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**INSTITUT
MITTAG-LEFFLER**

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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S. Ok, R. B. Richter and C. Thomassen

REPORT No. 39, 2013/2014, spring

ISSN 1103-467X

ISRN IML-R- -39-13/14- -SE+spring

Liftings in finite graphs and linkages in infinite graphs with prescribed edge-connectivity*

Seongmin Ok[†], R. Bruce Richter^{+‡}, and Carsten Thomassen^{†°}

L^AT_EX-ed: March 31, 2015

Abstract

Let G be a graph and let s be a vertex of G . We consider the structure of the set of all lifts of two edges incident with s that preserve edge-connectivity. Mader proved that two mild hypotheses imply there is at least one pair that lifts, while Frank showed (with the same hypotheses) that there are at least $(\deg(s) - 1)/2$ disjoint pairs that lift. We consider the *lifting graph*: its vertices are the edges incident with s , two being adjacent if they form a liftable pair. We have three main results, the first two with the same hypotheses as for Mader's Theorem.

(i) Let F be a subset of the edges incident with s . We show that F is independent in the lifting graph of G if and only if there is a single edge-cut C in G of size at most $r + 1$ containing all the edges in F , where r is the maximum number of edge-disjoint paths from a vertex (not s) in one component of $G - C$ to a vertex (not s) in another component of $G - C$.

(ii) In the k -lifting graph, two edges incident with s are adjacent if their lifting leaves the resulting graph with the property that any two vertices different from s are joined by k pairwise edge-disjoint paths. If both $\deg(s)$ and k are even, then the k -lifting graph is a connected complete multipartite graph. In all other cases, there are at most two components. If there are exactly two components, then each component is a complete multipartite graph. If $\deg(s)$ is odd and there are two components, then one component is a single vertex.

(iii) Huck proved that if k is odd and G is $(k + 1)$ -edge-connected, then G is weakly k -linked (that is, for any k pairs $\{x_i, y_i\}$, there are k edge-disjoint paths P_i , with P_i joining x_i and y_i). We use our results to extend a slight weakening of Huck's theorem to some infinite graphs: if k is odd, every $(k + 2)$ -edge-connected, locally finite, 1-ended, infinite graph is weakly k -linked.

Keywords: edge-connectivity, lifting

AMS Classification (2000): 05C40

seok@dtu.dk, brichter@uwaterloo.ca, and ctho@dtu.dk

*Some of this work was done by the second and third authors while at the Institut Mittag-Leffler (Djursholm, Sweden). Some of this work was done during a visit by the first author to the University of Waterloo.

[†]Technical University of Denmark, and ⁺University of Waterloo.

Supported in part by [‡]NSERC and [°]ERC Advanced Grant GRACOL.

1 Introduction

For distinct vertices x and y in a graph G , $\lambda_G(x, y)$ denotes the maximum number of pairwise edge-disjoint xy -paths in G . We shall assume that x and y have a *target connectivity* $\tau_G(x, y) \leq \lambda_G(x, y)$. In the cases of immediate interest, either $\tau_G \equiv \lambda_G$ or τ_G is constant, but the target unifies and generalizes both these particular cases.

Let s be a vertex of G and let sv and sw be two edges incident with s . The *lift of G at sv and sw* is the graph $G_{v,w}$ obtained from $G - \{sv, sw\}$ by adding the edge vw .

The lift of G at sv and sw is τ_G -feasible if, for every pair x, y of distinct vertices in $G - s$, $\lambda_{G_{v,w}}(x, y) \geq \tau_G(x, y)$. We will just say *feasible*, since τ_G will always be understood.

Let s be a vertex in a graph G that does not have degree 3 and is not incident with an isthmus. (An *isthmus* is an edge whose deletion from G increases the number of components.) Mader [5] proved (for target λ_G and therefore for any target) that there is always a feasible lift in G using two edges incident with s . Frank [3] extended this to show that there are $\lfloor \deg(s)/2 \rfloor$ pairwise disjoint such feasible pairs.

For any subset A of $V(G)$, we set $\delta_G(A)$ to be the set of edges of G having one end in A and one end not in A . By Menger's Theorem, the obstruction to sv and sw yielding a feasible lift is that there is a pair a, b of vertices and a set A of vertices so that $a \in A$, $b, s \notin A$, and $|\delta_{G_{v,w}}(A)| < \tau_G(a, b)$. Since obviously $|\delta_G(A)| \geq \tau_G(a, b)$ and $|\delta_{G_{v,w}}(A)| \geq |\delta_G(A)| - 2$, we see that $|\delta_G(A)| \leq |\tau_G(a, b)| + 1$. Thus motivates the following important notion.

Let A be a subset of $V(G) \setminus \{s\}$. Then $r(A)$ is defined to be $\max\{\tau_G(a, b) \mid a \in A, b \notin A \cup \{s\}\}$. Also, A is a *dangerous set* if $|\delta_G(A)| \leq r(A) + 1$. The preceding paragraph readily implies the observation that sv and sw do not have a feasible lift if and only if there is a dangerous set A such that $v, w \in A$.

Henceforth, all considerations are in G , so we write $\delta(A)$ instead of $\delta_G(A)$.

The first of our three main results is the following. The "if" part of the statement is trivial; the "only if" is proved in the next section.

Theorem 1.1 *Let G be a graph and let s be a vertex of G that does not have degree 3 and is not incident with an isthmus. Let F be any set of at least two edges, all incident with s . Then no pair of edges in F yields a feasible lift if and only if there is a dangerous set A so that, for every $sv \in F$, $v \in A$.*

Let G be a graph, let s be a vertex of G , and let τ be the edge-connectivity target function. The *lifting graph* $L(G, s, \tau)$ has as its vertices the edges of G incident with s and two edges are adjacent in $L(G, s, \tau)$ if they form a τ -feasible pair. If there is a positive

35 integer k so that $\tau \equiv k$, then we write $L(G, s, k)$ for $L(G, s, \tau)$; $L(G, s, k)$ is the k -lifting
36 graph.

37 Thomassen [8] proved that the k -lifting graph of an Eulerian graph has a disconnected
38 complement. This was used to prove a decomposition theorem for infinite graphs that
39 implies, among other things, a conjecture from 1989: every $8k$ -edge-connected infinite
40 graph has a k -arc-connected orientation.

41 Part (1.2.4) of our second main result generalizes Thomassen's Eulerian result to the
42 k -lifting graph when $\deg(s)$ and k are both even.

43 **Theorem 1.2** *Let G be a graph with a vertex s and let k be a positive integer such that*
44 *any distinct vertices different from s are joined by k pairwise edge-disjoint paths. If s is*
45 *not incident with an isthmus and $\deg(s) \geq 4$, then:*

46 (1.2.1) *the k -lifting graph $L(G, s, k)$ has at most two components;*

47 (1.2.2) *if $\deg(s)$ is odd and $L(G, s, k)$ has two components, then one has only one*
48 *vertex and the other component is complete multipartite;*

49 (1.2.3) *if $\deg(s)$ is even and $L(G, s, k)$ has two components, then each component is*
50 *complete multipartite with an even number of vertices; and*

51 (1.2.4) *if $\deg(s)$ and k are both even, then $L(G, s, k)$ is a connected, complete multi-*
52 *partite graph (in particular, it has a disconnected complement).*

53 *If either $L(G, s, k)$ is not connected or both $\deg(s)$ and k are even, then any component*
54 *of $L(G, s, k)$ with at least 4 vertices is not a star $K_{1,r}$.*

55 A graph G is *weakly k -linked* if, for any sequences x_1, x_2, \dots, x_k and y_1, y_2, \dots, y_k of
56 (not necessarily distinct) vertices of G , there are k edge-disjoint paths P_1, P_2, \dots, P_k such
57 that P_i has ends x_i and y_i . By choosing all the x_i to be the same vertex and all the y_i to be
58 the same vertex, we see that any weakly k -linked graph is k -edge-connected. Thomassen
59 [7] conjectured that, when k is odd, the converse holds. Okamura [6] obtained the first
60 significant result about this conjecture (roughly: if G is $\frac{4}{3}k$ -edge-connected, then G is
61 weakly k -linked). Then Huck [4] proved that, if k is odd and G is $(k+1)$ -edge-connected,
62 then G is weakly k -linked.

63 We use Huck's Theorem and Theorem 1.2 (1.2.4) to prove the following. Recall that
64 an infinite graph G is *locally finite* if, for every vertex v of G , $\deg(v)$ is finite. Also, a
65 graph G is *1-ended* if, for every finite set S of vertices, $G - S$ has at most one infinite
66 component.

67 **Theorem 1.3** *Let k be an odd positive integer. If G is a $(k+2)$ -edge-connected, 1-ended,*
68 *locally finite graph, then G is weakly k -linked.*

69 We remark that we can prove that the hypothesis of Theorem 1.3 implies that any
70 $(k+2)$ -edge-connected, infinite, locally finite graph with only finitely many ends is weakly
71 k -linked. There are some technicalities that are not germane to the application of Huck's
72 Theorem and Theorem 1.2. We believe the following much stronger statement is true and
73 so choose not to include this intermediate result.

74 **Conjecture 1.4** *Let k be an odd positive integer. If G is a $(k+2)$ -edge-connected (infi-*
75 *nite) graph, then G is weakly k -linked.*

76 2 Characterizing independent sets in the lifting graph

77 Our goal in this section is to prove Theorem 1.1. It is evident that, if there is a dangerous
78 set A such that, for every $sv \in F$, $v \in A$, then no two edges in F give a feasible lift. It
79 was the converse that attracted us.

80 Chan et al [2] give a very closely related argument, presented very efficiently. Our
81 theorem is used significantly in the next section, so we include our slightly modified
82 version of their proof.

83 For the proof, it will be helpful to set $\sigma(A) = |\delta(A)| - r(A)$ and $\delta(A, B)$ as the set of
84 edges with one end in A and other end in B . We note that A is dangerous if and only if
85 $\sigma(A) \leq 1$. The following observation is due to Frank.

86 **Lemma 2.1** [3, Prop. 2.3] *Let s be a vertex in a graph G and let A and B be subsets of*
87 *$V(G) \setminus \{s\}$. Then either*

88 (2.1.1) $\sigma(A \cup B) + \sigma(A \cap B) + 2|\delta(A \setminus B, B \setminus A)| \leq \sigma(A) + \sigma(B)$ or

89 (2.1.2) $\sigma(A \setminus B) + \sigma(B \setminus A) + 2|\delta(A \cap B, V(G) \setminus (A \cup B))| \leq \sigma(A) + \sigma(B)$. ■

90 The key lemma for our proof is the following variant of [2, Lemma 2.7]. The proof
91 requires only very minor modifications from that in [2].

92 **Lemma 2.2** *Let G be a graph and s a vertex of G . Suppose sa , sb , and sc are three edges*
93 *incident with s so that none of the lifts of $\{sa, sb\}$, $\{sa, sc\}$, and $\{sb, sc\}$ is τ -feasible.*
94 *For $\{x, y, z\} = \{a, b, c\}$, let D_x be a dangerous set containing y and z . Then either s has*
95 *degree 3, or s is incident with an isthmus, or there is a dangerous subset of $D_a \cup D_b \cup D_c$*
96 *containing all three of a , b , and c and at least one of D_a , D_b , and D_c .*

97 **Proof.** If any two of a, b, c are the same, then the result is trivial, so we assume $a, b,$
 98 and c are all distinct. We consider two cases.

99 **Case 1:** For at least one of the pairs (A, B) from $(D_a, D_b), (D_a, D_c),$ or $(D_b, D_c),$ (2.1.1)
 100 holds in Lemma 2.1.

101 We may choose the labelling of $a, b,$ and $c,$ so that

$$102 \quad \sigma(D_a \cup D_b) + \sigma(D_a \cap D_b) + 2|\delta(D_a \setminus D_b, D_b \setminus D_a)| \leq \sigma(D_a) + \sigma(D_b).$$

103 As each term on the right side is at most 1, the left-hand side is at most 2. If $D_a \cup D_b$ is
 104 dangerous, then we are done, so we may assume $\sigma(D_a \cup D_b) \geq 2.$ Therefore, the right-hand
 105 side is exactly 2, $\sigma(D_a \cup D_b) = 2,$ $\sigma(D_a \cap D_b) = 0,$ and $|\delta(D_a \setminus D_b, D_b \setminus D_a)| = 0.$

106 Suppose Lemma 2.1 (2.1.1) holds for $A = D_a \cap D_b$ and $B = D_c;$ that is,

$$107 \quad \sigma((D_a \cap D_b) \cup D_c) + \sigma((D_a \cap D_b) \cap D_c) + 2|\delta((D_a \cap D_b) \setminus D_c, D_c \setminus (D_a \cap D_b))| \\ 108 \quad \leq \sigma(D_a \cap D_b) + \sigma(D_c).$$

109 Since $\sigma(D_a \cap D_b) = 0,$ the right side is at most 1 and, therefore, $(D_a \cap D_b) \cup D_c$ is
 110 dangerous, and we are done. Therefore, we may assume Lemma 2.1 (2.1.2) applies to
 111 $A = D_a \cap D_b$ and $B = D_c.$ In particular, $\sigma(D_c \setminus (D_a \cap D_b)) \leq \sigma(D_a \cap D_b) + \sigma(D_c),$
 112 showing $D_c \setminus (D_a \cap D_b)$ is dangerous. (It is evidently not empty, as it contains a and $b.$)

113 Set $D'_c = D_c \setminus (D_a \cap D_b).$ The edges sa and sb show that $|\delta((D_a \cup D_b) \cap D'_c, V(G) \setminus$
 114 $(D_a \cup D_b \cup D'_c))| \geq 2.$ On the other hand, the labelling for this case shows $\sigma(D_a \cup D_b) \leq$
 115 $\sigma(D_a) + \sigma(D_b) \leq 2$ and the preceding paragraph shows $\sigma(D'_c) \leq 1.$ Thus,

$$116 \quad 2|\delta((D_a \cup D_b) \cap D'_c, V(G) \setminus (D_a \cup D_b \cup D'_c))| \geq 4 > 3 \geq \sigma(D_a \cup D_b) + \sigma(D'_c).$$

117 Consequently, Lemma 2.1 implies

$$118 \quad \sigma((D_a \cup D_b) \cup D'_c) + \sigma((D_a \cup D_b) \cap D'_c) + 2|\delta((D_a \cup D_b) \setminus D'_c, D'_c \setminus (D_a \cup D_b))| \leq \sigma(D_a \cup D_b) + \sigma(D'_c).$$

119 If $(D_a \cup D_b) \cup D'_c$ is dangerous, then we are done, so we may assume $\sigma((D_a \cup D_b) \cup D'_c) \geq$
 120 2. As $\sigma(D_a \cup D_b) = 2$ and $\sigma(D'_c) \leq 1,$ we conclude that $\sigma((D_a \cup D_b) \cap D'_c) \leq 1$ and
 121 $|\delta((D_a \cup D_b) \setminus D'_c, D'_c \setminus (D_a \cup D_b))| = 0.$ The inequality shows $(D_a \cup D_b) \cap D'_c$ is dangerous,
 122 while $|\delta(D_a \setminus D_b, D_b \setminus D_a)| = 0$ implies $|\delta((D_a \cap D'_c) \setminus (D_b \cap D'_c), (D_b \cap D'_c) \setminus (D_a \cap D'_c))| = 0.$

123 We claim that either sa or sb is an isthmus of $G.$ We have just seen that $(D_a \cup D_b) \cap D'_c$
 124 is dangerous, so,

$$125 \quad 1 \geq \sigma((D_a \cup D_b) \cap D'_c) \\ 126 \quad = |\delta((D_a \cup D_b) \cap D'_c)| - r((D_a \cup D_b) \cap D'_c) \\ 127 \quad \geq |\delta(D_a \cap D'_c)| + |\delta(D_b \cap D'_c)| - \max\{r(D_a \cap D'_c), r(D_b \cap D'_c)\} \\ 128 \quad \geq \min\{|\delta(D_a \cap D'_c)|, |\delta(D_b \cap D'_c)|\}.$$

129 Therefore, either $|\delta(D_a \cap D'_c)| \leq 1$ or $|\delta(D_b \cap D'_c)| \leq 1$. We may choose the labelling of a
130 and b so that the former holds. Since $b \in D_a \cap D'_c$, sb shows $|\delta(D_a \cap D'_c)| \geq 1$, so we have
131 $|\delta(D_a \cap D'_c)| = 1$. Therefore, sb is an isthmus, completing the proof in Case 1.

132 **Case 2:** For every one of the pairs (D_a, D_b) , (D_a, D_c) , and (D_b, D_c) , (2.1.2) holds in
133 Lemma 2.1.

134 The assumption of the case implies that, for example,

$$135 \quad \sigma(D_a \setminus D_b) + \sigma(D_b \setminus D_a) + 2|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| \leq \sigma(D_a) + \sigma(D_b) \leq 2.$$

136 Since $c \in D_a \cap D_b$ and $s \in V(G) \setminus (D_a \cup D_b)$, $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| \geq 1$. We
137 conclude that $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| = 1$, $\sigma(D_a \setminus D_b) = 0$, and $\sigma(D_b \setminus D_a) = 0$.

138 As in the preceding paragraph, since $b \in (D_a \setminus D_b) \cap D_c$, we see that $|\delta((D_a \setminus D_b) \cap$
139 $D_c, V(G) \setminus ((D_a \setminus D_b) \cup D_c))| \geq 1$. Also, $\sigma(D_a \setminus D_b) = 0$ and $\sigma(D_c) \leq 1$. Thus, Lemma
140 2.1 (2.1.2) does not hold for $A = D_a \setminus D_b$ and $B = D_c$. Therefore (2.1.1) holds in Lemma
141 2.1; in particular, $\sigma((D_a \setminus D_b) \cup D_c) \leq \sigma(D_a \setminus D_b) + \sigma(D_c) \leq 1$. That is, $(D_a \setminus D_b) \cup D_c$
142 is dangerous. Since this does not contain c , we could set $D'_c = (D_a \setminus D_b) \cup D_c$ and
143 conduct this argument over again. When we do this, $D_a \setminus D_b \subseteq D'_c$, so we may assume
144 this happens in the first place. That is, we may assume $D_a \setminus D_b \subseteq D_c$; likewise, we may
145 assume $D_c \setminus D_a \subseteq D_b$, and $D_b \setminus D_c \subseteq D_a$.

146 We still have $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| = 1$. Likewise both $|\delta(D_a \cap D_c, V(G) \setminus$
147 $(D_a \cup D_c))| = 1$ and $|\delta(D_b \cap D_c, V(G) \setminus (D_b \cup D_c))| = 1$ hold. In particular, we know
148 there is only one edge from s to each of a , b , and c . Also, it follows that $|\delta(D_a \cup D_b \cup$
149 $D_c, V(G) \setminus (\{s\} \cup D_a \cup D_b \cup D_c))| = 0$.

150 If s is not incident with an isthmus, then, for every component K of $G - s$, $|\delta(V(K))| \geq$
151 2. Since $|\delta(D_a \cup D_b \cup D_c)| = 3$ and all edges in $\delta(D_a \cup D_b \cup D_c)$ are also incident with s ,
152 we conclude that $G[D_a \cup D_b \cup D_c]$ is connected and is a component of $G - s$. Therefore,
153 there are two edge-disjoint as -paths in $G[\{s\} \cup D_a \cup D_b \cup D_c]$.

154 If the degree of s is not 3, then we conclude that $G - s$ has at least two components.
155 If K is a component of $G - s$ other than $G[D_a \cup D_b \cup D_c]$ and s is not incident with
156 an isthmus, then, for any neighbour t of s in K , there are two edge-disjoint ts -paths in
157 $G[\{s\} \cup V(K)]$. It follows that there are two edge-disjoint at -paths in G , showing that
158 $r(D_a \cup D_b \cup D_c) \geq 2$.

159 Since $|\delta(D_a \cup D_b \cup D_c)| = 3$, we conclude that $\sigma(D_a \cup D_b \cup D_c) \leq 1$. Thus, $D_a \cup D_b \cup D_c$
160 is dangerous, as required. ■

161 The proof of Theorem 1.1 is now quite simple.

162 **Proof of Theorem 1.1.** We proceed by induction on $|F|$, with the cases $|F| = 2$ and
 163 3 being, respectively, trivial and an immediate consequence of Lemma 2.2. So assume
 164 $|F| \geq 4$, with $F = \{su_1, su_2, \dots, su_k\}$. By induction, there are dangerous sets A_{k-1} and
 165 A_k containing, respectively, all of $F \setminus \{su_{k-1}\}$ and $F \setminus \{su_k\}$. If either $u_{k-1} \in A_{k-1}$ or
 166 $u_k \in A_k$, then we are done, so we may assume neither of these containments occurs.

167 Because su_{k-1} and su_k do not make a feasible lift, there is a dangerous set A containing
 168 both u_{k-1} and u_k ; among all such dangerous sets, we choose A to be maximal. If, for every
 169 $i \in \{1, 2, \dots, k-2\}$, $u_i \in A$, then we are done. Otherwise, there is some $i \in \{1, 2, \dots, k-2\}$
 170 such that $u_i \notin A$.

171 We apply Lemma 2.2 to the pairs $\{u_i, u_{k-1}\}$, $\{u_i, u_k\}$, and $\{u_{k-1}, u_k\}$ and the sets A ,
 172 A_{k-1} , and A_k . We conclude that there is a dangerous set A^* containing all of u_i , u_{k-1} ,
 173 and u_k and also containing one of A , A_{k-1} , and A_k .

174 If $A \subseteq A^*$, then, since $u_i \in A^* \setminus A$, we contradict the maximality of A . Therefore,
 175 either A_{k-1} or A_k is contained in A^* , from which we conclude that every u_j is in A^* , as
 176 required. ■

177 3 Connection in the lifting graph

178 In this section, we prove Theorem 1.2 dealing with the structure of the k -lifting graph
 179 $L(G, s, k)$.

180 The proofs are inductive and the base cases $\deg(s) = 4$ or 5 require some effort. There
 181 is one special argument needed for $\deg(s) = 6$ when k is odd. The inductive arguments
 182 are based on the following simple observation and its contrapositive.

183 **Observation 3.1** *If, after lifting the feasible pair $\{e_1, e_2\}$, the pair $\{e_3, e_4\}$ is feasible,*
 184 *then $\{e_3, e_4\}$ is feasible in the original graph.* □

185 3.1 Some general arguments

186 In this subsection, we give a few elementary general arguments used later for describing
 187 the lifting graph. The first arguments are based on standard methods for “crossing cuts”.
 188 Let A_1 and A_2 be two subsets of $V(G)$. It is an easy exercise to verify that, where
 189 $\overline{A} = V(G) \setminus A$,

$$190 \quad 2 \left[|\delta(A_1)| + |\delta(A_2)| - (|\delta(A_1 \cap A_2, \overline{A_1 \cup A_2})| + |\delta(A_2 \setminus A_1, A_1 \setminus A_2)|) \right] = \quad (3.1)$$

$$|\delta(A_1 \cap A_2)| + |\delta(A_2 \setminus A_1)| + |\delta(A_1 \setminus A_2)| + |\delta(\overline{A_1 \cup A_2})|.$$

191 A typical application will be when all four sets $A_1 \cap A_2$, $A_2 \setminus A_1$, $A_1 \setminus A_2$, and $\overline{A_1 \cup A_2}$
192 are non-empty and G is k -edge-connected. In that case, the right-hand side is at least $4k$.
193 If, for example, both $\delta(A_1)$ and $\delta(A_2)$ have size k , we deduce that $\delta(A_1 \cap A_2, \overline{A_1 \cup A_2})$
194 and $\delta(A_2 \setminus A_1, A_1 \setminus A_2)$ are both empty. Furthermore, it is a routine exercise to verify
195 that this extreme case can only occur with k even.

196 We will apply a slightly more sophisticated consequence of Equation 3.1.

197 **Lemma 3.2** *Let k be a natural number, and let G be a graph with a vertex s such that*
198 *any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G . For $i = 1, 2$,*
199 *let F_i be an independent set in $L(G, s, k)$ of size r_i and suppose there is a dangerous set*
200 *A_i so that $F_i = \delta(\{s\}) \cap \delta_G(A_i)$. Set $\alpha = |F_1 \cap F_2|$. If $\alpha > 0$, $r_1 > \alpha$, $r_2 > \alpha$, and*
201 *$\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$, then $r_1 + r_2 \leq \lfloor \deg(s)/2 \rfloor + 2$.*

202 **Proof.** Observe that: $|\delta_{G-s}(A_1)| \leq k + 1 - r_1$; $|\delta_{G-s}(A_2)| \leq k + 1 - r_2$; $|\delta_{G-s}(A_1 \cap$
203 $A_2)| \geq k - \alpha$; $|\delta_{G-s}(A_2 \setminus A_1)| \geq k - (r_2 - \alpha)$; $|\delta_{G-s}(A_1 \setminus A_2)| \geq k - (r_1 - \alpha)$; and
204 $|\delta_{G-s}(V(G - s) \setminus (A_1 \cup A_2))| \geq k - (\deg(s) - (r_1 + r_2 - \alpha))$.

205 From Equation 3.1, we deduce that

$$206 \quad 2(k+1-r_1+k+1-r_2) \geq (k-\alpha)+(k-(r_2-\alpha))+(k-(r_1-\alpha))+(k-(\deg(s)-(r_1+r_2-\alpha))).$$

207 Rearranging, we see that $\deg(s) + 4 \geq 2(r_1 + r_2)$. Since every term except possibly $\deg(s)$
208 is even, $\lfloor \deg(s)/2 \rfloor + 2 \geq r_1 + r_2$, as required. \blacksquare

209 Our final preliminary result gives our first glimpse of some structure in $L(G, s, k)$.

210 **Lemma 3.3** *Let k be a natural number, and let G be a graph with a vertex s such that*
211 *any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G . If $\deg(s)$ is*
212 *at least 4, then:*

213 (3.3.1) *every independent set in $L(G, s, k)$ has size at most $\lfloor \frac{1}{2} \deg(s) \rfloor$; and*

214 (3.3.2) *if $\deg(s)$ is even and at least 6, then any two distinct independent sets in*
215 *$L(G, s, k)$ of size $\frac{1}{2} \deg(s)$ are disjoint.*

216 **Proof.** By Theorem 1.1, an independent set F corresponds to a dangerous set A con-
217 taining all the non- s ends of the edges in F , so $|\delta(A)| \leq k + 1$. If $|\delta(\{s\}) \setminus F| < |F| - 1$,
218 then $\delta(A \cup \{s\})$ has size at most $k - 1$, a contradiction. Thus, $|\delta(\{s\}) \setminus F| \geq |F| - 1$, as
219 required for (3.3.1).

220 Suppose F_1 and F_2 are non-disjoint independent sets of size $\frac{1}{2} \deg(s)$, with correspond-
221 ing dangerous sets A_1 and A_2 . At most $\deg(s) - 1$ of the edges of $\delta(\{s\})$ have one end in

222 $A_1 \cup A_2$, so $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$. Also, each of $A_1 \cap A_2$, $A_2 \setminus A_1$, and $A_1 \setminus A_2$ has an end
223 of an edge in $F_1 \cup F_2$. Since, for $i = 1, 2$, Lemma 3.3 (3.3.1) implies $F_i = \delta(\{s\}) \cap \delta(A_i)$,
224 the hypotheses of Lemma 3.2 are satisfied. However, $r_1 = \frac{1}{2} \deg(s) = r_2$, showing the
225 conclusion of Lemma 3.2 fails, a contradiction that proves (3.3.2). ■

226 3.2 $\deg(s) = 4$

227 In this subsection, we treat the case $\deg(s) = 4$. Let e_1, e_2, e_3, e_4 be the four edges incident
228 with s . It is a triviality that if some pair, say e_1, e_2 is feasible, then so is the complementary
229 pair e_3, e_4 . It follows that $L(G, s, k)$ is a union of perfect matchings; Mader's Theorem
230 already shows there is at least one such matching in $L(G, s, k)$. Since it has only four
231 vertices, it can only be one of: a perfect matching; a 4-cycle C_4 ; and K_4 . These are all
232 realizable. However, when k is even, the perfect matching is not achievable, as we show
233 next.

234 **Proposition 3.4** *Let k be a natural number, and let G be a graph with a vertex s such that*
235 *any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G . If $\deg(s) = 4$,*
236 *then $L(G, s, k)$ is one of: a perfect matching; C_4 ; and K_4 . If k is even, then $L(G, s, k)$ is*
237 *not a perfect matching.*

238 **Proof.** We only prove the second assertion. Suppose both pairs e_1, e_2 and e_1, e_3 are not
239 feasible. Then there are dangerous sets A_2 and A_3 so that the non- s ends of e_1, e_2 are in
240 A_2 and the non- s ends of e_1, e_3 are in A_3 .

241 By definition, $|\delta_G(A_2)| \leq k + 1$, while the hypothesis implies $|\delta_G(A_2 \cup \{s\})| \geq k$.
242 Therefore, e_3 and e_4 have their non- s ends in $\overline{A_2} = V(G) \setminus (A_2 \cup \{s\})$. The analogous
243 statement holds for A_3 .

244 It follows that $|\delta_{G-s}(A_2 \cap A_3)|$, $|\delta_{G-s}(A_2 \setminus A_3)|$, $|\delta_{G-s}(A_3 \setminus A_2)|$, and $|\delta_{G-s}(\overline{A_2 \cup A_3})|$ are
245 all at least $k - 1$, while $|\delta_{G-s}(A_2)|$ and $|\delta_{G-s}(A_3)|$ are both at most $k - 1$. But $k - 1$ is odd,
246 so Equation 3.1 cannot be realized (as mentioned in the paragraph following Equation
247 3.1). ■

248 We comment that the proofs of Proposition 3.4 and Equation 3.1 also imply that,
249 when k is odd, there is only one pattern for G for which $L(G, s, k)$ is a perfect matching;
250 this is illustrated in Figure 3.5, where there are four edges incident with s and the thick
251 edges represent $(k - 1)/2$ edges. No two edges consecutive in the illustrated cyclic rotation
252 at s form a feasible pair.

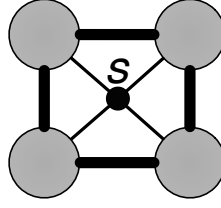


Figure 3.5: Each thick edge represents $(k - 1)/2$ edges.

253 **3.3** $\deg(s) = 5$

254 In this subsection, we prove the following, dealing with the case $\deg(s) = 5$.

255 **Proposition 3.6** *Let k be a natural number, and let G be a graph with a vertex s such that*
 256 *any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G . If $\deg(s) = 5$,*
 257 *then $L(G, s, k)$ is either an isolated vertex plus a 4-cycle or a connected graph. If k is*
 258 *even and $L(G, s, k)$ is connected, then G is a complete multipartite graph.*

259 **Proof.** Lemma 3.3 (3.3.1) implies the largest independent set in $L(G, s, k)$ has size at
 260 most 3. We break the proof into two cases.

261 **Case 1:** $L(G, s, k)$ contains an independent set of size 3.

262 Let F be an independent set in $L(G, s, k)$ of size 3 and let A_1 be a dangerous set in
 263 G so that the non- s ends of the edges in F are all in A_1 . As there are only two edges
 264 incident with s and not in F , they both have their non- s ends in $\bar{A}_1 = V(G) \setminus (A_1 \cup \{s\})$.
 265 In particular, $|\delta_G(A_1)| = k + 1$ and $|\delta_G(A_1 \cup \{s\})| = k$, so the two edges in $\delta(\{s\}) \setminus F$ are
 266 also independent in $L(G, s, k)$.

267 Suppose $e_1 \in F$ and $e_2 \in \delta(\{s\}) \setminus F$ do not form a feasible pair and let A_2 be a
 268 dangerous set that witnesses this. As in the preceding paragraph, there are at least two
 269 edges in $\delta(\{s\}) \setminus \{e_1, e_2\}$ having their non- s ends in \bar{A}_2 ; at least one of these is in $F \setminus \{e_1\}$.

270 Thus, there is at least one edge from s to each of $A_1 \cap A_2$ (namely, e_1), $A_2 \setminus A_1$ (e_2),
 271 and $A_1 \setminus A_2$ (the one at the end of the preceding paragraph).

272 If $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$, then Lemma 3.2 implies $3 + |\delta(\{s\}) \cap \delta(A_2)| \leq 4$. But $e_1, e_2 \in$
 273 $\delta(\{s\}) \cap \delta(A_2)$, so we deduce that $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$.

274 It follows that both edges in $\delta(\{s\}) \setminus F$ have their non- s ends in $A_2 \setminus A_1$. Thus,
 275 $|\delta(\{s\}) \cap \delta(A_2)| \geq 3$. Since A_2 is dangerous, Lemma 3.3 implies $|\delta(\{s\}) \cap \delta(A_2)| \leq 3$.
 276 Therefore there are also two edges in $\delta(\{s\})$ with ends in $A_1 \setminus A_2$.

277 An immediate consequence of the preceding is that e_1 has no feasible lift with any
 278 other edge in $\delta(\{s\})$. Frank's Theorem implies that there is at most one edge incident

279 with s that is not in any feasible pair. It follows that e_1 is the only such edge; now
 280 applying the above argument to another edge e'_1 in $F \setminus \{e_1\}$ and an edge e_2 in $\delta(\{s\}) \setminus F$
 281 shows e'_1, e_2 is a feasible pair.

282 We conclude that, in the event there is an independent set of size 3 in $L(G, s, k)$,
 283 $L(G, s, k)$ is either $K_{2,3}$ or an isolated vertex plus C_4 .

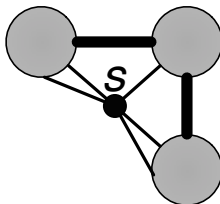


Figure 3.7: If each thick edge represents $k - 2$ edges, then $L(G, s, k)$ is an isolated vertex and C_4 . Changing one thick edge to $k - 1$ edges turns $L(G, s, k)$ into $K_{2,3}$.

284 **Case 2:** every independent set in $L(G, s, k)$ has size at most 2.

285 Suppose there are three edges e_0, e_1, e_2 in $\delta(\{s\})$ such that neither e_0, e_1 nor e_0, e_2 is a
 286 feasible pair.

287 (F1) The assumption of this case implies e_1, e_2 is a feasible pair.

288 For $i = 1, 2$, let A_i be a dangerous set containing the non- s ends of both e_0 and e_i .
 289 Because we are in Case 2, none of the three edges in $\delta(\{s\}) \setminus \{e_0, e_i\}$ has an end in A_i .
 290 Thus, each of these three edges has an end in $\overline{A_i \cup \{s\}}$. Since these three edges do not
 291 make an independent set in $L(G, s, k)$, $|\delta(\overline{A_i \cup \{s\}})| > k + 1$. Evidently, $|\delta(A_i)| \leq k + 1$,
 292 so $|\delta(A_i)| = k + 1$.

293 Moreover, there is precisely one edge from $\delta(\{s\})$ having an end in each of $A_1 \cap A_2$
 294 (e_0), $A_2 \setminus A_1$ (e_2), and $A_1 \setminus A_2$ (e_1). Therefore, the remaining two edges have their non- s
 295 ends in $\overline{A_1 \cup A_2 \cup \{s\}}$.

296 Since $\{e_0, e_1, e_2\}$ is not an independent set of size 3, $|\delta_G(A_1 \cup A_2)| \geq k + 2$. Thus, each
 297 of $\delta_{G-s}(A_1 \cap A_2)$, $\delta_{G-s}(A_2 \setminus A_1)$, $\delta_{G-s}(A_1 \setminus A_2)$, and $\delta_{G-s}(\overline{A_1 \cup A_2 \cup \{s\}})$ has size at least
 298 $k - 1$ (as this is trivially true for the first three). Since $\delta_{G-s}(A_1)$ and $\delta_{G-s}(A_2)$ have size
 299 precisely $k - 1$, as before from Equation 3.1, $k - 1$ is even.

300 It follows that, for k even, e_0, e_1 , and e_2 do not exist, so $L(G, s, k)$ is complete multi-
 301 partite.

302 In the case k is odd, $|\delta_G(\overline{A_1 \cup A_2 \cup \{s\}})| = k + 1$, showing the following.

303 (F2) The pair e_3, e_4 of edges in $\delta(\{s\}) \setminus \{e_0, e_1, e_2\}$ is not feasible.

304 **Subcase 2.1:** e_1, e_3 is not feasible.

305 Applying (F1) to e_1, e_0 and e_1, e_3 , we see that e_0, e_3 is a feasible pair.

306 On the other hand, (F2) implies the pair of edges e_2, e_4 in $\delta(\{s\}) \setminus e_1, e_0, e_3$ is not
 307 feasible. Now using e_2, e_0 and e_2, e_4 , we conclude from (F1) that e_0, e_4 is feasible.

308 Finally, (F1) and the infeasible pairs e_3, e_1 and e_3, e_4 show e_1, e_4 is feasible, and anal-
 309 ogously e_2, e_3 is feasible. In this case, $L(G, s, k)$ is C_5 .

310 **Subcase 2.2:** no version of Subcase 2.1; that is, $\{e_1, e_2, e_3, e_4\}$ induces $K_4 - e_3e_4$ in
 311 $L(G, s, k)$.

312 (We remark that this subcase occurs in the version of Figure 3.8 with one thick edge
 313 being $(k+1)/2$ edges.) Suppose e_0, e_3 is not a feasible pair. Then (F2) applied to e_0, e_1, e_3
 314 yields the contradiction that e_2, e_4 is not feasible. Therefore, e_0, e_3 and, symmetrically,
 315 e_0, e_4 , are feasible pairs. In this final case, $L(G, s, k)$ is $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$. ■

316 Figure 3.8 gives two examples for odd k . One has $L(G, s, k)$ being a 5-cycle, while, for
 317 the other, $L(G, s, k)$ is $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$.

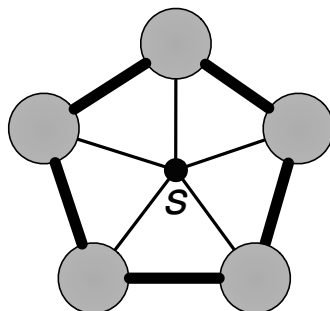


Figure 3.8: If each thick edge represents $(k-1)/2$ edges, then $L(G, s, k) = C_5$. Changing one thick edge to $(k+1)/2$ edges turns $L(G, s, k)$ into $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$.

318 3.4 The inductive step

319 In this subsection, we proceed with the induction to complete the proof of Theorem 1.2.

320 **Proof of Theorem 1.2.** For (1.2.1), we observe that if $\deg(s) = 4$ or 5 , then $L(G, s, k)$
 321 has at most two components. For the induction, suppose $\deg(s) \geq 6$. If $L(G, s, k)$ has
 322 more than two components, then it is the union of three subgraphs J_1, J_2, J_3 , with each
 323 J_i a union of components of $L(G, s, k)$.

324 Suppose, for some $i \in \{1, 2, 3\}$, J_i has at least three vertices. Frank's Theorem implies
 325 J_i has an edge e_1e_2 . Lifting e_1e_2 produces a graph G' with $\deg_{G'}(s) = \deg_G(s) - 2$ and

326 there is no edge of $L(G', s, k)$ between any two of the $J_j \cap L(G', s, k)$. This contradicts
 327 the inductive assumption that $L(G', s, k)$ has at most two components.

328 Therefore, each J_i has at most two vertices; since $\deg(s) \geq 6$, each J_i has precisely
 329 two vertices and $\deg(s) = 6$. However, in this case, there are 8 different independent sets
 330 of size 3, each consisting of one vertex from each of the J_i . This contradicts Lemma 3.3
 331 (3.3.2), completing the proof of (1.2.1).

332 For (1.2.2), the claim holds for $\deg(s) = 5$, so suppose $\deg(s) \geq 7$. Let H and J be the
 333 components of $L(G, s, k)$ with $|V(H)| < |V(J)|$. Then $|V(J)| \geq 4$ and if we lift an edge
 334 from J to get the graph G' , there is still no edge between $H \cap L(G', s, k)$ and $J \cap L(G', s, k)$
 335 and the latter has at least two vertices. Thus, $H \cap L(G', s, k)$, and therefore H , has only
 336 one vertex, as required.

337 To see that J is complete multipartite, suppose there exist e_0, e_1, e_2 in $V(J)$ such that
 338 e_0 is not adjacent in J to either of e_1 and e_2 , while $e_1e_2 \in E(J)$. Lift the pair e_1, e_2 to get
 339 the graph G' . Since J has at least 6 vertices, $J \cap L(G', s, k)$ is a component of $L(G', s, k)$
 340 with at least 4 vertices. By the inductive assumption, it is not a star, so it has an edge
 341 e_3e_4 not incident with e_0 . Then e_3e_4 is an edge of J .

342 Lift e_3, e_4 in G to get G'' ; the pair e_1, e_2 is feasible in G'' (the resulting graph is the
 343 same as first lifting e_1, e_2 and then lifting e_3, e_4), so e_1e_2 is an edge in $J \cap L(G'', s, k)$. But
 344 neither e_0e_1 nor e_0e_2 is an edge in $J \cap L(G'', s, k)$, contradicting the inductive assumption
 345 applied to $L(G'', s, k)$. Thus, J is both complete multipartite and not a star, as required.

346 For (1.2.3), we first prove that every component of $L(G, s, k)$ has an even number of
 347 vertices; this is trivial if there is only one component. This is known for $\deg(s) = 4$, so
 348 we suppose $\deg(s) \geq 6$. Let H and J be the two components with $|V(H)| \leq |V(J)|$. Let
 349 e_1e_2 be an edge of J and let G' be the result of lifting the pair e_1, e_2 . Then $H \cap L(G', s, k)$
 350 and $J \cap L(G', s, k)$ are the two components of $L(G', s, k)$. By induction they each have
 351 an even number of vertices, so this also holds for $L(G, s, k)$.

352 If $\deg(s) = 6$, then the induction and Lemma 3.3 (3.3.2) imply that $L(G, s, k)$ is the
 353 disjoint union of K_2 and either C_4 or K_4 . Therefore, we may assume $\deg(s) \geq 8$.

354 **Case 1:** *both components of $L(G, s, k)$ have at least four vertices.*

355 Suppose by way of contradiction that there are vertices e_0, e_1, e_2 in the component K
 356 of $L(G, s, k)$ such that neither e_0e_1 nor e_0e_2 is an edge of K , while e_1e_2 is an edge of K .
 357 Let J be the other component of $L(G, s, k)$.

358 Lift e_1, e_2 to get G' . Then $K \cap L(G', s, k)$ and $J \cap L(G', s, k)$ are the two components
 359 of $L(G', s, k)$. Thus, there is an edge e_3e_4 in $J \cap L(G', s, k)$. Now lift e_3, e_4 in G to get G'' .
 360 Then $K \cap L(G'', s, k)$ is a component of $L(G'', s, k)$. The edge e_1e_2 is in $K \cap L(G'', s, k)$,
 361 while neither e_0e_1 nor e_0e_2 is an edge of $K \cap L(G'', s, k)$. This contradicts the inductive

362 assumption that $K \cap L(G'', s, k)$ is complete multipartite.

363 **Case 2:** *one component of $L(G, s, k)$ has precisely two vertices.*

364 Let J and K be the components of $L(G, s, k)$ so that J has precisely two vertices; thus
 365 K has at least six vertices. Suppose e_0, e_1, e_2 are vertices of K such that neither e_0e_1 nor
 366 e_0e_2 is an edge of K , yet e_1e_2 is an edge of K .

367 Lift e_1, e_2 to obtain the graph G' . By the induction, $K \cap L(G', s, k)$ is a component of
 368 $L(G', s, k)$, and it has at least 4 vertices, so it is not a star. Therefore, it has an edge e_3e_4
 369 disjoint from e_0 ; we lift e_3, e_4 in G to obtain G'' . Induction tells us that $K \cap L(G'', s, k)$ is
 370 complete multipartite, which contradicts the fact that e_0, e_1, e_2 are all in $K \cap L(G'', s, k)$,
 371 e_0e_1 and e_0e_2 are not edges, and e_1e_2 is an edge.

372 Lastly, we prove (1.2.4). Proposition 3.4 gives the result for $\deg(s) = 4$, so we assume
 373 $\deg(s) \geq 6$. Suppose e_0, e_1, e_2 are vertices in $L(G, s, k)$ such that e_0 is not adjacent to
 374 either e_1 or e_2 , but e_1e_2 is an edge of $L(G, s, k)$. Lifting e_1, e_2 yields a graph G' for
 375 which $L(G', s, k)$ has at least 4 vertices. By induction, $L(G', s, k)$ is connected, complete
 376 multipartite, and not a star; in particular, it has an edge e_3e_4 disjoint from e_0 .

377 Lifting e_3, e_4 in G produces a graph G'' ; by induction $L(G'', s, k)$ is complete multi-
 378 partite. However, e_0 is still not adjacent to either e_1 or e_2 , while e_1e_2 is an edge. This
 379 contradiction shows $L(G, s, k)$ is complete multipartite and Frank's Theorem [3] shows it
 380 is not a star, as required. ■

381 We conclude this section with Figure 3.9. This is an example having $\deg(s) = 6$
 382 and $k = 5$ so that $L(G, s, k)$ is $K_{3,3}$ minus an edge; in particular, it is connected and not
 383 complete multipartite. The three edges incident with s on the left side are one independent
 384 set, the three on the right are a second, and the two going to the bottom are not feasible.

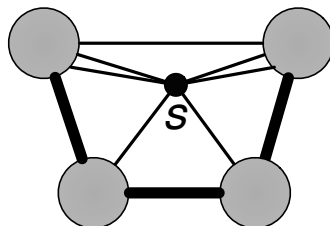


Figure 3.9: Each thick edge represents 2 edges and $k = 5$.

385 4 Weakly k -linked infinite graphs

386 In this section we prove Theorem 1.3: if k is odd, then a $(k + 2)$ -edge-connected, locally
 387 finite, 1-ended, infinite graph G is weakly k -linked.

388 If $\mathbf{x} = (x_1, x_2, \dots, x_k)$ and $\mathbf{y} = (y_1, y_2, \dots, y_k)$ are sequences of (not necessarily dis-
 389 tinct) vertices in graph G , then an \mathbf{xy} -linkage is a set $\{P_1, P_2, \dots, P_k\}$ of pairwise edge-
 390 disjoint paths in G such that, for $i = 1, 2, \dots, k$, P_i is an $x_i y_i$ -path.

391 Before we prove Theorem 1.3, we require extensions of the theorems of Mader and
 392 Frank and of our Theorem 1.2 to locally finite graphs. These extensions may all be
 393 proved as follows. Let G_d be the subgraph of a locally finite graph G consisting of those
 394 vertices at distance at most d from the specified vertex s . Let G'_d be the graph obtained
 395 from G by contracting each component of $G - V(G_d)$ to a vertex. For infinitely many d ,
 396 the lifting graph $L(G'_d, s, \tau)$ is the same graph; this is the the lifting graph $L(G, s, \tau)$.

397 **Proof of Theorem 1.3.** Let \mathbf{x} and \mathbf{y} be any sequences of k (not necessarily distinct)
 398 vertices of G . Let A be the set of vertices that occur in \mathbf{x} and \mathbf{y} .

399 Let S be a finite set of vertices containing A . There is a unique infinite component K
 400 of $G - S$. Let \mathcal{P} be a largest set of pairwise edge-disjoint, 1-way infinite paths (or *rays*),
 401 that begin with an edge in $\delta(V(K))$ and are otherwise contained in K . It is a standard
 402 fact that there is a finite set S' containing S such that $|\delta(S')| = |\mathcal{P}|$. We are interested
 403 only in S' , which we relabel as S , and restrict the rays in \mathcal{P} to begin at their edge in
 404 $\delta(S')$.

405 Because G is $(k + 2)$ -edge-connected, $|\delta(S)| \geq k + 2$. We consider three cases.

406 **Case 1:** $|\delta(S)| = k + 2$.

407 Contract $G - S$ to a single vertex v_S , yielding a finite $(k + 2)$ -edge-connected graph
 408 $G/(G - S)$. Huck's Theorem shows there is a weak \mathbf{xy} -linkage \mathcal{L} in $G/(G - S)$.

409 Let v be any vertex of $G - S$. There is a set \mathcal{L}' of $(k + 2)$ pairwise edge-disjoint paths
 410 with origin v whose other end is in S and incident with an edge of $\delta(S)$. Evidently, we
 411 can replace any passage of a path in \mathcal{L} through v_S with an appropriate pair of paths in
 412 \mathcal{L}' . Simplifying the resulting walks as needed, we convert \mathcal{L} into a weak \mathbf{xy} -linkage in G .

413 **Case 2:** $|\delta(S)|$ is odd and at least $k + 4$.

414 In this case, let e be any edge of $\delta(S)$ and let $G' = G - e$. Now G' is $(k + 1)$ -edge-
 415 connected and $|\delta(S)|$ is even. We now proceed as in Case 3.

416 **Case 3:** $|\delta(S)|$ is even.

417 In this case, we need only that G is $(k + 1)$ -edge-connected (so Case 2 continues
 418 smoothly here). Contract $G - S$ to a single vertex v_S resulting in the finite graph G^S .

419 We claim that $\delta(S)$ partitions into $|\delta(S)|/2$ pairs $\{e_i, e'_i\}$, $i = 1, 2, \dots, |\delta(S)|/2$, such
 420 that, letting $G_0^S = G^S$ and, for $i = 1, 2, \dots, |\delta(S)|/2$, G_i^S is the graph obtained from lifting
 421 $\{e_i, e'_i\}$ in G_{i-1}^S :

- 422 1. for $i \geq 1$, the pair $\{e_i, e'_i\}$ is $(k+1)$ -liftable in G_{i-1}^S ; and
- 423 2. for $i = 1, 2, \dots, |\delta(S)|/2$, there is a path P_i joining e_i and e'_i with only its end vertices
 424 and e_i, e'_i not in $G - S$ such that P_i is edge-disjoint from $P_1 \cup \dots \cup P_{i-1}$ and from
 425 all the rays in \mathcal{P} containing $e_{i+1}, e'_{i+1}, \dots, e_{|\delta(S)|/2}, e'_{|\delta(S)|/2}$.

426 Suppose we have the pairs $\{e_1, e'_1\}, \dots, \{e_{i-1}, e'_{i-1}\}$ and paths P_1, \dots, P_{i-1} . We show
 427 the existence of $\{e_i, e'_i\}$ and P_i .

428 Set $\delta_i(S)$ to be $\delta(S) \setminus \{e_1, e'_1, \dots, e_{i-1}, e'_{i-1}\}$. These are the edges in $G - \{e_1, e'_1, \dots, e_{i-1},$
 429 $e'_{i-1}\}$ having precisely one end in S . Let \mathcal{P}_i denote the paths in \mathcal{P} that do not contain
 430 any of the edges in $\{e_1, e'_1, \dots, e_{i-1}, e'_{i-1}\}$.

431 There are two graphs with vertex set $\delta_i(S)$ that are relevant to completing the proof.

432 In the *end graph* \mathcal{E}_i , distinct edges e, e' in $\delta_i(S)$ are adjacent if there are infinitely
 433 many vertex-disjoint paths in $G - S$ that: (i) join the two paths in \mathcal{P}_i containing e and
 434 e' ; and (ii) are edge-disjoint from all the other paths in \mathcal{P}_i . Since all the paths in \mathcal{P}_i are
 435 in the same end, \mathcal{E}_i is connected.

436 The other graph is the $(k+1)$ -lifting graph \mathcal{L}_i for v_S in G_{i-1}^S . By Theorem 1.2 (1.2.4),
 437 \mathcal{L}_i is a complete multipartite graph. Therefore, its complement is disconnected.

438 Since \mathcal{E}_i is connected, there is an edge $e_i e'_i$ of \mathcal{E}_i that is not in the complement of \mathcal{L}_i ;
 439 that is, $e_i e'_i$ is an edge of \mathcal{L}_i . This is the required next pair of edges.

440 Let Q and Q' be the rays in \mathcal{P} containing e_i and e'_i , respectively. Because $e_i e'_i$ is an
 441 edge of \mathcal{E}_i , there are infinitely many vertex-disjoint paths in G joining Q and Q' that are
 442 edge-disjoint from the other rays in \mathcal{P}_i . Let P be one of these contained in $G - S$ that
 443 is also disjoint from all of the finitely many finite paths P_1, \dots, P_{i-1} . Then $Q \cup P \cup Q'$
 444 contains a path P_i containing e_i , and e'_i . This is the required next path.

445 The choices of the lifts $\{e_i, e'_i\}$ show that $G_{|\delta(S)|/2}^S$ is $(k+1)$ -connected. Huck's Theorem
 446 shows that $G_{|\delta(S)|/2}^S$ has an **xy**-linkage \mathcal{Q} .

447 An occurrence of the lift of $\{e_i, e'_i\}$ in some path in \mathcal{Q} can be replaced by P_i . This
 448 converts \mathcal{Q} into an **xy**-linkage in G , as required. ■

449 References

- 450 [1] R. Aharoni and C. Thomassen, Infinite, highly connected digraphs with no two arc-
 451 disjoint spanning trees. J. Graph Theory **13** (1989), 71–74.

- 452 [2] Y.H. Chan, W.S. Fung, L.C. Lau, and C.K Yung, Degree bounded network design
453 with metric costs, *SIAM J. Comput.* **40** (2011), no. 4, 953–980.
- 454 [3] A. Frank, On a theorem of Mader, *Ann. of Disc. Math.* **101** (1992), 49–57.
- 455 [4] A. Huck, A sufficient condition for graphs to be weakly k -linked, *Graphs and Combi-*
456 *natorics* **7** (1991), 323–351.
- 457 [5] W. Mader, A reduction method for edge-connectivity in graphs, *Ann. of Disc. Math.*
458 **3** (1978), 145–164.
- 459 [6] H. Okamura, Every $4k$ -Edge-Connected Graph is weakly- $3k$ -linked. *Graphs and Com-*
460 *binatorics* **6** (1990), 179185.
- 461 [7] C. Thomassen, 2-linked graphs, *Europ. J. Comb.* **1** (1980), 371–378.
- 462 [8] C. Thomassen, Orientations of infinite graphs with prescribed edge-connectivity, *Com-*
463 *binatorica*, to appear.