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Restricted cycle factors and arc-decompositions of digraphs

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Abstract. We study the complexity of finding 2-factors with various restrictions as well as edge-decompositions in (the underlying graphs of) digraphs. In particular we show that it is $\mathcal{NP}$-complete to decide whether the underlying undirected graph of a digraph D has a 2-factor with cycles $C_1, C_2, \ldots, C_k$ such that at least one of the cycles C_i is a directed cycle in D (while the others may violate the orientation back in D). This solves an open problem from [3]. Our other main result is that it is also $\mathcal{NP}$-complete to decide whether a 2-edge-coloured bipartite graph has two edge-disjoint perfect matchings such that one of these is monochromatic (while the other does not have to be). We also study the complexity of a number of related problems. In particular we prove that for every even $k \geq 2$, the problem of deciding whether a bipartite digraph of girth k has a k-cycle free cycle factor is $\mathcal{NP}$-complete.

Keywords: Cycle factor, 2-factor, mixed problem, NP-complete, Complexity, cycle factors with no short cycles

1 Introduction

Notation not introduced here follows [1, 5]. We distinguish between *(non-directed) cycles* and *directed cycles* in digraphs, where the former is a subgraph that corresponds to a cycle in the underlying graph of a digraph. The notions of a *(non-directed) path* and a *directed path* are defined in a similar way. A *cycle factor* in a digraph is a spanning subgraph consisting of directed cycles, and a 2-*factor* in a digraph (or graph) is a spanning subgraph consisting of cycles. We denote by $UG(D)$ the underlying graph of a directed graph D.

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In this paper we consider several variations on the problem of finding cycle factors of digraphs. The problems of deciding if a given graph has a 2-factor and if a given digraph has a cycle factor are fundamental problems in combinatorial optimization, and both these problems are well-known to be solvable in polynomial time, see e.g. [1, 5]. Here, we are particularly interested in problems concerning the complexity of deciding existence of spanning subgraphs that in a sense lies "in-between" 2-factors and cycle factors. In particular, we answer the question of the complexity of the following two problems by the first author:

Problem 1.1. [3, Problem 3] **2-factor with at least one directed cycle.**

Instance: A digraph D.

Question: Does D have a 2-factor F such that at least one cycle in F is a directed cycle, while the rest of the cycles do not have to respect the orientations of arcs in D?

Problem 1.2. [7] **Disjoint perfect matchings one of which is monochromatic**

Instance: A 2-edge colored bipartite graph $B = (U, V; E)$.

Question: Does B have two edge-disjoint perfect matchings M_1, M_2 so that every edge of M_1 has color 1, while M_2 may use edges of both colors?

This problem is equivalent to the following problem (see [1, Section 16.7]).

Problem 1.3. Semi-directed 2-factors of bipartite digraphs

Instance: A bipartite digraph $B = (X, Y; A)$.

Question: Does $UG(B)$ have a 2-factor which is the union of a perfect mathching from X to Y in D (respecting the orientation) and a perfect matching in $UG(B)$?

Thus we are asking for a collection of cycles covering all vertices of B such that every second edge (starting from X) is oriented from X to Y in D, whereas the remaining edges do not have to respect the orientation of D.

The motivation for studying such "mixed" problems for digraphs, that is, problems concerning structures in a digraph D where only part of the structure has to respect the orientation of the arcs of D, is that this way one can obtain new insight into the complexity of various problems which have natural analogues for graphs and digraphs. As an example, in [2, 3] the problem of deciding for a digraph D the existence of a directed cycle C in D and a cycle C' in $UG(D)$ which are vertex disjoint was studied. It was shown that this problem is polynomially decidable for the class of digraphs with a bounded number of cycle transversals of size 1 (vertices whose removal eliminates all directed cycles) and $\mathcal{NP}$-complete if we allow arbitrarily many transversal vertices. For (di)graphs one can decide the existence of two disjoint (directed) cycles in polynomial time [12, 13].

Note that the variant of Problem 1.1 where we ask if D has a 2-factor F such that at most one cycle in F is not directed is $\mathcal{NP}$-complete. This can easily be proved as follows: It is $\mathcal{NP}$-complete to decide if a given graph G is Hamiltonian. Let D' be a digraph with a cycle factor, and let D'' be an acyclic orientation of the given graph G. Next, let D be the disjoint union of D' and D'' (or add an arbitrary arc between a vertex of D' and a vertex of D'' if one wants D to be connected). Then G is Hamiltonian if and only if D has a 2-factor where at most one cycle is not a directed cycle.

We show that Problem 1.1 and 1.2 are both $\mathcal{NP}$-complete.

Theorem 1.4. 2-factor with at least one directed cycle *is $\mathcal{NP}$-complete.*

Theorem 1.5. Disjoint perfect matchings one of which is monochromatic *is $\mathcal{NP}$-complete.*

In fact, in the latter case, we shall prove that this problem is $\mathcal{NP}$-complete already for bipartite graphs with maximum degree 3.

Given an $n \times n$ array A where each cell contains a (possibly empty) subset of $\{1, 2, \dots, n\}$, we say that A is *avoidable* if there is an $n \times n$ Latin square L such that each cell of L does not contain a symbol that appears in the corresponding cell of A. We also say that L *avoids* A. If a cell in A is empty, then we also say that this cell contains entry $\emptyset$. In [6] it was proved that determining whether a given array where each cell contains a subset of $\{1, 2\}$ is avoidable is $\mathcal{NP}$-complete. Using Theorem 1.5 we can prove the following strengthening of that result.

Corollary 1.6. *The problem of determining whether an array where each cell contains either the symbol* 1*, the set* $\{1, 2\}$ *or* $\emptyset$ *is avoidable is* $\mathcal{NP}$*-complete.*

Proof. We reduce Problem 1.2 to the problem of avoiding an array where each cell contains symbol 1, the set $\{1, 2\}$ or is empty.

Let G be a balanced bipartite graph on $n+n$ vertices with a edge coloring f using colors 1 and 2. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be the parts of G. We form an $n \times n$ array A from G as follows:

- If $x_i y_j \notin E(G)$, then we set $A(i, j) = \{1, 2\}$.
- If $x_i y_j \in E(G)$ and $f(x_i y_j) = 2$, then we set $A(i, j) = 1$.
- If $x_i y_j \in E(G)$ and $f(x_i y_j) = 1$, then the cell (i, j) of A is empty.

It is straightforward to verify that there are two disjoint generalized diagonals[1] D_1 and D_2 in A, where D_1 has no cell with symbol 1 and D_2 has no cell with symbol 2 (and therefore a Latin square avoiding A) if and only if G has disjoint perfect matchings one of which does not contain any edge with color 2. □

Next, we consider a variant of a problem studied by Hartvigsen: in [10] he proved that the problem of deciding if a bipartite graph has a 4-cycle-free 2-factor (i.e. a 2-factor with no 4-cycle) can be solved in polynomial time. The problem of determining if a general graph has a 2-factor without 3-cycles is also solvable in polynomial time [11]. The analogous question of the complexity of existence of 2-factors in general graphs where no cycle has length 4 or less is open to the best of our knowledge, while the problem of determining if a graph G has a 2-factor where all cycles have length at least 6 is $\mathcal{NP}$-complete if G is bipartite, and thus also in the general case (see e.g. [4, 10]).

For digraphs, the problem of determining whether a general digraph has a cycle factor where every cycle has length at least 3 is $\mathcal{NP}$-complete, see e.g. [4]. Here we consider the analogous question for bipartite digraphs. We strengthen the result of [4] as follows:

Proposition 1.7. *The problem of determining if a given bipartite digraph D has a* 2*-cycle-free cycle factor is* $\mathcal{NP}$*-complete.*

Using this result we show how to prove the following:

Theorem 1.8. *For each even $k \geq 2$, the problem of determining if a bipartite digraph D with no directed cycles of length at most k has a $(k+2)$-cycle-free cycle factor is* $\mathcal{NP}$*-complete.*

[1] A generalized diagonal of an $n \times n$ matrix A is a collection of elements $a_{1,\pi(1)}, \dots, a_{n,\pi(n)}$ where π is a permutation of $[n]$.

In particular, the problem of determining if a given oriented bipartite graph has a cycle factor with no 4-cycle is $\mathcal{NP}$-complete; so the natural analogue of Hartvigsen's positive result on 4-cycle-free 2-factors in bipartite graphs does not hold in the digraph setting.

Finally, we consider decompositions of digraphs. It is well-known that the problem of determining if the edge set of a given graph has a decomposition into edge-disjoint cycles is solvable in polynomial time, as is also the analogous problem of deciding if the arc set of a given digraph has a decomposition into arc-disjoint directed cycles. Here we shall prove the following:

Theorem 1.9. *It is $\mathcal{NP}$-complete to decide for a given digraph $D = (V, A)$ whether there is a decomposition of A into arc-disjoint cycles $C_1, \ldots, C_k$ of D with the property that at most one of these cycles is not directed.*

Note that the opposite problem, where we ask for a decomposition of $A(D)$ into arc-disjoint cycles $C_1, \ldots, C_k$ of D such that C_1 is a directed cycle but all the other cycles do not have to respect the orientation of the arcs, is trivial: As we want a decomposition into arc-disjoint cycles of D it follows that $UG(D)$ must be Eulerian. Hence if D contains any directed cycle C, this can play the role of C_1 above, showing that the answer is no if and only if either D is acyclic or $UG(D)$ is not Eulerian.

The rest of the paper is organized as follows: In section 2 we first prove Proposition 1.7 and using this result, we prove Theorem 1.8 and Theorem 1.4. Section 2 is concluded by the proof of Theorem 1.5. Section 3 contains the proof of Theorem 1.9 and in Section 4 we give an instance of Problem 1.2 which can be solved in polynomial time.

2 Restricted cycle factors

For the proof of Proposition 1.7 we will use a result on avoiding arrays. As mentioned above, in [6] it was proved that the following problem is $\mathcal{NP}$-complete:

Problem 2.1. Avoiding multiple-entry arrays with 2 symbols

Instance: An $n \times n$ array A, such that each cell is either empty or contains a subset of the symbols in $\{1, 2\}$.

Question: Is A avoidable?

Proof of Proposition 1.7. We shall reduce Problem 2.1 to the problem of deciding if a given bipartite digraph has a 2-cycle-free cycle factor.

Let A be an $n \times n$ array, where each cell is either empty or contains a subset of $\{1, 2\}$. We form an $n \times n$ bipartite digraph D with parts

$$X = \{x_1, \ldots, x_n\} \text{ and } Y = \{y_1, \ldots, y_n\}$$

by for all i, j including the arc (x_i, y_j) in $A(D)$ if and only if the symbol 1 does not appear in cell (i, j) of A, and including the arc (y_j, x_i) if and only if the symbol 2 does not appear in the cell (i, j) of A. We shall prove that A is avoidable if and only if D has a 2-cycle-free cycle factor.

Suppose that A is avoidable, which means that there is a Latin square L avoiding A. Consider the generalized diagonals B_1, B_2 in L, containing all cells with entries 1 and 2, respectively. Since L avoids A, any cell in A corresponding to a cell in B_1 does not contain symbol 1; and any cell in A corresponding to a cell in B_2 does not contain the symbol 2. Hence, if $(i, j) \in B_1$, then $(x_i, y_j) \in D$; and if $(i, j) \in B_2$, then $(y_j, x_i) \in D$. Thus the set of arcs

$$\{(x_i, y_j) : (i, j) \in B_1\} \cup \{(y_j, x_i) : (i, j) \in B_2\}$$

induces a cycle factor F in D, and since B_1 and B_2 are disjoint, F is 2-cycle-free.

Conversely, suppose that D has a 2-cycle-free cycle factor F. Let F_1 be the set of arcs going from X to Y in F, and let F_2 be the arcs going from Y to X. Define an array P by putting the symbol 1 in cell (i, j) if and only if $(x_i, y_j) \in F_1$, and putting the symbol 2 in cell (i, j) if and only if $(y_j, x_i) \in F_2$. Since F is 2-cycle-free, each cell in P has at most one entry. In fact, P is an $n \times n$ partial Latin square where both symbols 1 and 2 occur exactly n times. Thus we may partition the unfilled cells of P into $n - 2$ generalized diagonals $B_3, \ldots, B_n$ and then assign symbol i to all cells of B_i, $i = 3, \ldots, n$. So P is completable to a Latin square L. Moreover, since $(x_i, y_j) \in A(D)$ if and only if the symbol 1 does not appear in cell (i, j) of A, and $(y_j, x_i) \in A(D)$ if and only if the symbol 2 does not appear in the cell (i, j) of A, L avoids A. □

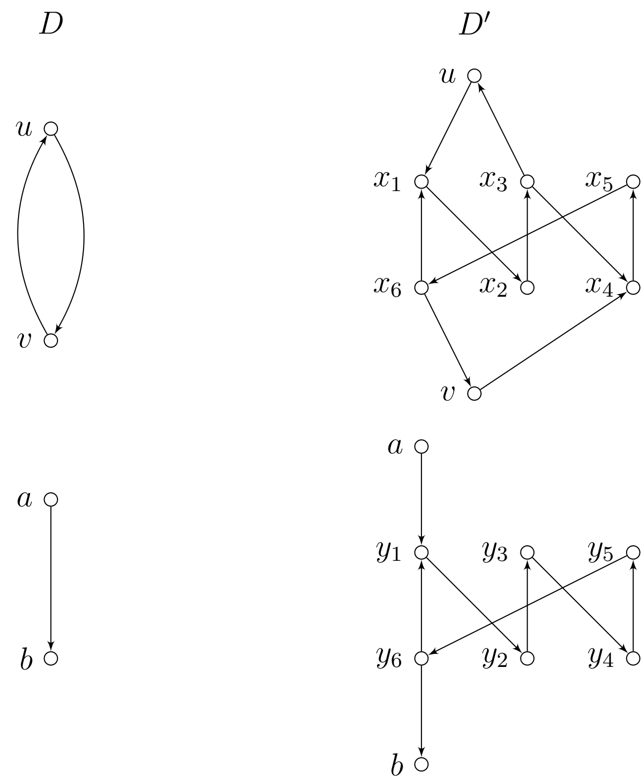

Figure 1: Constructing D' from D in the proof of Theorem 1.8.

Proof of Theorem 1.8. We shall prove the theorem in the case when $k = 2$, by reducing the problem of existence of 2-cycle-free cycle factors in bipartite digraphs to the problem of existence of 4-cycle-free cycle factors in oriented bipartite graphs. For $k > 2$, there is a similar reduction.

Let D be a bipartite digraph. From D we shall form a bipartite digraph D' that does not have any 2-cycles as follows:

(i) For each (directed) 2-cycle uvu of D we do the following: Remove the arcs (u, v) and (v, u), add 6 new vertices $x_1, \ldots, x_6$, let $Q = x_1x_2x_3x_4x_5x_6x_1$ be a directed 6-cycle joining these vertices, and add arcs $(u, x_1), (x_3, u), (v, x_4), (x_6, v)$.

(ii) For each arc (a, b) of D which is not in any 2-cycle we do the following: Remove the arc (a, b), add 6 new vertices $y_1, \ldots, y_6$, let $Q' = y_1y_2y_3y_4y_5y_6y_1$ be a directed 6-cycle joining these vertices, and add arcs $(a, y_1), (y_6, b)$.

The graph obtained by repeating the procedure (i) for each 2-cycle of D, and the procedure (ii) for each arc of D which is not in any 2-cycle, we denote by D' (see Figure 1). It is easily verified that D' is bipartite and contains no 2-cycles.

For the construction (i) we say that the arcs $(u, x_1), (x_3, u), (v, x_4), (x_6, v)$ are the arcs of D' *corresponding to* (u, v) *and* (v, u). We also say that the 6-cycle Q is the 6-cycle of D' *associated with* uvu.

For the construction (ii) we say that the arcs $(a, y_1), (y_6, b)$ are the arcs of D' *corresponding to* (a, b). We also say that the directed 6-cycle Q' is the directed 6-cycle of D' *associated with* (a, b).

Let us now prove that D has a 2-cycle-free cycle factor if and only if D' has a 4-cycle-free cycle factor. Suppose first that D has a 2-cycle-free cycle factor F, and let C be a directed cycle of F. Then C defines a directed cycle C' in D' in the following way:

Let (a, b) be an arc of C that is not in any 2-cycle of D, and let $y_1y_2 \ldots y_6y_1$ be its associated directed 6-cycle in D' along with the arcs $(a, y_1), (y_6, b)$ of D' corresponding to (a, b); we define a directed path

$$P_1 = ay_1y_2y_3y_4y_5y_6b$$

in D' corresponding to (a, b).

Since F does not contain any directed 2-cycles, we can proceed similarly for any arc in C that is in a 2-cycle: suppose that (u, v) is in the 2-cycle uvu and that (u, v) is in C. Assume further that $x_1x_2 \ldots x_6x_1$ is the associated directed 6-cycle in D' and that $(u, x_1), (x_3, u), (v, x_4), (x_6, v)$ are the arcs of D' corresponding to (u, v) and (v, u); we define a directed path

$$P_2 = ux_1x_2x_3x_4x_5x_6v$$

in D' corresponding to (u, v).

By concatenating all directed paths in D' corresponding to arcs of C in D, we obtain a directed cycle C' in D'. Moreover, if C_1 and C_2 are disjoint directed cycles in F, then the corresponding cycles C_1' and C_2' clearly traverses different vertices of D in D', so they are disjoint. Let F' be a subgraph of D' consisting of the collection of cycles in D' arising from cycles in F via the above construction. Then F' covers all vertices in $V(D') \cap V(D)$ and each cycle of F' clearly has length at least 6. Moreover, the subgraph of D' induced by $V(D') \setminus V(D)$ is a collection of disjoint directed 6-cycles, and if some vertex of such a directed 6-cycle is in F', then all vertices of this directed 6-cycle is in F'. Therefore, F' together with the subgraph of D' induced by $V(D') \setminus V(F')$ forms a cycle factor of D'. Moreover, each cycle in this cycle factor has length at least 6.

Suppose now that D' has a 4-cycle-free cycle factor F'. Let (a, b) be an arc of D that is not in any 2-cycle, $y_1y_2 \ldots y_6y_1$ be the associated directed 6-cycle in D', and (a, y_1) and (y_6, b) be the corresponding arcs in D'. We need the following easy claim, the proof of which is omitted.

Claim 2.2. *It holds that* $(a, y_1) \in A(F')$ *if and only if* $(y_6, b) \in A(F')$.

We define E_1 to be the set of all arcs (a, b) in D that are not in any 2-cycles and satisfying that there are vertices $y_1, y_6 \in V(D') \setminus V(D)$ such that $(a, y_1) \in A(F')$ and $(y_6, b) \in A(F')$. Let E_1' be the set of all corresponding arcs (a, y_1) and (y_6, b) in D' that are in F'. It follows from Claim 2.2 that E_1 induces a subgraph with maximum in- and outdegree at most 1 in D, and that a vertex in D has the same in- and outdegree in $D[E_1]$ as in $D'[E_1']$.

Next, let uvu be a directed 2-cycle in D, $x_1x_2\dots x_6x_1$ be the associated 6-cycle in D', and $(u,x_1),(x_3,u),(x_6,v),(v,x_4)$ be the corresponding arcs in D'. We will use the following claim which follows easily from the fact that F' is a 4-cycle-free cycle factor.

Claim 2.3. *(I) If $(u,x_1)\in A(F')$, then $(x_6,v)\in A(F')$, and $\{(x_3,u),(v,x_4)\}\cap A(F')=\emptyset$.*

(II) If $(x_6,v)\in A(F')$, then $(u,x_1)\in A(F')$ and $\{(x_3,u),(v,x_4)\}\cap A(F')=\emptyset$.

(III) If $(x_3,u)\in A(F')$, then $(v,x_4)\in A(F')$ and $\{(x_6,v),(u,x_1)\}\cap A(F')=\emptyset$.

(IV) If $(v,x_4)\in A(F')$, then $(x_3,u)\in A(F')$ and $\{(x_6,v),(u,x_1)\}\cap A(F')=\emptyset$.

We define E_2 to be the set of all arcs (u,v) in D such that (u,v) is in some 2-cycle, and there are vertices $x,x'\in V(D')\setminus V(D)$ satisfying that $(u,x)\in A(F')$ and $(x',v)\in A(F')$. Let E_2' be the set of all corresponding arcs (u,x) and (x',v) in D' that are in F'. It follows from Claim 2.3 that E_2 induces a subgraph with maximum in- and outdegree at most 1 in D, and that a vertex in D has the same in- and outdegree in $D[E_2]$ as in $D'[E_2']$. Moreover, there is no 2-cycle in $D[E_2]$.

Furthermore, the sets E_1 and E_2 are disjoint, and $E_1'\cup E_2'$ contains every arc of $A(F')$ that is incident with a vertex of D; so it follows that the subgraph of D induced by $E_1\cup E_2$ is a cycle factor with no cycle of length 2. □

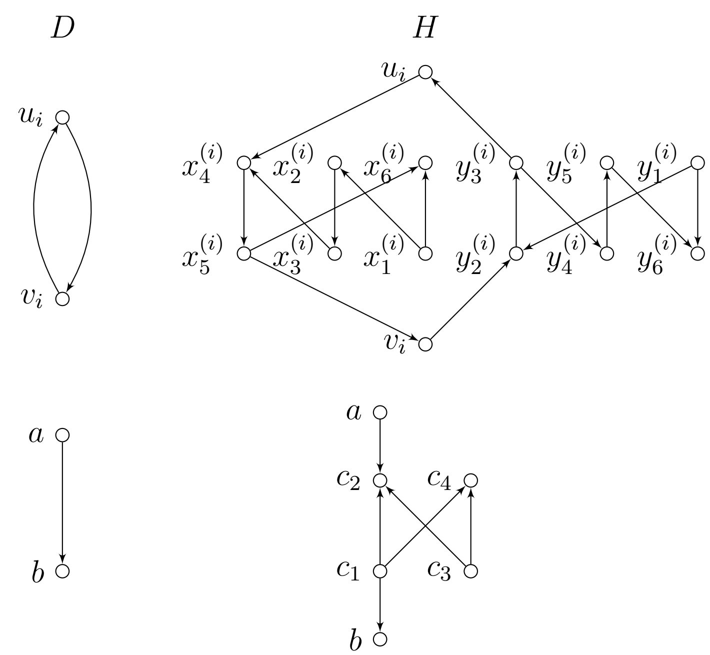

Figure 2: Constructing H from D in the proof of Theorem 1.4. Step 1.

Proof of Theorem 1.4. We shall reduce the problem of deciding existence of 2-cycle-free cycle factors in bipartite digraphs (the problem in Theorem 1.7) to the problem 2*-factor with at least one directed cycle.* Given a bipartite digraph D, we shall construct a digraph D' from D and then show that D has a 2-cycle-free cycle factor F if and only if D' has a 2-factor F' such that at least one cycle of F' is a directed cycle of D'.

So let D be a bipartite digraph. First we construct the auxiliary digraph H from D. Suppose that D has r directed 2-cycles and denote them by $T_i = u_i v_i u_i$, $i = 1, \ldots, r$.

(i) For each arc (a, b) of D that is not in any 2-cycle we do the following: remove the arc (a, b), and let Q_{ab} be an orientation of a non-directed 4-cycle with 4 new vertices c_1, c_2, c_3, c_4 divided into partite sets $\{c_1, c_3\}$ and $\{c_2, c_4\}$ where we orient all edges towards c_2 or c_4, and add the arcs (a, c_2) and (c_1, b) (see Figure 2).

(ii) For each 2-cycle $u_i v_i u_i$ of D we do the following: remove the arcs (u_i, v_i) and (v_i, u_i) and add two disjoint directed 6-vertex paths

$$L_i^{(x)} = x_1^{(i)} x_2^{(i)} x_3^{(i)} x_4^{(i)} x_5^{(i)} x_6^{(i)}$$

and

$$L_i^{(y)} = y_1^{(i)} y_2^{(i)} y_3^{(i)} y_4^{(i)} y_5^{(i)} y_6^{(i)}$$

on 12 new vertices, along with the additional arcs $(x_1^{(i)}, x_6^{(i)})$ and $(y_1^{(i)}, y_6^{(i)})$. Moreover, add the arcs $(u_i, x_4^{(i)})$, $(y_3^{(i)}, u_i)$, $(x_5^{(i)}, v_i)$, $(v_i, y_2^{(i)})$ (see Figure 2).

(iii) For each $i = 1, \ldots, r$, let $Z_i = \{z_1^{(i)}, z_2^{(i)}, z_3^{(i)}\}$ be a set of three new vertices, and for $i = 1, \ldots, r$ (indices taken modulo r) add the arcs in

$$\{(z_1^{(i)}, z_2^{(i)}), (z_2^{(i)}, z_3^{(i)}), (x_6^{(i)}, z_1^{(i)}), (y_6^{(i)}, z_1^{(i)}), (z_3^{(i)}, x_1^{(i+1)}), (z_3^{(i)}, y_1^{(i+1)})\}$$

(see Figure 3).

Denote the resulting graph by H. For the construction (i), we say that Q_{ab} is the *4-cycle associated with* (a, b).

Next, we shall construct the graph D' from H by proceeding as follows for each vertex v of $V(H) \cap V(D)$: let $\{q_1, \ldots, q_l\}$ be the in-neighbors of v in H. For $i = 1, \ldots, l$, add a set $\{w_1^{(i)}, w_2^{(i)}, w_3^{(i)}\}$ of new vertices and replace the arc (q_i, v) with the directed path

$$q_i w_1^{(i)} w_2^{(i)} w_3^{(i)} v;$$

these are called *connecting paths between* $N_H^-(v)$ *and* v; moreover, add $l - 1$ additional directed 3-vertex paths

$$p_1^{(1)} p_2^{(1)} p_3^{(1)}, \ldots, p_1^{(l-1)} p_2^{(l-1)} p_3^{(l-1)},$$

on altogether $3(l - 1)$ new vertices

$$\{p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, \ldots, p_1^{(l-1)}, p_2^{(l-1)}, p_3^{(l-1)}\};$$

these are called *non-connecting paths between between* $N_H^-(v)$ *and* v. Next, for each $i = 1, \ldots, l - 1$ we add the arcs $(p_1^{(i)}, w_3^{(i)}), (p_1^{(i)}, w_3^{(i+1)}), (w_1^{(i+1)}, p_3^{(i)}), (w_1^{(i)}, p_3^{(i)})$.

By repeating this process for every vertex of $V(H) \cap V(D)$ we obtain the digraph D' (see Figure 4 for an example). For a vertex v of $V(H) \cap V(D)$, denote by J_v, the subgraph of D' induced by $N_H^-(v), v$ and all connecting and non-connecting paths between v and $N_H^-(v)$.

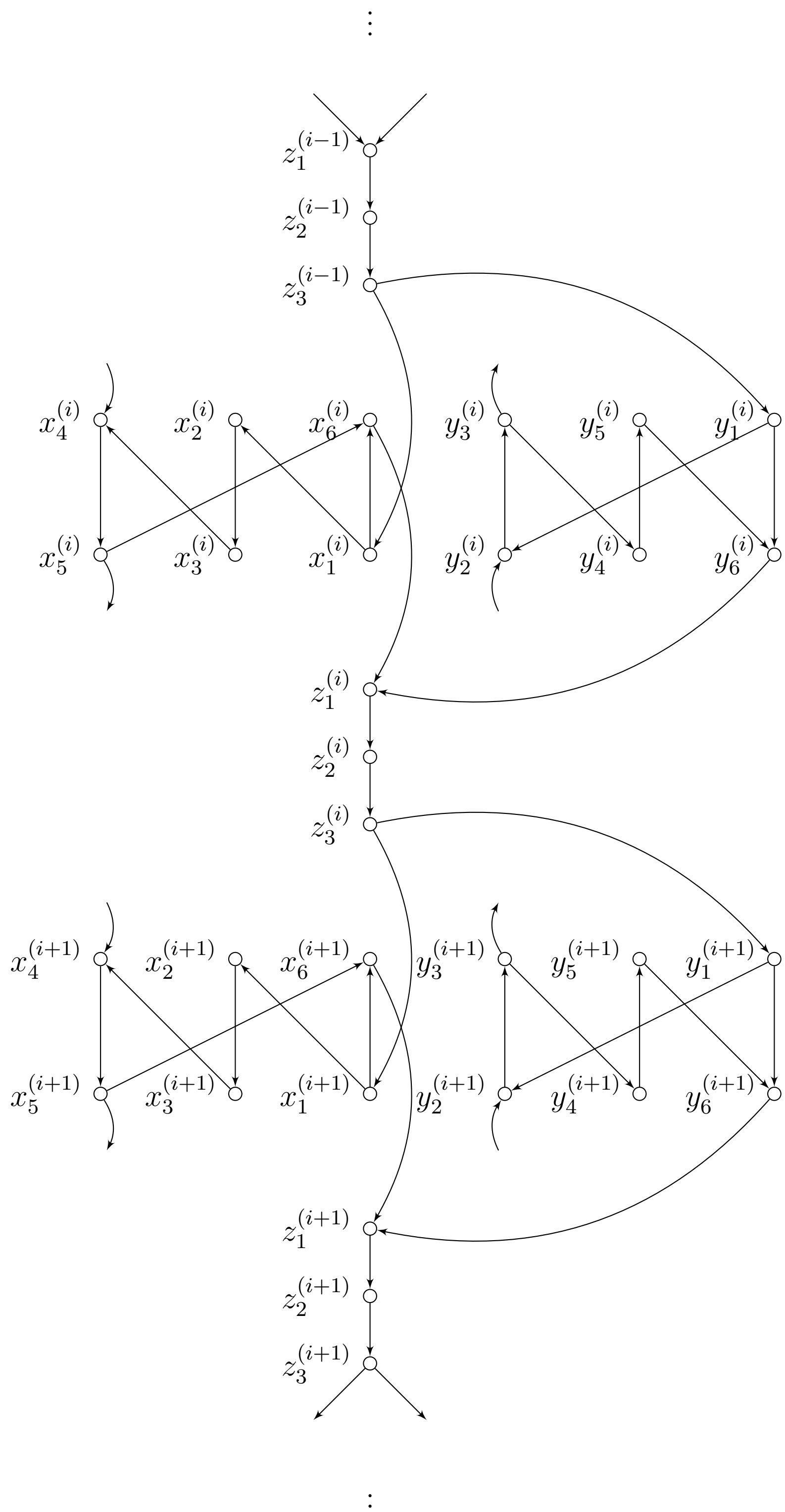

Figure 3: Constructing H from D in the proof of Theorem 1.4. Step 2.

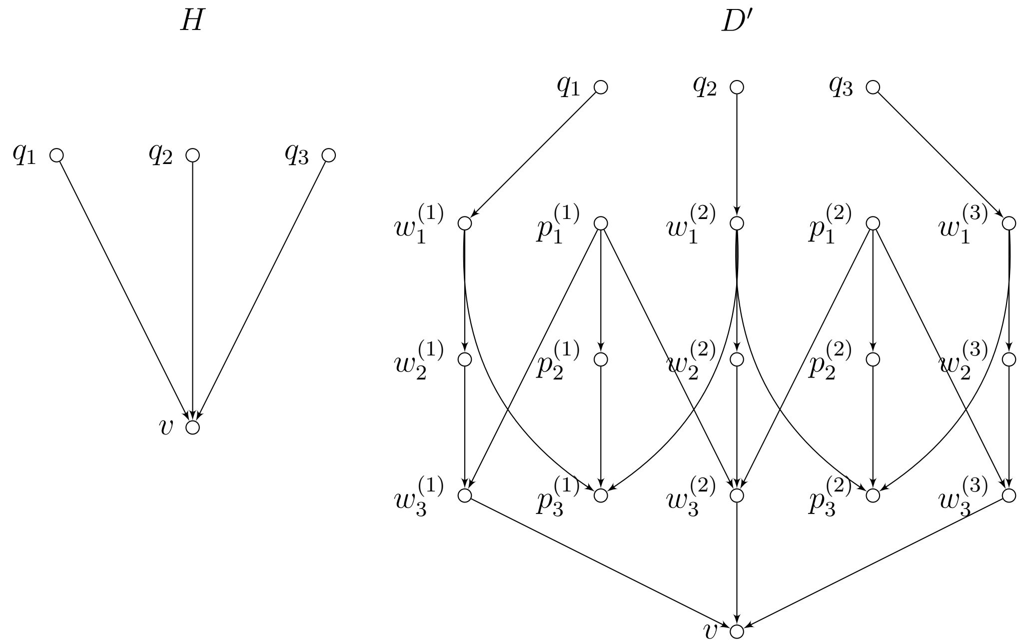

Figure 4: Constructing D' from H in the proof of Theorem 1.4.

Suppose now that D has a 2-cycle-free cycle factor F. We shall form the required 2-factor F' in D', by first showing how F induces a 2-factor F_H in H, and then demonstrating how F_H yields the required factor F' in D'.

Consider a cycle C in F. Then C defines a corresponding cycle C_H in H in the following way:

- Suppose that (a, b) is an arc of D that lies on C and is not in any 2-cycle, and let $\{c_1, c_2, c_3, c_4\}$ be the vertices of the associated 4-cycle of (a, b) as in Figure 2. Then we define a corresponding (non-directed) path $P_1 = ac_2c_3c_4c_1b$ in H.

- Suppose that (u_i, v_i) is an arc of a 2-cycle that lies on C. Then $(v_i, u_i) \notin A(C)$, and we define a corresponding (non-directed) path $P_2 = u_i x_4^{(i)} x_3^{(i)} x_2^{(i)} x_1^{(i)} x_6^{(i)} x_5^{(i)} v_i$ in H. (The case when (v_i, u_i) is in C is analogous.)

By concatenating all paths in H corresponding to arcs of C in D, we obtain a cycle C_H in H covering the same vertices of $V(D)$ as C.

It should be clear that the cycles in F in the way described above defines a subgraph $\hat{F}_H$ of H that consists of a collection of disjoint cycles, such that none of the cycles in $\hat{F}_H$ is a directed cycle in H, and each vertex in D has in- and outdegree 1 in $\hat{F}_H$. We shall now define a directed cycle C_{dir} in H.

Note that for each j, if some vertex of $L_j^{(x)}$ is in $\hat{F}_H$, then all vertices of $L_j^{(x)}$ are in $\hat{F}_H$ and no vertex of $L_j^{(y)}$ is in $\hat{F}_H$. Let I be a maximal subset of indices i of $\{1, \ldots, r\}$ such that no vertex of $L_i^{(x)}$ is in $\hat{F}_H$. Now, we define $\hat{C}_{\text{dir}}$ to be the subgraph of H induced by

$$\bigcup_{i \in I} V(L_i^{(x)}) \cup \bigcup_{i \in \{1, \ldots, r\} \setminus I} V(L_i^{(y)}) \cup Z_1 \cup \cdots \cup Z_r.$$

Let us now define

$$C_{\text{dir}} = \hat{C}_{\text{dir}} - \{(x_1^{(1)}, x_6^{(1)}), \ldots, (x_1^{(r)}, x_6^{(r)})\} - \{(y_1^{(1)}, y_6^{(1)}), \ldots, (y_1^{(r)}, y_6^{(r)})\}.$$

C_{dir} is a directed cycle of H disjoint from $\hat{F}_H$. Now consider the graph $H' = H - V(C_{\text{dir}}) - V(\hat{F}_H)$. Since each vertex of D has in- and out degree 1 in $\hat{F}_H$ and all vertices of $Z_1 \cup \cdots \cup Z_r$ are in $V(C_{\text{dir}})$, H' is a collection of disjoint (non-directed) 4- and 6-cycles. Define $F_H = \hat{F}_H \cup C_{\text{dir}} \cup H'$. Then F_H is a spanning subgraph of H where each component is a cycle, and exactly one cycle of F_H is directed. Moreover, each vertex in $V(D)$ has in- and outdegree 1 in F_H.

Let us now construct F' from F_H. A cycle C in $\hat{F}_H$ translates into a cycle C' in D' in the following way: If (s,t) is in C and $(s,t) \in A(D')$, then $t \notin V(D)$, because any in-neighbor in D' of a vertex in $V(D)$ is not in H (by the construction of D' from H); and we include (s,t) in C'; otherwise, if (s,t) is in C and $(s,t) \notin A(D')$, then $t \in V(D)$ and there is a connecting path P in J_t with origin s and terminus t. We include the path P in C'.

It should be clear that the cycles in $\hat{F}_H$ in this way defines a subgraph $\hat{F}_{D'}$ of D' which is a collection of disjoint cycles, and which covers all vertices of D. Note further that the cycle C_{dir} is in D' as are also all the cycles in H'. So it follows from the construction of D' from H that $\hat{D} = D' - V(\hat{F}_{D'}) - V(C_{\text{dir}}) - V(H')$ is a subgraph of D' consisting of equally many connecting and non-nonnecting paths between vertices $v \in V(D)$ and their in-neighbors in H. It is easy to see that $\hat{D}$ contains a 2-factor M where every cycle has length 6, and each cycle contains exactly one non-connecting path and some vertices of exactly one connecting path. The graph $M \cup C_{\text{dir}} \cup H' \cup \hat{F}_{D'}$ is the required 2-factor of D'.

Suppose now conversely that F' is a 2-factor of D' such that at least one cycle of F' is a directed cycle in D'. Denote by C_{dir} the directed cycle of D' that is in F'. We prove a series of claims concerning F'.

Claim 2.4. *Let $v \in V(D)$ and consider the subgraph J_v of D'. Let $\{q_1, \ldots, q_l\}$ be the in-neighbors of v in H. Then exactly one of the arcs in J_v that are incident with a vertex in $\{q_1, \ldots, q_l\}$ is in F'; and exactly one of the arcs in J_v that are incident with v is in F'.*

The proof of this claim is omitted. It is easily deduced by doing some case analysis using e.g. Figure 4 and the fact that F' is a 2-factor.

It follows from Claim 2.4 that F' induces a 2-factor F_H in H in the following way:

For each arc of F' that is in H, we include this arc in F_H; for each vertex $v \in V(D)$, include the arc (q_i, v) in F_H, where q_i is the unique in-neighbor of v in H that has outdegree 1 in F'. Obviously, we have that each vertex of $V(D)$ has in- and outdegree 1 in F_H. Moreover, the directed cycle C_{dir} clearly corresponds to a directed cycle $C_{\text{dir}}^{(H)}$ in F_H.

Consider an arc (a,b) in D that is not in any 2-cycle and its associated 4-cycle Q_{ab} in H and label the vertices of Q_{ab} according to Figure 2. It is easy to see that if (a, c_2) is in F_H, then (c_3, c_2) is in F_H, and thus (a, c_2) is not in $C_{\text{dir}}^{(H)}$.

Now consider an arc (u_i, v_i) that is in a 2-cycle of D. In H, this 2-cycle is replaced by the directed paths $L_i^{(x)}$ and $L_i^{(y)}$ and some additional arcs (see Figure 2). It is easy to see that if $(u_i, x_4^{(i)})$ is in F_H, then $(x_3^{(i)}, x_4^{(i)})$ is in F_H, and thus $(u_i, x_4^{(i)})$ is not in $C_{\text{dir}}^{(H)}$. Since each vertex of $V(D)$ has in- and outdegree 1 in F_H we have the following:

Claim 2.5. *No vertex of $V(D)$ is in $C_{dir}^{(H)}$.*

Claim 2.5 implies that

$$V(C_{\text{dir}}^{(H)}) \subseteq V(Z_1) \cup \cdots \cup V(Z_r) \cup V(L_1^{(x)}) \cup \cdots \cup V(L_r^{(x)}) \cup V(L_1^{(y)}) \cup \cdots \cup V(L_r^{(y)}).$$

Claim 2.6. *(i)* $Z_i \subseteq V(C_{dir}^{(H)})$, *for some* $i \in \{1, \ldots, r\}$.

(ii) If $x_1^{(i)} \in V(C_{dir}^{(H)})$, *then* $\{x_1^{(i)}, \ldots, x_6^{(i)}\} \subseteq V(C_{dir}^{(H)})$.

(iii) If $y_1^{(i)} \in V(C_{dir}^{(H)})$, *then* $\{y_1^{(i)}, \ldots, y_6^{(i)}\} \subseteq V(C_{dir}^{(H)})$.

(iv) If $Z_i \subseteq V(C_{dir}^{(H)})$, *then* $Z_{i+1} \subseteq V(C_{dir}^{(H)})$, *where indices are taken modulo* r.

Proof. Since

$$L_1^{(x)} \cup \cdots \cup L_r^{(x)} \cup L_1^{(y)} \cup \cdots \cup L_r^{(y)}$$

is a collection of disjoint (non-directed) 6-cycles, (i) is true. The statements (ii) and (iii) are straightforward to verify e.g. from Figure 3, and statement (iv) follows easily from (ii) and (iii). □

Since $C_{\text{dir}}^{(H)}$ is a directed cycle of H it follows from Claim 2.6 that for each $i = 1, \ldots, r$, $C_{\text{dir}}^{(H)}$ contains all vertices of $L_i^{(x)}$ or $L_i^{(y)}$. Moreover, $C_{\text{dir}}^{(H)}$ contains all the vertices $Z_1 \cup \cdots \cup Z_r$, and no vertices of $V(D)$.

Now consider the graph $\hat{H} = H - V(C_{\text{dir}}^{(H)})$. Clearly, $\hat{F}_H = F_H - V(C_{\text{dir}}^{(H)})$ is a 2-factor of $\hat{H}$ such that each vertex of $V(D)$ has in and out-degree 1 in $\hat{F}_H$.

Consider an arc (a, b) of D that is not in any 2-cycle and the corresponding associated 4-cycle Q_{ab} in H, and label the vertices of Q_{ab} according to Figure 2. We need the following easy claim, the proof of which is omitted.

Claim 2.7. *It holds that* (a, c_2) *is in* $\hat{F}_H$ *if and only if* (c_1, b) *is in* $\hat{F}_H$.

Now consider an arc (u_i, v_i) that is in a 2-cycle of D. We shall also need the following claim which follows easily from the facts that $\hat{F}_H$ is a 2-factor of $\hat{H}$ and that for each $i = 1, \ldots, r$, $C_{\text{dir}}^{(H)}$ contains all vertices of $L_i^{(x)}$ or $L_i^{(y)}$.

Claim 2.8. *(I) If* $(u_i, x_4^{(i)}) \in A(\hat{F}_H)$, *then* $(x_5^{(i)}, v_i) \in A(\hat{F}_H)$, *and* $\{(y_3^{(i)}, u_i), (v_i, y_2^{(i)})\} \cap A(\hat{F}_H) = \emptyset$.

(II) If $(x_5^{(i)}, v_i) \in A(\hat{F}_H)$, *then* $(u_i, x_4^{(i)}) \in A(\hat{F}_H)$ *and* $\{(y_3^{(i)}, u_i), (v_i, y_2^{(i)})\} \cap A(\hat{F}_H) = \emptyset$.

(III) If $(y_3^{(i)}, u_i) \in A(\hat{F}_H)$, *then* $(v_i, y_2^{(i)}) \in A(\hat{F}_H)$ *and* $\{(x_5^{(i)}, v_i), (u_i, x_4^{(i)})\} \cap A(\hat{F}_H) = \emptyset$.

(IV) If $(v_i, y_2^{(i)}) \in A(\hat{F}_H)$, *then* $(y_3^{(i)}, u_i) \in A(\hat{F}_H)$ *and* $\{(x_5^{(i)}, v_i), (u_i, x_4^{(i)})\} \cap A(\hat{F}_H) = \emptyset$.

Now, using the two claims above and the fact that each vertex of $V(D)$ has in and out-degree 1 in $\hat{F}_H$, it is straightforward to verify that $\hat{F}_H$ yields a 2-cycle-free cycle factor in D as in the last part of the proof of Theorem 1.8. This completes the proof of the theorem. □

For the proof of Theorem 1.5, we shall use the fact that the following problem is $\mathcal{NP}$-complete [8]. An *(edge) precoloring* of a graph G is a coloring of some of the edges of G.

Problem 2.9. Edge precoloring extension.

Instance: A 3-regular bipartite graph G, a precoloring f of $E' \subseteq E(G)$.

Question: Can f be extended to a proper edge coloring of G using precisely 3 distinct colors?

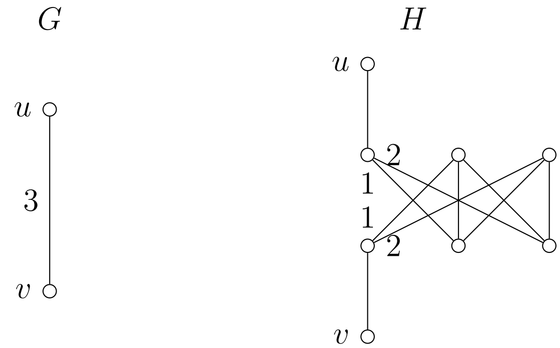

Figure 5: An edge precolored 3 in G and the corresponding subgraph of H.

Suppose now that G is a cubic bipartite graph with a precoloring using 3 colors. By replacing every edge precolored 3 with the gadget in Figure 5, we obtain the graph H. It is easy to check that the precoloring of G can be extended to a proper 3-edge coloring if and only if the precoloring of H can be extended to a proper 3-edge coloring. Hence, the following problem is also $\mathcal{NP}$-complete.

Problem 2.10. Edge precoloring extension with only two colors in the precoloring.

Instance: A 3-regular bipartite graph G, a precoloring f of $E' \subseteq E(G)$ using only two distinct colors.

Question: Can f be extended to a proper edge coloring of G using precisely 3 distinct colors?

Proof of Theorem 1.5. We shall reduce Problem 2.10 above to the problem of determining if a 2-edge-colored bipartite graph has two disjoint perfect matchings M_1 and M_2 so that every edge of M_1 has color 1.

Let G be a 3-regular bipartite graph with some edges colored 1 and some edges colored 2. Denote this precoloring by f. We shall construct a bipartite graph H with maximum degree 3 from G, where every edge in H is colored 1 or 2. Then we will argue that the precoloring f can be extended to a proper 3-edge coloring of G if and only if there are edge-disjoint perfect matchings M_1 and M_2 in H, such that M_1 only contains edges colored 1.

To this end, we define two 2-edge colored bipartite graphs B and C depicted in Figure 6, together with their compact notation. We say that w and z, and x and y are *endpoints* of the graphs B and C respectively.

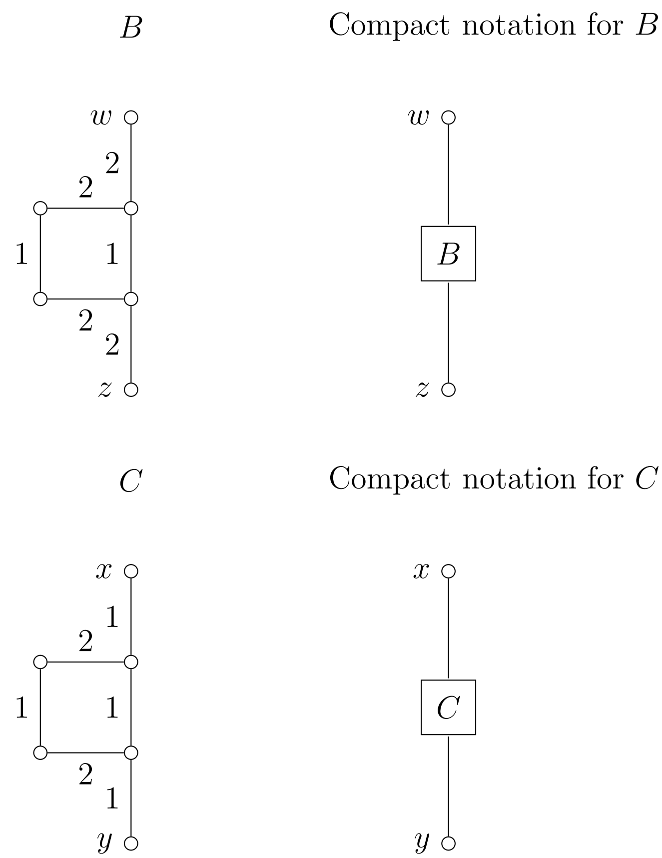

Figure 6: The graphs B and C and their compact notation.

Now we define the graph H from G by replacing all edges of G according to the following procedure (see Figure 7). Let $e = uv$ be an edge of G. If

(a) e is colored 1, then uv is replaced by a subgraph isomorphic to C by identifying the vertices x and y of C (see Figure 6) with the vertices u and v, respectively;

(b) e is colored 2, then the edge uv is in H and is colored 2 in H as well;

(c) e is uncolored and not adjacent to any colored edge of G, then uv is in H and we color it with 1;

(d) e is uncolored and adjacent to some edge colored 2 but not adjacent to any edge colored 1, then uv is replaced by a subgraph isomorphic to C by identifying the vertices x and y of C (see Figure 6) with the vertices u and v, respectively;

(e) e is uncolored and adjacent to some edge colored 1 but not adjacent to any edge colored 2, then uv is in H and it is colored 2;

(f) e is uncolored and adjacent to some edge colored 2 and also adjacent to some edge colored 1, then uv is replaced by a subgraph isomorphic to B by identifying w and z of B (see Figure 6) with the vertices u and v, respectively.

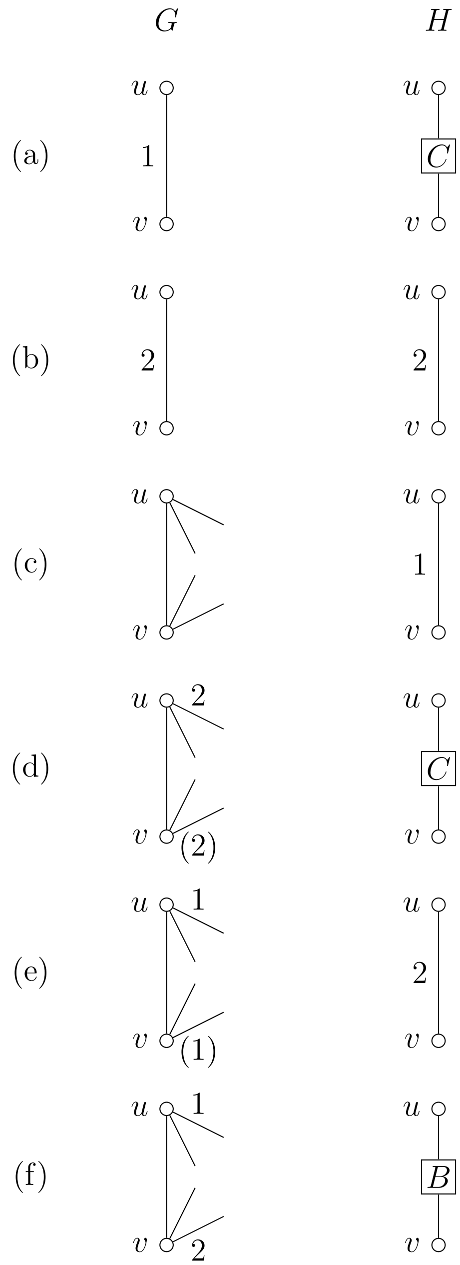

Figure 7: Constructing H from G.

The graph resulting from this process we denote by H and we denote its edge coloring by g. We say that a subgraph isomorphic to B (C) is a *B-subgraph (C-subgraph)* of H if it arises in H by replacing an edge of G.

Suppose first that the precoloring f of G can be extended to a proper 3-edge coloring f' of G. We shall define the required matchings M_1 and M_2 in H.

We first define a set $\hat{M}_1$. Let $e_1 = u_1v_1$ be an edge of G with $f'(u_1v_1) = 1$. It follows from the construction of H, that either $u_1v_1 \in E(H)$ and is colored 1, or u_1 and v_1 are endpoints of a C-subgraph C_1 in H. In the first case we include u_1v_1 in $\hat{M}_1$, and in the second case we include the edges of C_1 incident with u_1 and v_1 in $\hat{M}_1$; additionally, a third edge of C_1 is included in $\hat{M}_1$, so that these three edges form a perfect matching of C_1. By repeating this process for each e in G with $f'(e) = 1$, we obtain the matching $\hat{M}_1$ of H that covers all vertices of G. It follows that the vertices of H not covered by $\hat{M}_1$ lies on B- or C-subgraphs of H. For each such subgraph with vertices uncovered by $\hat{M}_1$ we include two edges with color 1 in the set M'_1, so that $\hat{M}_1 \cup M'_1$ is a perfect matching in H. We set $M_1 = \hat{M}_1 \cup M'_1$.

Let us now construct M_2. Suppose that $e_2 = u_2v_2$ is colored 2 under f'. It follows from the construction of H, that $u_2v_2 \in E(H)$ and is colored 1 or 2 under g. We include all edges $e \in E(G) \cap E(H)$ with $f'(e) = 2$ in M_2, and for each C- and B-subgraph of H we also include two edges with color 2 in M_2 so that M_2 is a perfect matching in H. Note that $M_1 \cap M_2 = \emptyset$, because M_1 only contains edges colored 1 (under g) from B- and C-subgraphs and edges from $E(G)$ with color 1 under f'. So M_1 and M_2 are the required perfect matchings of H.

Suppose now conversely that H has disjoint perfect matchings M_1 and M_2 such that M_1 contains no edges colored 2 under g. We shall prove that there are disjoint perfect matchings M'_1 and M'_2 in G such that M'_1 contains all edges colored 1 under f, and M'_2 contains all edges colored 2 under f, which yields the desired conclusion.

Consider a B-subgraph B_1 of H. Denote by s and t, respectively, the vertices of B_1 of degree 2 that are not endpoints of B_1. Since M_1 is a perfect matching of H that contains no edges colored 2, the two edges colored 1 in B_1 are both in M_1. Furthermore, since M_2 is perfect, the two edges colored 2 that are incident with s or t are in M_2. Thus, we have the following:

Claim 2.11. *If u is an endpoint of a B-subgraph B_1 in H, then no edge of $(M_1 \cup M_2) \cap E(B_1)$ is incident with u.*

Now consider a C-subgraph C_1 of H. Denote by a and b the vertices of degree 2 in C_1 that are not endpoints of C_1; and by a' and b' the two vertices of degree three in C_1, where a and a' are adjacent. Since M_1 is a perfect matching of H that contains no edges colored 2, $ab \in M_1$. Moreover, since M_2 is perfect, the two edges in C_1 colored 2 are in M_2. Now consider the three edges of C_1 that are incident with a' or b'. Since M_1 is perfect, it follows that either exactly two or one of these edges are in M_1. Thus, we have the following:

Claim 2.12. *Let u and v be endpoints of one C-subgraph C_1 in H. Then no edge of $E(C_1) \cap M_2$ is incident with u or v, and, moreover if u is incident with an edge from $E(C_1) \cap M_1$ then v is incident with an edge from $E(C_1) \cap M_1$.*

We now construct M'_2 from M_2 as follows: let $u_2 \in V(G)$ and suppose that $e_2 = u_2v_2 \in M_2$. It follows from Claims 2.11 and 2.12 that u_2v_2 is not in any B- or C-subgraph of H, so $u_2v_2 \in E(G)$. Hence, by the construction of H, either u_2v_2 is uncolored under f, or has color 2 under f. Let $\hat{M}_2$ be the set of all edges e of M_2 that are in B- or C-subgraphs of H. We simply define M'_2 by setting $M'_2 = M_2 \setminus \hat{M}_2$. Let us verify that M'_2 contains all edges precolored 2 in G. If $f(e) = 2$, then $g(e) = 2$, and any edge adjacent to e in G is replaced by a B- or C-subgraph in H. So Claims 2.11 and 2.12 imply that $e \in M_2$, and thus $e \in M'_2$.

Now we construct M_1' from M_1 as follows: let $u_1 \in V(G)$ and suppose that $e_1 = u_1v_1 \in M_1$. Let $\hat{M}_1$ be the set of all edges e of M_1 that are in B- or C-subgraphs, and which are not incident to endpoints of such subgraphs. It follows from Claim 2.11 that e_1 is not in any B-subgraph of H, and if u_1v_1 is in a C-subgraph, and thus u_1 is an endpoint of a C-subgraph C_1, then the other endpoint w_1 of C_1 is also incident with an edge from $M_1 \cap E(C_1)$. For each such edge in M_1 we include the edge u_1w_1 in M_1'. Note further that any edge of $M_1 \setminus \hat{M}_1$ that is not in any B- or C-subgraph of H is in G. Thus, we include all such edges in M_1'. Clearly, M_1' covers all vertices of G and is disjoint from M_2'. Let us verify that all edges precolored 1 in G are in M_1'. Let $e \in E(G)$ with $f(e) = 1$. Then e is replaced by a C-subgraph C_1 in H, and any edge that is adjacent to e in G is replaced by an edge colored 2 or a B-subgraph in H. Since M_1 only contains edges colored 1 under g and contains no edge of a B-subgraph B_1 that is incident with an endpoint of B_1, it follows that the endpoints of C_1 are covered by edges from $M_1 \cap E(C_1)$. Thus $e \in M_1'$. □

3 Restricted decompositions of digraphs

In this section we prove Theorem 1.9. The following is a well-known fact, see e.g. [1, Excercise 4.8].

Proposition 3.1. *The arc set of a digraph $D = (V, A)$ can be decomposed into arc-disjoint directed cycles $C_1, \dots, C_p$ for some p if and only if we have $d^+_D(v) = d^-_D(v)$ for all vertices $v \in V$.*

Proof of Theorem 1.9. Observe that if the arc set A can be decomposed into arc-disjoint directed cycles, then the problem is trivial, so we can assume that is not the case. Similarly, if any vertex v has $|d^+(v) - d^-(v)| > 2$, there can be no solution as all directed cycles contribute the same to the in- and out-degree of any vertex. So by Proposition 3.1, D is a yes-instance if and only if there is a cycle C which covers all the vertices in $V^+ \cup V^-$ where $V^+ = \{v \in V : d^+(v) = d^-(v) + 2\}$ and $V^- = \{v \in V; d^-(v) = d^+(v) + 2\}$ and satisfying that every vertex $v \in V^+$ has outdegree 2 in C, every vertex $v \in V^-$ has indegree 2 in C, and every vertex $v \in V(C) \setminus (V^+ \cup V^-)$ has in- and outdegree 1 in C. Note that after removing the arcs of such a cycle C the resulting digraph D' will satisfy the condition in Proposition 3.1.

Now we show how to reduce the hamiltonian cycle problem for cubic bipartite graphs to our problem (this problem is well-known to be NP-complete [9])

Let $B = (X, Y; E)$ be a cubic bipartite graph (note that $|X| = |Y|$) and form the directed graph $D(B)$ as follows: Orient all edges of E from X to Y. Add one new vertex s and the following arcs: all arcs from s to X as well as all arcs from Y to s. The digraph $D(B)$ satisfies that $d^+(s) = d^-(s)$, all vertices $x \in X$ have $d^+(x) = d^-(x) + 2$ and all vertices $y \in Y$ satisfy $d^-(y) = d^+(y) + 2$. By the remarks above, if D has the desired decomposition into arc-disjoint cycles $C_1, C_2, \dots, C_k$ such that C_1 is the only non-directed cycle, then the arcs of C_1 must correspond to a hamiltonian cycle back in B. Conversely, if C is a hamiltonian cycle of B, then by removing the corresponding arcs in $D(B)$ the resulting digraph D' has a decomposition into arc-disjoint 3-cycles all of which contain s and one arc of a perfect matching from X to Y. □

4 A polynomial instance of Problem 1.2

An easy consequence of Hall's marriage theorem (see e.g. [1, Theorem 4.11.3]) is that every balanced bipartite graph on $n+n$ vertices with minimum degree at least $\frac{n}{2}$ has a perfect matching. It is worth noting that this implies that Problem 1.2 is solvable in polynomial time if the balanced bipartite graph B on $n + n$ vertices has minimum degree at least $\frac{n}{2} + 1$: if this holds, then Problem 1.2 has a positive answer if and only if there is a matching M_1 containing only edges colored 1, because if there is such a matching M_1, then $B - M_1$ has a perfect matching by applying Hall's theorem to $B - M_1$.

With a little more effort we can prove the following strengthening of this result:

Proposition 4.1. *Problem 1.2 is solvable in polynomial time for balanced bipartite graphs on $n+n$ vertices with minimum degree at least $\frac{n}{2}$.*

Proof of Proposition 4.1. Let B be a bipartite graph with parts V_1 and V_2 both of which have size n. We may assume that $n \geq 6$, since smaller instances can surely be checked in polynomial time. We will deal with the cases that n is even and n is odd separately. By the above remark we may assume that $\delta(B) = \lceil n/2 \rceil$.

In the case when n is odd we will prove that if Problem 1.2 has a negative answer for B, then either B has no perfect matching with only edges of color 1, or if B has such a matching, then there are subsets $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ of size $\lceil \frac{n}{2} \rceil$ such that

(A.1) $B[W_1 \cup W_2]$ is a one-factor,

(A.2) $B[W_1 \cup V_2 \setminus W_2]$ is complete bipartite,

(A.3) $B[W_2 \cup V_1 \setminus W_1]$ is complete bipartite, and

(A.4) for every perfect matching M containing only edges of color 1 in B, every edge of $B[W_1 \cup W_2]$ is contained in M.

In the case when n is even we will prove that if Problem 1.2 has a negative answer for B, then either B has no perfect matching M_1 with only edges of color 1, or if B has such a matching, then, by renaming V_1 and V_2 if necessary, there is a subset W of V_1 of size $\frac{n}{2}+1$, and a maximal subset X of V_2 of size $n/2$ or $n/2+1$ containing all vertices x of V_2 with exactly one edge joining x with a vertex of W. Moreover, W and X satisfy the following:

(B.1) the edge set of $B[W \cup X]$ is a matching of size $n/2$ or $n/2+1$,

(B.2) The subgraph of B induced by $V_1 \setminus W$ and X is complete bipartite,

(B.3) if X has size $n/2+1$, then the subgraph of B induced by W and $V_2 \setminus X$ is complete bipartite,

(B.4) if X has size $n/2$, then every perfect matching M with only edges colored 1 contains each edge of $B[W \cup X]$,

(B.5) if X has size $n/2+1$, then every perfect matching M with only edges colored 1 contains at least $n/2$ edges from $B[W \cup X]$.

We first deal with the case when n is odd. If B has no perfect matching with only edges of color 1, then there is nothing to prove, so suppose that B has such a matching M_1. Consider the graph $B' = B - M_1$, and suppose that Problem 1.2 has a negative answer for B, in particular, there is no perfect matching in B'. Then, by Hall's theorem there is a subset W_1 of V_1 such that $|N_{B'}(W_1)| < |W_1|$. Since $\delta(B') = \lfloor n/2 \rfloor$, we have that $|W_1| \geq \lceil n/2 \rceil$. It is easy to see that if we choose a minimal set W_1 with this property, then $|W_1| \leq \lceil n/2 \rceil$. Hence, there is such a set W_1 of size exactly $\lceil n/2 \rceil$. We set $W_2 = V_2 \setminus N_{B'}(W_1)$. Then $|W_2| = \lceil n/2 \rceil$, and it is easy to see that $|N_{B'}(W_2)| = \lfloor n/2 \rfloor$. Since $\delta(B') = \lfloor n/2 \rfloor$, this implies that the subgraphs $B'[W_1 \cup N_{B'}(W_1)]$ and $B'[W_2 \cup N_{B'}(W_2)]$ of B are complete bipartite; and thus each vertex of W_1 is matched to a vertex in W_2 under M_1, and thus $B[W_1 \cup W_2]$ is a one-factor. Hence, we have proved that (A.1), (A.2) and (A.3) holds.

Suppose now that there is some matching M_1' in B that does not contain every edge of $B[W_1 \cup W_2]$. Then $B'' = B - M_1'$ has some edge connecting a vertex w_1 of W_1 and a vertex w_2 of W_2. Since $B[W_1 \cup (V_2 \setminus W_2)]$ and $B[W_2 \cup (V_1 \setminus W_1)]$ are complete bipartite it follows that $B'' - \{w_1 w_2\}$ has a perfect matching, which contradicts that Problem 1.2 has a negative answer. We conclude that (A.4) holds.

Note that deciding whether B has a perfect matching M_1 with only edges of color 1 can be done in polynomial time e.g. using flows (see [1, Section 4.11]). Assuming that B has such a matching, the problem of finding either a perfect matching in $B - M_1$ or a set W_1 as in the argument above can be solved in polynomial time by well-known algorithms for constructing maximum matchings in bipartite graphs. Moreover, given the set W_1, the problem of verifying whether there is a set W_2, such that (A.1)-(A.4) hold, is clearly solvable in polynomial time.

We now consider the case when n is even. If B has no matching with only edges of color 1, then there is nothing to prove, so suppose that B has such a matching M_1. Consider the graph $B' = B - M_1$, and suppose that Problem 1.2 has a negative answer, that is, there is no perfect matching in B'. Then, by Hall's theorem there is a set A that is a subset of V_1 or V_2 such that $|N_{B'}(A)| < |A|$. Since $\delta(B') = n/2 - 1$, we have that $|A| \geq n/2$. On the other hand, if we choose a minimal set A with this property, then $|A| \leq n/2$. Hence, there is such a set A of size exactly $n/2$ with $|N_{B'}(A)| = n/2 - 1$. Assume without loss of generality that $A \subseteq V_2$. Clearly $B'[A \cup N_{B'}(A)]$ is complete bipartite. We set $W = V_1 \setminus N_{B'}(A)$. Then W has size $n/2 + 1$ and $|N_{B'}(W)| = n/2$. Since $\delta(B') = n/2 - 1$, it follows that all vertices in A are matched to vertices in W under M_1. Hence, the edge set of $B[A \cup W]$ is a matching.

Let $X \subseteq V_2$ be a maximal subset of vertices such that every vertex of X has degree 1 in $B[X \cup W]$. Clearly X contains A, and thus has size at least $n/2$. On the other hand, since B has minimum degree $n/2$, $|X| \leq n/2 + 1$.

Suppose first that X has size $n/2$, and thus $X = A$. Then (B.1) holds, and (B.2) as well, because $B[A \cup N_{B'}(A)]$ is complete bipartite. Let us prove that every perfect matching M_1 with only edges colored 1 in B contains all edges of $B[X \cup W]$, i.e. that (B.4) holds.

Suppose that there is a perfect matching M_1' in B with only edges colored 1 such that $wx \notin M_1'$, where $w \in W$ and $x \in X$. Set $B'' = B - M_1'$. It follows that the bipartite graph $B_1 = B''[W \setminus \{w\} \cup V_2 \setminus X]$ is balanced, and contains no isolated vertex. Moreover, every vertex of W has degree at least $n/2 - 2$ in B_1, and thus B_1 contains at least $n^2/4 - n$ edges. If B_1 has no perfect matching, then by Hall's theorem there is a subset $W' \subseteq W$ of size $n/2 - 1$, such that $|N_{B_1}(W')| = n/2 - 2$. Consequently, there are two vertices $u_1, u_2 \in V_2 \setminus X$ of degree 1 in B_1. This implies that B_1 has at most $(n/2 - 2)(n/2 - 1) + 2$ edges, a contradiction because B_1 has at least $n^2/4 - n$ edges and $n \geq 6$. Thus B_1 has a perfect matching $\hat{M}_1$.

Now consider the balanced bipartite graph $B_2 = B''[X \setminus \{x\} \cup N_{B'}(X)]$. Every vertex of X has degree at least $n/2 - 2$ in B_2. So provided that $n \geq 6$, B_2 has a perfect matching $\hat{M}_2$. This means that $\hat{M}_1 \cup \hat{M}_2 \cup \{xw\}$ is a perfect matching in B'', contradicting that Problem 1.2 has a negative answer. Thus we conclude that (B.4) holds.

Suppose now that X has size $n/2 + 1$. Recall that the edge set of $B[W \cup A]$ is a matching, and we have $A \subseteq X$. Then (B.1) holds, since suppose that $x \in X \setminus A$ is adjacent to the same vertex in W as a vertex $a \in A$. Since $|W| = n/2 + 1$, this means that there is some vertex $w \in W$ that has no neighbor in X, which means that w has degree $n/2 - 1$ in B, a contradiction. Hence, $B[W \cup X]$ is a one-factor.

That (B.2) and (B.3) hold follows easily from the facts that $\delta(B) = n/2$ and $|V_1 \setminus W| = |V_2 \setminus X| = n/2 - 1$. Let us now prove that (B.5) holds.

Suppose that there is a perfect matching M_1' with only edges of color 1 such that $w_1x_1 \notin M_1'$ and $w_2x_2 \notin M_1'$, where $w_i \in W$ and $x_i \in X$, $i = 1, 2$. Set $B'' = B - M_1'$. It follows from (B.2) and (B.3) that $B_3 = B''[W \setminus \{w_1, w_2\} \cup V_2 \setminus X]$ and $B_4 = B''[V_1 \setminus W \cup X \setminus \{x_1, x_2\}]$ are balanced bipartite graphs with parts of size $n/2 - 1$ where each vertex has degree at least $n/2 - 2$. Hence, B_3 has a perfect matching $\hat{M}_3$ and B_4 has a perfect matching $\hat{M}_4$. Then $\hat{M}_3 \cup \hat{M}_4 \cup \{x_1w_1, x_2w_2\}$ is a perfect matching in B'', contradicting that Problem 1.2 has a negative answer. Hence, (B.5) holds.

As mentioned above, deciding whether B has a perfect matching with only edges of color 1 can be done in polynomial time. Assuming that B has such a matching, the problem of finding the set W in the proof above can be solved in polynomial time by well-known algorithms for constructing maximum perfect matchings in bipartite graphs. The set X can clearly also be found in polynomial time. Moreover, given the sets W and X, the problem of verifying whether W and X satisfy that (B.1)-(B.5) hold is clearly solvable in polynomial time.

We conclude that Problem 1.2 is solvable in polynomial time for a balanced bipartite graph on $n + n$ vertices with minimum degree at least $n/2$. □

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