$ and $\alpha \geq a_m$, and for which there is an $\alpha\beta$-edge in $G_{\geq b_i}$ but no cycle containing an $\alpha\beta$-edge". Similarly, one might think of the set $\mathcal{A}_{i+1}$ as "the set of all constants $\alpha \geq a_m$ for which there is an α-vertex in $G_{\geq b_i}$ but no cycle containing an α-vertex". Note further that since G is 2-connected, for each $\alpha \in \mathcal{A}_i$, there is some minimum integer j such that $G_{\geq b_j}$ contains a cycle with an α-vertex. A similar statement holds for the elements of $\mathcal{S}_i$.

Let us now give a formal description of the algorithm:

Algorithm 1

Step 0: Define $\mathcal{A}_1 := \mathcal{A}$, $\mathcal{S}_1 := \mathcal{S}$, and $\mathcal{T}_1 := \varnothing$.

Step 1:

(a) For each element (a_{j_1}, a_{j_2}) of $\mathcal{S}_1$, if there is an $a_{j_1}a_{j_2}$-edge on a cycle in $G_{\geq b_1}$, then include (a_{j_1}, a_{j_2}) in $\mathcal{S}'_1$.

(b) For each element a_j of $\mathcal{A}_1$:

- If there is an a_j-vertex on a cycle in $G_{\geq b_1}$, then include a_j in $\mathcal{A}'_1$.
- If there is an a_j-vertex on a cycle in $G_{\geq b_1}$ *and* no element $a_{j_1} \in \{a_1, \ldots, a_m\}$ such that there is an $a_j a_{j_1}$-edge on a cycle in $G_{\geq b_1}$, then include (a_j, b_1, b_1) in $\mathcal{T}'_1$.
- If there is *no* a_j-vertex on a cycle in $G_{\geq b_1}$ but there is an $a_j b_1$-edge in $G_{\geq b_1}$, then include a_j in $\mathcal{A}''_1$.

(c) Define

- $\mathcal{S}_2 := (\mathcal{S}_1 \setminus \mathcal{S}'_1) \cup \{(a_j, b_1) \mid a_j \in \mathcal{A}''_1\}$,
- $\mathcal{A}_2 := \mathcal{A}_1 \setminus \mathcal{A}'_1$,
- $\mathcal{T}_2 := \mathcal{T}_1 \cup \{(a_{j_1}, a_{j_2}, b_1) \mid (a_{j_1}, a_{j_2}) \in \mathcal{S}'_1\} \cup \mathcal{T}'_1 \cup \{(b_1, b_1, b_1)\}$,

and go to Step 2.

Step i ($2 \leq i \leq n$):

(a) For each element (α, β) of $\mathcal{S}_i$, if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_i}$, then include (α, β) in $\mathcal{S}'_i$.

(b) For each element a_j of $\mathcal{A}_i$:

- If there is an a_j-vertex on a cycle in $G_{\geq b_i}$, then include a_j in $\mathcal{A}'_i$.
- If there is an a_j-vertex on a cycle in $G_{\geq b_i}$ *and* no element $\alpha \in \{a_1, \ldots, a_m, b_1, \ldots, b_{i-1}\}$ such that there is an $a_j\alpha$-edge on a cycle in $G_{\geq b_i}$, then include (a_j, b_i, b_i) in $\mathcal{T}'_i$.
- If there is *no* a_j-vertex on a cycle in $G_{\geq b_i}$, but there is an $a_j b_i$-edge in $G_{\geq b_i}$, then include a_j in the set $\mathcal{A}''_i$.

(c) Define

- $\mathcal{S}_{i+1} := (\mathcal{S}_i \setminus \mathcal{S}'_i) \cup \{(a_j, b_i) \mid a_j \in \mathcal{A}''_i\}$,
- $\mathcal{A}_{i+1} := \mathcal{A}_i \setminus \mathcal{A}'_i$,
- $\mathcal{T}_{i+1} := \mathcal{T}_i \cup \{(\alpha, \beta, b_i) \mid (\alpha, \beta) \in \mathcal{S}'_i\} \cup \mathcal{T}'_i$,

and go to Step $(i+1)$ if $i \leq n-1$, otherwise Stop.

We now prove some properties of the algorithm. The algorithm stops after Step n when the sets $\mathcal{S}_{n+1}, \mathcal{A}_{n+1}$ and $\mathcal{T}_{n+1}$ have been defined.

Lemma 3.

(1) $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots \supseteq \mathcal{A}_{n+1}$.

(2) *The sets* $\mathcal{A}'_1, \dots, \mathcal{A}'_{n-1}$, *and* $\mathcal{A}'_n$ *are all disjoint.*

(3) $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}_{n+1}$.

Proof. (1): The inclusions follow directly from the description of the algorithm, in particular, part (c) of Steps $0, 1, \dots, n$.

(2): Suppose that a is some element of $\mathcal{A}'_{j_1} \cap \mathcal{A}'_{j_2}$ with $j_1 < j_2$. By part (b) of Step j_2, $a \in \mathcal{A}_{j_2}$. Since also $a \in \mathcal{A}'_{j_1}$, it follows from part (c) of Step j_1 that a is not in $\mathcal{A}_{j_1+1}$ and so, by (1), a is not in $\mathcal{A}_{j_2}$, a contradiction.

(3) The inclusions follow directly from part (c) of Steps $0, 1, \dots, n$.

□

Lemma 4. *At the end of Step i of the algorithm, the following holds:*

$$\begin{aligned}\chi(G'_{\geq b_i}) \leq \quad & \max(\{\alpha + \beta + \gamma \mid (\alpha, \beta, \gamma) \in \mathcal{T}_{i+1}\} \\ & \cup \{\alpha + \beta + b_{i+1} \mid (\alpha, \beta) \in \mathcal{S}_{i+1}\} \\ & \cup \{a_j + 2b_{i+1} \mid a_j \in \mathcal{A}_{i+1}\}). \qquad (3)\end{aligned}$$

Proof. For each $i \in [n]$, let E_i denote the set of all $\alpha\beta$-edges in $G_{\geq b_i}$ with $(\alpha, \beta) \in \mathcal{S}_{i+1}$. For each $i \in [n]$, let V_i denote the set of all α-vertices of $G_{\geq b_i}$ with $\alpha \in \mathcal{A}_{i+1}$. Define $H_i := G_{\geq b_i} - E_i - V_i$.

The following three claims are easily deduced from the description of the algorithm.

Claim 1. *For each $i \in [n]$ and each $\alpha\beta$-edge e of $G_{\geq b_i}$ with $(\alpha, \beta) \in \mathcal{S}_i$, the edge e is in H_i if and only if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_i}$.*

Claim 2. *For every $i \in [n]$, each edge of E_i is a cut-edge of $G_{\geq b_i}$.*

Claim 3. *For each $i \in [n]$ and each α-vertex v of $G_{\geq b_i}$ with $\alpha \in \mathcal{A}_i$, the vertex v is in H_i if and only if there is an α-vertex on a cycle in $G_{\geq b_i}$.*

Claim 4. *For each $i \in [n]$, each vertex of V_i is an isolated vertex of $G_{\geq b_i} - E_i$.*

Proof of Claim 4. Suppose that there is some edge uv in $E(G_{\geq b_i}) \setminus E_i$ with $v \in V_i$. Since $v \in V_i$, v is an α-vertex for some $\alpha \in \mathcal{A}_{i+1}$, and, by Claim 3, this means that there is no α-vertex on a cycle in $G_{\geq b_i}$.

The edge uv is an $\alpha\beta$-edge where neither (α, β) nor (β, α) is in $\mathcal{S}_{i+1}$, since otherwise uv would be in E_i. If $\alpha, \beta \geq a_m$, then $(\alpha, \beta) \in \mathcal{S}_1$ or $(\beta, \alpha) \in \mathcal{S}_1$, by the definition of $\mathcal{S}_1 = \mathcal{S}$, and if $\beta = b_r$ for some $1 \leq r < j$, then $(\alpha, \beta) \in \mathcal{S}_{r+1}$ according to part (b) and (c) of Step r. In both cases we must have that either (α, β) or (β, α) is in $\mathcal{S}'_j$ for some $j < i + 1$, because otherwise (α, β) or (β, α) is in $\mathcal{S}_{i+1}$. However, according to part (a) of Step j, this happens only if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_j}$. This clearly contradicts the fact that there is no α-vertex on a cycle in $G_{\geq b_i}$. □

Next, we define H''_i to be the graph obtained from $G''_{\geq b_i}$ by removing all edges corresponding to edges in E_i and all vertices corresponding to vertices in V_i. So H''_i is the subgraph of $G''_{\geq b_i}$ corresponding to the subgraph H_i of $G_{\geq b_i}$.

Instead of proving (3), we prove, by induction, that the following stronger statement holds for every integer $i \in [n]$:

$$\begin{aligned}\chi(G''_{\geq b_i}) \leq \max(&\{\alpha+\beta+\gamma-3b_{i+1} \mid (\alpha,\beta,\gamma) \in \mathcal{T}_{i+1}\} \\ &\cup\ \{\alpha+\beta-2b_{i+1} \mid (\alpha,\beta) \in \mathcal{S}_{i+1}\} \\ &\cup\ \{a_j - b_{i+1} \mid a_j \in \mathcal{A}_{i+1}\}),\end{aligned} \tag{4}$$

and

$$\chi(H''_i) \leq \max\{\alpha+\beta+\gamma-3b_{i+1} \mid (\alpha,\beta,\gamma) \in \mathcal{T}_{i+1}\}. \tag{5}$$

The subgraph $G'_{\geq b_i} - V(G''_{\geq b_i})$ of $G'_{\geq b_i}$ is a b_{i+1}-inflation of a 3-colorable graph, and so, by Lemma 1, $\chi(G'_{\geq b_i} - V(G''_{\geq b_i})) \leq 3b_{i+1}$. This along with (4) implies that (3) holds.

We first prove that (4) and (5) hold for $i = 1$.

Claim 5. *The upper bounds* (4) *and* (5) *hold for* $i = 1$.

Proof of Claim 5. We shall first give an upper bound on $\chi(G''_{\geq a_m} \cap H''_1)$ and then extend this to an upper bound on $\chi(H''_1)$, thus establishing that (5) hold for $i = 1$. Of course, $G_{\geq a_m} \cap H_1$ is a subgraph of $G_{\geq a_m}$, and $G_{\geq a_m}$ is a 2-colorable graph, in particular, it is a perfect graph. Thus, by Theorem 2, $G''_{\geq a_m} \cap H''_1$ is a perfect graph, and so $\chi(G''_{\geq a_m} \cap H''_1) = \omega(G''_{\geq a_m} \cap H''_1)$. Since $G''_{\geq a_m} \cap H''_1$ is an inflation of the triangle-free graph $G_{\geq a_m} \cap H_1$ it follows that any largest clique in $G''_{\geq a_m} \cap H''_1$ corresponds to a single vertex or a pair of adjacent vertices in $G_{\geq a_m} \cap H_1$. Thus, $\chi(G''_{\geq a_m} \cap H''_1)$ is at most

$$\max\left(\{\alpha+\beta-2b_1 \mid \alpha\beta\text{-edge in } G_{\geq a_m} \cap H_1\} \cup \{\alpha - b_1 \mid \alpha\text{-vertex in } G_{\geq a_m} \cap H_1\}\right). \tag{6}$$

By Claim 1, an $\alpha\beta$-edge e of $G_{\geq a_m}$ with $(\alpha,\beta) \in \mathcal{S}_1$ is in H_1 if and only if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_1}$. According to part (a) of Step 1, an element $(\alpha,\beta) \in \mathcal{S}_1 = \mathcal{S}$ is in $\mathcal{S}'_1$ if and only if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_1}$.

Similarly, by Claim 3, an α-vertex v of $G_{\geq a_m}$ with $\alpha \in \mathcal{A}_1$ is in H_1 if and only if there is an α-vertex on a cycle in $G_{\geq b_1}$. According to part (b) of Step 1, an element $\alpha \in \mathcal{A}_1 = \mathcal{A}$ is in $\mathcal{A}'_1$ if and only if there is an α-vertex on a cycle in $G_{\geq b_1}$. Hence, by (6) we may now conclude that $\chi(G''_{\geq a_m} \cap H''_1)$ is at most

$$\max\left(\left\{\alpha+\beta-2b_1 \mid (\alpha,\beta) \in \mathcal{S}'_1\right\} \cup \left\{\alpha - b_1 \mid \alpha \in \mathcal{A}'_1\right\}\right). \tag{7}$$

For any element α of $\mathcal{A}'_1$ for which there is an element $a_j \in \{a_1,\dots,a_m\}$ such that there is an αa_j-edge on a cycle in $G_{\geq b_1}$, (α, a_j) or (a_j, α) is included in $\mathcal{S}'_1$, and so, since $\alpha + a_j - 2b_1 \geq \alpha - b_1$, the value of (7) is unaffected by removing such an element $\alpha - b_1$ from the second set in (7). By part (b) of Step 1, for any element α of $\mathcal{A}'_1$ for which there is *no* $a_j \in \{a_1,\dots,a_m\}$ such that $G_{\geq b_1}$ contains an αa_j-edge on a cycle, the element (α, b_1, b_1) is included in $\mathcal{T}'_1$. Thus, $\chi(G''_{\geq a_m} \cap H''_1)$ is at most

$$\max\left(\left\{\alpha+\beta-2b_1 \mid (\alpha,\beta) \in \mathcal{S}'_1\right\} \cup \left\{\alpha - b_1 \mid (\alpha, b_1, b_1) \in \mathcal{T}'_1\right\}\right). \tag{8}$$

Since $G''_{\geq b_1} - V(G''_{\geq a_m})$ is an inflation of a 3-colorable graph with inflation sizes at most $b_1 - b_2$, it follows from Lemma 1 that $\chi(G''_{\geq b_1} - V(G''_{\geq a_m})) \leq 3(b_1 - b_2)$. Thus, since H''_1 is a subgraph of $G''_{\geq b_1}$, we also have $\chi(H''_1 - V(G''_{\geq a_m})) \leq 3(b_1 - b_2)$. Thus, combining optimal colorings of $H''_1 - V(G''_{\geq a_m})$ and $G''_{\geq a_m} \cap H''_1$ using disjoint sets of colors for a coloring of H''_1, we deduce that

$$\chi(H''_1) \leq \max\left(\{\alpha+\beta+b_1-3b_2 \mid (\alpha,\beta) \in \mathcal{S}'_1\} \cup \{\alpha+2b_1-3b_2 \mid (\alpha,b_1,b_1) \in \mathcal{T}'_1\} \cup \{3b_1-3b_2\}\right). \tag{9}$$

Since, by part (c) of Step 1,

$$\mathcal{T}_2 = \mathcal{T}_1 \cup \{(a_{j_1}, a_{j_2}, b_1) \mid (a_{j_1}, a_{j_2}) \in \mathcal{S}_1'\} \cup \mathcal{T}_1' \cup \{(b_1, b_1, b_1)\},$$

it follows that

$$\chi(H_1'') \le \max\{\alpha + \beta + \gamma - 3b_2 \mid (\alpha, \beta, \gamma) \in \mathcal{T}_2\}, \tag{10}$$

which means that (5) holds for $i = 1$.

Let I'' denote the subgraph of $G''_{\ge b_1}$ corresponding to the edge-induced subgraph $G_{\ge b_1}[E_1]$ of $G_{\ge b_1}$. By Claim 2, each edge in E_1 is a cut-edge of $G_{\ge b_1}$, and so, $G_{\ge b_1}[E_1]$ is a forest, in particular, it is a 2-colorable graph and, hence a perfect graph. Thus, by Theorem 2, I'' is a perfect graph, and so the chromatic number of I'' is equal to the clique number of I''. This implies that

$$\chi(I'') \le \max\{\alpha + \beta - 2b_2 \mid \alpha\beta\text{-edge in } E_1\}. \tag{11}$$

Recall that E_1 is the set of all $\alpha\beta$-edges in $G_{\ge b_1}$ with $(\alpha, \beta) \in \mathcal{S}_2$. Thus,

$$\chi(I'') \le \max\{\alpha + \beta - 2b_2 \mid (\alpha, \beta) \in \mathcal{S}_2\}. \tag{12}$$

Let J'' denote the subgraph $G''_{\ge b_1} - V(H_1'') - V(I'')$ of $G''_{\ge b_1}$. Note that any component in J'' corresponds to an isolated vertex of $G_{\ge b_1}$ that is in V_1. Recall that V_1 is the set of all α-vertices in $G_{\ge b_1}$ with $\alpha \in \mathcal{A}_2$. This implies that the chromatic number of J'' is at most

$$\max\{a_j - b_2 \mid a_j \in \mathcal{A}_2\}. \tag{13}$$

Putting (10), (12), and (13) together and using Lemma 2, we may now deduce that (4) holds for $i = 1$ in the following way:

First we properly color the graph H_1'' with at most the number of colors in the right hand side of (10). Then by using Lemma 2 for the edges of I'', which correspond to edges of E_1, we may properly color the graph $H_1'' \cup I''$ using at most

$$\max(\{\alpha + \beta + \gamma - 3b_2 \mid (\alpha, \beta, \gamma) \in \mathcal{T}_2\} \cup \{\alpha + \beta - 2b_2 \mid (\alpha, \beta) \in \mathcal{S}_2\})$$

colors. Finally, we can color the vertices of J'' using at most

$$\max(\{a_j - b_2 \mid a_j \in \mathcal{A}_2\})$$

colors. This completes the proof of the claim. □

We now prove that (4) and (5) hold in the general case.

Claim 6. *The upper bounds* (4) *and* (5) *hold for any* $i \in [n]$.

Proof of Claim 6. Our induction hypothesis is that the following holds:

$$\begin{aligned}\chi(G''_{\ge b_{i-1}}) \le \max(&\{\alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in \mathcal{T}_i\} \\ &\cup \{\alpha + \beta - 2b_i \mid (\alpha, \beta) \in \mathcal{S}_i\} \\ &\cup \{a_j - b_i \mid a_j \in \mathcal{A}_i\}),\end{aligned} \tag{14}$$

and

$$\chi(H_{i-1}'') \le \max\{\alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in \mathcal{T}_i\}. \tag{15}$$

The basis for the induction was established in Claim 5. We are going to be using much the same approach as in the proof of Claim 5. First we give an upper bound on $\chi(G''_{\geq b_{i-1}} \cap H''_i)$ and then extend this to an upper bound on $\chi(H''_i)$.

Recall that E_i is the set of all $\alpha\beta$-edges in $G_{\geq b_i}$ with

$$(\alpha, \beta) \in \mathcal{S}_{i+1} = \left(\mathcal{S}_i \setminus \mathcal{S}'_i\right) \cup \{(\gamma, b_i) \mid \gamma \in \mathcal{A}''_i\}$$

and that V_i is the set of all α-vertices v of $G_{\geq b_i}$ with $\alpha \in \mathcal{A}_{i+1} = \mathcal{A}_i \setminus \mathcal{A}'_i$. Furthermore, we have that $H_i = G_{\geq b_i} - E_i - V_i$, and H''_i is the subgraph of $G''_{\geq b_i}$ corresponding to H_i.

Consider the graph $G_{\geq b_{i-1}} \cap H_i$. Since $\mathcal{A}_{i+1} \subseteq \mathcal{A}_i$ and $\mathcal{S}_{i+1} \subseteq \mathcal{S}_i \cup \{(\gamma, b_i) \mid \gamma \in \mathcal{A}''_i\}$, it holds that $H_{i-1} \subseteq H_i$, and thus H_{i-1} is a subgraph of $G_{\geq b_{i-1}} \cap H_i$.

Suppose that e is an $\alpha\beta$-edge of E_{i-1}. This means that $(\alpha, \beta) \in S_i$; also by Claim 2 there is no $\alpha\beta$-edge on a cycle in $G_{\geq b_{i-1}}$. Moreover, by part (c) of Step i, $(\alpha, \beta) \in \mathcal{S}_{i+1}$, and thus $e \in E_i$, unless $(\alpha, \beta) \in \mathcal{S}'_i$, which by part (a) means that there is an $\alpha\beta$-edge in a cycle of $G_{\geq b_i}$. Hence $e \in E(H_i)$ if and only if $(\alpha, \beta) \in \mathcal{S}'_i$.

Now consider an α-vertex $v \in V_{i-1}$. Clearly, $\alpha \in \mathcal{A}_i$; also by Claims 2 and 4, there is no α-vertex on a cycle of $G_{\geq b_{i-1}}$. Moreover, by part (c) of Step i, $\alpha \in \mathcal{A}_{i+1}$, and thus $v \in V_i$, unless $\alpha \in \mathcal{A}'_i$, which by part (b) means that there is an α-vertex on a cycle in $G_{\geq b_i}$. Hence $v \in V(H_i)$ if and only if $\alpha \in \mathcal{A}'_i$.

Now, any edge of E_{i-1} is a cut-edge of $G_{\geq b_{i-1}}$, and any vertex of V_{i-1} is an isolated vertex of $G_{\geq b_{i-1}} - E_{i-1}$ (by Claims 2 and 4). So for the inflation $G''_{\geq b_{i-1}} \cap H''_i$ it now follows from (15) and by applying Lemma 2 that

$$\begin{aligned}\chi(G''_{\geq b_{i-1}} \cap H''_i) \;\leq\; & \max(\{\alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in \mathcal{T}_i\} \\ & \cup \{\alpha + \beta - 2b_i \mid (\alpha, \beta) \in \mathcal{S}'_i\} \\ & \cup \{a_u - b_i \mid a_u \in \mathcal{A}'_i\}). \end{aligned} \tag{16}$$

Let us now prove the following:

Subclaim 1. *If $a_u \in \mathcal{A}'_i$, then either (a_u, b_i, b_i) in $\mathcal{T}'_i$, or there is an $\alpha \in \{a_1, \ldots, a_m, b_1, \ldots, b_{i-1}\}$, such that $(a_u, \alpha) \in \mathcal{S}'_i$ or $(\alpha, a_u) \in \mathcal{S}'_i$.*

Proof of Subclaim 1. Suppose that a_u is some element of $\mathcal{A}'_i$. By part (b) of Step i, $a_u \in \mathcal{A}_i$, and so, by Lemma 3 (1), $a_u \in \mathcal{A}_q$ for any $q < i$. By Lemma 3 (2), $a_u \notin \mathcal{A}'_q$ for any $q < i$. Thus i is the minimum integer q such that there is an a_u-vertex on a cycle in $G_{\geq b_q}$.

Suppose that $(a_u, b_i, b_i) \notin \mathcal{T}'_i$. Then it follows from part (b) of Step i that there is an $a_u\alpha$-edge e on a cycle in $G_{\geq b_i}$ for some $\alpha \in \{a_1, \ldots, a_m, b_1, \ldots, b_{i-1}\}$. We shall prove that $(a_u, \alpha) \in \mathcal{S}'_i$ or $(\alpha, a_u) \in \mathcal{S}'_i$. Since there is an $a_u\alpha$-edge e on a cycle in $G_{\geq b_i}$ for some $\alpha \in \{a_1, \ldots, a_m, b_1, \ldots, b_{i-1}\}$, the desired result will follow from part (a) of Step i if we can prove that $(a_u, \alpha) \in \mathcal{S}_i$ or $(\alpha, a_u) \in \mathcal{S}_i$.

Thus in the following we will argue that (a_u, α) or (α, a_u) is in $\mathcal{S}_i$. We shall distinguish between two cases: $\alpha \in \{a_1, \ldots, a_m\}$ and $\alpha \in \{b_1, \ldots, b_{i-1}\}$.

(i) Suppose $\alpha \in \{a_1, \ldots, a_m\}$. Then, at least one of the elements (a_u, α) and (α, a_u) must be in $\mathcal{S}_1$, since $\mathcal{S}_1 = \mathcal{S}$ and, by definition, $\mathcal{S}$ contains all ordered pairs (a_i, a_j) of $\mathcal{A}$ with $a_i \geq a_j$ for which there is an $a_i a_j$-edge in G.

Suppose (a_u, α) is in $\mathcal{S}_1$. Assume that (a_u, α) is not in $\mathcal{S}_i$. Then (a_u, α) is included in $\mathcal{S}'_p$ at Step p of the algorithm for some $p < i$. By part (a) of Step p, this means that there is an $a_u\alpha$-edge on a cycle in $G_{\geq b_p}$. This, however, is a contradiction to the fact that i is the minimum integer q for which there is an a_u-vertex on a cycle in $G_{\geq b_q}$. Hence $(a_u, \alpha) \in \mathcal{S}_i$.

If $(\alpha, a_u) \in \mathcal{S}_1$, then a similar argument shows that $(\alpha, a_u) \in \mathcal{S}_i$.

(ii) Suppose $\alpha \in \{b_1, \dots, b_{i-1}\}$, say $\alpha = b_p$ for some $p \in [i-1]$.

The integer i is the minimum integer q such that there is an a_u-vertex on a cycle in $G_{\geq b_q}$ and thus $G_{\geq b_p}$ has no cycle with an $a_u b_p$-edge. Moreover, by part (b) of Step p, a_u is included in $\mathcal{A}''_p$. Now, by part (c) of Step p, (a_u, b_p) is included in $\mathcal{S}_{p+1}$.

The rest of the argument goes along the same lines as in (i): Assume that (a_u, b_p) is not in $\mathcal{S}_i$. Then (a_u, b_p) is included in $\mathcal{S}'_k$ at some step k of the algorithm for some integer k satisfying $p < k < i$. But this means that there is an $a_u b_p$-edge on a cycle in $G_{\geq b_k}$. This, however, is a contradiction to the fact that i is the minimum integer q for which there is an a_u-vertex on a cycle in $G_{\geq b_q}$. Hence $(a_u, b_p) \in \mathcal{S}_i$. □

Subclaim 1 along with (16) implies

$$\begin{aligned}\chi(G''_{\geq b_{i-1}} \cap H''_i) \quad \leq \quad & \max(\{\alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in \mathcal{T}_i\} \\ & \cup \{\alpha + \beta - 2b_i \mid (\alpha, \beta) \in \mathcal{S}'_i\} \\ & \cup \{a_u - b_i \mid a_u \in \mathcal{T}'_i\}),\end{aligned} \tag{17}$$

It follows from Lemma 1 that any proper coloring of $G''_{\geq b_{i-1}} \cap H''_i$ can be extended to a proper coloring of H''_i by using at most $3(b_i - b_{i+1})$ new colors, because the graph $H''_i - V(G''_{\geq b_{i-1}} \cap H''_i)$ is an inflation of a 3-colorable graph with inflation sizes at most $3(b_i - b_{i+1})$. That fact along with (17) implies

$$\begin{aligned}\chi(H''_i) \quad \leq \quad & \max(\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in \mathcal{T}_i\} \cup \{\alpha + \beta + b_i - 3b_{i+1} \mid (\alpha, \beta) \in \mathcal{S}'_i\} \\ & \cup \{a_j + 2b_i - 3b_{i+1} \mid (a_j, b_i, b_i) \in \mathcal{T}'_i\}),\end{aligned} \tag{18}$$

Note that, since $\mathcal{T}_{i+1} = \mathcal{T}_i \cup \{(\alpha, \beta, b_i) \mid (\alpha, \beta) \in \mathcal{S}'_i\} \cup \mathcal{T}'_i$, (18) implies that (5) holds. By Claim 2, every edge in E_i is a cut-edge of $G_{\geq b_i}$, so the edge-induced subgraph $G_{\geq b_i}[E_i]$ is a forest, in particular, it is a perfect graph. Thus, by Theorem 2, the subgraph I''_i of $G''_{\geq b_i}$ corresponding to $G_{\geq b_i}[E_i]$ satisfies

$$\chi(I''_i) = \omega(I''_i) \leq \max\{\alpha + \beta - 2b_{i+1} \mid (\alpha, \beta) \in \mathcal{S}_i \setminus \mathcal{S}'_i\} \cup \{\alpha + b_i - 2b_{i+1} \mid \alpha \in \mathcal{A}''_i\}. \tag{19}$$

Finally, let J''_i denote the subgraph $G''_{\geq b_i} - V(H''_i) - V(I''_i)$. Clearly, any component of J'' corresponds to an isolated vertex of $G_{\geq b_i}$ that is in V_i. Thus

$$\chi(J''_i) \leq \max\{\alpha - b_{i+1} \mid \alpha \in \mathcal{A}_i \setminus \mathcal{A}'_i\}. \tag{20}$$

Putting (18)-(20) together and applying Lemma 2 we now deduce that

$$\begin{aligned}\chi(G''_{\geq b_i}) \quad \leq \quad & \max(\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in \mathcal{T}_{i+1}\} \\ & \cup \{\alpha + \beta - 2b_{i+1} \mid (\alpha, \beta) \in \mathcal{S}_{i+1}\} \\ & \cup \{\alpha - b_{i+1} \mid \alpha \in \mathcal{A}_{i+1}\},\end{aligned}$$

which implies that (4) holds.

It now follows by induction that (4) and (5) hold for every $i \in [n]$. □

The statement of the lemma now follows from (4), since, as pointed out above, for any $i \in [n]$, the inequality (3) follows from (4). □

Lemma 5. *At the end of Step n, the sets $\mathcal{S}_{n+1}$ and $\mathcal{A}_{n+1}$ are empty.*

Proof. We first consider the sets $\mathcal{S}_1, \dots, \mathcal{S}_{n+1}$. According to the description of the algorithm, $\mathcal{S}_{i+1}$ is constructed from $\mathcal{S}_i$ at Step i by removing any element (α, β) from $\mathcal{S}_i$ for which there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_i}$, and adding any element (α, b_i) for which

(i) $\alpha \in \mathcal{A}_i$

(ii) there is an αb_i-edge of $G_{\geq b_i}$, and

(iii) there is no αb_i-edge on a cycle in $G_{\geq b_i}$.

Note that by part (b) and (c) of Step $1, \dots, i$, α is in $\mathcal{A}_i$ if and only if $\alpha \in \{a_1, \dots, a_m\}$ and there is no α-vertex in a cycle of $G_{\geq b_i}$. Since G is 2-connected, every edge (and vertex) of G lies on a cycle in G, and since $G = G_{\geq b_n}$, this means that $\mathcal{S}_{n+1}$ is empty.

According to the description of the algorithm, $\mathcal{A}_{i+1}$ is constructed from $\mathcal{A}_i$ at Step i by removing any element a_j from $\mathcal{A}_i$ such that there is an a_j-vertex that lies on a cycle in $G_{\geq b_i}$. Again, since G is 2-connected, any vertex of $G = G_{\geq b_n}$ lies on a cycle, which implies the desired result. □

Lemma 6. *For each $(\alpha, \beta, \gamma) \in \mathcal{T}_{n+1}$, G' contains a complete minor of size $\alpha + \beta + \gamma$.*

Proof. By Lemma 3 (3), $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_{n+1}$. Let j be the minimum integer such that $(\alpha, \beta, \gamma) \in \mathcal{T}_j$.

By the definition of $\mathcal{T}_j$ in part (c) of Step $(j-1)$, (α, β, γ) must be in one of the sets

$$\{(\alpha', \beta', b_{j-1}) \mid (\alpha', \beta') \in \mathcal{S}'_{j-1}\}$$

and $\mathcal{T}'_{j-1}$. Moreover, by the definition of these sets in part (a) and (b) of Step $(j-1)$, γ is not greater than α or β.

Suppose $(\alpha, \beta, \gamma) \in \{(\alpha', \beta', b_{j-1}) \mid (\alpha', \beta') \in \mathcal{S}'_{j-1}\}$, that is, $(\alpha, \beta) \in \mathcal{S}'_{j-1}$. Then, $\gamma = b_{j-1}$ and there exists an $\alpha\beta$-edge on a cycle C of $G_{\geq b_{j-1}}$. The inflated cycle in $G'_{\geq b_{j-1}}$ corresponding to C can be contracted to a complete graph on $\alpha + \beta + b_{j-1}$ vertices, and so $\eta(G') \geq \alpha + \beta + \gamma$.

Now suppose $(\alpha, \beta, \gamma) \in \mathcal{T}'_{j-1}$. Then, by definition of $\mathcal{T}'_{j-1}$, we have $\beta = \gamma = b_{j-1}$ and $\alpha \in \mathcal{A}'_{j-1}$, that is, there is an α-vertex on a cycle C in $G_{\geq b_{j-1}}$. The inflated cycle in $G'_{\geq b_{j-1}}$ corresponding to C can be contracted to a complete graph on $\alpha + 2\beta_{j-1}$ vertices, and so $\eta(G') \geq \alpha + \beta + \gamma$. □

By Lemma 4 and 5, $\chi(G') \leq \max\{\alpha+\beta+\gamma \mid (\alpha, \beta, \gamma) \in \mathcal{T}_{n+1}\}$, and so, by Lemma 6, $\eta(G') \geq \chi(G')$. Thus, G' is not a counterexample to Hadwiger's Conjecture, and we have obtained a contradiction from which the theorem follows. □

Algorithm 1 together with the proof of Lemma 4 can be used to produce a proper coloring φ of any inflation of any 2-connected 3-chromatic graph such that the number of colors used in φ is at most $\max\{\alpha + \beta + \gamma \mid (\alpha, \beta, \gamma) \in \mathcal{T}_{n+1}\}$. (The case when the graph is not 2-connected can be handled by Lemma 2.) Since a triple (α, β, b_j) is in $\mathcal{T}_{i+1}$ at Step i of the algorithm, where $j \leq i$, if and only there is an α-vertex and a β-vertex in G that are adjacent and lie on a cycle C of G, which satisfies that every vertex in C is replaced by a clique of size at least b_j in G', we in fact have that the number of colors used in φ is at most

$$\max\{\alpha + \beta + \gamma \mid \text{there is an } \alpha\beta\text{-edge in } G \text{ that lies on a cycle where every vertex is replaced by a clique of size at least } \gamma\}.$$

References

[1] K. Appel and W. Haken, Every planar map is four colorable. Part I. Discharging, *Illinois. J. Math.* 21 (1977), 429–490.

[2] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.* 21 (1977), 491–567.

[3] M. O. Albertson, D. L. Boutin, E. Gethner, The thickness and chromatic number of r-inflated graphs, *Discrete Math.* 3010 (2010), 2725–2734.

[4] M. O. Albertson, D. L. Boutin, E. Gethner, More results on r-inflated graphs: arborocity, thickness, chromatic number and fractional chromatic number, *Ars Math. Contemp.* 4 (2011), 5–24.

[5] B. Bollobás, P.A. Catlin, P. Erdős, Hadwiger's Conjecture is True for Almost Every Graph, *Eur. J. Combin.* 1 (1980), 195–199.

[6] P. A. Catlin, Hajós graph-coloring conjecture: variations and counterexamples, *J. Combin. Theory Ser.* B 26 (1979), 268–274.

[7] H. Hadwiger, Über eine klassifikation der streckenkomplexe, em Vierteljschr. Naturforsch. Ges. Zürich, 88 (1943), 133–143.

[8] L. Lovasz, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* 2 (1972) 253–267.

[9] A. S. Pedersen, Hadwiger's Conjecture and inflations of the Petersen graph, *Discrete Math.* 312 (2012), 3537–3543.

[10] M. D. Plummer, M. Stiebitz, B. Toft, On a special case of Hadwiger's conjecture *Discuss. Math. Graph Theory* 23 (2003), 333–363.

[11] B. Reed and P. Seymour, Hadwiger's conjecture for line graphs, *Eur. J. Combin.* 25(6) (2004), 873–876.

[12] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* 13 (1993), 279–361.

[13] M. Chudnovsky and A. Ovetsky Fradkin Hadwiger's conjecture for quasi-line graphs, *J. Graph Theory* 59 (1) (2008), 17–33.

[14] C. Thomassen, Some remarks on Hajós conjecture, *J. Combin. Theory Ser. B* 93 (2005), 95–105.

[15] B. Toft, A survey of Hadwiger's conjecture, *Congr. Numer.* 115 (1996), 249–283.

[16] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* 114 (1937).