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NONDEGENERACY IN THE PARABOLIC OBSTACLE PROBLEM WITH A DEGENERATE FORCE TERM

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Abstract. In this paper we prove the optimal nondegeneracy inequality for the solution of the parabolic variational inequality

$$\Delta u - \partial_t u = f\chi_{\{u>0\}} \text{ in } D \times (0, T],$$

where the main assumption on the force term $f(x)$ is the lower bound $f(x) \geq \lambda|x'|^\alpha$ where $\alpha > 0$, $\lambda > 0$, $1 \leq p \leq n$ is an integer, $x' = (x_1, \cdots, x_p)$ and $x = (x_1, \cdots, x_n)$. We show that there exists a $C > 0$ such that if $u(x^0, t^0) = 0$ and $Q_r^-(x^0, t^0) \subset\subset D \times (0, T]$ then

$$\sup_{\Omega \cap \partial_P Q_r^-(x^0,t^0)} u \geq C\lambda r^2\big(|(x^0)'|^\alpha + r^\alpha\big)$$

where $Q_r^-(x, t) = B_r(x) \times (t - r^2, t]$ and $\partial_P Q_r^-(x, t) = \big(\partial B_r(x) \times [t - r^2, t]\big) \cup \big(B_r(x) \times \{t - r^2\}\big)$. Using the nondegeneracy result we investigate the asymptotic behavior of the noncoincidence set as time grows.

1. Introduction

In [6] the optimal nondegeneracy of the solution of an elliptic obstacle problem with a degenerate force term has been established. This result plays an important role in the study of the regularity of the free boundary in such problems, cf. [7]. In this paper our aim is to extend this optimal nondegeneracy result to the case of parabolic obstacle problem.

Let $n \geq 1$ be an integer and $D \subset \mathbb{R}^n$ a bounded domain. Let $f \in L^\infty(D)$ such that $f \geq 0$ a.e. in D. Let $u_0 \in L^2(D)$ such that $u_0 \geq 0$ a.e. in D. Let $g \in L^\infty(0, T, W^{2,\infty}(D))$, $\partial_t g \in L^\infty(D \times (0, T))$ and $0 \leq g \leq M$ a.e. in D for some $M > 0$.

As described in Section 3, there exists a unique $u \in g + L^2(0, T, H_0^1(D))$ such that $\partial_t u \in L^2(0, T, H^{-1}(D))$, $u(0) = u_0$, $u \geq 0$ a.e. in $D \times (0, T]$ and

$$\langle \partial_t u, v - u\rangle + \int_D \nabla u \cdot \nabla(v - u)dx + \int_D f(x)(v - u)dx \geq 0$$

for a.e. $0 < t < T$ where $v \in g + L^2(0, T, H_0^1(D))$ and $v \geq 0$ a.e. in $D \times (0, T]$. Also we have that $u \in C(D \times (0, T])$.

Let us denote by Ω the noncoincidence set

$$\Omega = \Big\{(x, t) \in D \times (0, T] \;\Big|\; u(x, t) > 0\Big\}.$$

In the case when the force term is bounded away from zero by a positive constant, i.e. $f \geq \lambda$ for some constant $\lambda > 0$ it is known that one has optimal quadratic nondegeneracy of the solution, cf. [1]. As for the elliptic case, in this paper by requiring f to grow away from its zeros as a power function, we still obtain the appropriate optimal nondegeneracy estimate.

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Let p be an integer such that $1 \leq p \leq n$, $\alpha > 0$ and $\lambda > 0$ be positive real numbers. For $x \in \mathbb{R}^n$ we have $x = (x_1, \cdots, x_n)$ and we denote $x' = (x_1, \cdots, x_p)$. We also define the parabolic lower half-cylinder $Q_r^-(x,t) = B_r(x) \times (t - r^2, t]$ and its parabolic boundary $\partial_P Q_r^-(x,t) = \big(\partial B_r(x) \times [t - r^2, t]\big) \cup \big(B_r(x) \times \{t - r^2\}\big)$.

Our main result is the following theorem.

Theorem 1 (Optimal Nondegeneracy)**.** *There exists a $C > 0$ such that if $f \geq \lambda |x'|^\alpha$ a.e. in D, u is a solution in $D \times (0,T]$, $(x^0,t^0) \in \Omega$ and $Q_r^-(x^0,t^0) \subset\subset D \times (0,T]$ then*

$$\sup_{\Omega \cap \partial_P Q_r^-(x^0,t^0)} u \geq u(x^0,t^0) + C\lambda r^2 \big(|(x^0)'|^\alpha + r^\alpha\big).$$

The proof is based on the construction of appropriate comparison functions using homogeneous harmonic and caloric polynomials.

This paper is structured as follows. In Section 2, the main notations used in this paper have been enlisted. In Section 3, the known existence, uniqueness and regularity of the solution of the parabolic obstacle problem studied in this paper has been briefly described. In Section 4, the nondegeneracy result is proved. In Section 5, in the case when the boundary data g is independent of t, as an application of the nondegeneracy result we investigate the asymptotic behavior of the set $A_\epsilon(t) \cup \Omega$ as t grows where $\epsilon > 0$,

$$A_\epsilon(t) = (D_\epsilon \times (t,T])^c \tag{1.1}$$

and for $\eta > 0$

$$D_\eta = \Big\{ x \in D \;\Big|\; \operatorname{dist}(x, D^c) > \eta \Big\}. \tag{1.2}$$

2. Notation

C, C_1, C_2	generic constants;
H	$\Delta - \partial_t$ (the heat operator);
$B_r(x)$, B	$\big\{ y \in \mathbb{R}^n \mid \lvert y - x\rvert < r \big\}$, $B_1(0)$;
$Q_r(x,t)$	$B_r(x) \times (t - r^2, t + r^2)$ parabolic cylinder;
$Q_r^-(x,t)$	$B_r(x) \times (t - r^2, t]$ parabolic lower half-cylinder;
$\partial_P Q_r^-(x,t)$	$\big(\partial B_r(x) \times [t - r^2, t]\big) \cup \big(B_r(x) \times \{t - r^2\}\big)$;
$d_P^-((x,t), A)$	$\sup\big\{ r \geq 0 \mid Q_r^-(x,t) \subset A^c \big\}$;
f^+, f^-	$\max(0, f)$, $\max(0, -f)$;
$\subset\subset$	compactly contained;

3. Existence, Uniqueness and Regularity

In this section we review the known results about the existence, uniqueness and regularity of solutions to our parabolic variational inequality.

3.1. **Existence and uniqueness.** Let $\epsilon > 0$ and $\chi_\epsilon \in C^\infty(\mathbb{R})$ such that $\chi_\epsilon(z) = 0$ for $z \leq -\epsilon$, $0 \leq \chi_\epsilon(z) \leq 1$ for $-\epsilon < z < 0$ and $\chi_\epsilon(z) = 1$ for $z \geq 0$.

Using that χ_ϵ is Lipschitz continuous and uniformly bounded, by Galerkin method (cf. [2]) together with the Aubin compactness theorem [5] we might establish the existence of a unique v_ϵ such that

$$v_\epsilon \in C([0,T], L^2(D)) \cap L^2(0,T,H_0^1(D)),\ \partial_t v_\epsilon \in L^2(0,T,H^{-1}(D)),$$

$v_\epsilon(0) = u_0 - g(0)$ and

$$Hv_\epsilon = -Hg + f\chi_\epsilon(v_\epsilon + g) \ \text{ in } \ H^{-1}(D) \ \text{ for a.e. } \ t \in (0,T). \tag{3.1}$$

By testing with the function $(v_\epsilon + g + \epsilon)^- \chi_{0<t<s}$ where $0 < s < T$ and using that $\chi_\epsilon(z) = 0$ for $z \leq -\epsilon$ we obtain that

$$v_\epsilon \geq -g - \epsilon \ \text{ a.e. in } \ D \times (0,T). \tag{3.2}$$

Let $w \in L^2(0,T,H_0^1(D))$ such that $w+g \geq 0$ a.e. in $D \times (0,T]$. Using $w+g \geq 0$, $f \geq 0$, $\chi_\epsilon \leq 1$ and $\chi_\epsilon(z) = 1$ for $z \geq 0$ we might show that

$$-\int_D f(x)\chi_\epsilon(v_\epsilon + g)(w - v_\epsilon)dx \geq -\int_D f(x)(w - v_\epsilon)dx. \tag{3.3}$$

Testing the equation (3.1) by $w - v_\epsilon$ and using (3.3) we obtain

$$\langle Hv_\epsilon + Hg, w - v_\epsilon\rangle \leq \langle f, w - v_\epsilon\rangle\,, \tag{3.4}$$

here $\langle \cdot,\cdot\rangle$ stands for the duality pairing between $L^2(0,T,H^{-1}(D))$ and $L^2(0,T,H_0^1(D))$.

Also from (3.1) it follows that for $\varphi \in L^2(0,T,H_0^1(D))$ such that $\varphi \geq 0$ a.e. in $D \times (0,T]$ we have

$$0 \leq \langle Hv_\epsilon + Hg, \varphi\rangle\,. \tag{3.5}$$

Using (3.1) it is easy to see that we have uniform in ϵ bounds for v_ϵ in $L^2(0,T,H_0^1(D))$ and for $\partial_t v_\epsilon$ in $L^2(0,T,H^{-1}(D))$. Also again by Aubin compactness theorem we have that $\{v_\epsilon \mid \epsilon > 0\}$ is a precompact set in $L^2(D \times (0,T))$. It follows that there exists a sequence $\epsilon_k \to 0$ and $v \in L^2(0,T,H_0^1(D))$ with $\partial_t v \in L^2(0,T,H^{-1}(D))$ such that $v_{\epsilon_k} \to v$ weakly in $L^2(0,T,H_0^1(D))$, strongly in $L^2(0,T,L^2(D))$ and $\partial_t v_{\epsilon_k} \to \partial_t v$ weakly in $L^2(0,T,H^{-1}(D))$.

We might now pass to the limit in (3.4) and (3.5) and obtain that for $w \in L^2(0,T,H_0^1(D))$ such that $w+g \geq 0$ a.e. in $D \times (0,T]$ we have

$$\langle Hv + Hg, w - v\rangle \leq \langle f, w - v\rangle \tag{3.6}$$

and for $\varphi \in L^2(0,T,H_0^1(D))$ such that $\varphi \geq 0$ a.e. in $D \times (0,T]$ we have

$$0 \leq \langle Hv + Hg, \varphi\rangle\,. \tag{3.7}$$

From (3.2) it follows that

$$v + g \geq 0 \ \text{ a.e. in } \ D \times (0,T). \tag{3.8}$$

Taking $\varphi \in L^2(0,T,H_0^1(D))$ such that $\varphi \geq 0$ a.e. in $D \times (0,T]$ and defining $w = v + \varphi$, we have that $w + g \geq 0$ a.e. in $D \times (0,T]$, thus from (3.6) we obtain

$$\langle Hv + Hg, \varphi\rangle \leq \langle f, \varphi\rangle\,. \tag{3.9}$$

From (3.7) and (3.9) we obtain

$$0 \leq Hv + Hg \leq f \ \text{ as distributions in } \ D \ \text{ for a.e. } \ t \in (0,T). \tag{3.10}$$

By the inequality $0 \leq Hv + Hg = H(v+g)$ and the bound $g \leq M$ we obtain that $v + g \leq M$ a.e. in $D \times (0,T]$.

Let $0 < \delta < T$. Then by interior L^p regularity estimates (cf. [3, p. 175]) for the heat equation and the inequalities in (3.10) it follows that for every $p > 2$ we have

$$\begin{aligned} \|\partial_t v\|_{L^p(D_\delta\times(\delta,T))} &+ \|v\|_{L^p(\delta,T,W^{2,p}(D_\delta))} \\ &\leq C_1\big(\|v\|_{L^p(D\times(0,T))} + \|Hv\|_{L^p(D\times(0,T))}\big) \\ &\leq C_2\big(\|v\|_{L^\infty(D\times(0,T))} + \|Hv\|_{L^\infty(D\times(0,T))}\big) \\ &\leq C_3\big(\|v\|_{L^\infty(D\times(0,T))} + \|Hg\|_{L^\infty(D\times(0,T))} + \|f\|_{L^\infty(D)}\big). \end{aligned} \tag{3.11}$$

Using the estimates in (3.11) one may show that $v \in C([2\delta,T],W^{1,p}(D_{2\delta}))$. In particular if $p > n$ then by Sobolev embeddings $v \in C([2\delta,T],C^{0,\alpha}(D_{2\delta}))$ for some $0 < \alpha < 1$ and thus $v \in C(D_{2\delta} \times [2\delta,T])$. Because $0 < \delta < T$ is arbitrary we obtain that $v \in C(D \times (0,T])$ with uniform continuity in $\overline{D_\delta} \times [\delta,T]$ for small enough $\delta > 0$.

Now let us denote $u = v + g$. Considering $\varphi \in C_c^\infty(\Omega)$ such that $u \geq \varphi \geq 0$ and taking $w = v - \varphi$ in (3.6), together with (3.9) we obtain $\langle Hv + Hg, \varphi\rangle = \langle f, \varphi\rangle$. Thus we have

$$Hu = f \ \text{ in } \ \Omega.$$

Now we have a well established solution of our parabolic variational inequality and also we have

$$f\chi_\Omega \le Hu \le f \text{ in } D\times(0,T].$$

3.2. **Regularity.** Let $0\le f\le\Lambda$ and $u\le M$ in $D\times(0,T]$.

Using (3.10) by a similar method as the optimal regularity estimate (cf. [4]) for elliptic obstacle problem there exists a $C>0$ (depending only on n) such that if $Q_r^-(x^0,t^0)\subset D\times(0,T]$ and $u(x^0,t^0)=0$ then

$$u(x,t)\le C\Lambda r^2 \text{ for } (x,t)\in Q^-_{\frac{r}{2}}(x^0,t^0). \tag{3.12}$$

As demonstrated in [1] using the quadratic growth (3.12) there exists a $C>0$ (depending only on n) such that if $Q_R^-(x^0,t^0)\subset D\times(0,T]$ and $u(x^0,t^0)=0$ then for $0<r<R$ we have

$$u(x,t)\le C\max\big(\Lambda,\frac{M}{R^2}\big)r^2 \text{ for } (x,t)\in Q_r(x^0,t^0)\cap(D\times(0,T]). \tag{3.13}$$

Using (3.13) one may show that there exists a $C>0$ (depending only on n) such that if $(x^1,t^1)\in\Omega$, $d=d_P^-\big((x^1,t^1),(D\times(0,T])\backslash\Omega\big)$, $\epsilon>0$, $Q_\epsilon^-(x^1,t^1)\subset D\times(0,T]$ and $3d<\epsilon$ then

$$u(x,t)\le C\max\big(\Lambda,\frac{M}{\epsilon^2}\big)d^2 \text{ for } (x,t)\in Q_d^-(x^1,t^1). \tag{3.14}$$

By (3.14) and interior regularity estimates for caloric functions one may show that there exists a $C>0$ (depending only on n) such that if there exists $\psi\in C^{1,1}(D)$ such that $\Delta\psi=f$ then $u\in C^{1,1}_{x,loc}(D\times(0,T])\cap C^{0,1}_{t,loc}(D\times(0,T])$ and for $0<\epsilon<1$ the following estimate holds

$$\|u\|_{C_t^{0,1}(D_\epsilon\times(\epsilon^2,T])}+\|u\|_{C_x^{1,1}(D_\epsilon\times(\epsilon^2,T])}\le C\max\big(\|\psi\|_{C^{1,1}(D)},\frac{M}{\epsilon^2}\big). \tag{3.15}$$

4. Optimal Nondegeneracy

In [6] we have defined

$$\psi(y)=\frac{1}{(\alpha+2)(\alpha+p)}|y|^{\alpha+2} \text{ for } y\in\mathbb{R}^p,$$

$$w_{y^0}(y)=\psi(y)-\psi(y^0)-\nabla\psi(y^0)\cdot(y-y^0) \text{ for } y^0,y\in\mathbb{R}^p,$$

and

$$p_{2k}(x)=\sum_{j=0}^k a_jx_1^{2j}|x|^{2k-2j} \text{ for } k\in\mathbb{N}\cup\{0\} \text{ and } x\in\mathbb{R}^n$$

where $a_0=1$ and for $1\le j\le k$ the coefficients a_j are chosen such that p_{2k} is harmonic.

In [6] we have proved that there exists $C>0$ such that for $y,y^0\in\mathbb{R}^p$ we have

$$w_{y^0}(y)\ge C|y-y^0|^2(|y^0|^\alpha+|y-y^0|^\alpha), \tag{4.1}$$

there exist $a>0$ and $C>0$ such that for x,$x^0\in\mathbb{R}^n$ we have

$$w_{(x^0)'}(x')+a|(x^0)'|^\alpha p_2(x-x^0)\ge C|(x^0)'|^\alpha|x-x^0|^2 \tag{4.2}$$

and for $k\in\mathbb{N}$ such that $2k\ge2+\alpha$ there exist $b>0$ and $C>0$ such that for $r>0$, $x^0\in\mathbb{R}^n$ and $x\in B_r(x^0)$ we have

$$w_{(x^0)'}(x')+\frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x^0)\ge C\frac{|x-x^0|^{2k}}{r^{2k-(2+\alpha)}}. \tag{4.3}$$

For $k\in\{0\}\cup\mathbb{N}$ let us define

$$\tilde{p}_{2k}(x_1,t)=\sum_{j=0}^k\tilde{a}_jx_1^{2j}t^{k-j} \text{ for } t<0 \text{ and } x_1\in\mathbb{R}$$

where $\tilde{a}_0 = 1$ and for $1 \le j \le k$

$$2j(2j-1)\tilde{a}_j - (k-j+1)\tilde{a}_{j-1} = 0.$$

By these choice of the coefficients we have that $\tilde{p}_{2k}$ is caloric.

Lemma 1. *Let $k \in \{0\} \cup \mathbb{N}$ then there exist $c > 0$ and $C > 0$ such that*

$$|x_1|^{2k} + c(-1)^k \tilde{p}_{2k}(x_1,t) \ge C\big(|x_1|^{2k} + |t|^k\big) \text{ for } t < 0 \text{ and } x_1 \in \mathbb{R}.$$

Proof. For $k = 0$ the claim is trivial. Let us assume that $k \ge 1$. We write

$$\tilde{p}_{2k}(x_1,t) = t^k + q_k(x_1,t)$$

where

$$q_k(x_1,t) = \sum_{j=1}^{k} \tilde{a}_j x_1^{2j} t^{k-j}.$$

For $x_1 = 0$ we have $q_k(0,t) = 0$ and for $x_1 \ne 0$ we estimate

$$\begin{aligned} (4.4)\quad |q_k(x_1,t)| &= \big|\sum_{j=1}^{k} \tilde{a}_j x_1^{2j} t^{k-j}\big| = |x_1|^{2k}\big|\sum_{j=1}^{k} \tilde{a}_j (\frac{t}{x_1^2})^{k-j}\big| \\ &\le |x_1|^{2k} \sum_{j=1}^{k} |\tilde{a}_j| |\frac{t}{x_1^2}|^{k-j} \le |x_1|^{2k}(C_1 + \frac{1}{2}|\frac{t}{x_1^2}|^k) = C_1|x_1|^{2k} + \frac{1}{2}|t|^k. \end{aligned}$$

Using (4.4) we estimate

$$\begin{aligned} (4.5)\quad (-1)^k \tilde{p}_{2k}(x_1,t) &= (-t)^k + (-1)^k q_k(x_1,t) \\ &= |t|^k + (-1)^k q_k(x_1,t) \ge |t|^k - |q_k(x_1,t)| \\ &\ge |t|^k - \big(C_1|x_1|^{2k} + \frac{1}{2}|t|^k\big) = \frac{1}{2}|t|^k - C_1|x_1|^{2k}. \end{aligned}$$

Now using (4.5) by choosing $\frac{1}{4C_1} \le c \le \frac{1}{2C_1}$ we have

$$\begin{aligned} |x_1|^{2k} + c(-1)^k \tilde{p}_{2k}(x_1,t) &\ge |x_1|^{2k} + c\big(\frac{1}{2}|t|^k - C_1|x_1|^{2k}\big) \\ &= (1 - cC_1)|x_1|^{2k} + \frac{c}{2}|t|^k \ge \frac{1}{2}|x_1|^{2k} + \frac{1}{8C_1}|t|^k \end{aligned}$$

and this proves the lemma. □

Lemma 2. *There exist $\tilde{a} > 0$ and $C > 0$ such that*

$$(4.6)\quad w_{(x^0)'}(x') - \tilde{a}|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0) \ge C|(x^0)'|^\alpha\big(|x_1 - x_1^0|^2 + (t^0 - t)\big)$$

for all $t < t^0$ and x,$x^0 \in \mathbb{R}^n$.

Proof. Let $C_1 > 0$ be the constant in (4.1) and $c > 0$ as in Lemma 1 for $k = 1$. We estimate

$$\begin{aligned} &w_{(x^0)'}(x') - C_1 c|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0) \\ &\quad\ge C_1|(x^0)'|^\alpha |x' - (x^0)'|^2 - C_1 c|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0) \\ &\quad= C_1|(x^0)'|^\alpha\big(|x' - (x^0)'|^2 - c\tilde{p}_2(x_1 - x_1^0, t - t^0)\big) \\ &\quad\ge C_2|(x^0)'|^\alpha\big(|x_1 - x_1^0|^2 + |t - t^0|\big) \\ &\quad= C_2|(x^0)'|^\alpha\big(|x_1 - x_1^0|^2 + (t^0 - t)\big) \end{aligned}$$

and this proves the lemma. □

Lemma 3. *Let $k \in \mathbb{N}$ be such that $2k \geq 2+\alpha$ then there exist $\tilde{b} > 0$ and $C > 0$ such that*

$$w_{(x^0)'}(x') + \frac{\tilde{b}(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0) \geq \frac{C}{r^{2k-(2+\alpha)}}\big((t^0 - t)^k + |x_1 - x_1^0|^{2k}\big) \tag{4.7}$$

for $t < t^0$, $r > 0$, $x^0 \in \mathbb{R}^n$ and $x \in B_r(x^0)$.

Proof. Because $x \in B_r(x^0)$ we have $|x_1 - x_1^0| < r$ and also because $2k \geq 2+\alpha$ we have

$$r^{2k-(2+\alpha)}|x_1 - x_1^0|^{2+\alpha} \geq |x_1 - x_1^0|^{2k}.$$

Let $C_1 > 0$ be the constant in (4.1) and $c > 0$ as in Lemma 1. We estimate

$$\begin{aligned}
w_{(x^0)'}(x') &+ \frac{C_1 c(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0) \\
&\geq C_1|x_1 - x_1^0|^{2+\alpha} + \frac{C_1 c(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0) \\
&= \frac{C_1}{r^{2k-(2+\alpha)}}\Big(r^{2k-(2+\alpha)}|x_1 - x_1^0|^{2+\alpha} + c(-1)^k\tilde{p}_{2k}(x_1 - x_1^0, t - t^0)\Big) \\
&\geq \frac{C_1}{r^{2k-(2+\alpha)}}\Big(|x_1 - x_1^0|^{2k} + c(-1)^k\tilde{p}_{2k}(x_1 - x_1^0, t - t^0)\Big) \\
&\geq \frac{C_2}{r^{2k-(2+\alpha)}}\big(|x_1 - x_1^0|^{2k} + |t - t^0|^k\big) \\
&= \frac{C_2}{r^{2k-(2+\alpha)}}\big(|x_1 - x_1^0|^{2k} + (t^0 - t)^k\big)
\end{aligned}$$

and this proves the lemma. □

Proof of Theorem 1. Let $k \in \mathbb{N}$ be such that $2k \geq \alpha + 2$. For $c, r > 0$, $0 < t < t^0$ and $x, x^0 \in \mathbb{R}^n$ let us define

$$\begin{aligned}
v(x,t) = w_{(x^0)'}(x') &+ \frac{a}{4}|(x^0)'|^\alpha p_2(x - x^0) + \frac{b}{4}\frac{1}{r^{2k-(2+\alpha)}}p_{2k}(x - x^0) \\
&- \frac{\tilde{a}}{4}|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0) + \frac{\tilde{b}}{4}\frac{(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0).
\end{aligned}$$

Then we have $Hv = |x'|^\alpha$ and $v(x^0, t^0) = 0$. Using (4.2), (4.3), (4.6) and (4.7) for $t < t^0$, $x \in B_r(x^0)$ and small enough $C > 0$ we have

$$\begin{aligned}
v(x,t) &= w_{(x^0)'}(x') + \frac{a}{4}|(x^0)'|^\alpha p_2(x - x^0) + \frac{b}{4}\frac{1}{r^{2k-(2+\alpha)}}p_{2k}(x - x^0) \\
&\quad - \frac{\tilde{a}}{4}|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0) + \frac{\tilde{b}}{4}\frac{(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0) \\
&= \frac{1}{4}\big(w_{(x^0)'}(x') + a|(x^0)'|^\alpha p_2(x - x^0)\big) \\
&\quad + \frac{1}{4}\big(w_{(x^0)'}(x') + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x - x^0)\big) \\
&\quad + \frac{1}{4}\big(w_{(x^0)'}(x') - \tilde{a}|(x^0)'|^\alpha \tilde{p}_2(x_1 - x_1^0, t - t^0)\big) \\
&\quad + \frac{1}{4}\big(w_{(x^0)'}(x') + \frac{\tilde{b}(-1)^k}{r^{2k-(2+\alpha)}}\tilde{p}_{2k}(x_1 - x_1^0, t - t^0)\big)
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\geq \frac{C}{4}|(x^0)'|^\alpha |x-x^0|^2 + \frac{C}{4}\frac{|x-x^0|^{2k}}{r^{2k-(2+\alpha)}}\\
&\quad + \frac{C}{4}|(x^0)'|^\alpha \big(|x_1-x_1^0|^2 + (t^0-t)\big)\\
&\quad + \frac{C}{4}\frac{1}{r^{2k-(2+\alpha)}}\big(|x_1-x_1^0|^{2k} + (t^0-t)^k\big)\\
&\geq \frac{C}{4}|(x^0)'|^\alpha |x-x^0|^2 + \frac{C}{4}\frac{|x-x^0|^{2k}}{r^{2k-(2+\alpha)}}\\
&\qquad + \frac{C}{4}|(x^0)'|^\alpha (t^0-t) + \frac{C}{4}\frac{1}{r^{2k-(2+\alpha)}}(t^0-t)^k.
\end{aligned}$$

From (4.8) it follows that for $(x,t)\in Q_r^-(x^0,t^0)$ we have $v(x,t)\geq 0$.

From (4.8) for $(x,t)\in \partial B_r(x^0)\times[t^0-r^2,t^0]$ we have

$$\begin{aligned}
(4.9)\quad v(x,t) &\geq \frac{C}{4}\Big(|(x^0)'|^\alpha |x-x^0|^2 + \frac{|x-x^0|^{2k}}{r^{2k-(2+\alpha)}}\Big)\\
&= \frac{C}{4}\Big(|(x^0)'|^\alpha r^2 + \frac{r^{2k}}{r^{2k-(2+\alpha)}}\Big)\\
&= \frac{C}{4}\big(|(x^0)'|^\alpha r^2 + r^{2+\alpha}\big) = \frac{C}{4}r^2\big(|(x^0)'|^\alpha + r^\alpha\big).
\end{aligned}$$

From (4.8) for $(x,t)\in B_r(x^0)\times\{t^0-r^2\}$ we have

$$\begin{aligned}
(4.10)\quad v(x,t) &\geq \frac{C}{4}\Big(|(x^0)'|^\alpha (t^0-t) + \frac{1}{r^{2k-(2+\alpha)}}(t^0-t)^k\Big)\\
&= \frac{C}{4}\Big(|(x^0)'|^\alpha r^2 + \frac{1}{r^{2k-(2+\alpha)}}r^{2k}\Big)\\
&= \frac{C}{4}\big(|(x^0)'|^\alpha r^2 + r^{2+\alpha}\big) = \frac{C}{4}r^2\big(|(x^0)'|^\alpha + r^\alpha\big).
\end{aligned}$$

From (4.9) and (4.10) for $(x,t)\in \partial_P Q_r^-(x^0,t^0)$ we have

$$v(x,t)\geq \frac{C}{4}r^2\big(|(x^0)'|^\alpha + r^\alpha\big).$$

Let us define

$$h(x,t) = u(x,t) - u(x^0,t^0) - \lambda v(x,t).$$

Then we have

$$Hh = f - \lambda |x'|^\alpha \geq 0 \text{ in } \Omega,$$
$$h(x^0,t^0) = 0$$

and for $(x,t)\in \{u=0\}\cap \overline{Q_r^-(x^0,t^0)}$

$$h(x,t) = -u(x^0,t^0) - \lambda v(x,t) \leq -u(x^0,t^0) < 0.$$

Applying the maximum principle we have

$$0 = h(x^0,t^0) \leq \sup_{\partial_P(\Omega\cap Q_r^-(x^0,t^0))} h.$$

It follows that

$$\begin{aligned}
0 \leq \sup_{\Omega\cap\partial_P Q_r^-(x^0,t^0)} h &\leq \sup_{\Omega\cap\partial_P Q_r^-(x^0,t^0)} u - u(x^0,t^0) - \lambda \inf_{\Omega\cap\partial_P Q_r^-(x^0,t^0)} v\\
&\leq \sup_{\Omega\cap\partial_P Q_r^-(x^0,t^0)} u - u(x^0,t^0) - \frac{C}{4}\lambda r^2\big(|(x^0)'|^\alpha + r^\alpha\big)
\end{aligned}$$

and this proves the theorem. □

5. Asymptotic Behaviour

In this section we assume that the boundary condition g is independent of t.

Let $u_\infty \in H^1(D)$ such that $u_\infty - g \in H^1_0(D)$ and $u_\infty \geq 0$ a.e. in D, be the solution of the elliptic obstacle problem written in the variational inequality form

$$\int_D \big(\nabla u_\infty \cdot \nabla(v - u_\infty) + f(x)(v - u_\infty)\big) dx \geq 0 \tag{5.1}$$

for all $v \in H^1(D)$ such that $v - g \in H^1_0(D)$ and $v \geq 0$ a.e. in D.

In this section we denote by $\lambda_1 > 0$ the first eigenvalue of the operator $-\Delta$ in the domain D with Dirichlet boundary condition on ∂D.

Lemma 4. *We have*

$$\|u(\cdot,t) - u_\infty\|_{L^2(D)} \leq e^{-\lambda_1 t}\|u_0 - u_\infty\|_{L^2(D)} \tag{5.2}$$

for $0 < t \leq T$.

Proof. Testing (5.1) by $u(\cdot,t)$ we obtain

$$\int_D \big(\nabla u_\infty(x) \cdot \nabla(u - u_\infty)(x,t) + f(x)(u(x,t) - u_\infty(x))\big) dx \geq 0 \tag{5.3}$$

for a.e. $t \in (0,T)$.

Taking $w = u_\infty + g$ in (3.6) we obtain

$$\begin{aligned} &\langle \partial_t u(\cdot,t), u_\infty - u(\cdot,t)\rangle_{H^{-1}(D),H^1_0(D)} \\ &\quad + \int_D \big(\nabla u(x,t) \cdot \nabla(u_\infty - u)(x,t) + f(x)(u_\infty(x) - u(x,t))\big) dx \geq 0 \end{aligned} \tag{5.4}$$

for a.e. $t \in (0,T)$.

Adding (5.3) and (5.4) together we obtain

$$\langle \partial_t u(\cdot,t), u(\cdot,t) - u_\infty\rangle_{H^{-1}(D),H^1_0(D)} + \int_D |\nabla(u - u_\infty)(x,t)|^2 dx \leq 0 \tag{5.5}$$

for a.e. $t \in (0,T)$.

By (5.5), independence of u_∞ from t and Poincaré inequality for the domain D we obtain

$$\frac{d}{dt}\|u(\cdot,t) - u_\infty\|^2_{L^2(D)} + 2\lambda_1\|u(\cdot,t) - u_\infty\|^2_{L^2(D)} \leq 0 \tag{5.6}$$

for a.e. $t \in (0,T)$.

By multiplying (5.6) by $e^{2\lambda_1 t}$ and integration in t we prove the lemma. □

Lemma 5. *There exists* $C > 0$ *such that for* $v \in C^{1,1}(B)$ *we have*

$$\|v\|_{C(B_{\frac{1}{2}})} \leq C\|v\|^{\frac{n}{n+4}}_{C^{1,1}(B)}\|v\|^{\frac{4}{n+4}}_{L^2(B)}. \tag{5.7}$$

Proof. There exists $C > 0$ such that for $x \in B_{\frac{1}{2}}$, $0 < r < 1$ and closed half-space V we have

$$Cr^n \leq \big|(x + V) \cap B_r(x) \cap B\big|. \tag{5.8}$$

Let us denote $M = [v]_{C^{1,1}(B)}$ then we have for $x, y \in B$

$$\big|v(y) - v(x) - \nabla v(x) \cdot (y - x)\big| \leq M|y - x|^2. \tag{5.9}$$

Let us assume that $v(x) > 0$ and let us denote

$$V_x = \Big\{z \in \mathbb{R}^n \ \Big|\ z \cdot \nabla v(x) \geq 0\Big\}.$$

For $y \in (x + V_x) \cap B$ because $(y - x) \cdot \nabla v(x) \geq 0$ using (5.9) we have

$$v(x) - v(y) \leq v(x) + \nabla v(x) \cdot (y - x) - v(y) \leq \big| v(x) + \nabla v(x) \cdot (y - x) - v(y) \big| \leq M|y - x|^2.$$

Thus for $x \in B$ and $y \in (x + V_x) \cap B$ we have

$$v(x) - M|y - x|^2 \leq v(y). \tag{5.10}$$

Let us denote

$$r_x = \min\big(1, (\frac{1}{2M} v(x))^{\frac{1}{2}}\big). \tag{5.11}$$

Then for $y \in B_{r_x}(x) \cap B$ we have

$$\frac{1}{2} v(x) \leq v(x) - M|y - x|^2. \tag{5.12}$$

From (5.10) and (5.12) it follows that for $x \in B$ and $y \in (x + V_x) \cap B_{r_x}(x) \cap B$ we have

$$\frac{1}{2} v(x) \leq v(y).$$

For $x \in B$ integrating for $y \in (x + V_x) \cap B_{r_x}(x) \cap B$ we have

$$\big| (x + V_x) \cap B_{r_x}(x) \cap B \big| (v(x))^2 \leq 4 \int_{(x+V_x)\cap B_{r_x}(x) \cap B} (v(y))^2 dy. \tag{5.13}$$

By (5.8) and (5.13) for $x \in B_{\frac{1}{2}}$ we have

$$C r_x^n (v(x))^2 \leq 4 \|v\|_{L^2(B)}^2. \tag{5.14}$$

By (5.11) and (5.14) we have

$$C\Big(\min\big(1, (\frac{1}{2M} v(x))^{\frac{1}{2}}\big)\Big)^n (v(x))^2 \leq 4 \|v\|_{L^2(B)}^2.$$

It follows that either

$$v(x) \leq C_1 \|v\|_{L^2(B)}$$

or

$$v(x) \leq C_1 M^{\frac{n}{4+n}} \|v\|_{L^2(B)}^{\frac{4}{4+n}}.$$

Thus we have

$$\begin{aligned} |v(x)| &\leq C_1 \max\big(\|v\|_{L^2(B)}, M^{\frac{n}{4+n}} \|v\|_{L^2(B)}^{\frac{4}{4+n}}\big) \\ &= C_1 \max\big(\|v\|_{L^2(B)}^{\frac{n}{4+n}}, M^{\frac{n}{4+n}}\big) \|v\|_{L^2(B)}^{\frac{4}{4+n}} \leq C_2 \|v\|_{C^{1,1}(B)}^{\frac{n}{4+n}} \|v\|_{L^2(B)}^{\frac{4}{4+n}}. \end{aligned} \tag{5.15}$$

In the case $v(x) = 0$ the inequality (5.15) holds trivially and in the case $v(x) < 0$ by considering $-v$ instead of v the inequality (5.15) follows.

From (5.15) by considering supremum for $x \in B_{\frac{1}{2}}$ we obtain (5.7) and this proves the lemma. □

Corollary 1. *There exists $C > 0$ such that for $\epsilon > 0$ and $v \in C^{1,1}(D)$ we have*

$$\|v\|_{C(D_\epsilon)} \leq \frac{C}{\epsilon^{\frac{2n}{n+4}}} \|v\|_{C^{1,1}(D)}^{\frac{n}{n+4}} \|v\|_{L^2(D)}^{\frac{4}{n+4}}$$

where D_ϵ is defined in (1.2).

Proof. Let $0 < r < 1$ and $v \in C^{1,1}(B_r)$. Let us define $v_r(x) = v(rx)$ for $x \in B$. Using (5.7) we compute

$$
\begin{aligned}
(5.16)\quad \|v\|_{C(B_{\frac{r}{2}})} = \|v_r\|_{C(B_{\frac{1}{2}})} &\le C\|v_r\|_{C^{1,1}(B)}^{\frac{n}{n+4}}\|v_r\|_{L^2(B)}^{\frac{4}{n+4}} \\
&\le C_1\big(C_2\max(1,r^2)\|v\|_{C^{1,1}(B_r)}\big)^{\frac{n}{n+4}}\big(\frac{1}{r^{\frac{n}{2}}}\|v\|_{L^2(B_r)}\big)^{\frac{4}{n+4}} \\
&\le \frac{C}{r^{\frac{2n}{n+4}}}\|v\|_{C^{1,1}(B_r)}^{\frac{n}{n+4}}\|v\|_{L^2(B_r)}^{\frac{4}{n+4}}.
\end{aligned}
$$

Now let $v \in C^{1,1}(D)$ and $x \in D_\epsilon$. Then we have $B_\epsilon(x) \subset D$ and using (5.16) we compute

$$
\begin{aligned}
|v(x)| \le \|v\|_{C(B_{\frac{\epsilon}{2}}(x))} &\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}\|v\|_{C^{1,1}(B_\epsilon(x))}^{\frac{n}{n+4}}\|v\|_{L^2(B_\epsilon(x))}^{\frac{4}{n+4}} \\
&\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}\|v\|_{C^{1,1}(D)}^{\frac{n}{n+4}}\|v\|_{L^2(D)}^{\frac{4}{n+4}}
\end{aligned}
$$

and this completes the proof of the corollary. □

Lemma 6. *For $\epsilon > 0$, $M > 0$ and $\Lambda > 0$ there exists $A > 0$ such that if u is a solution in $D \times (0, T]$, $0 \le f \le \Lambda$ and $u \le M$ in $D \times (0, T]$ then for $\epsilon^2 < t \le T$ we have*

$$
\|u(\cdot,t) - u_\infty\|_{C(D_{2\epsilon})} \le A\|u_0 - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} e^{-\frac{4}{n+4}\lambda_1 t}.
$$

Proof. By the known regularity result outlined in Subsection 3.2 there exists $A_1 > 0$ depending on ϵ, M and Λ such that $\|u_\infty\|_{C_x^{1,1}(D_\epsilon)} \le A_1$ and

$$
\|u(\cdot,t)\|_{C_x^{1,1}(D_\epsilon)} \le A_1 \ \text{ for } \ \epsilon^2 < t \le T.
$$

For $\epsilon^2 < t \le T$ using Corollary 1 and Lemma 4 we compute

$$
\begin{aligned}
\|u(\cdot,t) - u_\infty\|_{C(D_{2\epsilon})} &\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}\|u(\cdot,t) - u_\infty\|_{C_x^{1,1}(D_\epsilon)}^{\frac{n}{n+4}}\|u(\cdot,t) - u_\infty\|_{L^2(D_\epsilon)}^{\frac{4}{n+4}} \\
&\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}\big(\|u(\cdot,t)\|_{C_x^{1,1}(D_\epsilon)} + \|u_\infty\|_{C_x^{1,1}(D_\epsilon)}\big)^{\frac{n}{n+4}}\|u(\cdot,t) - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} \\
&\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}(2A_1)^{\frac{n}{n+4}}\|u(\cdot,t) - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} \\
&\le \frac{C}{\epsilon^{\frac{2n}{n+4}}}(2A_1)^{\frac{n}{n+4}}e^{-\frac{4}{n+4}\lambda_1 t}\|u_0 - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} \\
&= A\|u_0 - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}}e^{-\frac{4}{n+4}\lambda_1 t}
\end{aligned}
$$

and this completes the proof of the lemma. □

Lemma 7. *There exists $C > 0$ such that if $\lambda > 0$, $f \ge \lambda|x'|^\alpha$, u_1 and u_2 are solutions in $D \times (0, T]$, $\epsilon > 0$ and*

$$
\|u_2 - u_1\|_{C(D\times(0,T])} < C\lambda\epsilon^{2+\alpha}
$$

then

$$
\Big\{(x,t) \in D_\epsilon \times (\epsilon^2, T] \;\Big|\; C\lambda(d_P^-((x,t),\{u_1 > 0\}))^2\big((d_P^-((x,t),\{u_1 > 0\}))^\alpha + |x'|^\alpha\big) > \|u_2 - u_1\|_{C(D\times(0,T])}\Big\} \subset \{u_2 = 0\}.
$$

Proof. By Theorem 1 there exists $C > 0$ such that if $(x,t) \in D \times (0,T]$, $u_2(x,t) > 0$ and $Q_r^-(x,t) \subset\subset D \times (0,T]$ then we have

$$\sup_{\Omega_{u_2} \cap \partial_P Q_r^-(x,t)} u_2 \geq u_2(x,t) + C\lambda r^2 \big(r^\alpha + |x'|^\alpha\big).$$

Thus if $(x,t) \in D \times (0,T]$, $u_2(x,t) > 0$, $Q_r^-(x,t) \subset\subset \{u_1 = 0\}$ and $C\lambda r^2(r^\alpha + |x'|^\alpha) \geq \|u_2 - u_1\|_{C(D\times(0,T])}$ then we have

$$\begin{aligned} 0 = \sup_{\Omega_{u_2} \cap \partial_P Q_r^-(x,t)} u_1 &= \sup_{\Omega_{u_2} \cap \partial_P Q_r^-(x,t)} (u_2 - (u_2 - u_1)) \\ &\geq \sup_{\Omega_{u_2} \cap \partial_P Q_r^-(x,t)} u_2 - \|u_2 - u_1\|_{C(D\times(0,T])} \\ &\geq u_2(x,t) + C\lambda r^2 \big(r^\alpha + |x'|^\alpha\big) - \|u_2 - u_1\|_{C(D\times(0,T])} \\ &> C\lambda r^2 \big(r^\alpha + |x'|^\alpha\big) - \|u_2 - u_1\|_{C(D\times(0,T])} \end{aligned}$$

a contradiction.

Thus if $(x,t) \in D \times (0,T]$, $Q_r^-(x,t) \subset\subset \{u_1 = 0\}$ and $C\lambda r^2(r^\alpha + |x'|^\alpha) \geq \|u_2 - u_1\|_{C(D\times(0,T])}$ then $u_2(x,t) = 0$.

For $(x,t) \in D \times (0,T]$ set

$$d = d_P^-\big((x,t), \{u_1 = 0\}^c\big).$$

Let $r = \frac{1}{2}d$ then it follows that if

$$\frac{C\lambda}{4} d^2 \big(\frac{1}{2^\alpha} d^\alpha + |x'|^\alpha\big) > \|u_2 - u_1\|_{C(D\times(0,T])}$$

then $u_2(x,t) = 0$.

This proves that

$$\Big\{(x,t) \in D \times (0,T] \;\Big|\; \frac{C\lambda}{2^{2+\alpha}} d^2 \big(d^\alpha + |x'|^\alpha\big) > \|u_2 - u_1\|_{C(D\times(0,T])}\Big\} \subset \big\{u_2 = 0\big\}. \tag{5.17}$$

Let $(x,t) \in D_\epsilon \times (\epsilon^2, T]$ then we compute

$$\begin{aligned} d = d_P^-\big((x,t), \{u_1 = 0\}^c\big) &= d_P^-\big((x,t), \{u_1 > 0\} \cup (D \times (0,T])^c\big) \\ &= \min\big(d_P^-((x,t), \{u_1 > 0\}), d_P^-((x,t), (D \times (0,T])^c)\big) \\ &\geq \min\big(d_P^-((x,t), \{u_1 > 0\}), \epsilon\big) \end{aligned}$$

so we have

$$\begin{aligned} d^2 \big(d^\alpha + |x'|^\alpha\big) \\ \geq \min\Big((d_P^-((x,t), \{u_1 > 0\}))^2 \big((d_P^-((x,t), \{u_1 > 0\}))^\alpha + |x'|^\alpha\big), \epsilon^{2+\alpha}\Big). \end{aligned} \tag{5.18}$$

So by (5.17) and (5.18), if

$$\|u_2 - u_1\|_{C(D\times(0,T])} < \frac{C\lambda}{2^{2+\alpha}} \epsilon^{2+\alpha}$$

then

$$\begin{aligned} \Big\{(x,t) \in D_\epsilon \times (\epsilon^2, T] \;\Big|\; \\ \frac{C\lambda}{2^{\alpha+2}} (d_P^-((x,t), \{u_1 > 0\}))^2 \big((d_P^-((x,t), \{u_1 > 0\}))^\alpha + |x'|^\alpha\big) \\ > \|u_2 - u_1\|_{C(D\times(0,T])}\Big\} \subset \{u_2 = 0\}. \end{aligned}$$

and this finishes the proof of the lemma. ☐

Definition 1 (Parabolic Hausdorff Distance)**.** *For $A, B \subset \mathbb{R}^{n+1}$ we define*

$$\mathrm{dist}_P^-(A,B) = \inf\Big\{ r \geq 0 \ \Big| \ A \subset B^r \ \text{and} \ B \subset A^r \Big\}$$

where

$$A^r = \Big\{ (x,t) \in \mathbb{R}^{n+1} \ \Big| \ d_P^-\big((x,t),A\big) \leq r \Big\}$$

and B^r is defined similarly.

Lemma 8. *There exists $C > 0$ such that if $\lambda > 0$, $f \geq \lambda |x'|^\alpha$, u_1 and u_2 are solutions in $D \times (0,T]$, $\epsilon > 0$ and*

$$\|u_2 - u_1\|_{C(D \times (0,T])} < C\lambda \epsilon^{2+\alpha} \tag{5.19}$$

then

$$\mathrm{dist}_P^-\big(A_\epsilon \cup \{u_1 > 0\}, A_\epsilon \cup \{u_2 > 0\}\big) \leq \Big(\frac{1}{C\lambda}\|u_2 - u_1\|_{C(D\times(0,T])}\Big)^{\frac{1}{2+\alpha}}$$

where $A_\epsilon = (D_\epsilon \times (\epsilon^2, T])^c$.

Proof. Let $C > 0$ be as in Lemma 7. Let us define

$$r_0 = \Big(\frac{1}{C\lambda}\|u_2 - u_1\|_{C(D\times(0,T])}\Big)^{\frac{1}{2+\alpha}}.$$

Because (5.19) holds, from Lemma 7 we have that

$$\Big\{ (x,t) \in A_\epsilon^c \ \Big| \ d_P^-\big((x,t),\{u_1 > 0\}\big) > r_0 \Big\} \subset \{u_2 = 0\}.$$

Considering the complement of both sides we obtain

$$\begin{aligned} A_\epsilon \cup \{u_2 > 0\} &\subset A_\epsilon \cup \{u_2 = 0\}^c \\ &\subset A_\epsilon \cup \Big\{ (x,t) \in A_\epsilon^c \ \Big| \ d_P^-\big((x,t),\{u_1>0\}\big) > r_0 \Big\}^c \\ &= A_\epsilon \cup \Big\{ (x,t) \in A_\epsilon^c \ \Big| \ d_P^-\big((x,t),\{u_1>0\}\big) \leq r_0 \Big\} \\ &\subset A_\epsilon \cup \Big\{ (x,t) \in A_\epsilon^c \ \Big| \ d_P^-\big((x,t),A_\epsilon \cup \{u_1>0\}\big) \leq r_0 \Big\} \\ &= \Big\{ (x,t) \in \mathbb{R}^{n+1} \ \Big| \ d_P^-\big((x,t),A_\epsilon \cup \{u_1>0\}\big) \leq r_0 \Big\} \\ &= \big(A_\epsilon \cup \{u_1 > 0\}\big)^{r_0}. \end{aligned}$$

Similar inclusion holds with place of u_1 and u_2 exchanged.
From the definition of dist_P^- it follows that

$$\mathrm{dist}_P^-\big(A_\epsilon \cup \{u_1 > 0\}, A_\epsilon \cup \{u_2 > 0\}\big) \leq r_0$$

and this finishes the proof of the lemma. □

Theorem 2. *Let $\epsilon > 0$, $M > 0$, $\lambda > 0$ and $\Lambda > 0$. There exists a $\tau_0 > 0$ and $C > 0$ (both depending on D, n, ϵ, M, λ, Λ, α, u_0 and u_∞) such that if $\tau_0 < \tau \leq T$, $\lambda|x'|^\alpha \leq f \leq \Lambda$, u is a solution in $D \times (0,T]$ and $u \leq M$ in $D \times (0,T]$ then*

$$\mathrm{dist}_P^-\big(A_\epsilon(\tau) \cup \{u > 0\}, A_\epsilon(\tau) \cup \{u_\infty > 0\}\big) \leq C e^{-\frac{4}{n+4}\frac{1}{2+\alpha}\lambda_1 \tau}$$

where $A_\epsilon(\tau)$ is defined in (1.1).

Proof. By Lemma 6 if $2(\frac{\epsilon}{3})^2 < \tau \leq T$ then for $\tau - (\frac{\epsilon}{3})^2 < t \leq T$ we have

$$\begin{aligned} \|u(\cdot,t) - u_\infty\|_{C(D_{\frac{2}{3}\epsilon})} &\leq C_1 \|u_0 - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} e^{-\frac{4}{n+4}\lambda_1 t} \\ &\leq C_1 \|u_0 - u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} e^{\frac{4}{n+4}\lambda_1 (\frac{\epsilon}{3})^2} e^{-\frac{4}{n+4}\lambda_1 \tau}. \end{aligned}$$

It follows that

$$\|u-u_\infty\|_{C(D_{\frac{2}{3}\epsilon}\times(\tau-(\frac{\epsilon}{3})^2,T])} \leq C_2\|u_0-u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} e^{-\frac{4}{n+4}\lambda_1\tau}. \tag{5.20}$$

By Lemma 8 if

$$\|u-u_\infty\|_{C(D_{\frac{2}{3}\epsilon}\times(\tau-(\frac{\epsilon}{3})^2,T])} < C\lambda\epsilon^{2+\alpha} \tag{5.21}$$

then

$$\begin{aligned}\operatorname{dist}_P^-\big(A_\epsilon(\tau)\cup\{u>0\}, A_\epsilon(\tau)\cup\{u_\infty>0\}\big)\\ \leq \big(\frac{1}{C\lambda}\|u-u_\infty\|_{C(D_{\frac{2}{3}\epsilon}\times(\tau-(\frac{\epsilon}{3})^2,T])}\big)^{\frac{1}{2+\alpha}}.\end{aligned} \tag{5.22}$$

One may see that if

$$\tau_1 = \frac{1}{\lambda_1}\frac{n+4}{4}\ln\big(\frac{C_2}{C\lambda}\frac{3^{2+\alpha}}{\epsilon^{2+\alpha}}\|u_0-u_\infty\|_{L^2(D)}^{\frac{4}{n+4}}\big) < \tau$$

then the right hand side of (5.20) is smaller than the right hand side of (5.21).

Thus if $\max\big(2(\frac{\epsilon}{3})^2, \tau_1\big) < \tau \leq T$ then from (5.20) and (5.22) it follows that

$$\begin{aligned}\operatorname{dist}_P^-\big(A_\epsilon(\tau)\cup\{u>0\}, A_\epsilon(\tau)\cup\{u_\infty>0\}\big)\\ \leq \big(\frac{C_2}{C\lambda}\|u_0-u_\infty\|_{L^2(D)}^{\frac{4}{n+4}} e^{-\frac{4}{n+4}\lambda_1\tau}\big)^{\frac{1}{2+\alpha}}\end{aligned}$$

and this proves the theorem. □

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