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REPORT No. 29, 2013/2014, spring

ISSN 1103-467X

ISRN IML-R- -29-13/14- -SE+spring

A NOTE ON VERTEX COLORING EDGE-WEIGHTED DIGRAPHS

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ABSTRACT. A coloring of a digraph with non-negative edge weights is a partition of the vertex set into independent sets, where a set is independent if the weighted in-degree of each node within the set is less than 1. We give constructive optimal bounds on the chromatic number in terms of maximum in-degree and inductiveness of the graph.

1. INTRODUCTION

Let $D = (V, E, w)$ be a digraph with an asymmetric weight function $w : E \rightarrow \mathbb{R}_0^+$ mapping edges to non-negative real numbers. Let $n = |V|$. The (weighted) *in-degree* of node v with respect to a set S is $d_S^-(v) = \sum_{u \in S} w(u, v)$. A subset S of V is an *independent set* (or *color*) if $d_S^-(v) < 1$ holds for every v in S , i.e., if the in-degree of each node in S is strictly less than 1. A *coloring* of D is a partition of V into independent sets and the *chromatic number* $\chi(D)$ is the minimum number of colors needed on D . Observe that these definitions properly generalize independent sets and colorings in ordinary graphs, which correspond to the special case of 0-1 weight functions.

We explore here bounds on the chromatic number of edge-weighted digraphs in terms of degree parameters of the graph. In particular, we consider the maximum in-degree $\Delta^-(D) = \max_{v \in V} d^-(v)$, where $d^-(v) = d_V^-(v)$, and the *inductiveness* $\tau^-(D) = \max_{H \subseteq D} \min_{v \in V(H)} d_{V(H)}^-(v)$.

Previous work. This problem has origin in the scheduling of wireless communication links under the SINR model of interference; Hoefer, Kesselheim and Vöcking [12] were the first to explicitly propose an edge-weighted graph formulation. Each node in a conflict graph corresponds to a communication *link* and the weight of the edge from u to v corresponds to the relative interference (or *affectance* [10]) of link u on link v . A set of links is feasible if all the links can successfully communicate simultaneously. In fact, feasibility of a link set corresponds precisely to independence in the link conflict graph¹.

The link scheduling problem is usually studied in a metric space. This naturally constrains the possible edge weightings, which significantly impacts the computational tractability of the problem. In fact, our independent set problem is constant-factor approximable in a metric setting [9], whereas in arbitrary ordinary graphs the problem is hard to approximate within $\Omega(n^{1-\epsilon})$ factor, for any $\epsilon > 0$ [11].

The link scheduling problem was first posed as an algorithmic problem by Moscibroda and Wattenhofer in 2006 [15]. The problem was shown to be NP-complete by Goussevskaia et al. [7], even to determine if $\chi(D) \leq 2$ (for a special subclass of metric instances). While most results known

Date: October 21, 2014.

Work done as part of the program “Graphs, hypergraphs and computing” at Institut Mittag-Leffler, spring 2014. The authors gratefully thank the institute for the hospitality and support.

Department of Mathematics and Computer Science, University of Southern Denmark. Also supported by the Danish research council under grant number 1323-00178B..

ICE-TCS, School of Computer Science, Reykjavik University, 101 Reykjavik, Iceland. Also supported by Icelandic Research Fund grant-of-excellence no. 120032011.

¹A minor caveat is that in scheduling studies, feasibility corresponds to a set where the weighted in-degree of each link is at most 1 (not necessarily strictly less).

apply to specific metric settings, there are some results known for the general case. A bound of $\chi(D) \leq \lfloor 2\Delta^- + 1 \rfloor^2$ was given in [10], attained by a simple sequential algorithm. When Δ^- is sufficiently large, a randomized distributed algorithm attains an asymptotically stronger bound: $\chi(D) = O(\Delta^- + \log^2 n)$ [6].

Some related problems have been studied in the graph literature. A graph G is said to be *T-improper k-colorable* (also known as *defective coloring*) if the vertices can be partitioned into k induced subgraphs each of maximum degree at most T . Numerous papers have been written on this problem, which can be viewed as a restriction of our problem to the case when all the discretized weights are equal and same in both directions.

Araujo et al. [2] recently addressed the weighted version of the improper coloring problem, which corresponds to the undirected (symmetric) version of our problem. Their main result was an application of an result of Lovász [14]. The directed version of the problem (i.e., the discretized version of our problem) was recently considered by Archetti et al. [3], who gave a branch-and-bound algorithm.

Our contributions. We give constructive bounds on the edge-weighted chromatic number in terms of the degree parameters of the graph: maximum in- and out-degree, and inductiveness. The bounds are essentially tight. The results have implications for the theory of wireless scheduling in the SINR model.

In Section 2, we build on a result of Alon to obtain an upper bound in terms of maximum in-degree. We then show in Section 3 that stronger lower bounds hold in terms of the other degree parameters, whereas a better bound can be obtained for the corresponding independence number of sparse instances. The applications to SINR theory are indicated in Section 4 before closing off with conclusions.

2. BOUNDS IN TERMS OF MAXIMUM IN-DEGREE

We obtain an essentially tight bound on $\chi(D)$ in terms of the maximum indegree. We need the following lemma of Alon [1] that generalizes a result that he attributes Keith Ball citing [4].

Lemma 1. *Let $A = (a_{ij})$ be an n by n real matrix, where $a_{ii} = 0$ for all i , $a_{ij} \geq 0$ for all $i \neq j$, and $\sum_j a_{ij} \leq 1$ for all i . Then, for every k and positive reals c_1, \dots, c_k whose sum is 1, there is a partition of $[n] = \{1, 2, \dots, n\}$ into pairwise disjoint sets S_1, S_2, \dots, S_k , such that for every r , $1 \leq r \leq k$ and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \leq 2c_r$.*

Using this lemma, we get the following.

Theorem 1. *For every digraph D , $\chi(D) \leq \lfloor 2\Delta^- + 1 \rfloor$.*

Proof. Given D with $|V(D)| = n$, form the matrix (a_{ij}) where $a_{ij} = w(v_j, v_i)/\Delta^-$. Let $k = \lfloor 2\Delta^- + 1 \rfloor$ and define $c_r = 1/k$, for $1 \leq r \leq k$. These parameters satisfy the conditions of Lemma 1. Let S_1, S_2, \dots, S_k be the partition of $V(D)$ resulting from applying Lemma 1 with these parameters. It then holds for each $1 \leq r \leq k$ and each $v_i \in S_r$ that

$$d_{S_r}^-(v_i) = \sum_{v_j \in S_r} w(v_j, v_i) = \Delta^- \sum_{v_j \in S_r} a_{ij} \leq \Delta^- \frac{2}{k} = \frac{2\Delta^-}{\lfloor 2\Delta^- + 1 \rfloor} < 1 .$$

Hence, the partition is a valid coloring. □

This turns out to be a tight bound.

Proposition 1. *For every natural number t , there is a digraph D with $\Delta^-(D) = t$ and $\chi(D) = 2\Delta^- + 1$.*

Proof. Consider a regular tournament T_n with $n = 2k + 1$, i.e., where each vertex has in- and out-degree k . Then, viewing the edges as having weight 1, we see that each node must receive a different color. \square

With a slight increase in the number of colors, we can obtain an algorithmic version.

Lemma 2. *Let $q > 0$ and let $A = (a_{ij})$ be an n by n real matrix, where $a_{ii} = 0$ for all i , $a_{ij} > q$ for all $i \neq j$, and $\sum_j a_{ij} \leq 1$ for all i . Also let k be a number and $\epsilon > 0$. There is an algorithm running in time polynomial in n , $1/q$, and $1/\epsilon$ that finds a partition of $[n] = \{1, 2, \dots, n\}$ into disjoint sets S_1, S_2, \dots, S_k , such that for every r , $1 \leq r \leq k$ and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \leq 2/k + \epsilon/q$.*

Proof. We follow closely Alon's proof of Lemma 1. By increasing some of the numbers a_{ij} , if needed, we may assume that $\sum_j a_{ij} = 1$ for all i . Thus, by the Perron-Frobenius Theorem, 1 is the largest eigenvalue of A , with right eigenvector $(1, 1, \dots, 1)$, and A has a left eigenvector (u_1, u_2, \dots, u_n) in which all entries are positive and $\sum_j u_j = 1$. It follows that for all j , $\sum_i u_i a_{ij} = u_j$. Observe that for all j , $u_j = \sum_i u_i a_{ij} \geq q \sum_i u_i = q$.

Define $b_{ij} = u_i a_{ij}$, and note that $\sum_i b_{ij} = u_j$ and $\sum_j b_{ij} = u_i (\sum_j a_{ij}) = u_i$. Define the potential function Φ that, given a partition $\Pi = (S_1, S_2, \dots, S_k)$ of $[n]$ into k disjoint sets, has value

$$\Phi(\Pi) = \sum_{r=1}^k \sum_{i \in S_r} \sum_{j \in S_r} b_{ij} .$$

In essence, Φ counts the total b -weight of ordered pairs contained in the same set S_r .

Now, suppose Π_ϵ is a partition such that every move of a single vertex to a different class decreases the potential function by at most ϵ . Consider i, r, t , where $i \in S_r$, and let Π' be the partition obtained from Π_ϵ by moving i from S_r to S_t . Then, using also the assumption about Π_ϵ ,

$$\Phi(\Pi_\epsilon) - \Phi(\Pi') = \sum_{j \in S_r} (b_{ij} + b_{ji}) - \sum_{j \in S_t} (b_{ij} + b_{ji}) \leq \epsilon .$$

Summing up over all t , we get that

$$k \sum_{j \in S_r} (b_{ij} + b_{ji}) \leq \sum_{j \in [n]} (b_{ij} + b_{ji}) + k\epsilon = 2u_i + k\epsilon .$$

Dividing by k and using the definition of b_{ij} , it follows that

$$\sum_{j \in S_r} u_i a_{ij} = \sum_{j \in S_r} b_{ij} \leq \sum_{j \in S_r} (b_{ij} + b_{ji}) \leq 2u_i/k + \epsilon .$$

Dividing by u_i , we get

$$\sum_{j \in S_r} a_{ij} \leq 2/k + \epsilon/u_i \leq 2/k + \epsilon/q ,$$

as desired.

To bound the time complexity within a polynomial factor, it suffices to bound the number of improvement operations performed. The maximum potential is $\sum_{i,j} b_{ij} = \sum_j u_j = 1$, and each iteration decreases the potential by at least ϵ . Thus, the number of iterations is ϵ^{-1} . A more efficient approach is to start with a greedy partition or a random partition, which have initial potential of only $1/k$, for a total of $1/(k\epsilon)$ iterations.

Each iteration can be implemented in $O(n \log n)$ time by maintaining the nk possible improvements in a binary heap, performing $O(n)$ change-key operations in each iteration. \square

Theorem 2. *Given $\delta > 0$, there is an algorithm running time polynomial in $1/\delta$ and n that colors edge-weighted digraphs using at most $\lfloor 2\Delta^- + 1 + \delta \rfloor$ colors.*

Proof. Let $D = (V, E)$ be an edge-weighted graph on n vertices. Assume without loss of generality that $2\Delta^- + 1 + \delta \leq n$, as we can otherwise assign each node a separate color.

Let $q = \delta/(4nX)$, $k = \lfloor 2\Delta^- + 1 + \delta \rfloor$, $X = \Delta^- + \delta/4$, $\epsilon' = \delta/(2X)$ and $\epsilon = \epsilon'q/k$. Form the matrix (a_{ij}) where $a_{ij} = \max(q, w(v_j, v_i)/X)$, for $i \neq j$ and $a_{ii} = 0$ for all i . Then,

$$\sum_j a_{ij} \leq qn + \frac{d^-(v_i)}{X} \leq \frac{\delta/4}{X} + \frac{\Delta^-}{X} = \frac{\delta/4 + \Delta^-}{\Delta^- + \delta/4} = 1,$$

satisfying the conditions of Lemma 2. Applying Lemma 2, we obtain a partition into k sets S_1, S_2, \dots, S_k with $\sum_{j \in S_r} a_{ij} \leq 2/k + \epsilon/q = (2 + \epsilon')/k$. We verify that the degree of each node within its subgraph is bounded:

$$d_{S_r}^-(v_i) = \sum_{v_j \in S_r} w(v_j, v_i) \leq X \sum_{v_j \in S_r} a_{ij} \leq X \frac{2 + \epsilon'}{k} = \frac{2X + \delta/2}{k} = \frac{2\Delta^- + \delta}{k} < 1.$$

Hence, the partition is a valid coloring.

Starting with a greedy partition, the number of iterations performed is at most

$$\frac{1}{k\epsilon} = \frac{8nX^2}{\delta^2} = O(n \cdot (\Delta^-)^2/\delta^2).$$

□

Note that to obtain a $O(\Delta^-)$ -coloring, the number of iterations is $O(n)$.

3. BOUNDS ON SPARSE INSTANCES

Many classes of graphs have large maximum degree but small average degree. If the average degree of every subgraph is low – which is captured by the inductiveness – then it may be possible to color the graph with much fewer colors than the maximum degree. For ordinary graphs, the chromatic number is at most one plus the inductiveness; note that, e.g., the inductiveness of planar graphs is at most 5 whereas their maximum degree is not bounded.

This is particularly important for link scheduling problems. While a constant-approximation algorithms are known for the independent set problem in metric spaces, the best results known for the corresponding coloring or scheduling problem give only logarithmic approximations [9, 8]. It is not known if it possible to do better, even with the known algorithms.

Let $\bar{d} = \bar{d}(D) = \frac{1}{n} \sum_{v,u \in V} w(v, u)$ be the average degree of a vertex in D ; the average in-degree and out-degree are the same. Observe that $\bar{d} \leq \tau^- \leq \Delta^-$. Let $\alpha(D)$ be the cardinality of the largest independent set in D .

It was argued in [5, Prop. 3] that for any digraph D , $\alpha(D) \geq n/(8\Delta^-)$. This can be turned into a Turán-like result.

Theorem 3. *For all digraphs D , $\alpha(D) = \Omega(n/\bar{d})$.*

Proof. We use the probabilistic method. Let $S = \{v \in V(D) : d^-(v) < 2\bar{d}\}$ be the subset of vertices of indegree less than twice the average. By Markov's inequality, $|S| \geq n/2$. Form a set R by selecting each vertex in S uniformly at random with probability $1/(4\bar{d})$. Let X_v denote the indicator random variable that v is in R . Then, $\mathbb{E}(X_v) = \mathbb{P}(X_v = 1) = 1/(4\bar{d})$.

Let D_v be a random variable denoting the in-degree of v in R . By linearity of expectation,

$$\mathbb{E}(D_v) = \sum_{u \in S} \mathbb{E}(X_u) \cdot w(u, v) = \frac{1}{4\bar{d}} \sum_{u \in S} w(u, v) = \frac{d_S^-(v)}{4\bar{d}} \leq \frac{1}{2}.$$

By Markov's inequality, the probability that D_v has value less than twice its average is at least half, i.e.,

$$\mathbb{P}(D_v < 1) \geq \mathbb{P}(D_v < 2 \cdot \mathbb{E}(D_v)) \geq \frac{1}{2}.$$

Let T be the set of vertices v in S with $D_v = d_{\bar{R}}(v) < 1$. The set $T \cap R$ is clearly independent. Observe that for a node v in S , the event that v is in R depends only on X_v , while the event that v is in T does not depend on X_v . Thus, the two events are independent, giving that

$$\mathbb{P}(v \in T \cap R) = \mathbb{P}(D_v < 1) \cdot \mathbb{P}(X_v = 1) \geq \frac{1}{8\bar{d}}.$$

By the linearity of expectation,

$$\mathbb{E}(T \cap R) \geq \frac{|S|}{8\bar{d}} \geq \frac{n}{16\bar{d}},$$

and the theorem follows. \square

The following is an easy corollary. The result was first shown to be attained by a distributed algorithm in [13].

Corollary 1. *For all digraphs D , $\chi(D) = O(\tau^- \cdot \log n)$.*

It turns out that this is close to best possible, at least for general instances.

Theorem 4. *For every positive number k , there is an infinite family of digraphs where each digraph D satisfies $\tau^-(D) = k$ and $\chi(D) = \Omega(k \cdot \log n)$.*

Proof. Let t be a positive integer. We form a complete graph D_t on the union of the vertex sets V_t, V_{t-1}, \dots, V_1 , where $|V_s| = n_s := 2^{s-1}(k+1)$. The weight of the edge (u_s, u_r) , with $u_s \in V_s$ and $u_r \in V_r$, is $w_s := k/(n_s - 1)$ if $s \geq r$ and 0 otherwise. This completes the specification of D_t . Note that $n = |V(D_t)| = \sum_{s=1}^t n_s < 2^t(k+1)$.

Observe that within the subgraph induced by $\cup_{i \leq s} V_i$, for $1 \leq s \leq t$, the nodes in V_s are of in-degree k , while nodes in V_r , $r < s$, have higher in-degree. Thus, within D_t , the nodes in V_t have the minimum in-degree of $(n_t - 1)w_t = k$. It follows that the instance is k -inductive, i.e., $\tau^-(D_t) = k$, with the ordering of the nodes given by V_t, V_{t-1}, \dots, V_1 .

Define the *potential* $p(v_s)$ of a node v_s in V_s as $p(v_s) = 1/n_s$, and the potential of a set S of vertices as $p(S) = \sum_{v \in S} p(v) = \sum_{i=1}^t \frac{|S \cap V_i|}{n_i}$. Observe that the potential of the whole graph is $p(V(D_t)) = t$.

Let S be an arbitrary feasible set. We claim that $p(S) < 2/k$. The claim implies that the chromatic number $\chi(D_t)$ is greater than $\frac{p(V(D_t))}{2/k} = \frac{tk}{2} \geq \frac{k}{2} \cdot \lg(n/(k+1))$, from which the theorem follows.

To prove the claim, let $R = \{i : S \cap V_i \neq \emptyset\}$ and let j be the smallest index in R . By the definition of the edge weights, for $v \in V_j$,

$$d_{\bar{S}}(v) = (|S \cap V_j| - 1) \cdot w_j + \sum_{i=j+1}^t |S \cap V_i| \cdot w_i.$$

Thus,

$$d_{\bar{S}}(v) + w_j = \sum_{i \neq j} w_i |S \cap V_i| = k \sum_i \frac{|S \cap V_i|}{|V_i| - 1} > k \sum_i \frac{|S \cap V_i|}{|V_i|} = k \cdot p(S).$$

Since S is feasible and contains v , it holds that $d_{\bar{S}}(v) < 1$. Thus, $p(S) < (1 + w_j)/k \leq 2/k$, as claimed.

Note that $t > \lg(n/(k+1)) = \Omega(\log n)$. It follows from the claim that $\chi(D_t) = \Omega(k \log n) = \Omega(\tau^- \log n)$. \square

The proof actually gives a stronger result. Let $d_S^+(v) = \sum_{u \in S} w(v, u)$ be the *out-degree* of v . and let $\Delta^+ = \Delta^+(D) = \max_{v \in V} d^+(v)$ be the maximum out-degree.

Theorem 5. *For every positive number k , there is an infinite family of digraphs where each digraph D satisfies $\chi(D) = \Omega(\Delta^+ \cdot \log n)$.*

Proof. Consider the graph D_t as in the proof of Thm. 4. Consider a node v_s in V_s , for $1 \leq s \leq t$. Observe that v_s has outgoing edges of non-zero weight only to V_s, \dots, V_1 and its out-degree is

$$d^+(v_s) = (n_s - 1)w_s + \sum_{i=1}^{s-1} n_i w_s = k + (n_s - 2)w_s < 2k .$$

The result then follows from Thm. 4. □

Thus, there are no upper bounds on χ in terms of either τ^- or Δ^+ alone.

4. IMPLICATIONS FOR WIRELESS SCHEDULING

The main theorem has two important applications to wireless algorithmics.

When many attempt to communicate at the same time, some sort of contention mechanism must be put into place. In wireless computing, the key issue is dealing with interference. Wireless scheduling is the problem of partitioning a given set of communication requests into the fewest number of feasible sets, i.e., where the links in each of the sets can communicate simultaneously without conflict due to interference. The input then consists of a collection of sender-receiver pairs, $(s_1, r_1), (s_2, r_2), \dots, (s_n, r_n)$.

In the SINR model of interference, a communication transmission is successful if the strength of the intended received signal is sufficiently larger than the cumulative signal strengths from other transmitting nodes. Signal decays as it travels, and the decay can vary from pair to pair. We here assume the most general version where arbitrary signal decay can be experienced between any (directed) pair, and denote the strength of the signal from s_i received at r_j by p_{ij} . Then, the SINR formula says that the signal from s_i is successfully received at r_i iff

$$p_{ii} \geq \beta \left(\sum_{j \in S_i} p_{ji} + N \right) ,$$

where β and N are fixed constants and S_i is the set of transmissions concurrent with transmission i .

We can formulate this as a weighted digraph, with a node for each of the communication requests. The weight of the directed edge (i, j) is given by

$$w(i, j) = \frac{p_{ji}}{p_{ii}/\beta - N} .$$

Then, a set S of communication links is feasible iff it holds for each vertex $v_j \in S$ that $\sum_{i \in S} w(i, j) \leq 1$.

There is therefore a one-to-one mapping between feasible schedules and colorings of the weighted digraph, except for one minor detail: the above relationship allows for equality, whereas the definition in Sec. 1 requires inequality. This can, of course, be fixed with minor effort, for instance by subtracting some minuscule value from each weight. It requires though a slight restatement of the results.

Signal strengthening. One basic question regards the importance of the exact value of the parameter β . Could it be that a slight change in β could have a major impact on the schedulability of the instance? This was answered negatively in [10]: increasing β by some factor ρ only affects the schedule length by a $O(\rho^2)$ -factor. This has proved very useful for algorithm analysis, as it facilitates the use of triangular inequalities. Thm. 1 strengthens this relationship to linear.

Let $\Gamma_{\beta'}$ be the optimal schedule length for a given set of communication requests with $\beta = \beta'$.

Corollary 2. *For any collection of communication requests and any $\rho > 0$, $\Gamma_{\rho\beta} \leq \lceil 2\rho \rceil \Gamma_{\beta}$.*

Scheduling with linear power. Consider the case of wireless scheduling with linear power. That is, links l_1, l_2, \dots, l_n are located in a metric space, where $l_i = (s_i, r_i)$, and the power P_v of a link l_v of length ℓ_v is proportional to ℓ_v^α , where α is a fixed constant. Then, the weight of a directed edge (v_i, v_j) is $w(i, j) = \frac{P_i/d(s_i, r_j)^\alpha}{P_j/\ell_j^\alpha} = (\ell_i/d(s_i, r_j))^\alpha$ (ignoring the effect of the noise term).

Fanghänel, Kesselheim and Vöcking [6, Thm. 1] showed that in this setting, $\chi(D) = \Omega(\Delta^-)$; i.e., the scheduling number is proportional to the maximum interference experienced by a link from all the other links in the instance. They proceeded to give an algorithm that uses $O(\Delta^- + \log^2 n)$ colors, obtaining an asymptotic constant factor approximation. Tonoyan gave a pure constant-factor approximation by a deterministic sequential algorithm for instances in the Euclidean plane [16].

With Thm. 2, we can now generalize this to arbitrary metric spaces. This is currently the only known class of link scheduling instances with links of varying lengths that allows a constant-factor approximation.

Corollary 3. *The link scheduling problem with linear power in arbitrary metric space is constant-approximable.*

5. CONCLUSIONS

We have given essentially tight constructive bounds on the chromatic number of edge-weighted graphs in terms of the maximum degree and their inductiveness. It would be interesting to determine if there exist simple, e.g., greedy, algorithms that achieve the bound of Thm. 2. Also, if the constant in the Turán-like bound can be improved.

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