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MODICA TYPE GRADIENT ESTIMATES FOR AN INHOMOGENEOUS VARIANT OF THE NORMALIZED p -LAPLACIAN EVOLUTION

AGNID BANERJEE AND NICOLA GAROFALO

Dedicated to Enzo Mitidieri, on the occasion of his 60th birthday

ABSTRACT. In this paper, we study an inhomogeneous variant of the normalized p -Laplacian evolution which has been recently treated in [BG1], [Do], [MPR] and [Ju]. We show that if the initial datum satisfies the pointwise gradient estimate (1.6) a.e., then the unique solution to the Cauchy problem (1.2) satisfies the same gradient estimate a.e. for all later times, see (1.7) below. A general pointwise gradient bound for the entire bounded solutions of the elliptic counterpart of equation (1.2) was first obtained in [CGS]. Such estimate generalizes one obtained by L. Modica for the Laplacian, and it has connections to a famous conjecture of De Giorgi.

1. INTRODUCTION

Recently, there has been increasing attention about the equation of the so-called normalized p -Laplacian evolution

$$(1.1) \quad |Du|^{2-p} \operatorname{div}(|Du|^{p-2} Du) = u_t, \quad 1 < p < \infty,$$

see [BG1], [Do], [MPR], [Ju], [BG2] and [JK]. The equation (1.1) is an evolution associated with the p -Laplacian that interpolates between the motion by mean curvature, which corresponds to the case $p = 1$, and the heat equation, corresponding to $p = 2$. In the interesting paper [MPR] solutions to (1.1) have been characterized by asymptotic mean value properties. These properties are connected with the analysis of tug-of-war games with noise in which the number of rounds is bounded. The value functions for these games approximate a solution to the PDE (1.1) when the parameter that controls the size of possible steps go to zero. The equation (1.1) also arises in image processing, see [Do], in which the Cauchy-Neumann problem was studied. In [BG1] we constructed viscosity solutions to (1.1) and derived properties such as comparison principles for solutions of (1.1), convergence of solutions as $p \rightarrow 1$, and the large-time behavior of solutions to a Cauchy-Dirichlet problem for (1.1). We also proved unweighted energy monotonicity and a generalized Struwe's monotonicity formula. In the paper [Ju] Juutinen studied the large-time behavior for $p > 2$ of solutions of (1.1). The case $p = \infty$ of the normalized ∞ -Laplacian evolution was studied in [JK]. The equation (1.1) has the advantage of being 1-homogeneous but it has the serious disadvantage of having a non-divergence structure.

In the present paper for a given $T > 0$ we consider the following Cauchy problem in $\mathbb{R}^n \times [0, T]$

$$(1.2) \quad \begin{cases} |Du|^{2-p} \{ \operatorname{div}(|Du|^{p-2} Du) - F'(u) \} = u_t, \\ u(\cdot, 0) = g. \end{cases}$$

We suppose that $F \in C_{loc}^{2,\beta}(\mathbb{R})$ for some $\beta > 0$ and $F \geq 0$. Throughout this paper we assume $1 < p \leq 2$. We observe that, because of its non-divergence structure, when $F \not\equiv 0$ the equation (1.2) does not make sense for $p > 2$ because the term $|Du|^{2-p} F'(u)$ cannot be interpreted in the viscosity sense at the critical point of a test function ϕ , see Definition 2.1 below. As a

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consequence, in the case $p > 2$ it presently remains an interesting open question what is the right evolution for which results similar to those in this paper can be established.

The equation in (1.2) can be considered as the parabolic counterpart of

$$(1.3) \quad \operatorname{div}(|Du|^{p-2}Du) = F'(u),$$

which is a special case of the class of equations $\operatorname{div}(\Phi'(|Du|^2)Du) = F'(u)$ treated in [CGS]. As a consequence of the results in [CGS], it follows that entire bounded (weak) solutions to (1.3) satisfy the following pointwise gradient estimate

$$(1.4) \quad |Du|^p \leq \frac{p}{p-1}F(u).$$

We recall that in the linear case $p = 2$ the estimate (1.4) was first proved by L. Modica in [Mo]. The estimate (1.4) (in fact, a generalization of it) was employed in [CGS] to provide a partial answer to a famous conjecture of De Giorgi, see [dG] (also known as the ε -version of the Bernstein theorem for minimal graphs) asserting that entire solutions to

$$(1.5) \quad \Delta u = u^3 - u,$$

such that $|u| \leq 1$ and $\frac{\partial u}{\partial x_n} > 0$, must be one-dimensional, i.e., must have level sets which are hyperplanes, at least in dimension $n \leq 8$. In [CGS] the estimate (1.4) was also used to establish a result on the propagation of the zeros of a solution to (1.3). We recall that the conjecture of De Giorgi has been fully solved for $n = 2$ in [GG1] and $n = 3$ in [AC], and it is known to fail for $n \geq 9$, see [dPKW]. For $4 \leq n \leq 8$ it is still an open question. Additional fundamental progress on De Giorgi's conjecture is contained in the papers [GG2], [Sa].

We mention that Modica type gradient estimates have been extended to more general integrands of the calculus of variations in [DG], to domains Ω which are epigraphs with boundary having nonnegative mean curvature in [FV], and also to compact Riemannian manifolds which nonnegative Ricci curvature, see [FV1] and the references therein. For results concerning the p -Laplacian version of De Giorgi's conjecture, we refer the reader to the interesting paper [SSV]. For further results, the state of art and recent progress on De-Giorgi's conjecture, we refer to [CNV], [FV2] and [FSV] and one can find the references therein.

In this paper, we study the parabolic analogue of the Modica type gradient estimate (1.4). Before stating our main results, we introduce the relevant class of solutions for the Cauchy problem (1.2):

$$H^T = \{u \in C(\mathbb{R}^n \times [0, T]) \mid x \rightarrow u(x, t) \in C^{0,1}(\mathbb{R}^n), \|u\|_{L^\infty(\mathbb{R}^n \times [0, T])}, \|Du\|_{L^\infty(\mathbb{R}^n \times [0, T])} < \infty\}.$$

The notation $C^{0,1}(\Omega)$ indicates the class of Lipschitz continuous functions on a given open set $\Omega \subset \mathbb{R}^n$. The following is our main result.

Theorem 1.1. *Let $g \in C^{0,1}(\mathbb{R}^n)$ with $\|g\|_{L^\infty(\mathbb{R}^n)}, \|Dg\|_{L^\infty(\mathbb{R}^n)} < \infty$. Moreover, corresponding to g , we assume that F satisfies the assumption (4.4) below. Then, for every $T > 0$ there exists a unique solution u to the Cauchy problem (1.2) in the class H^T . Furthermore, if the initial datum g satisfies the following gradient estimate for a.e. $x \in \mathbb{R}^n$*

$$(1.6) \quad |Dg(x)|^p \leq \frac{p}{p-1}F(g(x)),$$

then, at any given time $t > 0$ one has for a.e. $x \in \mathbb{R}^n$

$$(1.7) \quad |Du(x, t)|^p \leq \frac{p}{p-1}F(u(x, t)).$$

Remark 1.2. *The assumption (4.4) below is used to assert the existence of solutions in the class H^T via a regularization scheme described in the subsequent sections, see Remark 4.1. The hypothesis (4.4) is however not needed when $1 < p < 2$, see Remark 4.2 below. In addition, such a regularization scheme is also crucially employed to justify the computations in Section*

5. Now, when $p = 2$, any solution in the class H^T is a classical solution, a fact which follows from the parabolic regularity theory. Hence, in this case one does not need to apply any further regularization scheme. In conclusion, if we a priori assume that the solution u belongs to the class H^T , then we obtain the following version of Theorem 1.1.

Theorem 1.3. *Let $1 < p \leq 2$, and for some $0 < T \leq \infty$ let $u \in H^T$ be a solution to*

$$(1.8) \quad |Du|^{2-p} \{ \operatorname{div}(|Du|^{p-2} Du) - F'(u) \} = u_t,$$

where $F \in C_{loc}^{2,\beta}(\mathbb{R})$ for some $\beta > 0$, and $F \geq 0$. If at some time level t_0 $u(\cdot, t_0)$ satisfies (1.7), then $u(\cdot, t)$ satisfies (1.7) for all $t_0 \leq t \leq T$ ($t < \infty$ if $T = \infty$).

Remark 1.4. *Note that unlike the hypothesis in Theorem 1.1, in Theorem 1.3 we do not require that F satisfy (4.4). See Remark 1.2 above.*

Theorem 1.1 and Theorem 1.3 can be considered as a parabolic analogue in the case $1 < p \leq 2$ of the above mentioned result in [CGS] (see also [Mo]) which states that an entire bounded solution to (1.3) satisfies the estimate (1.4) except that in our situation we are only able to assert that the estimate (1.7) holds a.e. in \mathbb{R}^n . It remains an open question as to whether the solution u in Theorem 1.1 has higher regularity so that one can assert that the estimate (1.7) holds pointwise everywhere. In the next result we show that, under an additional assumption on the initial datum g , this is true when $n = 2$.

Theorem 1.5. *Let $n = 2$, and let u, g be as in Theorem 1.1. Furthermore, if the initial datum g has bounded derivatives up to order two there exists $\alpha \in (0, 1)$ depending only on p such the solution $u(\cdot, t) \in C^{1,\alpha}$ for every $t > 0$. Consequently, the gradient estimate (1.7) holds pointwise everywhere.*

We conclude with an application of the estimate (1.7). The following result can be thought of as theorem on the propagation of zeros for solutions of the Cauchy problem (1.2).

Theorem 1.6. *Suppose that the initial datum g satisfies (1.6), and let u be the solution as in Theorem 1.1. If $F(u(x_0, t_0)) = 0$ for some point (x_0, t_0) , then $u(\cdot, t_0)$ is constant.*

In closing we mention that concerning Theorems 1.1 and 1.6 and for the standard heat equation ($p = 2$), there is an unpublished note of Caffarelli, Garofalo, Segala, [CGSpc]. There, the authors could successfully employ subsolution estimates and compactness arguments based on $C^{1,\alpha}$ estimates. In our situation, however, in the genuinely nonlinear case when $p \neq 2$, the subsolutions arguments work, but only in a restricted sense, i.e., we do all our calculations at the point of extrema and substitute the undesirable higher order term in terms of lower-order terms which can be controlled. The higher- order terms show up in our situation because of the normalization of the equation which is obviously evident when $p \neq 2$. Secondly, we do not have $C^{1,\alpha}$ estimates, and we compensate for this with the use of barriers.

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2. PRELIMINARIES

Suppose that u is a solution to the equation (1.2). We begin by observing that, after some formal computations, we have the following equation in non-divergence form

$$(2.1) \quad \left(\delta_{ij} + (p-2) \frac{u_i u_j}{|Du|^2} \right) u_{ij} = |Du|^{2-p} f(u) + u_t,$$

where $f = F'$ (see [BG1] for similar formal computations in the homogeneous case $F \equiv 0$). Following [CGG], we now introduce the following notion of viscosity solution to the equation in (1.2).

Definition 2.1. *A function $u \in C(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T])$ is called a viscosity subsolution of (2.1), provided that for every $\phi \in C^2(\Omega \times (0, T))$ such that*

$$(2.2) \quad u - \phi \quad \text{has a local maximum at} \quad z_0 \in \Omega \times (0, T),$$

then either

$$(2.3) \quad \begin{cases} \phi_t + |D\phi|^{2-p} f(u) \leq \left(\delta_{ij} + (p-2) \frac{\phi_i \phi_j}{|D\phi|^2} \right) \phi_{ij} & \text{at } z_0, \\ \text{if } D\phi(z_0) \neq 0, \end{cases}$$

or

$$(2.4) \quad \begin{cases} \inf_{|a|=1} \{ \phi_t + |D\phi|^{2-p} f(u) - (\delta_{ij} + (p-2) a_i a_j) \phi_{ij} \} \leq 0 & \text{at } z_0, \\ \text{if } D\phi(z_0) = 0. \end{cases}$$

A function u is a viscosity supersolution if $v = -u$ is a viscosity subsolution. Finally, u is a viscosity solution if it is at the same time a subsolution and a supersolution.

Similarly to the case $F = 0$, by arguing as in Proposition 2.8 in [BG1] we have the following equivalent definition.

Definition 2.2. *A function $u \in C(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T])$ is called a viscosity subsolution of (2.1), provided that for every $\phi \in C^2(\Omega \times (0, T))$ such that*

$$(2.5) \quad u - \phi \quad \text{has a local maximum at} \quad z_0 \in \Omega \times (0, T),$$

then

$$(2.6) \quad \begin{cases} \phi_t + |D\phi|^{2-p} f(u) \leq \left(\delta_{ij} + (p-2) \frac{\phi_i \phi_j}{|D\phi|^2} \right) \phi_{ij} & \text{at } z_0, \\ \text{if } D\phi(z_0) \neq 0, \end{cases}$$

or

$$(2.7) \quad \begin{cases} \phi_t + |D\phi|^{2-p} f(u) \leq (\delta_{ij} + (p-2) a_i a_j) \phi_{ij} & \text{at } z_0, \\ \text{for some } a \in \mathbb{R}^n \quad \text{with } |a| \leq 1, \text{ if } D\phi(z_0) = 0. \end{cases}$$

Analogous definitions for supersolutions, and for solution.

3. MAXIMUM MODULUS PRINCIPLE

In this short section we establish a maximum modulus theorem for viscosity solutions of (1.2) which will be needed subsequently.

Theorem 3.1. *Let u and v be two bounded continuous solutions in $\mathbb{R}^n \times [0, T]$ to (1.2) which are globally Lipschitz in the space variable. Let*

$$(3.1) \quad \|u\|_{L^\infty(\mathbb{R}^n \times (0, T))}, \|Du\|_{L^\infty(\mathbb{R}^n \times (0, T))}, \|v\|_{L^\infty(\mathbb{R}^n \times (0, T))}, \|Dv\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq C.$$

Then, there exists a constant $M = M(C)$ such that

$$(3.2) \quad \|u - v\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq e^{MT} \|u(\cdot, 0) - v(\cdot, 0)\|_{L^\infty(\mathbb{R}^n \times (0, T))}.$$

Proof. First, we let $G \in C^2(\mathbb{R})$ be a compactly supported real-valued function such that $G(w) = F(w)$ when $|w| \leq 2C + 1$. Let now ϕ be a test function such that $u - \phi$ has a local extremum at a point $z_0 = (x_0, t_0)$. From (3.1), it follows that $|D\phi| \leq C$, and a similar conclusion is also true

when u is replaced by v . Therefore, if we define $Q(y) = |y|^{2-p}$ if $|y| \leq 2C$ and $Q(y) = 2^{2-p}C^{2-p}$ when $|y| \geq 2C$, we have that both u and v are viscosity solutions to

$$(3.3) \quad w_t + Q(Dw)G'(w) = (\delta_{ij} + (p-2)\frac{w_i w_j}{|Dw|^2})w_{ij}.$$

This equation obeys the hypothesis of Theorem 4.1 in [GGIS]. As a consequence, (3.2) follows from a slight modification of the arguments in the proof of Theorem 4.1 in [GGIS] which can be found for instance in Theorem 1.2.1 in [Zh]. Note that the modification is similar to the one employed for the case $F = 0$ in proof of Theorem 3.4 in [BG1]. \square

4. EXISTENCE OF SOLUTIONS

In this section we establish the solvability of the Cauchy problem (1.2) when the initial datum $g \in C^{0,1}(\mathbb{R}^n)$, i.e., g is globally Lipschitz and bounded. With this objective in mind, for any $\varepsilon > 0$ we consider the approximating Cauchy problem

$$(4.1) \quad \begin{cases} u_t^\varepsilon + (\varepsilon^2 + |Du^\varepsilon|^2)^{1-p/2} f(u^\varepsilon) = a_{ij}^\varepsilon(Du^\varepsilon)u_{ij}^\varepsilon \\ u^\varepsilon(\cdot, 0) = g, \end{cases}$$

where we have let $f = F'$, and

$$(4.2) \quad a_{ij}^\varepsilon(\sigma) = \delta_{ij} + (p-2)\frac{\sigma_i \sigma_j}{\varepsilon^2 + |\sigma|^2}, \quad i, j = 1, \dots, n.$$

It is easily seen that for every $\sigma \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ the following uniform ellipticity condition is satisfied, independently of $\varepsilon > 0$,

$$(4.3) \quad \min\{1, p-1\} |\xi|^2 \leq a_{ij}^\varepsilon(\sigma)\xi_i \xi_j \leq \max\{1, p-1\} |\xi|^2.$$

Proceeding as follows we first obtain a unique bounded classical solution u^ε to (4.1).

We let $M = \|g\|_{L^\infty(\mathbb{R}^n)}$. In correspondence of the initial datum g we assume that the nonlinearity F in (1.2) satisfy the following hypothesis: there exist constants q, M_1, M_2 , all depending on M , such that one has

$$(4.4) \quad \begin{cases} -q \leq M_1 \leq -M \text{ and } M \leq M_2 \leq q, \\ f(M_1) \leq 0 \leq f(M_2). \end{cases}$$

We remark immediately that assumption (4.4) will be needed only in the case $p = 2$, but not when $1 < p < 2$. We also note that for the typical representatives of nonlinearities $f(u) = u^3 - u$, $f(u) = \sin u$ in (1.2) the assumption (4.4) is satisfied.

Assuming (4.4) let now \tilde{F} be a compactly supported, $C^{2,\beta}(\mathbb{R})$ function such that $\tilde{F} = F$ for $|u| \leq 2q + 1$. We first suppose additionally that g is smooth and has bounded derivatives of all orders. We take a sequence of smooth domains $\Omega^N \nearrow \mathbb{R}^n$. Given any $T > 0$, we consider the finite cylinders $\Omega_T^N = \Omega^N \times (0, T)$, and indicate with $\partial_p \Omega_T^N = (\partial \Omega^N \times (0, T)) \cup (\Omega^N \times \{0\})$ its parabolic boundary. For each $N \in \mathbb{N}$, and $\varepsilon > 0$, we solve the Cauchy-Dirichlet problem

$$(4.5) \quad \begin{cases} u_t^{\varepsilon,N} + (\varepsilon^2 + |Du^{\varepsilon,N}|^2)^{1-p/2} \tilde{F}'(u^{\varepsilon,N}) = a_{ij}^\varepsilon(Du^{\varepsilon,N})u_{ij}^{\varepsilon,N} \text{ in } \Omega_T^N, \\ u^{\varepsilon,N} = g \text{ on } \partial_p \Omega_T^N \text{ (one should keep in mind that } g(x, t) = g(x)). \end{cases}$$

The existence of classical solutions $u^{\varepsilon,N}$, such that $\sup_{\Omega_T^N} \|Du^{\varepsilon,N}\| < \infty$, is guaranteed by Theorem

4.2, p. 559 in [LU]. Because of the boundedness of gradient, one can see that $u^{\varepsilon,N}$ satisfies an equation which obeys the hypothesis of the comparison principle, Theorem 9.1 in [Li]. Moreover, because of (4.4) M_1 is a subsolution and M_2 is a supersolution to such an equation. Therefore, from the comparison principle Theorem 3.1 above we conclude that $|u^{\varepsilon,N}|$ is bounded from above by q , which is independent of N and ε . Since $\tilde{F}'(s) = f'(s)$ when $|s| \leq 2q$, we infer that

$u^{\varepsilon, N}$ solves the Cauchy-Dirichlet problem with \tilde{F}' replaced by f . The rest of the proof for the existence of solutions u^ε to the Cauchy problem corresponding to (4.1) remains the same as for the case $F = 0$, see [BG1]. Since $F \in C_{loc}^{2, \beta}(\mathbb{R})$, it follows from the Schauder theory (see Chapter 4 and Chapter 12 in [Li]), that $u^\varepsilon \in H_{3+\alpha}(\mathbb{R}^n \times [0, T])$ for some $\alpha > 0$ which depends on ε, p, n, q and β . We refer to Chapter 4 in [Li] for relevant notion of $H_{3+\alpha}$ spaces.

We note that the solutions u^ε 's have spatial gradient bounds, depending only on n, p, q and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$, which are uniform in ε for $\varepsilon \leq 1$. This follows from Theorem 11.3 b) in [Li]. For this, one needs to observe that the limit behavior in (11.17) in [Li] is uniform in ε , similarly to the case $F = 0$. Now, as in the case $F = 0$, the uniform bounds on the time derivatives of u^ε , which depend only on the C^2 norm of g , can be obtained by differentiating the approximating equations (4.1) with respect to the time variable and by applying Theorem 3.1 above. Therefore, in the same way as for the case $F = 0$, one can assert the existence of u to (1.2) in the class H^T when g is smooth and has bounded derivatives of all orders.

In the case when g is only globally Lipschitz, we take ε_k -mollifications of g for a sequence $\varepsilon_k \rightarrow 0$, and call them g_k . Then, g_k has bounded derivatives of all order and

$$(4.6) \quad \|g_k\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^\infty(\mathbb{R}^n)}, \quad \|Dg_k\|_{L^\infty(\mathbb{R}^n)} \leq \|Dg\|_{L^\infty(\mathbb{R}^n)}.$$

Let u_k be the solution to the Cauchy problem corresponding to the initial datum g_k . As mentioned above, thanks to Theorem 11.3 in [Li] ensures that $\|Du_k\|_{L^\infty(\mathbb{R}^n \times (0, T))}$ is bounded uniformly in k by constants which depends only on $\|Dg\|_{L^\infty(\mathbb{R}^n)}$, q, p and n . Since $g_k \rightarrow g$ uniformly in \mathbb{R}^n , by the maximum modulus principle Theorem 3.1 above we conclude that $u_k \rightarrow u$ uniformly in $\mathbb{R}^n \times [0, T]$, where u is the unique solution to the Cauchy problem (1.2) in the class H^T corresponding to the initial datum g .

Remark 4.1. *We note that the assumption (4.4) is only used to assert a bound on $u^{\varepsilon, N}$ independent of ε and N as an intermediate step. If we instead assume that f is bounded, it turns out that $w = \|g\|_{L^\infty(\mathbb{R}^n)} + M_1 t$ is a supersolution to the equation satisfied by $u^{\varepsilon, N}$ when $\varepsilon \leq 1$ and M_1 is chosen large enough depending only on $\|f\|_{L^\infty(\mathbb{R}^n)}$. Hence, such w can be used as a barrier for $u^{\varepsilon, N}$ from above and one can similarly bound $u^{\varepsilon, N}$ from below by using $-w$ which is a subsolution to the same equation.*

Remark 4.2. *When $1 < p < 2$, the assumption (4.4) is not needed. In that case, let \tilde{F} be a $C^{2, \beta}$ compactly supported function such that $\tilde{F}(s) = F(s)$ when $|s| \leq \|g\|_{L^\infty(\mathbb{R}^n)} + 2$. Then, for each $\varepsilon > 0$, we solve the corresponding Cauchy-Dirichlet problem as before in Ω_T^N with \tilde{F}' instead of f and denote the corresponding solutions by $u^{\varepsilon, N}$. For all ε small enough depending only on p, f, T and $\|g\|_{L^\infty(\mathbb{R}^n)}$, it turns out that $w = \|g\|_{L^\infty(\mathbb{R}^n)} + \frac{t}{T}$ is a supersolution to the equation satisfied by $u^{\varepsilon, N}$ and hence can be used to assert boundedness of $u^{\varepsilon, N}$ from above. Similarly, the subsolution $-w$ can be used to assert boundedness for $u^{\varepsilon, N}$ from below. Therefore, for all such small enough ε , it follows from the definition of \tilde{F} that $u^{\varepsilon, N}$ solves the Cauchy-Dirichlet problem with \tilde{F}' replaced with f . The rest of the proof remains the same. This procedure does not work in the case $p = 2$. This is because when the approximating equation (4.5) is computed for w , the term $\varepsilon^{2-p} \tilde{F}'(w)$ does not go to zero as $\varepsilon \rightarrow 0$ in the case $p = 2$ and therefore one cannot assert that w is a supersolution to (4.5). Therefore, one interesting aspect is that for $1 < p < 2$, one has existence of solution to the Cauchy problem (1.2) without any growth assumption on f due to the special structure of the equation unlike what one needs in the general theory of uniformly parabolic equations, see for instance Theorem 12.16 in [Li].*

5. PROOF OF THE MAIN RESULTS

We first prove an intermediate crucial result which asserts gradient estimates for solutions to the approximating Cauchy problems (4.1). For each $\varepsilon > 0$, we define

$$(5.1) \quad P_\varepsilon(u^\varepsilon)(x, s) = \xi_\varepsilon(|Du^\varepsilon(x, s)|^2) - 2F(u^\varepsilon(x, s)),$$

where u^ε is a solution to (4.1), and we have let

$$(5.2) \quad \xi_\varepsilon(s) = 2s\phi'_\varepsilon - \phi_\varepsilon, \quad \text{with } \phi_\varepsilon(s) = \frac{2}{p}(\varepsilon^2 + s)^{p/2}.$$

Theorem 5.1. *Let u^ε be a solution of the approximating equation (4.1) such that $u^\varepsilon \in H_{3+\alpha}(\mathbb{R}^n \times [0, T])$ for some $\alpha > 0$. If $P_\varepsilon(u^\varepsilon)(\cdot, 0) \leq 0$, then $P_\varepsilon(u^\varepsilon(x, t)) \leq 0$ for all $x \in \mathbb{R}^n$ and all $t \geq 0$.*

Remark 5.2. *Note that, when the initial datum g has bounded derivatives of sufficiently high order (up to order five), then the solutions u^ε constructed in Section 4 satisfy the hypothesis of Theorem 5.1.*

Proof of Theorem 5.1. Henceforth, we will routinely omit ε -subscripts and superscripts, and suppress the dependence of P on u . Thus, for instance, we will write u instead of u^ε , P instead of $P_\varepsilon(u^\varepsilon)$. We will also write ϕ and ξ , instead of ϕ_ε and ξ_ε like in (5.2). Note that the approximating equation can be rewritten as

$$(5.3) \quad \operatorname{div}(\phi'(|Du|^2)Du) = f(u) + \phi'(|Du|^2)u_t.$$

We let $\Lambda = \xi'$, and note that for each $\varepsilon > 0$ we have from (5.2)

$$(5.4) \quad \Lambda = (\varepsilon^2 + |Du|^2)^{p/2-2}(\varepsilon^2 + (p-1)|Du|^2) > 0.$$

We next write (5.3) in the following manner

$$a_{ij}(Du) u_{ij} = f(u) + \phi' u_t,$$

where

$$(5.5) \quad a_{ij} = 2\phi'' u_i u_j + \phi' \delta_{ij}.$$

Therefore, u satisfies

$$(5.6) \quad d_{ij} u_{ij} = \frac{f}{\Lambda} + \frac{\phi'}{\Lambda} u_t,$$

where $d_{ij} = \frac{a_{ij}}{\Lambda}$. By differentiating (5.5) with respect to x_k , we obtain

$$(5.7) \quad (a_{ij} (u_k)_i)_j = f' u_k + \phi' u_{tk} + 2\phi'' u_{hk} u_h u_t.$$

From the definition of P in (5.1) we have,

$$(5.8) \quad P_i = 2\Lambda u_{ki} u_k - 2f' u_i, \quad P_t = 2\Lambda u_{kt} u_k - 2f u_t.$$

We now consider the following auxiliary function

$$w = w_R = P - \frac{M}{R} \sqrt{|x|^2 + 1} - \frac{ct}{R^{1/2}},$$

where $R > 1$ and M, c are to be determined subsequently. Note that $P \geq w$ for $t \geq 0$. Consider the cylinder $Q_R = B(0, R) \times [0, T]$. One can see that if M is chosen large enough, depending on the L^∞ norm of u and its first derivatives, then $w < 0$ on the lateral boundary of Q_R . In this situation we see that if w has a strictly positive maximum at a point (x_0, t_0) , then such point cannot be on the parabolic boundary of Q_R . In fact, since $w < 0$ on the lateral boundary, the point cannot be on such set. But it cannot be on the bottom of the cylinder either since at $t = 0$ we have $w(\cdot, 0) \leq P(u(\cdot, 0)) = P(g) \leq 0$, where in the last inequality we have used the hypothesis.

Our objective is to prove the following claim:

$$(5.9) \quad w \leq K \stackrel{\text{def}}{=} R^{-\frac{p}{2}}, \quad \text{in } Q_R,$$

provided that M and c are chosen appropriately. This claim will be established in (5.30) below. We first fix a point (y, s) in \mathbb{R}^n . Now for all R sufficiently large enough, we have that $(y, s) \in Q_R$.

We would like to emphasize over here that finally we let $R \rightarrow \infty$. Therefore, once (5.9) is established, we obtain from it and the definition of w that

$$(5.10) \quad P(u)(y, s) \leq \frac{K'}{R^{1/2}},$$

where K' depends on ε , (y, s) and the bounds of the derivatives of u of order three. By letting $R \rightarrow \infty$ in (5.10), we find that

$$(5.11) \quad P(u)(y, s) \leq 0.$$

The sought for conclusion thus follows from the arbitrariness of the point (y, s) .

In order to prove the claim (5.9) we argue by contradiction and suppose that there exist $(x_0, t_0) \in \overline{Q}_R$ at which w attains its maximum and for which

$$w(x_0, t_0) > K.$$

This implies that (x_0, t_0) is not on the parabolic boundary of Q_R . Note that from the definition (5.1) of P , we have

$$\frac{1}{2}P = (\varepsilon^2 + |Du|^2)^{\frac{p}{2}-1} \left[\frac{1}{p'} |Du|^2 - \frac{\varepsilon^2}{p} \right] - F(u).$$

Since $1 < p \leq 2$, we have $2 \leq p' < \infty$, and so $\frac{1}{p'} \leq \frac{1}{2} < 1$. Thus, at every point of Q_R we have

$$\frac{1}{2}w \leq \frac{1}{2}P \leq \frac{1}{p'} (\varepsilon^2 + |Du|^2)^{\frac{p}{2}-1} |Du|^2 < (\varepsilon^2 + |Du|^2)^{\frac{p}{2}-1} |Du|^2.$$

It follows that at (x_0, t_0) we must have

$$(5.12) \quad (\varepsilon^2 + |Du(x_0, t_0)|^2)^{\frac{p}{2}-1} |Du(x_0, t_0)|^2 \geq \frac{1}{2}P(x_0, t_0) \geq \frac{1}{2}w(x_0, t_0) > \frac{1}{2}K,$$

which implies, in particular, that $Du(x_0, t_0) \neq 0$. Therefore, since $1 < p \leq 2$, we obtain from (5.12)

$$(5.13) \quad |Du(x_0, t_0)|^p \geq (\varepsilon^2 + |Du(x_0, t_0)|^2)^{p/2-1} |Du(x_0, t_0)|^2 \geq \frac{1}{2}P(x_0, t_0) > \frac{1}{2}K.$$

On the other hand, since (x_0, t_0) does not belong to the parabolic boundary, from the hypothesis that w has its maximum at such point, we conclude that $w_t(x_0, t_0) \geq 0$ and $Dw(x_0, t_0) = 0$. These conditions translate into

$$(5.14) \quad P_t \geq \frac{c}{R^{1/2}},$$

and

$$(5.15) \quad P_i = \frac{M}{R} \frac{x_{0,i}}{(|x_0|^2 + 1)^{1/2}}.$$

Now

$$(d_{ij}w_i)_j = (d_{ij}P_i)_j - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j,$$

where

$$(5.16) \quad (d_{ij}P_i)_j = 2 \left(\frac{a_{ij}}{\Lambda} (\Lambda u_{ki} u_k - f u_i) \right)_j = 2(a_{ij} (u_k)_i u_k)_j - 2(f d_{ij} u_i)_j.$$

After a simplification, (5.16) equals

$$2a_{ij} (u_{ki})_j u_k + 2a_{ij} u_{ki} u_{kj} - 2f' d_{ij} u_i u_j - 2f d_{ij} u_{ij} - 2f (d_{ij})_j u_i.$$

We notice that

$$d_{ij}u_i u_j = \frac{2\phi'' u_i u_j u_i u_j + \phi' \delta_{ij} u_i u_j}{\Lambda} = |Du|^2.$$

Now by using (5.7) and by cancelling the term $2f'|Du|^2$, we get that the right-hand side in (5.16) equals

$$2\phi' u_{tk} u_k + 4\phi'' u_{hk} u_h u_k u_t + 2a_{ij} u_{ki} u_{kj} - 2fd_{ij} u_{ij} - 2fd_{ij,j} u_i.$$

Therefore by using the equation (5.6), we obtain

$$(5.17) \quad \begin{aligned} (d_{ij}P_i)_j &= 2a_{ij} u_{ki} u_{kj} + 2\phi' u_{tk} u_k + 4\phi'' u_{hk} u_h u_k u_t \\ &\quad - 2\frac{f^2}{\Lambda} - 2\frac{f\phi'}{\Lambda} u_t - 2fd_{ij,j} u_i. \end{aligned}$$

By using the extrema conditions (5.14), (5.15), we have the following two conditions at (x_0, t_0)

$$(5.18) \quad u_{kh} u_k u_h = \frac{f}{\Lambda} |Du|^2 + \frac{M}{2R\Lambda} \frac{x_h u_h}{(|x|^2 + 1)^{1/2}},$$

$$(5.19) \quad 2\Lambda u_{kt} u_k \geq 2fu_t + \frac{c}{R^{1/2}}.$$

Using the extrema conditions and by canceling $2\phi' u_{tk} u_k$ we obtain,

$$(5.20) \quad \begin{aligned} (d_{ij}w_i)_j &\geq 2a_{ij} u_{ki} u_{kj} + \frac{4\phi'' f}{\Lambda} |Du|^2 u_t - \frac{2f^2}{\Lambda} - 2fd_{ij,j} u_i \\ &\quad + \frac{2\phi'' M x_h u_h u_t}{R \Lambda (|x|^2 + 1)^{1/2}} + \frac{c\phi'}{R^{1/2}\Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j. \end{aligned}$$

Now we have the following structure equation, whose proof is lengthy but straightforward,

$$(5.21) \quad d_{ij,j} u_i = \frac{2\phi''}{\Lambda} (|Du|^2 \Delta u - u_{hk} u_h u_k).$$

Using (5.19) in (5.21), we find

$$d_{ij,i} u_i = \frac{2\phi'' |Du|^2}{\Lambda} (\Delta u - \frac{f}{\Lambda} - \frac{M x_h u_h}{2R |Du|^2 \Lambda (|x|^2 + 1)^{1/2}}).$$

Using the equation (5.3), we have

$$2\phi'' u_{hk} u_h u_k + \phi' \Delta u = f + \phi' u_t.$$

Therefore,

$$(5.22) \quad \Delta u = \frac{f + \phi' u_t - 2\phi'' u_{hk} u_h u_k}{\phi'}.$$

Substituting the value for Δu in (5.22) and by using the extrema condition (5.19), we have the following equality at (x_0, t_0) ,

$$(5.23) \quad \begin{aligned} d_{ij,j} u_i &= \frac{2\phi'' |Du|^2}{\Lambda \phi'} \left[f + u_t \phi' - 2\phi'' \frac{|Du|^2}{\Lambda} f - f \frac{\phi'}{\Lambda} \right. \\ &\quad \left. - \frac{\phi'' M x_h u_h}{R\Lambda (|x|^2 + 1)^{1/2}} - \frac{M x_h u_h \phi'}{2R |Du|^2 \Lambda (|x|^2 + 1)^{1/2}} \right]. \end{aligned}$$

Using the definition of Λ and cancelling terms in (5.23), we have that the right-hand side in (5.23) equals

$$(5.24) \quad 2\phi'' \frac{|Du|^2 u_t}{\Lambda} - \frac{\phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}} - \frac{2(\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}}.$$

Therefore, by canceling the terms $4\phi'' f \frac{|Du|^2 u_t}{\Lambda}$ in (5.20), we obtain the following differential inequality at (x_0, t_0) ,

$$(5.25) \quad (d_{ij} w_i)_j \geq \frac{c \phi'}{R^{1/2} \Lambda} - \frac{2 f^2}{\Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j + \frac{2\phi'' M x_h u_h u_t}{R \Lambda (|x|^2 + 1)^{1/2}} \\ + \frac{2f \phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}} + \frac{4f (\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}} + 2a_{ij} u_{ki} u_{kj}.$$

Now by using the identity for DP in (5.8) above, we have

$$(5.26) \quad u_{ki} u_{kj} u_i u_j = \frac{(P_k + 2f u_k)^2}{4\Lambda^2}.$$

Also,

$$a_{ij} u_{kj} u_{ki} = \phi' u_{ik} u_{ik} + 2\phi'' u_{ik} u_i u_{jk} u_j.$$

Therefore, by Schwarz inequality, we have

$$a_{ij} u_{kj} u_{ki} \geq \phi' \frac{u_{ik} u_{jk} u_i u_j}{|Du|^2} + 2\phi'' u_{ik} u_i u_{jk} u_j = \frac{\Lambda u_{ik} u_i u_{jk} u_j}{|Du|^2}.$$

Then, by using (5.26) we find

$$(5.27) \quad a_{ij} u_{kj} u_{ki} \geq \frac{(P_k + 2f u_k)^2}{4\Lambda |Du|^2} = \frac{|DP|^2 + 4f^2 |Du|^2 + 2f \langle Du, DP \rangle}{4|Du|^2 \Lambda}.$$

At this point, using (5.27) in (5.25), we can cancel off $\frac{2f^2}{\Lambda}$ and consequently obtain the following inequality at (x_0, t_0) ,

$$(5.28) \quad (d_{ij} w_i)_j \geq \frac{c\phi'}{R^{1/2} \Lambda} + \frac{f \langle Du, DP \rangle}{|Du|^2 \Lambda} - \frac{M}{R} (d_{ij} \frac{x_i}{(|x|^2 + 1)^{1/2}})_j + \frac{2\phi'' M x_h u_h u_t}{R \Lambda (|x|^2 + 1)^{1/2}} \\ + \frac{4f (\phi'')^2 |Du|^2 M x_h u_h}{R \Lambda^2 \phi' (|x|^2 + 1)^{1/2}} + \frac{2f \phi'' M x_h u_h}{\Lambda^2 R (|x|^2 + 1)^{1/2}}.$$

By assumption, since $w(x_0, t_0) \geq K$, we have that

$$|Du| \geq \frac{1}{2^{1/p} R^{1/2}}.$$

Moreover, since u has bounded derivatives of upto order 3, for a fixed $\varepsilon > 0$, we have that ϕ' and Λ are bounded from below by a positive constant. Therefore by (5.15), the term $\frac{f \langle Du, DP \rangle}{|Du|^2 \Lambda}$ can be controlled from below by $-\frac{M''}{R^{1/2}}$ where M'' depends on ε and the bounds of the derivatives of u . Consequently, from (5.28), we have at (x_0, t_0) ,

$$(5.29) \quad (d_{ij} w_i)_j \geq \frac{C(c)}{R^{1/2}} - \frac{L(M)}{R} - \frac{M''}{R^{1/2}}.$$

Now in the very first place, if c is chosen large enough depending only on ε and the bounds of the derivatives of u up to order three, we would have the following inequality at (x_0, t_0) ,

$$(d_{ij} w_i)_j > 0.$$

This contradicts the fact that w has a maximum at (x_0, t_0) . Therefore, either $w(x_0, t_0) < K$, or the maximum of w is achieved on the parabolic boundary where $w < 0$. In either case, for an arbitrary point (y, s) such that $|y| \leq R$, we have that

$$(5.30) \quad w(y, s) \leq \frac{1}{R^{p/2}}.$$

□

Proof of Theorem 1.1. Let g_k be the ε_k mollifications of g which converges to g uniformly in \mathbb{R}^n as $k \rightarrow \infty$. Note that g_k has bounded derivatives of all orders with bounds depending on ε_k . Given any $\delta > 0$, we note that for large enough k , g_k satisfies (1.6) with F replaced by $G = F + \delta$. This can be seen as follows:

$$(5.31) \quad |Dg_k(x)| = \left| \int_{\mathbb{R}^n} Dg(x-y)\rho_{\varepsilon_k}(y)dy \right| \leq \int_{\mathbb{R}^n} |Dg(x-y)|\rho_{\varepsilon_k}(y)dy$$

See for instance Theorem 6.25 in [R]. We choose to cite this reference since the integrals considered in (5.31) are vector valued and we need to make sure that no additional constants are incurred in front of the last integral in (5.31). Therefore,

$$(5.32) \quad |Dg_k(x)|^p = \left| \int_{\mathbb{R}^n} Dg(x-y)\rho_{\varepsilon_k}(y)dy \right|^p \leq \int_{\mathbb{R}^n} |Dg(x-y)|^p \rho_{\varepsilon_k}(y)dy$$

The last inequality in (5.32) follows from (5.31) and Jensen inequality. Now since $|Dg|^p \leq \frac{p}{p-1}F(g)$ a.e., we have for all k large enough,

$$(5.33) \quad |Dg_k(x)|^p \leq \frac{p}{p-1} \int_{\mathbb{R}^n} F(g(x-y))\rho_{\varepsilon_k}(y)dy \leq \frac{p}{p-1} \sup_{B_{\varepsilon_k}(x)} F(g) \leq \frac{p}{p-1}(F(g_k(x)) + \delta).$$

In the last inequality in (5.33), we have made use of the fact that g_k converges to g uniformly in \mathbb{R}^n since g is globally Lipschitz. This justifies the claim above.

Now for each such k , let u_k^ε be the solution to the Cauchy problem corresponding to equation (4.1) with initial datum g_k . We furthermore assume that for $1 < p < 2$, ε is small enough so that the conditions in Remark 4.2 is satisfied. We now note that for $1 < p \leq 2$,

$$(5.34) \quad \tilde{P}_\varepsilon \leq \tilde{P},$$

where $\tilde{P}, \tilde{P}_\varepsilon$ are defined as in (5.1). Therefore, since $\tilde{P}(g_k) \leq 0$, we have that $\tilde{P}_\varepsilon(g_k) \leq 0$. Theorem 5.1 applied to u_k^ε implies that $\tilde{P}_\varepsilon(u_k^\varepsilon) \leq 0$ for all positive times. Now, by Dini's theorem the functions $h_\varepsilon(x) = (\varepsilon^2 + |x|^2)^{p/2-1}|x|^2 \rightarrow |x|^p$, uniformly on compact sets. Thus, because of uniform bounds on the gradients, given any $\gamma > 0$ for all small enough ε we have that at each time level t ,

$$(5.35) \quad \frac{p-1}{p} |Du_k^\varepsilon|^p \leq G(u_k^\varepsilon) + \gamma.$$

Integrating (5.35) over an open ball $B_r = B_r(x)$ where x is any arbitrary point, by using lower semicontinuity on the left-hand side, and by passing to the limit in ε on the right-hand side, and then by letting $\gamma \rightarrow 0$, we find

$$(5.36) \quad \frac{p-1}{p} \int_{B_r} |Du_k|^p \leq \int_{B_r} G(u_k).$$

Now by the maximum modulus principle, Theorem 3.1, $u_k \rightarrow u$ uniformly in \mathbb{R}^n and weakly in $W_{loc}^{1,p}(\mathbb{R}^n)$ at any given time t where u is the solution to the Cauchy problem with initial datum g . Therefore in (5.36), by using lower semicontinuity on the left hand side and by passing to the limit in k on the right hand side, we have that (5.36) holds for u . Then from the Lebesgue differentiation theorem, it follows that at a given time level t ,

$$(5.37) \quad \frac{p-1}{p} |Du|^p \leq F(u) + \delta \quad \text{a.e. in } \mathbb{R}^n.$$

By letting $\delta \rightarrow 0$, we reach the desired conclusion. □

Proof of Theorem 1.5. Since by hypothesis, g has bounded derivatives of up to order 2, we have by an application of maximum principle as described in Section 4, that the solutions u^ε to the approximating equations (4.1) are such that $|Du^\varepsilon|$ and $|u_t^\varepsilon|$ are bounded from above by constants

which are independent of ε . Therefore, in the case $n = 2$, we see that, because of uniform bounds on the space and time derivatives, at each time level t the solutions u^ε to the approximating equations (4.1) solve a uniformly elliptic linear PDE in non-divergence form with right-hand side uniformly bounded in ε . Therefore, from Theorem 12.4 in [GT] (see also [T]), it follows that $Du^\varepsilon(\cdot, t)$ has uniform Hölder bounds independent of ε with an exponent α which only depends on p (note that Theorem 12.4 is very specific to $n = 2$ because in higher dimensions the Krylov-Safonov estimates only ensure uniform C^α , but not $C^{1,\alpha}$ bounds). Consequently, $Du(\cdot, t)$ is Hölder continuous in x and the conclusion follows. \square

Now we turn our attention to the proof of Theorem 1.6 which is similar to that for the elliptic case in [CGS].

Proof of Theorem 1.6. Via an approximation argument as used before in the proof of Theorem 1.1, we can assume that the initial datum g has bounded derivatives of sufficiently high order. Let u^ε be the solution to (4.1) corresponding to initial datum g . We consider the function

$$\psi_\varepsilon(s) = u^\varepsilon(x_1 + s\omega, t_0) - u^\varepsilon(x_1, t_0)$$

for some $x_1 \in \mathbb{R}^n$ where ω is some unit direction. The point x_1 is going to be chosen appropriately later. From the definition, we have that $\psi_\varepsilon(0) = 0$ and

$$|\psi'_\varepsilon(s)| \leq |Du^\varepsilon(x_1 + s\omega, t_0)|.$$

We now define the function

$$\xi^\varepsilon(s) = 2s\phi'_\varepsilon - \phi_\varepsilon + \frac{2}{p}\varepsilon^2.$$

For δ small enough, let

$$G_\varepsilon = \xi^\varepsilon - \delta(\varepsilon^2 + s)^{p/2}.$$

Clearly, $G_\varepsilon(0) = -\delta\varepsilon^p$ and by the ellipticity it is easily seen that $G'_\varepsilon \geq 0$. This implies that

$$(5.38) \quad G_\varepsilon(s) \geq -\delta\varepsilon^p$$

Therefore from (5.38) and the definition of G_ε , given any $\gamma > 0$, for small enough ε ,

$$|Du^\varepsilon(x_1 + s\omega, t_0)|^p \leq C\xi^\varepsilon(|Du^\varepsilon(x_1 + s\omega, t_0)|^2) + \delta\varepsilon^p + \gamma.$$

By applying Theorem 5.1, we thus obtain

$$(5.39) \quad |\psi'_\varepsilon(s)|^p \leq C(F(u^\varepsilon(x_1 + s\omega, t_0)) + k(\varepsilon) + \gamma),$$

where $k(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Repeating the arguments in the proof of Theorem 1.1, and finally letting $\gamma \rightarrow 0$, we obtain

$$(5.40) \quad |\psi'(s)|^p \leq CF(u(x_1 + s\omega, t_0)) \quad \text{a.e. in } s$$

where

$$\psi(s) = u(x_1 + s\omega, t_0) - u(x_1, t_0).$$

Now suppose that $F(u(x_0, t_0)) = 0$, and let $u_0 = u(x_0, t_0)$. Indicating with Π_x the projection onto the x -component, consider the set $V = \Pi_x(u^{-1}(u_0) \cap \mathbb{R}^n \times \{t_0\})$, and let $x_1 \in V$. Clearly, V is closed. Since $F \geq 0$ and $F(u_0) = F(u(x_1, t_0)) = 0$, we have that

$$(5.41) \quad F(u - u_0) = O((u - u_0)^2)$$

Hence for s small enough,

$$F(u(x_1 + s\omega, t_0)) \leq K|\psi(s)|^2.$$

Therefore from (5.40), we have for all such s in a small enough interval which does not depend on ω ,

$$|\psi'(s)| \leq C|\psi(s)| \quad \text{a.e.}$$

This implies $\psi = 0$ in that same interval. Since ω is arbitrary, this implies that V is open and hence equals the whole of \mathbb{R}^n . The desired conclusion thus follows. \square

Remark 5.3. *We would like the reader to note that the reason for which we employ the regularization scheme u^ε 's which are solutions to (4.1) in the proof of Theorem 1.6 as an intermediate step is because we can only assert that the corresponding gradient estimate (1.7) for u holds a.e. in \mathbb{R}^n . Therefore, it does not follow from Theorem 1.1 that the gradient estimate holds on the 1- dimensional line $\{x + s\omega \mid s \in \mathbb{R}\}$, which is of Lebesgue measure zero.*

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