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Abstract

For $1 \leq \ell < k$, an *ℓ-overlapping k-cycle* is a k-uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices.

A k-uniform hypergraph H is *ℓ-Hamiltonian saturated* if H does not contain an ℓ-overlapping Hamiltonian k-cycle but every hypergraph obtained from H by adding one edge does contain such a cycle. Let $\mathrm{sat}(n, k, \ell)$ be the smallest number of edges in an ℓ-Hamiltonian saturated k-uniform hypergraph on n vertices. In the case of graphs Clark and Entringer showed in 1983 that $\mathrm{sat}(n, 2, 1) = \lceil \frac{3n}{2} \rceil$. The present authors proved that for $k \geq 3$ and $\ell = 1$, as well as for all $0.8k \leq \ell \leq k - 1$, $\mathrm{sat}(n, k, \ell) = \Theta(n^{\ell})$. In this paper we prove two upper bounds which cover the remaining range of ℓ. The first, quite technical one, restricted to $\ell \geq \frac{k+1}{2}$, implies in particular that for $\ell = \frac{2}{3}k$ and $\ell = \frac{3}{4}k$ we have $\mathrm{sat}(n, k, \ell) = O(n^{\ell+1})$. Our main result provides an upper bound $\mathrm{sat}(n, k, \ell) = O(n^{\frac{k+\ell}{2}})$ valid for all k and ℓ. In the smallest open case we improve it further to $\mathrm{sat}(n, 4, 2) = O(n^{\frac{14}{5}})$.

1 Introduction

Given integers $1 \leq \ell < k$, we define an *ℓ-overlapping k-cycle* as a k-uniform hypergraph (*k-graph* for short) in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. The notion of an *ℓ-overlapping k-path* is defined similarly, that is, with vertices ordered $v_1, \ldots, v_s$, the edges of the path are $\{v_1, \ldots, v_k\}$, $\{v_{k-\ell+1}, \ldots, v_{k+\ell}\}$, $\ldots$, $\{v_{s-k+1}, \ldots, v_s\}$, Note that the number of edges of an ℓ-overlapping k-cycle with s vertices is $s/(k-\ell)$ (and thus, s is divisible by $k-\ell$). Similarly, it can be easily seen that the number of vertices s of an ℓ-overlapping k-path equals ℓ modulo $k-\ell$.

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We denote an *ℓ-overlapping k-cycle* on s vertices by $C_s^{(k,\ell)}$. We further denote by $g := g(k,\ell)$ the number of vertices between any two consecutive *disjoint* edges belonging to an ℓ-overlapping path (or cycle) and notice that

$$0 \le g = \left\lceil \frac{k}{k-\ell} \right\rceil (k-\ell) - k < k - \ell < k, \tag{1}$$

and that $g = 0$ if and only if $k - \ell$ divides k.

An ℓ-overlapping Hamiltonian k-cycle in a n-vertex k-graph H is defined as any subhypergraph of H isomorphic to $C_n^{(k,\ell)}$. If H contains an ℓ-overlapping Hamiltonian k-cycle then H itself is called *ℓ-Hamiltonian*.

Given a k-graph H and a k-element set $e \in H^c$, where $H^c = \binom{V}{k} \setminus H$ is the complement of H, we denote by $H + e$ the hypergraph obtained from H by adding e to its edge set. A k-graph H is *ℓ-Hamiltonian saturated*, $1 \le \ell \le k-1$, if H is not ℓ-Hamiltonian but for every $e \in H^c$ the k-graph $H + e$ is such. The largest number of edges in an ℓ-Hamiltonian saturated k-graph on n vertices is called the Turán number for the cycle $C_n^{(k,\ell)}$. In [2] this number has been determined in terms of the Turán number of a $(k-1)$-uniform path with a constant number of vertices.

In this paper we are interested in the other extreme. For n divisible by $k - \ell$, let $sat(n,k,\ell)$ be the *smallest* number of edges in an ℓ-Hamiltonian saturated k-graph on n vertices. In the case of graphs, Clark and Entringer proved in 1983 that $sat(n,2,1) = \lceil \frac{3n}{2} \rceil$ for $n \ge 52$.

For k-graphs with $k \ge 3$ the problem was first mentioned in [3, 4]. It seems to be quite hard to obtain such precise results as for graphs. Therefore, the emphasis has been put on the order of magnitude of $sat(n,k,\ell)$. The present authors proved in [5] that for $k \ge 3$ and $\ell = 1$, as well as for all $0.8k \le \ell \le k-1$,

$$\mathrm{sat}(n,k,\ell) = \Theta(n^\ell), \tag{2}$$

see also [6] for the case $\ell = k-1$.

The facts that (2) holds for very small and very large (with respect to k) values of ℓ and that no better lower bound is known suggest, as conjectured already in [5], that (2) holds for all $1 \le \ell \le k-1$ and $k \ge 2$.

Our first result provides an upper bound on $\mathrm{sat}(n,k,\ell)$ higher than the conjectured $O(n^\ell)$, but for a broader range of ℓ than in [5].

Theorem 1 *For all $k \ge 3$ and $\ell \ge \frac{k+1}{2}$*

$$\mathrm{sat}(n,k,\ell) = O\left(n^{\ell+2g+1}\right).$$

Of course, this bound is good only when g is small, and when $g = 0$ it is only by a factor of n worse than the conjectured optimum. All cases of Theorem 1 which are not covered by the result from [5], but for which $g = 0$, are given in the following corollary.

Corollary 2 *For every k divisible by three and $\ell = \frac{2}{3}k$, as well as for every k divisible by four and $\ell = \frac{3}{4}k$, we have $sat(n,k,\ell) = O(n^{\ell+1})$.*

In the remaining range of ℓ, that is, for $2 \le \ell \le k/2$, nothing else than the trivial upper bound

$$\mathrm{sat}(n,k,\ell) = O(n^k)$$

and the easy lower bound ([5, Prop. 2.1])

$$\mathrm{sat}(n,k,\ell) = \Omega\left(n^{\ell}\right)$$

have been known. Our main result in this paper provides a first, non-trivial, general upper bound on $\mathrm{sat}(n,k,\ell)$.

Theorem 3 *For all $k \geq 3$ and $2 \leq \ell \leq k-1$,*

$$\mathrm{sat}(n,k,\ell) = O\left(n^{\frac{k+\ell}{2}}\right).$$

One consequence of Theorem 3, combined with the case $\ell = k-1$ of (2), is that for all ℓ and k we have

$$\mathrm{sat}(n,k,\ell) = O\left(n^{k-1}\right).$$

In view of Theorem 3, the bound in Theorem 1 is not overwritten only when $\ell + 2g + 1 \leq \frac{k+\ell-1}{2}$, equivalently, when $g \leq (k-\ell-1)/4$.

Theorems 1 and 3 are proved, respectively, in Sections 3 and 4. In the smallest open case, $k=4$, $\ell=2$, we improve Theorem 3 a bit by showing the following result in Section 5.

Theorem 4 $\mathrm{sat}(n,4,2) = O\left(n^{\frac{14}{5}}\right)$.

Our proofs expand and refine a general approach to this type of problems first developed in [6] and modified in [5]. In short, we begin with constructing two k-graphs, H' and H'', such that H' is not ℓ-Hamiltonian, while $H'' \supset H'$ contains some "troublemaking" edges. Then we define H as a maximal non-ℓ-Hamiltonian k-graph satisfying $H' \subseteq H \subseteq H''$. It then remains to show that for every $e \notin H$, $H+e$ is ℓ-Hamiltonian, but, what is crucial, in doing so we may restrict ourselves to $e \notin H''$.

In [6] the constructions of H' and H'' were based on a special partition of the vertex set, while in [5] we used blow-ups of sparse Hamiltonian saturated graphs. In this paper we return to both these ideas: we use the approach from [5] in the proof of Theorem 1, and the approach from [6] in the proofs of Theorems 3 and 4.

2 Preliminaries

Our proofs utilize the following special construction of a k-graph. Given a partition of the vertex set $V = \bigcup_{i=1}^{h} U_i$, for a subset $S \subseteq V$, let

$$tr(S) = \{i : U_i \cap S \neq \emptyset\}$$

and

$$\min(S) = \min\{i : i \in tr(S)\} = \min\{i : U_i \cap S \neq \emptyset\}.$$

Let

$$H_{k,\ell}(U_1,\ldots,U_h) := H_{k,\ell} = \left\{e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geq k-\ell+1\right\}.$$

For further use, note that

$$|tr(e)| \leq \ell \qquad \text{for every } e \in H_{k,\ell}. \tag{3}$$

For $i = 1, \ldots, h$, let

$$C_i = \{e \in H_{k,\ell} \,:\, \min(e) = i\}.$$

Obviously, $H_{k,\ell} = C_1 \cup \cdots \cup C_h$.

Define *an ℓ-component* of a k-graph H as a minimal subset of edges $C \subseteq H$ such that for all $e \in C$ and $f \in H \setminus C$, we have $|e \cap f| < \ell$.

Proposition 5 *For each $i = 1, \ldots, h$, the set C_i is an ℓ-component of $H_{k,\ell}$.*

Proof. By the definition of $H_{k,\ell}$, for every $e \in C_i$ and $f \in C_j$, where $i < j$, we have $|e \cap U_i| \geq k - \ell + 1$ and $f \cap U_i = \emptyset$, and so $|e \cap f| < \ell$. Moreover, for every $e \in C_i$ there is an $f \in C_i$, $f \neq e$ such that $|e \cap f| \geq k - 1 \geq \ell$ (just switch one vertex without violating the membership in C_i), so that C_i satisfies the minimality condition in the definition of an ℓ-component. □

Since every ℓ-overlapping k-path in a k-graph H must be entirely contained in one the ℓ-components of H, we have the following corollary of Proposition 5.

Corollary 6 *For every ℓ-overlapping k-path P in $H_{k,\ell}$ there is an $i \in \{1, \ldots, h\}$ such that $P \subseteq C_i$, or equivalently, for every edge e of P, we have* $\min(e) = i$.

We now investigate the maximum length of an ℓ-overlapping k-path in C_i, $i < h$, which traverses through exactly x vertices of U_i. Our next, purely combinatorial, result provides an easy upper bound, independent of ℓ. Given a positive integer x, let A and B be two disjoint sets, with $|A| = x$ and $|B| = \infty$. Let $\nu(x) = \max_P |V(P)|$, where the maximum is taken over all ℓ-overlapping paths P with $A \subset V(P) \subset A \cup B$ and $|e \cap A| \geq k - \ell + 1$ for all $e \in P$.

Proposition 7 *For every $x \geq k - 2$, we have $\nu(x) \leq kx$.*

Proof. Suppose there is a path P with $A \subset V(P) \subset A \cup B$, $|e \cap A| \geq k - \ell + 1$ for all $e \in P$, and $|V(P)| \geq kx + 1$. Let us view $V(P)$ as a binary sequence, where each vertex of A is replaced by symbol a and each vertex of $V(P) \cap B$ is replaced by symbol b. If there is a pair of consecutive symbols a in the sequence then, by averaging, there is a run (=a sequence of consecutive symbols) of at least

$$\frac{(k-1)x + 1}{x} > k - 1,$$

that is, of at least k symbols b. But then there is an edge of P with at most $k - \ell$ vertices of A – a contradiction. If, on the other hand, there are no consecutive symbols a in the sequence then, again by averaging, there is a run of at least

$$\frac{(k-1)x + 1}{x + 1} > k - 2,$$

that is, of at least $k - 1$ symbols b (here we use the assumption $x \geq k - 2$). Thus, there is a segment $b \cdots bab$ where the run of $b's$ is of length $k - 1$. The first (from the left) edge of P whose leftmost end is in this run may have at most $k - \ell$ symbols a – a contradiction, again. □

We also have the following lower bound on $\nu(x)$.

Proposition 8 *For every $x \geq (k-3)(k-1)$*

$$\nu(x) \geq x + \left\lfloor \frac{x}{k-1} \right\rfloor + 3 - k.$$

Proof. Let a sequence Q begin with a vertex in B and then traverse, alternately, groups of $k-1$ vertices of A followed by one vertex of B until fewer than $k-1$ vertices of A are left. The remaining vertices of A are placed all at one end of Q. Clearly, every k-tuple of consecutive vertices of Q contains $k-1 \geq k-\ell+1$ vertices of A. To turn Q into an ℓ-overlapping path, the number of vertices of Q must equal ℓ modulo $k-\ell$. Therefore, we may be forced to drop up to $k-\ell-1 \leq k-2$ vertices of B from Q. This is possible as

$$|Q \cap B| = \left\lfloor \frac{x}{k-1} \right\rfloor + 1 \geq k-2,$$

by our assumption on x. The obtained path has the required properties and the claimed number of vertices. □

Note that $\nu(x)$ is a nondecreasing function of x (just replace any vertex of B with a new vertex of A). Our next observation shows that it cannot increase too fast.

Proposition 9 *For all $x \geq 1$ we have $\nu(x-1) \geq \nu(x) - k$.*

Proof. Consider a longest path P of length $\nu(x)$ and remove its first (from the left) s vertices, where $\ell \leq s \leq k$ and $s = \nu(x) \mod k-\ell$. As there must be a vertex of A among the first ℓ vertices of any edge, the remaining path P' satisfies $x' := |V(P') \cap A| \leq x-1$ and, by the monotonicity of $\nu(x)$ we have

$$\nu(x) - k \leq \nu(x) - s \leq \nu(x') \leq \nu(x-1).$$

□

Returning to the hypergraph $H_{k,\ell}$, Propositions 7-9 imply the following corollary.

Corollary 10 *Let $i < h$, $k^2 \leq x \leq |U_i|$, $A \subset U_i$, $|A| = x$, and $B \subset \bigcup_{j>i} U_j$, $|B| \geq (k-1)x$. Then the length of a longest path P in C_i such that $A \subset V(P) \subset A \cup B$ equals $\nu(x)$. Moreover, we have $\nu(x) - k \leq \nu(x-1) \leq \nu(x)$ and*

$$\frac{k}{k-1}x - k < \nu(x) \leq kx.$$

In addition to the basic construction $H_{k,\ell}$, the proof of Theorem 1 relies on the notion of a (hypergraph) blow-up of a graph which will be defined soon. First, however, we recall a simple fact about graphs proved in [5, Fact 2.2]. For a graph G, let $c(G)$ denote the number of components of G. Given a subset $T \subseteq V(G)$, let $G[T]$ be the subgraph of G induced by T.

Fact 11 ([5]) *Let k, ℓ, and Δ be constants, and for $n = 1, 2, \ldots$, let G_n be a graph with n vertices and $\Delta(G_n) \leq \Delta$. Then the number of k-element subsets $T \subseteq V(G_n)$ with $c(G[T_n]) \leq \ell$ is $O(n^\ell)$.*

Given a graph G and an integer sequence $\mathbf{a} = (a_1, \ldots, a_n)$, the $\mathbf{a}$-*blow-up* of G is the k-graph $H := H[G]$ with

$$V(H) = \bigcup_{i=1}^{n} U_i, \quad |U_i| = a_i,$$
$$H = \bigcup_{ij \in G} K^{(k)}(U_i \cup U_j)$$

where $K^{(k)}(U)$ is the complete k-graph on U and the sets U_i are pairwise disjoint. For a subset $S \subset V(H)$, let

$$tr(S) = \{i \in V(G) : U_i \cap S \neq \emptyset\}.$$

Furthermore, set

$$c(S) = c\left(G[tr(S)]\right).$$

The following immediate corollary of Fact 11 has been already noted in [5, Cor. 2.3].

Corollary 12 ([5]) *Let $a_1, \ldots, a_n$, k, ℓ, and Δ be constants. If $\Delta(G_n) \leq \Delta$ and $H_n = H[G_n]$ is the $\mathbf{a}$-blow-up of G_n then the number of k-element subsets $S \subseteq V(H_n)$ with $c(S) \leq \ell$ is $O(n^\ell)$.* □

3 Proof of Theorem 1

In this section we prove Theorem 1, where the construction of an ℓ-Hamiltonian saturated k-graph is based on a blow-up of a suitably chosen Hamiltonial saturated graph.

Our proof is a substantial modification of the proof of Theorem 1.1 in [5]. Specifically, we have made the range of ℓ in (7) broader (it used to be $2k - \ell + 1 \leq a_i \leq 4\ell - 2k + 1$) and, at the same time, we altered the definition of H_2 (by introducing the cores $\overline{U}_i$). In what follows, we assume that

$$g \leq \frac{k - \ell - 1}{4}, \tag{4}$$

since otherwise $\ell + 2g + 1 \geq (k + \ell)/2$ and Theorem 1 follows from Theorem 3.

We begin with a technical inequality.

Proposition 13 *If $\frac{k+1}{2} \leq \ell \leq k - 1$ then $2k - \ell - 2g - 2 \leq 2\ell - 2$.*

Proof. The inequality in question is equivalent to

$$3\ell + 2g \geq 2k, \tag{5}$$

To prove (5), note that, by the assumptions on ℓ, there exists some integer $a \geq 1$ such that

$$\frac{ak + 1}{a + 1} \leq \ell < \frac{(a + 1)k + 1}{(a + 1) + 1} \leq \frac{2ak + 1}{2a + 1}.$$

Then, by the lower bound on ℓ,

$$\begin{aligned} g = \left\lceil \frac{k}{k - \ell} \right\rceil (k - \ell) - k &\geq \left\lceil \frac{k}{k - (ak + 1)/(a + 1)} \right\rceil (k - \ell) - k \\ &= \left\lceil \frac{k}{k - 1}(a + 1) \right\rceil (k - \ell) - k \geq (a + 2)(k - \ell) - k. \end{aligned}$$

Hence, by the upper bound on ℓ, we finally have

$$3\ell + 2g \geq (2a + 2)k - (2a + 1)\ell > 2k - 1,$$

which implies (5). □

It follows from Proposition 13, as in [5], that every sufficiently large integer N can be expressed as a sum

$$N = a_1 + \cdots + a_n, \tag{6}$$

for some n, where

$$2k - \ell - 2 - 2g \leq a_i \leq 2\ell - 1, \qquad i = 1, \ldots, n. \tag{7}$$

(This is because the range of a_i in (7) has at least two consecutive values.)

Fix a large integer N which is divisible by $(k - \ell)$ and let $\mathbf{a} = (a_1, \ldots, a_n)$, where the a_i's and n are as in (7). Note that $N = \Theta(n)$. Let G_n be an n-vertex Hamiltonian saturated graph with $\Delta(G_n) = O(1)$, and let

$$H_1 = H[G_n]$$

be the $\mathbf{a}$-blow-up k-graph of G_n (see the definition in Section 2) with

$$V = V(H_1) = \bigcup_{i=1}^{n} U_i, \text{ where } |U_i| = a_i, \quad i = 1, \ldots, n.$$

Thus, by (6),

$$|V| = N = \sum_{i=1}^{n} a_i.$$

It is easy to check that (4) implies that $a_i \geq k - \ell$, for all $i = 1, \ldots, n$. Fix a $(k - \ell)$-subset $\overline{U}_i$ of U_i, $i = 1, \ldots, n$, and let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geq k - l + 1, e \supset \overline{U}_{\min(e)} \text{ and } c(e) \geq g + 2 \right\}.$$

Since $H_2 \subseteq H_{k,\ell}$, by (3), for every $e \in H_2$ we have, in fact,

$$2 \leq g + 2 \leq c(e) \leq |tr(e)| \leq \ell. \tag{8}$$

(Note that (4) implies that, indeed, $g \leq \ell - 2$, which guarantees that H_2 is nonempty.) We have the following immediate consequence of the definition of H_2 and Corollary 6.

Corollary 14 *If P is a path in H_2, then there is $i \in \{1, \ldots, n\}$ such that for every $e \in P$ we have $|e \cap U_i| \geq k - \ell + 1$ and $e \supset \overline{U}_i$. In particular, each path in H_2 has at most $\lfloor \frac{k}{k-\ell} \rfloor$ edges.* □

Observe also that for each $e \in H_1$, the set $tr(e)$ is either a vertex or an edge of G. Consequently, $c(e) = 1$ and the k-graphs H_1 and H_2 are edge-disjoint. Set $H' = H_1 \cup H_2$

Lemma 15 *H' is not ℓ-Hamiltonian.*

Proof. Suppose that H' contains an ℓ-Hamiltonian k-cycle $C_H = (e_1, \ldots, e_m)$. Unlike in [5], the proof breaks only into two cases:

Case 1. $C_H \subseteq H_1$**:** We omit the proof in this case, as it is identical to Case 1 of the proof of Lemma 4.1 in [5] (Indeed that proof relied only on the assumption that $a_i \leq 2\ell - 1$.)

Case 2. $H_2 \cap C_H \neq \emptyset$**:** Let (w.l.o.g.) $e_1, \ldots, e_{s-1}$ be a maximal segment in C_H of consecutive edges from H_2. By Corollary 14, $s - 1 \leq \lfloor \frac{k}{k-\ell} \rfloor$ and there exists an index $i \in \{1, \ldots, n\}$ such that

$$e_1 \cap e_{s-1} \supseteq \overline{U}_i, \quad \text{and thus} \quad |e_1 \cap e_{s-1}| \geq |\overline{U}_i| = k - \ell. \tag{9}$$

Let Z be the set of vertices that lie between e_m and e_s on C_H. Formally,

$$Z = \left(\bigcup_{t=1}^{s-1} e_t \right) \setminus (e_m \cup e_s).$$

Then $e_1 \subseteq e_m \cup Z \cup e_s$ and, consequently,

$$\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(Z) \cup tr(e_s). \tag{10}$$

What is more, $e_m \cap U_i \neq \emptyset$ and $e_s \cap U_i \neq \emptyset$. Since $e_m \in H_1$ and $e_s \in H_1$, by the definition of H_1, each of $tr(e_m)$ and $tr(e_s)$ is either the singleton $\{i\}$ or an edge of G containing vertex i. Hence, by (10), $c(e_1) \leq 1 + |Z|$, which combined with the bound $g + 2 \leq c(e_1)$ from the definition of H_2, yields

$$|Z| \geq g + 1. \tag{11}$$

This further implies that e_m and e_s are disjoint, but more importantly, that e_1 and e_s are disjoint too (since e_m and e_s cannot be consecutive disjoint edges). Thus, $s \geq 3$ and

$$|Z| \leq 2(k - \ell) - |e_1 \cap e_{s-1}| \leq k - \ell, \tag{12}$$

by (9). Note, however, that due to the structure of ℓ-overlapping k-paths,

$$|Z| = g + t(k - \ell) \text{ for some } t \geq 0. \tag{13}$$

Therefore, by (13), (12) and (11), $|Z| = k - \ell$ (and $g = 0$). Consequently, by (12), $|e_1 \cap e_{s-1}| = k - \ell$, implying that, in fact, $e_1 \cap e_{s-1} = Z = \overline{U}_i$. But then (10) becomes

$$\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(e_s),$$

and hence, $c(e_1) = 1$ – a contradiction with the definition of H_2. □

Let

$$H'' = \left\{ e \in \binom{V}{k} : c(e) \leq \ell + 2g + 1 \right\}.$$

Recall that $H_1 = H[G_n]$ is the **a**-blow-up k-graph of a Hamiltonian saturated n-vertex graph G_n. It means that for all $e \in H_1$ we have $c(e) = 1$, while, by (8), for all $e \in H_2$ we have $c(e) \leq |tr(e)| \leq \ell$. Thus, $H' = H_1 \cup H_2 \subseteq H''$.

Finally, let H be a maximal non-ℓ-Hamiltonian k-graph on V such that $H' \subseteq H \subseteq H''$. In view of Lemma 25, H does exist. By Corollary 12,

$$|H| \leq |H''| = O(N^{\ell+2g+1}). \tag{14}$$

Thus, to complete the proof of Theorem 1, it remains to show the following lemma.

Lemma 16 *For every $e \in H^c$, $H + e$ is ℓ-Hamiltonian.*

Proof. By the maximality of H, $H + e$ is ℓ-Hamiltonian for each $e \in H'' \setminus H$. Hence, we may restrict ourselves only to $e \in (H'')^c$, that is, such that $c(e) \geq \ell + 2g + 2$. Let us fix one such e. Let $j_1, j_2, \ldots, j_{\ell+2g}, y$, and $x = \min(e)$ belong to $\ell + 2g + 2$ different components of $G[tr(e)]$ and satisfy

$$\min\{j_1, j_2, \ldots, j_{\ell+2g}\} > y > x. \tag{15}$$

Let $r_x = |e \cap U_x|$ and $r_y = |e \cap U_y|$. Note that, since $|tr(e)| \geq c(e) \geq \ell + 2g + 2$,

$$\max\{r_x, r_y\} \leq \max_{1 \leq i \leq n} |e \cap U_i| \leq k - (|tr(e)| - 1) \leq k - \ell - 2g - 1. \tag{16}$$

We will build an ℓ-overlapping Hamiltonian cycle C_H in $H + e$ using the Hamiltonian saturation of G_n. Let $(u_1, \ldots, u_N)$ be the vertices of V in the order as they will appear on the C_H under construction. Our goal is to define this ordering so that each segment of k consecutive vertices which begins at u_i, where $i \equiv 1 \pmod{k - \ell}$, is an edge of $H + e$. We will denote by e_1 the edge beginning at u_1, by e_2 – the edge beginning at $u_{1+k-\ell}$ and so on, until the last edge e_m of C_H which begins at $u_{N-k+\ell+1}$, where $m = \frac{N}{k-\ell}$.

To achieve our goal, we will first construct an ℓ-overlapping path $P \subseteq H_2 + e$, extending e in both directions, and using only the vertices of U_x and U_y, one type at each end of e. Then, we will connect the endsets of P by an ℓ-overlapping path $P' \subseteq H_1$, covering all the remaining vertices and, thus, creating, together with P, an ℓ-overlapping Hamiltonian cycle in $H + e$. The construction of P' will be facilitated by tracing a Hamiltonian path in G connecting x and y.

To construct P, let $e_1 := e$ and order the vertices of $e_1 = (u_1, \ldots, u_k)$ so that the first r_x vertices belong to U_x, the last r_y vertices belong to U_y, and the $\ell - r_y$ vertices immediately preceding the r_y vertices of $U_y \cap e_1$ all belong to sets U_j with $j > y$. (We know from (15) that there are more than enough such vertices in e_1.) In other words, we request that

$$\{u_1, \ldots, u_{r_x}\} \subset U_x, \tag{17}$$

$$\{u_{k-r_y+1}, \ldots, u_k\} \subset U_y, \tag{18}$$

$$\min\left(\{u_{k-\ell+1}, \ldots, u_{k-r_y}\}\right) > y. \tag{19}$$

The remaining vertices of e_1 are labeled arbitrarily by $u_{r_x+1}, \ldots, u_{k-\ell}$.

Our plan is to extend e_1 in either direction, but only for as long as the new edges still intersect e_1. This means that we will have in P precisely

$$\kappa := \left\lceil \frac{l}{k - \ell} \right\rceil$$

new edges, and thus, precisely

$$\kappa(k - \ell) = g + \ell$$

new vertices on each side of e_1, where the last equality follows from (1).

Formally, we set

$$V(P) = \{u_{N-\ell-g+1}, \ldots, u_N, u_1, \ldots, u_k, u_{k+1}, \ldots, u_{k+g+\ell}\}$$

and

$$E(P) = \{e_1\} \cup \{e_{m+1-i} : i = 1, \ldots, \kappa\} \cup \{e_{1+i} : i = 1, \ldots, \kappa\},$$

where, recall, the edge e_j begins at the vertex $u_{1+(j-1)(k-\ell)}$.

We request that all vertices of P to the left of e_1 belong to U_x and all vertices to the right of e_1 belong to U_y, that is,

$$\{u_{N-\ell-g+1}, \ldots, u_N, u_1, \ldots, u_{r_x}\} \subseteq U_x \quad \text{and} \quad \{u_{k-r_x+1}, \ldots, u_k, u_{k+1} \ldots, u_{k+g+\ell}\} \subseteq U_y, \tag{20}$$

This is possible, since, by (16) and (7).

$$\min(|U_x \setminus e|, |U_y \setminus e|) \geq 2k - \ell - g - 2 - (k - \ell - 2g - 1) = k + g - 1 \geq \ell + g.$$

We also request that

$$\{u_{N-k+\ell+1},\dots,u_{r_x}\}\supseteq \overline{U}_x \quad \text{and} \quad \{u_{k-r_y+1},\dots,u_{2k-\ell}\}\supseteq \overline{U}_y. \tag{21}$$

This can be easily accommodated, as each of these sets contains precisely $k-\ell$ vertices from outside of e_1. Note that P is, trivially, an ℓ-overlapping path *in the complete k-graph on V*. We will show that, in fact, $P\subseteq H_2+e$.

Suppose first that $m+1-\kappa\le j\le m$. Then, by the definition of x, $\min(e_j)=x$. By our construction (see (17), (20), and (21)), $|e_j\cap U_x|\ge k-\ell+1$ and $e_j\supseteq \overline{U}_x$. The same is true for e_j with $j=2,\dots,\kappa+1$, if we replace x by y (see (18), (19),(20), and (21)).

To conclude that $P\subseteq H_2+e$, it remains to show that $c(e_j)\ge g+2$ for each e_j, $j\ne 1$. As, clearly, $|e_j\setminus e_1|\le \ell+g$, we also have

$$|e_1\setminus e_j|\le \ell+g. \tag{22}$$

Trivially, $c(e_1)\le c(e_1\setminus e_j)+c(e_1\cap e_j)$. Moreover, $tr(e_j)=tr(e_1\cap e_j)$. Therefore, by the choice of $e=e_1$ and (22),

$$c(e_j)=c(e_1\cap e_j)\ge c(e_1)-c(e_1\setminus e_j)\ge c(e_1)-|e_1\setminus e_j|\ge \ell+2g+2-(\ell+g)=g+2.$$

Thus $e_j\in H_2$ for each $e_j\in P$, $j\ne 1$.

Now we will build the rest of C_H using only the edges of H_1. Recall that x and y belong to different components of $tr(e)$ and, hence, $xy\notin G$. Therefore, by the Hamiltonian saturation of G, there is a Hamiltonian path $Q=(v_1=y,v_2,\dots,v_{n-1},v_n=x)$ from y to x in G. We connect the two ℓ-element endsets of P by an ℓ-overlapping path $P'=(e_{\kappa+2},\dots,e_{m-\kappa})$ in $H_1\subseteq H$ which, by tracing Q, "swallows" all the remaining $N-|V(P)|$ vertices of V.

Set $U'_v=U_v\setminus V(P)$, $v\in V(G)$, and

$$R:=\bigcup_{v\in V(G)} U'_v.$$

Observe that

$$|R|=N-|V(P)|=N-2\kappa(k-\ell)-k=N-2(g+\ell)-k.$$

Let us order the elements R so that all elements of U'_{v_i} precede all elements of $U'_{v_{i+1}}$, for $i=1,\dots,n-1$, and denote this ordering by $(u_{k+g+\ell+1},\dots,u_{N-g-\ell})$. The vertex set of P' is then defined as

$$V(P')=\{u_{k+g+1},\dots,u_{k+g+\ell},u_{k+g+\ell+1},\dots,u_{N-g-\ell},u_{N-g-\ell+1},\dots,u_{N-g}\}.$$

Note that for $v\notin\{x,y\}$, by (7) and (16),

$$|U'_v|\ge |U_v|-(k-\ell-2g-1)\ge k-1.$$

Hence, every edge of P' stretches over at most two sets U_v and each such two sets are always indexed by adjacent vertices of G. This implies that $P'\subseteq H_1$. □

4 Proof of Theorem 3

In this section we prove Theorem 3, where the construction of an ℓ-Hamiltonian saturated k-graph is based on a special partition of the vertex set into $q+1$ sets $U_1, \dots, U_{q+1}$ (q to be chosen), and the associated with it notion of the hypergraph $H_{k,\ell}(U_1, \dots, U_{q+1})$, introduced at the beginning of Section 2.

Recall that the function $\nu(x)$ has been defined in Section 2. Given a large integer n divisible by $k-\ell$, choose integers $\alpha = \Theta\left(n^{1/2}\right)$, $\beta = \Theta\left(n^{1/2}\right)$, $p = \Theta\left(n^{1/2}\right)$, and

$$q = \left\lfloor \frac{p(k+2g) + (p-1)\nu}{\alpha} \right\rfloor + 2, \tag{23}$$

where $g = g(k,\ell)$ is given by (1) and $\nu := \nu(\alpha)$, such that

$$\alpha \geq 10k^3 p, \tag{24}$$

$$\beta \geq k\alpha,$$

and

$$n = (q-1)\alpha + \beta + p(k-2) + k - 3. \tag{25}$$

To see that such a choice is feasible, one may set, for instance, $\alpha = \lceil 2k^2\sqrt{n} \rceil$. Recall that, by Proposition 7, $\alpha \leq \nu \leq k\alpha$. Next, choose $p = \lfloor n/\nu \rfloor - k - 1$. Then, first of all, (24) holds. Furthermore, using (23) and the estimates $g \leq k$, $2p \geq k-3$, and $4kp \leq \alpha$ among others, we can sandwich the quantity

$$n - \beta = (q-1)\alpha + p(k-2) + k - 3$$

as follows:

$$n - (k+3)\nu \leq \nu(p-1) \leq n - \beta \leq 4kp + \alpha + n - (k+2)\nu \leq n - k\alpha.$$

Thus, there exists an integer β, $k\alpha \leq \beta \leq (k+3)\alpha$, which satisfies (25). Note that, in particular, by (23) and Proposition 8,

$$q \geq p + 2k + 1. \tag{26}$$

Let

$$V = \bigcup_{i=1}^{q+1} U_i,$$

where

$$|U_i| = \alpha \quad \text{for} \quad i = 1, \dots, q-1, \quad |U_q| = \beta \quad \text{and} \quad |U_{q+1}| = p(k-2) + k - 3,$$

and all sets U_i, $i = 1, \dots, q+1$, are pairwise disjoint.

We begin our construction of the required ℓ-Hamiltonian saturated k-graph H, by letting

$$H_1 = H_{k,\ell}(U_1, \dots, U_{q+1}).$$

Recall from Section 2 that H_1 breaks naturally into $q+1$ ℓ-components, that is, $H_1 = C_1 \cup \dots \cup C_{q+1}$. Thus, every path in H_1 is entirely contained in some C_i, and, by Corollary 10, for all $i \leq q-1$ such paths are no longer than $k\nu \leq k^2\alpha$. On the other hand, by the definition of C_i, the vertex set of every path contained in $C_q \cup C_{q+1}$ must be a subset of $U_q \cup U_{q+1}$. Therefore, in view of our assumptions on β, p and α, we have the following conclusion.

Corollary 17 *The length of a longest path in H_1 is $O(\sqrt{n})$. In particular, H_1 is not ℓ-Hamiltonian.* $\square$

Following the outline described in the Introduction, we build a k-graph H' by slightly enriching H_1, but so that it still remains non-ℓ-Hamiltonian. Let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{q+1}| \geq k-2 \right\} \tag{27}$$

and $H' = H_1 \cup H_2$.

Lemma 18 *H' is not ℓ-Hamiltonian.*

Proof. Suppose that C is an ℓ-overlapping Hamiltonian cycle in H'. Let M be a maximal set of disjoint edges in $C \cap H_2$. By Corollary 17, $M \neq \emptyset$. Set $t := |M|$. Since

$$|U_{q+1}| = p(k-2) + k - 3 < (p+1)(k-2),$$

we have $t \leq p$.

From C we now extract t vertex disjoint paths, all contained in H_1, as follows. For every $e \in M$, denote by $N(e)$ the union of the set of vertices of e, the set of g consecutive vertices lying just before e, and the set of g consecutive vertices lying just after e (here, 'before' and 'after' refer to an arbitrarily fixed direction of traversing C). Let $W = \bigcup_{e \in M} N(e)$. Then $C[V \setminus W]$ consists of at most t paths (we treat a nonempty set of fewer than k consecutive isolated vertices as a single trivial path). Observe that

$$|W| \leq t(k + 2g). \tag{28}$$

Since each obtained path P is contained in H_1, either $\min(V(P)) \leq q-1$ or $V(P) \subseteq U_q \cup U_{q+1}$. If all t paths are of the former kind, then their total number of vertices is at most $t\nu$, and otherwise, it is at most $(t-1)\nu + |U_q| + |U_{q+1}|$. Note that, since $|U_q| = \beta \geq k\alpha \geq \nu$, we have

$$\max\{t\nu, (t-1)\nu + |U_q| + |U_{q+1}|\} \leq (t-1)\nu + |U_q| + |U_{q+1}|. \tag{29}$$

Finally, by (23), (28), and (29), and using $t \leq p$, we get

$$\begin{aligned} n = |V(C)| &\leq |W| + (t-1)\nu + |U_q| + |U_{q+1}| \\ &\leq p(k+2g) + (p-1)\nu + |U_q| + |U_{q+1}| \\ &< (q-1)\alpha + |U_q| + |U_{q+1}| = n, \end{aligned}$$

which is a contradiction. Hence, there is no ℓ-overlapping Hamiltonian cycle in H'. $\square$

Before we finalize our construction, we need one more piece of notation. For each $e \in \binom{V}{k}$ with $|tr(e)| \geq 2$, let

$$\min\nolimits_2(e) = \min\{i : (e \setminus U_{\min(e)}) \cap U_i \neq \emptyset\}. \tag{30}$$

Finally, set

$$H_3 = \left\{ e \in \binom{V}{k} : |tr(e)| \geq 2 \quad \text{and} \quad \min\nolimits_2(e) \geq q - 2k \right\},$$

$$H'' = H_1 \cup H_2 \cup H_3,$$

and let H be a maximal non-ℓ-Hamiltonian k-graph such that $H' \subseteq H \subseteq H''$. By Lemma 18, such a k-graph H exists.

Fact 19

$$|H| = O(n^{(k+\ell)/2})$$

Proof. By the definitions of H and H'',

$$|H| \leq |H''| \leq |H_1| + |H_2| + |H_3|.$$

Now, noticing that $\max_{1\leq i\leq q+1} |U_i| = \beta$, we have

$$\begin{aligned}
|H_1| &\leq \sum_{i=1}^{q+1} \binom{|U_i|}{k-\ell+1} \cdot \binom{n}{\ell-1} \leq (q+1) \cdot \beta^{k-\ell+1} \cdot n^{\ell-1} = O\left(n^{(k+\ell)/2}\right), \\
|H_2| &\leq \binom{|U_q|}{k-2} \cdot \binom{n}{2} \leq \beta^{k-2} \cdot n^2 = O\left(n^{(k+2)/2}\right), \text{ and} \\
|H_3| &\leq \sum_{i=1}^{q} \sum_{t=1}^{k-1} \binom{|U_i|}{t} \cdot \binom{|U_{q-2k}| + \cdots + |U_{q+1}|}{k-t} = O\left(q \cdot \alpha^t \cdot \beta^{k-t}\right) = O\left(n^{(k+1)/2}\right),
\end{aligned}$$

where $i = \min(e)$ and $t = |e \cap U_{\min(e)}|$. □

To complete the proof of Theorem 3, it remains to show the following lemma.

Lemma 20 *For every $e \in \binom{V}{k} \setminus H$ the k-graph $H + e$ is ℓ-Hamiltonian.*

Proof. Fix $e \in \binom{V}{k} \setminus H$. If $e \in H''$, then, by the definition of H, $H+e$ is ℓ-Hamiltonian. Therefore, we may assume that $e \notin H''$. This implies that $|tr(e)| \geq 2$, since otherwise $e \in H_1$. Define

$$x = \min(e) \quad \text{and} \quad y = \min_2(e).$$

Since $e \notin H_1 \cup H_3$, we have $|U_x \cap e| \leq k - \ell$ and $x < y \leq q - 2k - 1$.

Our ultimate goal is to construct in H an ℓ-overlapping Hamiltonian cycle C. Recalling (26), let $J = \{j_1, \ldots, j_{p-2}\}$ be the set of the $p-2$ smallest indices in the set $\{1, \ldots, q-2k-1\} \setminus \{x, y\}$. Further, let

$$r_i = |e \cap U_i|, \qquad i = 1, \ldots, q+1.$$

Since $e \notin H_2$, we have $r_{q+1} \leq k-3$. Thus $|U_{q+1} \setminus e| \geq p(k-2)$. Let us now set aside p disjoint $(k-2)$-element subsets $B_1, \ldots, B_p$ of $U_{q+1} \setminus e$ and let

$$B = \bigcup_{i=1}^{p} B_i.$$

Note that

$$|U_{q+1} \setminus (B \cup e)| = k - 3 - r_{q+1} \leq k. \tag{31}$$

Furthermore, let us also put aside a set $Q = A_q \cup A'_q$ of $2(g+1)$ elements of $U_q \setminus e$, where $|A_q| = |A'_q| = g+1$. The vertices in B and Q will be used later in our construction.

First, however, we construct p vertex disjoint paths $P_{j_1}, \ldots, P_{j_{p-2}}$, P_{xy} and P_q. Together, these p paths will contain all elements of V, except for some $k-\ell+g+1$ vertices of U_x, the same number of vertices of U_y, twice as many vertices of each U_j, $j \in J$, and except for the vertices in $B \cup Q$. Using these exceptional vertices, the paths will be connected by p 'bridges', made mostly of the edges of H_2, to form an ℓ-overlapping Hamiltonian cycle C in H.

Construction of P_{xy}. Order the vertices of e so that the set $e \cap U_x$ constitutes the leftmost segment of e, while the rightmost vertex of e belongs to U_y. Next, we will extend e in both directions (see Fig. 1). Let A'_x be a set of arbitrary $k-\ell+g$ vertices of $U_x \setminus e$ and A_y be a set of arbitrary $k-\ell+g$ vertices of $U_y \setminus e$ (the reader should not worry, we will later construct sets A_x and A'_y too). Let

$$R = \bigcup_{i=q-2k}^{q-1} U_i \setminus e.$$

Further, for each $z \in \{x, y\}$, let $P_z \subseteq C_z$ be a path containing precisely

$$\alpha_z := \alpha - r_z - (2k - 2\ell + 2g + 1)$$

vertices of $U_z \setminus (e \cup A'_x \cup A_y)$ and $\nu(\alpha_z) - \alpha_z$ vertices of R, where $V(P_x) \cap V(P_y) = \emptyset$. Since, by Proposition 7, each of P_x and P_y requires no more than $(k-1)\alpha$ vertices of R, while $|R| \geq 2k\alpha - k$, we will not run out of the vertices of R.

To finish the construction of P_{xy}, we extend e

- to the left, by adding the set A'_x, followed by P_x, and
- to the right, by adding the set A_y, followed by P_y.

Thus,

$$V(P_{xy}) = V(P_x) \cup A'_x \cup e \cup A_y \cup V(P_y) \subset U_x \cup U_y \cup e \cup R\ .$$

Set

$$A_x = U_x \setminus V(P_{xy}) \qquad \text{and} \qquad A'_y = U_y \setminus V(P_{xy})$$

and observe that

$$|A_x| = |A'_y| = k - \ell + g + 1\ . \tag{32}$$

Fact 21

$$P_{xy} \subseteq H_1 + e$$

Proof. The path P_{xy} consists, besides the edges of P_x, P_y, and e itself, also of a set A of $2\lceil \frac{k}{k-\ell} \rceil$ additional edges, $\lceil \frac{k}{k-\ell} \rceil$ on each side of e. These are precisely those edges of P_{xy} which intersect the set $A'_x \cup A_y$. Thus, to prove that $P_{xy} \subseteq H_1 + e$, it remains to show that each edge from A belongs to H_1.

Let us consider an edge e' intersecting A'_x. Obviously, $\min(e') = x$. Also, $|e' \cap A'_x| \geq k - \ell$, and so $|e' \cap U_x| \geq k - \ell$. Furthermore, if $|e' \cap A'_x| = k - \ell$ then either e' contains also the leftmost vertex of e (which belongs to U_x), or $|e' \cap V(P_x)| = \ell$. In the latter case, recall that each edge of

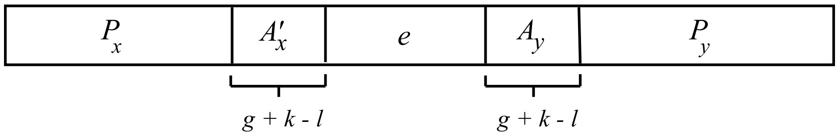

Figure 1: Construction of P_{xy}

P_x contains at least $k-\ell+1$ vertices from U_x, and consequently there is always a vertex form U_x among any ℓ vertices of such an edge. In either case, this implies that $|e' \cap U_x| \geq k-\ell+1$, thus $e' \in H_1$. If an edge e' intersects A_y then, by the same argument, we also have $|e' \cap U_y| \geq k-\ell+1$. Finally, note that $\min(e') = y$. Indeed, since $|U_x \cap e| \leq k-\ell$, none of the ℓ rightmost vertices of e is in U_x, and hence, we have $e' \cap U_x = \emptyset$. □

Construction of P_q. Let P_q be a longest path with $V(P_q) \subset U_q \setminus (e \cup Q)$. Clearly, at most $k-\ell-1$ vertices of U_q will be left out, that is,

$$|U_q \setminus (V(P_q) \cup e \cup Q))| \leq k-\ell-1 \leq k. \tag{33}$$

Trivially, $P_q \subset H_1$.

Construction of P_j, $j \in J$. Set

$$W := \left(\bigcup_{i \in \{1,\dots,q+1\} \setminus (J \cup \{x,y\})} U_i \right) \setminus (V(P_{xy}) \cup V(P_q) \cup B \cup Q \cup e),$$

and, for each $j \in J$, let $P_j \subseteq C_j \subseteq H_1$ be a path with $V(P_j) \subseteq U_j \cup W$ which uses precisely

$$\alpha_j := \alpha - r_j - (2k - 2\ell + 2g + 2)$$

vertices of $U_j \setminus e$ and *as many as possible* vertices from W (we maintain that all paths P_j, $j \in J$, are pairwise vertex-disjoint). Since $i > j$ for every $i \in [q+1] \setminus (J \cup \{x,y\})$, we do have $\min(V(P_j)) = j$. Also,

$$|U_j \setminus (V(P_j) \cup e)| = 2(k-\ell+g+1) \qquad \text{for each } j \in J\,. \tag{34}$$

Split arbitrarily the set $U_j \setminus (V(P_j) \cup e)$ into two sets A_q and A'_q of equal size $|A_q| = |A'_q| = k-\ell+g+1$.

Next, we perform crucial calculations showing that we have, indeed, used all the vertices of W, that is, there are no vertices outside the constructed paths except for those listed in (32,34) and those put aside in $B \cup Q$.

Fact 22

$$W \subseteq \bigcup_{j \in J} V(P_j)$$

Proof. We have, by the definition of P_{xy}, and by (31) and (33),

$$\begin{aligned} |W| &= (q-1-p)\alpha - |R \cap V(P_{xy})| + |U_q \setminus (V(P_q) \cup e \cup Q)| + |U_{q+1} \setminus (B \cup e)|, \\ &\leq (q-1-p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k. \end{aligned}$$

Recall that each path P_j, $j \in J$, may have the maximum length $\nu(\alpha_j)$, and thus cover up to $\nu(\alpha_j) - \alpha_j$ vertices of W. Therefore, to complete the proof it suffices to show that

$$(q-1-p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k \leq \sum_{j \in J} (\nu(\alpha_j) - \alpha_j),$$

or, equivalently,

$$\sum_{j \in J \cup \{x,y\}} (\nu(\alpha_j) - \alpha_j) \geq (q-1-p)\alpha + 2k.$$

Note that for each $j \in J \cup \{x, y\}$

$$r_j + 2k - 2\ell + 2g + 2 \leq 5k. \tag{35}$$

Hence, by the monotonicity of the function $\nu(\cdot)$ and by Proposition 9, we have

$$\nu(\alpha_j) - \alpha_j \geq \nu(\alpha - 5k) - \alpha \geq \nu - 5k^2 - \alpha,$$

and it remains to show that

$$p(\nu - 5k^2 - \alpha) \geq (q-1-p)\alpha + 2k. \tag{36}$$

To this end,

$$\begin{aligned} p(\nu - 5k^2) - p\alpha &\geq (p-1)\nu + (\alpha + \alpha/(k-1) - k) - 5k^2 p - p\alpha \quad \text{(by Corollary 10)} \\ &\geq (p-1)\nu + \alpha + p(k+2g) + 2k - p\alpha \quad \text{(by (24))} \\ &\geq (q-1-p)\alpha + 2k \quad \text{(by (23))}. \end{aligned}$$

(Since there is some margin in the above estimates, it means that not all the paths P_j, $j \in J$, are of maximum length.) □

Now comes the final stage of our construction, where we glue together the paths $P_{j_1}, \ldots, P_{j_{p-2}}$, P_q, and P_{xy}, in this order, to form a Hamiltonian cycle C. We do it as indicated in Fig. 4, with the set A_x placed at the left end of P_{xy}, that is, next to the end of the path P_x (see Fig. 4).

Clearly, every edge of $\bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q$ belongs to $H + e$. As the last ingredient of our proof of Theorem 3, we now show that every other edge of C belongs to $H_1 \cup H_2 \subseteq H$.

Fact 23

$$C \setminus \left(\bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q \right) \subseteq H_1 \cup H_2$$

Proof. Let

$$\mathcal{A} := \left\{ A_{j_i}, A'_{j_i} : i = 1, \ldots, p-2 \right\} \cup \{A_q, A'_q, A_x, A'_y\}.$$

Note that each edge of $C \setminus \left(\bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q \right)$ intersects some set $A \in \mathcal{A}$. recall that between any two disjoint edges of C there are exactly $g + t(k-\ell)$ vertices on C, for some $t \geq 0$. In that case we say that the edge to the right (in some fixed ordering of C) *t-follows* the other edge. Let f_1, be the edge of C which 1-follows the rightmost edge of P_{xy}. Similarly, for $i = 1, \ldots, p-2$, let f_{i+1} be the edge of C which 1-follows the rightmost edge of P_{j_i}. Finally, let f_p be the edge of C which

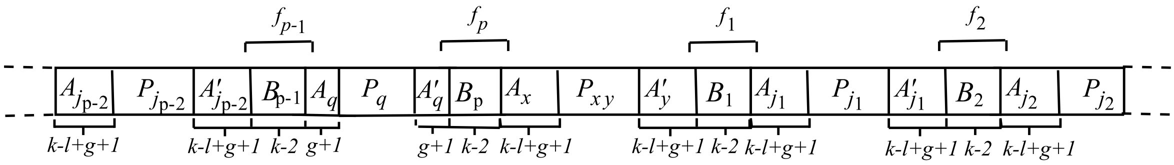

Figure 2: Construction of C

1-follows the rightmost edge of P_q, see Fig. 4. Note that for each $i = 1, \ldots, p$, we have $B_i \subset f_i$, and thus $f_i \in H_2$. Furthermore, these are the only edges of C which intersect more than one set from $\mathcal{A}$.

Consider now some $f \in C$, $f \neq f_i$ intersecting A_{j_i}. Obviously $\min(f) = j_i$. Also $|f \cap A_{j_i}| \geq k - \ell$. However, if $|f \cap A_{j_i}| = k - \ell$, then $|f \cap V(P_{j_i})| = \ell$. Recall that each edge of P_{j_i} contains at least $k - \ell + 1$ vertices of U_{j_i}, and consequently there is always a vertex of U_{j_i} among any ℓ vertices of such an edge. This implies that $|f \cap U_{j_i}| \geq k - \ell + 1$ and so, $f \in H_1$. The same argument works for any $f \in C$ intersecting some set $A \in \mathcal{A}$. □

Thus, we have constructed an ℓ-overlapping Hamiltonian cycle C in $H + e$, which completes the proof of Lemma 20, which, in turn, together with Fact 19, implies Theorem 3.

5 The smallest open case: $k = 4$ and $\ell = 2$

In this section we prove Theorem 4. Our ultimate goal is, given large even integer n, to construct a maximally non-2-Hamiltonian 4-graph H. In doing so we refine the technique used in the proof of Theorem 3.

Choose integers $\alpha = \Theta\left(n^{2/5}\right)$, $\alpha \equiv 1 \bmod 3$, $\beta = O\left(n^{3/5}\right)$, $p = \Theta\left(n^{3/5}\right)$, and

$$q = \left\lfloor \tfrac{4(\alpha-1)}{3\alpha}(p-1) \right\rfloor + 1 \tag{37}$$

such that

$$n = q\alpha + 3p + \beta. \tag{38}$$

To see that such a choice is feasible, one may set, for instance, $\alpha = \lceil n^{2/5} \rceil + \epsilon$ where $\epsilon \in \{0, 1, 2\}$ is such that $\alpha \equiv 1 \bmod 3$. Next choose $p = \left\lceil \frac{3n}{4\alpha+8} \right\rceil + 1$. Then, using (37,38) we have

$$\begin{aligned} n - \beta &> \frac{4}{3}(\alpha-1)(p-1) \geq n - \frac{3n}{\alpha+2} \quad \text{and} \\ n - \beta &\leq \frac{4}{3}(\alpha-1)(p-1) + \alpha + 3p = (p-2)\left(\frac{4}{3}(\alpha-1)+4\right) - \left(p - \frac{7}{3}(\alpha-1) - 9\right) \\ &\leq n - \left(p - \frac{7}{3}(\alpha-1) - 9\right), \end{aligned}$$

which shows that a choice of an appropriate β is possible.

Let $V = \bigcup_{i=1}^{q+1} U_i$, where $|U_i| = \alpha$, $i = 1, \ldots, q$, while $|U_{q+1}| = 3p + \beta$, and all sets U_i, $i = 1, \ldots, q+1$, are pairwise disjoint. Furthermore, let $G \cong pK_3 + \beta K_1$ be a graph with vertex set $V(G) = U_{q+1}$ consisting of p vertex disjoint triangles and β isolated vertices.

We define H_1 in the same way as in the general case, while H_2 is defined smaller:

$$\begin{aligned} H_1 &= \left\{ e \in \binom{V}{4} : |e \cap U_{\min(e)}| \geq 3 \right\}, \\ H_2 &= \left\{ e \in \binom{V}{4} : |e \cap U_{q+1}| = 2,\ |tr(e)| = 2 \text{ and } G[e \cap U_{q+1}] = K_2 \right\}. \end{aligned} \tag{39}$$

The improvement of the upper bound on $\mathrm{sat}(n, 4, 2)$ is possible mainly because in this particular case one can compute (quiet easily) the value of $\nu(x)$. Below we give only a (sharp) upper bound in some special case.

Proposition 24 *Let $x \equiv 0$ mod 3. Then*

$$\nu(x) \leq 4\frac{x}{3}.$$

Proof. Let $P = (e_1, \ldots, e_r)$, $P \subseteq H_1$ and $|V(P) \cap U_{\min(P)}| = x$. Recall that each e_i, $i = 1, \ldots, r$, contains at least 3 vertices from $U_{\min(P)}$. Since the e_i's with odd indices are disjoint,

$$\lceil r/2 \rceil \leq \frac{x}{3}.$$

If r is odd then

$$|V(P)| \leq 4\lceil r/2 \rceil \leq 4\frac{x}{3}$$

and the statement follows. Similarly, if r is even and $r/2 \leq \frac{x}{3} - 1$ then

$$|V(P)| \leq 2r + 2 \leq 4\frac{x}{3} - 2$$

and the statement follows again. Suppose, finally, that $r/2 = \frac{x}{3}$, r even. Since e_r contains at least 3 vertices from $U_{\min(P)}$, at least one of them is not in e_{r-1}, however there are no more available vertices in $U_{\min(P)}$, meaning that this case is vacuous. □

Lemma 25 *$H' = H_1 \cup H_2$ is not 2-Hamiltonian.*

Proof. Suppose that C is a 2-overlapping Hamiltonian cycle in H'. As before (cf. Corollary 17), one can easily show that H_1 cannot be 2-Hamiltonian. Let M be a maximal set of edges in $C \cap H_2$ with the property that if $e_1, e_2 \in M$ then $(e_1 \cap e_2) \cap U_{q+1} = \emptyset$. In view of the above remark $M \neq \emptyset$. Set

$$V_2 = \bigcup_{e \in M} e \cap U_{q+1}.$$

Clearly, $t := |M| \leq p$ and $|V_2| = 2t$. We divide C into t vertex disjoint paths P_j, $j = 1, \ldots, t$, by cutting through the middle of every edge from M (we treat a set ot 2 consecutive isolated vertices as a single trivial path). More precisely, we keep all vertices in and take the edge set $C - M$. We number the obtained paths so that, for some $1 \leq s \leq t$, we have $\min(P_j) \leq q$ for all $j = 1, \ldots, s$ and $V(P_j) \subseteq U_{q+1}$ for all $j = s+1, \ldots, t$. Note that, because $M \neq \emptyset$, at least one path must be of the first kind, but possibly $s = t$. Let

$$V_2' = V_2 \cap \bigcup_{j=1}^{s} V(P_j).$$

Since $V(P_j) \subseteq U_{q+1}$ for all $j = s+1, \ldots, t$, we have

$$\sum_{j=s+1}^{t} |V(P_j)| \leq |U_{q+1}| - |V_2'|. \tag{40}$$

Claim For every $j = 1, \ldots, s$

$$|V(P_j) \setminus V_2'| \leq 4\frac{\alpha - 1}{3}.$$

Proof. If some P_j consists of only two vertices then the claim obviously holds. Thus, we may assume that each P_j is non-trivial. For $j \leq s$, consider the path $P_j = (e_1, \ldots, e_r)$. Let $e_m \in M$ with $|e_m \cap e_1| = 2$. That is e_m precedes e_1 on C. Similarly, let $e_{r+1} \in M$ with $|e_{r+1} \cap e_r| = 2$, which means that e_{r+1} follows e_r on C.

Note that the edges from H_2 can occur in P_j only at the ends. Thus $(e_2, \ldots, e_{r-1}) =: P_j' \subset H_1$. If $e_1 \in H_1$ then $|e_1 \cap U_{\min(P_j)}| \geq 3$, meaning that $|e_m \cap U_{\min(P_j)}| \geq 1$. Thus, by the definition of H_2, $|e_m \cap U_{\min(P_j)}| = 2$. If $e_1 \in H_2$ then, since $e_1 \notin M$, we have $|e_1 \cap V_2'| \in \{1, 2\}$. If $|e_1 \cap V_2'| = 1$ then $|e_m \cap U_{\min(P_j)}| \geq 1$ because $|e_m \cap e_1| = 2$ and $|tr(e_1)| = 2$. Thus, again, $|e_m \cap U_{\min(P_j)}| = 2$. To sum up

$$\text{if } e_1 \in H_1 \text{ or } |e_1 \cap V_2'| = 1 \text{ then } |e_m \cap U_{\min(P_j)}| = 2. \tag{41}$$

The same holds for e_r and e_{r+1}

$$\text{if } e_r \in H_1 \text{ or } |e_r \cap V_2'| = 1 \text{ then } |e_{r+1} \cap U_{\min(P_j)}| = 2. \tag{42}$$

Suppose first that the assumptions on both e_1 and e_r from (41,42), respectively, holds. Thus, $|V(P_j') \cap U_{\min(P_j)}| \leq \alpha - 4$. Since $\alpha - 4 \equiv 0 \bmod 3$, by Proposition 24 and the monotonicity of the function ν,

$$|V(P_j)| = |V(P_j')| + 4 \leq 4\frac{\alpha - 4}{3} + 4 = 4\frac{\alpha - 1}{3}$$

and the claim follows.

Suppose now that $e_1 \in H_2$ with $|e_1 \cap V_2'| = 2$, while e_r satisfies the assumptions from (42). Let P_j'' be defined by $(e_3, \ldots, e_{r-1})$. By the definition of H_2, $|e_1 \cap U_{\min(P_j)}| = 2$. This together with (42) implies that $|V(P_j'') \cap U_{\min(P_j)}| \leq \alpha - 4$. Hence, by Proposition 24 and the assumption on e_1,

$$|V(P_j) \setminus V_2'| = (|V(P_j'')| + 6) - 2 \leq 4\frac{\alpha - 4}{3} + 4 = 4\frac{\alpha - 1}{3}$$

and the claim follows again.

The case when e_1 satisfies the assumption of (41) and $|e_r \cap V_2'| = 2$, is analogous (with $P_j'' = (e_2, \ldots, e_{r-2})$).

Finally, if $|e_1 \cap V_2'| = 2$ and $|e_r \cap V_2'| = 2$ then let $P_j'' = (e_3, \ldots, e_{r-2})$. Since $e_1, e_r \in H_2$ (and $e_2, e_{r-1} \in H_1$), we have $|e_1 \cap U_{\min(P_j)}| = 2$ and $|e_r \cap U_{\min(P_j)}| = 2$. Therefore,

$$|V(P_j) \setminus V_2'| = (|V(P_j'')| + 8) - 4 \leq 4\frac{\alpha - 4}{3} + 4 = 4\frac{\alpha - 1}{3}$$

and the claim follows. □

Returning to the proof of Lemma 25, notice that $|V_2'| \leq |V_2| = 2t \leq 2p$. Thus

$$|U_{q+1}| = 3p > |V_2'| + 4\frac{\alpha - 1}{3}, \tag{43}$$

because $p >> \alpha$. Recalling that $q > \frac{4(\alpha-1)}{3\alpha}(p-1)$ and using the above claim as well as (40,43), we finally argue that

$$\begin{aligned}
n &= |V(C_H)| = \sum_{j=1}^{s} |V(P_j)| + \sum_{j=s+1}^{t} |V(P_j)| \\
&\le \max\{|V_2'| + 4t\frac{\alpha-1}{3}, |V_2'| + 4(t-1)\frac{\alpha-1}{3} + |U_{q+1}| - |V_2'|\}, \quad \text{according to wheather } s = t \text{ or } s \le t-1 \\
&= |V_2'| + 4(t-1)\frac{\alpha-1}{3} + |U_{q+1}| - |V_2'| \quad \text{ by (43)} \\
&\le 4(p-1)\frac{\alpha-1}{3} + 3p < q\alpha + 3p \le n,
\end{aligned}$$

which is a contradiction. Hence, no 2-overlapping Hamiltonian cycle exists in $H_1 \cup H_2$. □

Let

$$H_3 = \left\{ e \in \binom{V}{4} : |tr(e)| \ge 2 \quad \text{and} \quad \min_2(e) \ge q \right\}$$

be the same as in the proof of Theorem 3. Finally, let $H'' = H_1 \cup H_2 \cup H_3$ and let H be a maximal non-2-Hamiltonian hypergraph such that $H' \subseteq H \subseteq H''$. By Lemma 25, such a 4-graph exists.

Fact 26

$$|H| = O(n^{14/5})$$

Proof. By the definitions of H and H'',

$$|H| \le |H''| \le |H_1| + |H_2| + |H_3|.$$

Furthermore,

$$\begin{aligned}
|H_1| &= O\left(q \cdot \alpha^3 \cdot n + p^4\right) = O\left(n^{14/5}\right), \\
|H_2| &= O\left(3p \cdot n \cdot n^{2/5}\right) = O\left(n^2\right) \text{ and} \\
|H_3| &= O\left(n \cdot p^3\right) = O\left(n^{14/5}\right).
\end{aligned}$$

□

To complete the proof of Theorem 4, it remains to show the following lemma.

Lemma 27 *For every $e \in \binom{V}{4} \setminus H$ the 4-graph $H + e$ is 2-Hamiltonian.*

Proof. Let $e = \{u_1, u_2, u_3, u_4\}$, where $u_j \in U_{i_j}$, $j = 1, 2, 3, 4$, and $i_1 \le i_2 \le i_3 \le i_4$. As $e \notin H_1$, we have $|tr(e)| \ge 2$. Let x and y stand for the two smallest *different* indices among i_1, i_2, i_3, i_4. Note that by the definition of H, $e \notin H_3$, and thus $y \le q - 1$.

Set $I = [q-1] \setminus \{x, y\}$, note that $p-2$ is (much) smaller than $q-3$, and let $J = \{j_1, \dots, j_{p-2}\}$ be the set of the $p-2$ smallest indices in I. We will construct p paths $P_{j_1}, \dots, P_{j_{p-2}}$, P_{xy}, and P_{q+1}, such that for each $j \in J$, we have $V(P_j) \supseteq U_j \setminus e$,

$$U_x \cup U_y \cup e \subseteq V(P_{xy}) \subset U_x \cup U_y \cup e \cup U_q,$$

and $V(P_{q+1}) \subset U_{q+1}$. Together, these paths will contain all vertices in V except some $2p$ vertices of U_{q+1}. Using these exceptional vertices, the paths will be connected by p 'bridges' made of the edges of H_2, to form a 2-Hamiltonian cycle in H.

For the ease of notation assume that $x = q-2$ and $y = q-1$. Then $J = [p-2]$. To display the structure of each path we will use a shorthand notation j for any element of U_j, $j = 1, \ldots, p-2, x, y, q, q+1$. Finally, we designate by $*$ each of the two unknown elements of $e = \{u_1, u_2, u_3, u_4\}$ (other than x and y); recall that $u_1 \in U_x$, while $\{u_2, u_3, u_4\} \subseteq \bigcup_{i=x}^{q+1} U_i$ and $|\{u_2, u_3, u_4\} \cap U_x| \leq 1$.

Construction of P_{xy}. We consider five cases with respect to the multiplicities of the vertices of V_x and V_y in e.

Case 1. In the case when $u_1 \in U_x$, $u_2 \in U_y$ and none of u_3, u_4 belongs to U_y, the path P_{xy} is constructed as follows:

$$xx|xx|xx|qx|xx|qx|xx|\ldots|qx|xx\underbrace{|x*|*y|}_{e}yy|yq|yy|yq|\ldots|yy|yq|yy|yy|yy$$

(the sequence begins with 3 blocks $|xx|$ followed by $(\alpha-7)/3$ pairs $|qx|xx|$ and the edge e; the right side is constructed similarly with y replacing x and the blocks being arranged in the opposite order), where every element of $U_x \cup U_y$ appears exactly once, while $\frac{2}{3}(\alpha-7) \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha-7)+2$ or equivalently $\frac{2}{3}(\alpha-1)-4 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha-1)-2$ (recall that $3|(\alpha-1)$). Note that each pair of consecutive blocks of size two forms an edge of H_1 (except the middle pair $x*|*y$, which is just the edge e) and $|V(P_{xy})| = 2\left(4\frac{\alpha-7}{3}+8\right) = \frac{8}{3}(\alpha-1)$.

Case 2. If $u_1 \in U_x$, $u_2 \in U_y$ and exactly one of u_3, u_4 belongs to U_y, the path P_{xy} is constructed as follows:

$$xx|xx|xx|qx|xx|\ldots|qx|xx\underbrace{|x*|yy|}_{e}yq|yy|yq|\ldots|yy|yq|yy|yy.$$

Again, $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$, while $\frac{2}{3}(\alpha-1)-3 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha-1)-2$.

Case 3. If $u_1 \in U_x$ and $u_2, u_3, u_4 \in U_y$ then we form P_{xy} as follows:

$$xx|xx|xx|qx|xx|\ldots|qx|xx\underbrace{|xy|yy|}_{e}yq|yy|yq|\ldots|yy|yq|yy|yy|yy.$$

This time $|V(P_{xy})| = \frac{8}{3}(\alpha-1)-2$ and $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha-1)-4$.

Case 4. If $u_1, u_2 \in U_x$, $u_3 \in U_y$ and $u_4 \notin U_y$, the path P_{xy} is constructed as follows:

$$xx|xx|qx|xx|\ldots|qx|xx|qx\underbrace{|xx|*y|}_{e}yy|yq|yy|\ldots|yq|yy|yy|yy.$$

Now $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$ and $\frac{2}{3}(\alpha-1)-3 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha-1)-2$.

Case 5. If $u_1, u_2 \in U_x$ and $u_3, u_4 \in U_y$, we form the path P_{xy} as follows:

$$xx|xx|qx|xx|\ldots|qx|xx|qx\underbrace{|xx|yy|}_{e}yq|yy|yq|\ldots|yy|yq|yy|yy.$$

We have again $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$, while $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha-1)-2$.

Let us now set aside p 2-element disjoint subsets $B_1, \ldots, B_p$ of U_{q+1} which correspond to disjoint edges of the graph G, one from each triangle of G. Set $B = \bigcup_{i=1}^{p} B_i$. These pairs will be used to glue together all p paths into a Hamiltonian 2-cycle.

To describe the remaining paths, let symbol w represent any element of the set

$$W := \bigcup_{i=p-1}^{q-3} U_i \cup U_q \cup (U_{q+1} \setminus B) \setminus V(P_{xy}).$$

Construction of P_j, $j = 1, \dots, p-2$. For $j = 1, \dots, p-2$, we build path P_j by splitting $\alpha - 4$ vertices of U_j into $(\alpha-4)/3$ blocks of length 3, separating them by arbitrary vertices from W and putting the remaining 4 vertices of U_j at the end. In a diagram form

$$P_j = jj|jw|jj|jw|\dots|jj|jw|jj|jj.$$

Because $j < \min\{i : U_i \cap W \neq \emptyset\}$, each pair of consecutive blocks of size two forms an edge of H_1. Also, $|V(P_j)| = \frac{4}{3}(\alpha - 1)$, which means that P_j can accommodate precisely $(\alpha-4)/3$ vertices from W. As, by our choice of q,

$$(p-2)\frac{\alpha-4}{3} \geq (q-p-1)(\alpha-1) + \frac{\alpha-1}{3} + 3, \tag{44}$$

we have

$$\bigcup_{r=1}^{p-2} V(P_j) \supseteq \bigcup_{i=p-1}^{q-3} U_i \cup (U_q \setminus V(P_{xy}))\,.$$

On the other hand, the difference between the L-H-S and R-H-S of (44) is less than $4\frac{\alpha}{3} << p$, so that the surplus w-spots can be filled with some elements of U_{q+1}.

Construction of P_{q+1}. The last path, P_{q+1}, consists of all the remaining vertices of U_{q+1} whose number is even, because n is even and every so far built path, as well as the set B, consists of an even number of vertices.

The constructed paths $P_1, \dots, P_{p-2}, P_{xy}$, and P_{q+1} are now connected together, in arbitrary order, by the 2-element blocks $B_1, \dots, B_p$. Note that each B_j makes edges of H_2 with arbitrary 2-element sets from some U_i, $i = 1, \dots, q$. This completes the construction of a 2-Hamiltonian cycle in $H + e$. □

The proof of Theorem 4 follows immediately from Lemma 27 and Fact 26.

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