THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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J. Grytczuk, K. Kosiński and M. Zmarz

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Nonrepetitive colorings of line arrangements

Jarosław Grytczuk*, Karol Kosiński, and Michał Zmarz
Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, 30-348 Kraków, Poland;
grytczuk@tcs.uj.edu.pl, kosinski@tcs.uj.edu.pl, zmarz@tcs.uj.edu.pl

Abstract

A sequence S is *nonrepetitive* if no two adjacent segments of S are identical. A famous result of Thue from 1906 asserts that there are arbitrarily long nonrepetitive sequences over 3 symbols. We study the following geometric variant of this problem. Given a set P of points in the plane and a set L of lines, what is the least number of colors needed to color P so that every line in L is nonrepetitive? If P consists of all intersection points of a prescribed set of lines L, then we prove that there is such coloring using at most 405 colors. The proof is based on a theorem of Thue and on a result of Alon and Marshall concerning homomorphisms of edge colored planar graphs. We also consider nonrepetitive colorings involving other geometric structures. For instance, a nonrepetitive analog of the famous Hadwiger-Nelson problem is formulated as follows: what is the least number of colors needed to color the plane so that every path of the unit distance graph whose vertices are colinear is nonrepetitive? Using a theorem of Thue we prove that this number is at most 36.

1 Introduction

A sequence $R = r_1 r_2 \dots r_{2t}$ is called a *repetition* if $r_i = r_{i+t}$ for all $i = 1, 2, \dots, t$. A *segment* in a sequence S is a subsequence consisting of consecutive terms of S. A sequence S is *nonrepetitive* if none of its segments is a repetition. In 1906 Thue [18] proved that there exist arbitrarily long nonrepetitive sequences over the set of three symbols [5]. This result has lots of interesting applications and generalizations (cf. [1], [6], [9], [11], [14], [15]). In particular, one may consider *nonrepetitive colorings* of graphs in which a color sequence of every simple path is nonrepetitive. The least number of colors in such coloring of a graph G is denoted by $\pi(G)$ and called the *Thue chromatic number* of G. It was proved that $\pi(G)$ is bounded for graphs of bounded degree [2] and for graphs of bounded treewidth [4], [13], but the following conjecture remains open (see [10]).

Conjecture 1 *There is a constant C such that every planar graph G satisfies $\pi(G) \leq C$.*

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In this note we study a geometric variant of nonrepetitive colorings inspired by this conjecture. Let L be a *line arrangement* consisting of a finite set of lines in the plane. Let $P = P(L)$ denote the set of all intersection points of these lines. A *nonrepetitive coloring* of L is a coloring of the set P such that a sequence of colors determined by consecutive points on every line in L is nonrepetitive. Our main result reads as follows.

Theorem 2 *Every line arrangement has a nonrepetitive coloring using at most* 405 *colors.*

The proof is based on a theorem of Alon and Marshall [3] on homomorphisms of edge colored graphs. We give it in Section 2. In Section 3 we consider a nonrepetitive analog of the famous Hadwiger-Nelson problem concerning the chromatic number of the plane (see [17]). In the last section we state a general conjecture on nonrepetitive colorings involving geometric graphs.

2 Proof of the main result

We start with recalling a definition of homomorphism of edge colored graphs. We say that an edge colored graph G has a *homomorphic embedding* into another edge colored graph H if there is a function $h : V(G) \to V(H)$ such that for every pair of adjacent vertices $u, v \in V(G)$, their images $h(u)$ and $h(v)$ are adjacent in H, and color of the edge $h(u)h(v)$ is the same as color of the edge uv. The following lemma is proved in [3].

Lemma 3 (Alon and Marshall [3])*Let k be a positive integer. There exists a graph H_k on at most $5k^4$ vertices with k-colored edges such that every planar graph G whose edges are colored arbitrarily with k colors embeds homomorphically into H_k.*

The proof of this result is based on the fact that planar graphs have bounded *acyclic chromatic number* $\chi_a(G)$, defined as the least number of colors in a proper vertex coloring of G with no 2-colored cycles. By a famous theorem of Borodin [7] every planar graph G satisfies $\chi_a(G) \leq 5$, which is best possible.

Another lemma we will need is a simple consequence of the theorem of Thue [18].

Lemma 4 *There exist arbitrarily long sequences over* 3 *symbols such that no two segments separated by exactly one term are identical.*

Proof. Let $S = s_1 s_2 \dots s_n$ be a nonrepetitive sequence over 3 symbols guaranteed by the result of Thue. Consider a sequence $T = t_1 t_2 \dots t_{2n}$ obtained by doubling each term of S, that is, $t_{2i-1} = t_{2i} = s_i$ for all $i = 1, 2, \dots, t$. We claim that the sequence T cannot contain segments of the form BxB, where $B = b_1 b_2 \dots b_k$ is any nonempty sequence, while x is a single symbol. To prove this suppose on the contrary that some segment of T has the form BxB, and let $x = t_j$, $(j \geq k+1)$. Then we have

$$t_{j-k} \dots t_{j-2} t_{j-1} = b_1 b_2 \dots b_k = t_{j+1} t_{j+2} \dots t_{j+k}.$$

We may assume that j is odd (the other case will follow by reversing the sequence T). This implies that $t_{j+1} = t_{j-k} = x$. If $j - k$ is odd, then also $t_{j-k+1} = x$, which in turn implies that

$t_{j+2} = x = t_j$. Actually, in this case we get that all terms of B are equal x which clearly contradicts the assumption on S. If $j-k$ is even, then

$$t_{j-k}t_{j-k+2}\dots t_{j-1} = t_{j+1}t_{j+3}\dots t_{j+k}$$

and we again get a contradiction with nonrepetitivity of S. ■

Now we give a proof of Theorem 2.

Proof of Theorem 2. Let L be a finite set of lines in the plane and let P be the set of all intersection points determined by L. Consider a graph $G = G(L)$ on the vertex set P with two points p and q adjacent if and only if there is a line in L containing both of them, and there is no other point of P lying between p and q on this line. Clearly G is a planar graph.

A path in G whose vertices are collinear will be called a *straight path*. Using Lemma 4 we may color the edges of G by three colors such that no straight path contains a pattern of the form BxB: just take sufficiently long sequence S with this property and color the edges of each straight path consecutively, accordingly to S. No conflicts between paths may arise as they are edge disjoint. Denote by $c : E(G) \to \{1, 2, 3\}$ any coloring satisfying this property.

Now we apply Lemma 3 to G with edge coloring c. Let H_3 and $h : P \to V(H_3)$ be a graph, and a homomorphism satisfying assertion of the lemma, respectively. We claim that h is a nonrepetitive coloring of L. Suppose that this is not the case and let $Q = q_1q_2\dots q_{2t}$ be a repetitively colored straight path in G with edges denoted by $e_i = q_iq_{i+1}$, $i = 1, 2, \dots, 2t-1$. This means that $h(q_i) = h(q_{i+t})$ for all $i = 1, 2, \dots, t$. Thus, by the edge color preservation property of h, we get that $c(e_i) = c(e_{i+t})$ for each $i = 1, 2, \dots t-1$. But this means that the color pattern of Q in coloring c has the form BxB, with $B = c(e_1)c(e_2)\dots c(e_{t-1})$ and $x = c(e_t)$. This contradicts the property of coloring c and finishes the proof. ■

Lemma 3 is a special case of the following more general result from [3].

Theorem 5 (Alon and Marshall [3])*For every pair of integers k and r there exists a graph $H_{k,r}$ with edges colored by k colors such that every k-edge colored graph G with $\chi_a(G) \leq r$ embeds homomorphically into $H_{k,r}$. Moreover, the least number of vertices in such graph $H_{k,r}$ is bounded by rk^{r-1}.*

By this result and Lemma 4 the proof of Theorem 2 generalizes easily giving the following statement.

Theorem 6 *Let G be a graph with $\chi_a(G) \leq r$ whose edge set is decomposed arbitrarily into simple paths $Q_1, Q_2, \dots, Q_m$. Then there exist a vertex coloring of G using at most $r3^{r-1}$ colors such that the sequence of colors on every path Q_i is nonrepetitive.*

3 Nonrepetitive coloring of the plane

The famous Hadwiger-Nelson problem asks for the chromatic number of the plane $\chi(\mathbb{R}^2)$, defined as the least number of colors needed to color the plane such that every pair of points at distance one apart is colored differently. We formulate a natural analog of this question in the spirit of nonrepetitive colorings of line arrangements.

A finite sequence of points $P_1, P_2, \ldots, P_n$ in the plane is called *nice* if the points are collinear and the distance between each pair P_i and P_{i+1} is one. A coloring of the plane is *nonrepetitive* if a sequence of colors determined by every nice sequence of points is nonrepetitive. Let $\pi(\mathbb{R}^2)$ denote the least number of colors needed for a nonrepetitive coloring of the plane. We prove below that this number is finite. As before we will need the following simple consequence of the theorem of Thue.

Lemma 7 *There exists a* 6*-coloring of the integers such that every finite sequence of integers whose consecutive terms differ by at most* 2 *is colored nonrepetitively.*

Proof. Let $A = \{a, b, c\}$ and $A' = \{a', b', c'\}$ be two disjoint sets of colors. By the result of Thue there is a coloring $f : \mathbb{Z} \to \{a, b, c\}$ such that every segment of integers is nonrepetitive. Let $f' : \mathbb{Z} \to \{a', b', c'\}$ be a true copy of coloring f. Define a new coloring $g : \mathbb{Z} \to \{a, b, c, a', b', c'\}$ by shuffling f and f', that is, by $g(2n) = f(n)$ and $g(2n+1) = f'(n)$ for every $n \in \mathbb{Z}$. It is not hard to see that g satisfies the assertion of the lemma. Indeed, let $M = m_1 m_2 \ldots m_{2t}$ be any sequence of integers satisfying $m_{i+1} - m_i \in \{1, 2\}$. Let M_0 and M_1 be subsequences of M consisting of even and odd terms, respectively. It is clear that both are arithmetic progressions of difference 2, and at least one of them is nonempty, say M_0. Suppose now that the sequence of colors on M is a repetition, that is, $g(m_i) = g(m_{i+t})$ for $i = 1, 2, \ldots, t$. It follows that restriction of g to M_0 must also form a repetition. But this contradicts nonrepetitivity of f and completes the proof of the lemma. ■

With this lemma at hand we may prove the following result.

Theorem 8 $\pi(\mathbb{R}^2) \leq 36$.

Proof. Let g be a coloring of the integers satisfying the assertion of Lemma 7. We extend g to a coloring of the real line as follows. First we split $\mathbb{R}$ into half open intervals $I_n = [a_n, b_n)$, $(n \in \mathbb{Z})$, each of length $1/\sqrt{2}$. Then we color each point of I_n with $g(n)$. Next we define the product coloring h of the plane by $h(x, y) = (g(x), g(y))$. We claim that h is a nonrepetitive coloring of $\mathbb{R}^2$. To see this let $S = P_1 P_2 \ldots P_n$ be a nice sequence of points, with $P_i = (x_i, y_i)$. Define a function $p : \mathbb{R} \to \mathbb{Z}$ by $p(x) = n$ if $x \in I_n$. Next consider two sequences $S_x = p(x_1)p(x_2)\ldots p(x_n)$ and $S_y = p(y_1)p(y_2)\ldots p(y_n)$. It is not hard to see that at least one of the sequences S_x or S_y has gaps precisely in the set $\{1, 2\}$ (or in $\{-1, -2\}$). This is because coloring h determines a tiling of $\mathbb{R}^2$ into squares of diameter slightly less than one and therefore any line cannot 'jump' more than two squares. It follows by the property of g that a color sequence of S_x or S_y is nonrepetitive, which implies the same for the color sequence of S. The proof is complete. ■

4 Problems and remarks

We naturally expect that the bound of 405 from Theorem 2 is not optimal. We propose the following (risky) conjecture.

Conjecture 9 *Every line arrangement has a nonrepetitive coloring with at most* 4 *colors.*

The conjecture is optimal as is seen in arrangement of 10 lines determined by 5 vertices of a regular pentagon (it is enough to consider points of intersection contained in the pentagon). We think the same for the problem of coloring the plane, but this time we will not try to guess the correct value of $\pi(\mathbb{R}^2)$.

Let us conclude with a conjecture generalizing our results. Recall that a *geometric graph* is a graph drawn on the plane such that each vertex corresponds to a point and every edge is a closed line segment connecting two vertices but not passing through a third. A *straight path* in a geometric graph G is a path whose vertices are collinear. A *straight nonrepetitive coloring* of a geometric graph is a coloring of its vertices such that the sequence of colors on any straight path is nonrepetitive. Let $\overline{\pi}(G)$ denote the least number of colors needed in such coloring of G.

Conjecture 10 *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every geometric graph G satisfy $\overline{\pi}(G) \leq f(\chi(G))$.*

Clearly the chromatic number $\chi(G)$ is a lower bound for $\overline{\pi}(G)$. For graphs arising from line arrangements $\chi(G) \leq 4$ by the Four Color Theorem. For unit distance graphs we have $\chi(G) \leq 7$. Another example in favor of our conjecture is given by the geometric (visibility) graph $\mathcal{V}(\mathbb{Z}^2)$ generated by integer lattice points in the plane (two lattice points are connected if there is no other lattice point on a line segment between them). It is not hard to see that that $\chi(\mathcal{V}(\mathbb{Z}^2)) = 4$, and it was proved by Carpi [8] that $\overline{\pi}(\mathcal{V}(\mathbb{Z}^2)) \leq 16$.

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