THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Some variants of Frobenius splitting

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Some variants of Frobenius splitting *

Kaneda Masaharu
Osaka City University
Department of Mathematics
kaneda@sci.osaka-cu.ac.jp

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Abstract

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic p, $F : G \to G^{(1)}$ the Frobenius morphism on G, $G_r = \ker F^r$, r a positive integer, and P a parabolic subgroup of G. If λ is a 1-dimensional P-module, G_rP-Verma module $\hat{\nabla}_P(\lambda) = \operatorname{ind}_P^{G_rP}(\lambda)$ of highest weight λ has a unique simple submodule $\hat{L}(\lambda)$. We show that the imbedding $i_P : \hat{L}(\lambda) \hookrightarrow \hat{\nabla}_P(\lambda)$ splits upon sheafification $\mathcal{L}_{G/G_rP}(i_P) : \mathcal{L}_{G/G_rP}(\hat{L}(\lambda)) \to \mathcal{L}_{G/G_rP}(\hat{\nabla}_P(\lambda))$ on G/G_rP.

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic p, B a Borel subgroup of G, and T a maximal torus of B. For simplicity we will assume that G is semi simple and simply connected. Let Λ be the character group of T, R the set of roots of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, and R^{s} the set of simple roots of R^+. We make Λ into a PO set with respect to R^+ such that $\lambda \geq \mu$, $\lambda, \mu \in \Lambda$, iff $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$. Let $I \subseteq R^{\mathrm{s}}$ and P the parabolic subgroup of G containing B associated to I. Put $\Lambda_P = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle = 0\}$, where $\alpha^\vee$ denotes the coroot of α. Put $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in \Lambda$ and $\rho_P = \frac{1}{2}\sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in \Lambda_P$ with $R_I^+ = R^+ \cap \mathbb{Z}I$ the set of positive roots of the standard Levi subgroup of P.

Let $F^r : G \to G^{(r)}$ be the r-th Frobenius morphism on G, r a positive integer, $G_r = \ker F^r$ the r-th Frobenius kernel of G. Let $\hat{\nabla}_P = \operatorname{ind}_P^{G_rP}$ be the induction functor from the category of P-modules to the category of G_rP-modules. For each $\lambda \in \Lambda_P$ we call $\hat{\nabla}_P(\lambda)$, induced from the 1-dimensional P-module afforded by λ, the G_rP-Verma module of highest weight λ. It has a unique simple submodule $\hat{L}(\lambda)$. We show that the imbedding $i_P : \hat{L}(\lambda) \hookrightarrow \hat{\nabla}_P(\lambda)$ splits upon sheafification $\mathcal{L}_{G/G_rP}(i_P) : \mathcal{L}_{G/G_rP}(i_P)(\hat{L}(\lambda)) \to \mathcal{L}_{G/G_rP}(i_P)(\hat{\nabla}_P(\lambda))$ on G/G_rP. If $q_P : G/P \to G/G_rP$ is the natural morphism, one has a commutative diagram

$$\begin{array}{ccc} G/P & \xrightarrow{\quad F^r \quad} & (G/P)^{(r)}. \\ {\scriptstyle q_P}\downarrow & \overset{\sim}{\nearrow}{\scriptstyle \phi} & \\ G/G_rP & & \end{array}$$

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In the case $\lambda = 0$ is the trivial P-module, under the identification via ϕ, $\mathcal{L}_{G/G_rP}(i_P)$ coincides with the comorphism $(F^r)^\sharp : \mathcal{O}_{(G/P)^{(r)}} \to (F^r)_*\mathcal{O}_{G/P}$ of the Frobenius morphism on G/P, the splitting of which has had many applications [MR], [BK].

Dually, the $\Bbbk$-linear dual $\hat{\nabla}_P(\lambda)^*$ of $\hat{\nabla}_P(\lambda)$ has a unique simple quotient $\hat{L}(2(p^r-1)\rho_P-\lambda)^*$ and the sheafification of the quotient $\hat{\nabla}_P(\lambda)^* \to \hat{L}(2(p^r-1)\rho_P-\lambda)^*$ splits. In the case $\lambda = 0$ the splitting gives a Frobenius cosplitting considered in [GK] and [K]. In particular, we remove the characteristic restriction assumed in the latter.

The author is grateful to the referee of [GK] who reminded him of a use of [AK].

1° The case of a Borel subgroup

For subgroup schemes $H \leq K$ of G we denote by ind_H^K the induction functor from the category of H-modules to the category of K-modules as in [J, I.3], and by $\mathcal{L}_{K/H}$ its sheafification, the functor from the category of H-modules to the category of quasi-coherent sheaves on K/H as in [J, I.5]. For a K-module M we call the sum of simple submodules (resp. the intersection of all maximal submodules) of M the socle (resp. radical) of M and denote it by $\mathrm{soc}M$ (resp. $\mathrm{rad}M$). We set $\mathrm{hd}M = M/\mathrm{rad}M$ and call it the head of M.

Fix a positive integer r. We will first deal with the case $P = B$. For short put $\hat{\nabla} = \mathrm{ind}_B^{G_rB}$, $\mathcal{B} = G/B$, $\mathcal{B}_r = G/G_rB$, and $q : \mathcal{B} \to \mathcal{B}_r$ the natural morphism.

(1.1) Set $\Lambda_r = \{\lambda \in \Lambda \mid \langle\lambda, \alpha^\vee\rangle \in [0, p^r[\ \forall\alpha \in R^{\mathrm{s}}\}$. For $\lambda \in \Lambda$ we let $\hat{L}(\lambda)$ denote the simple G_rB-module of highest weight λ. If we write $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda \in \Lambda_r$ and $\lambda^1 \in \Lambda$, one has an isomorphism $\hat{L}(\lambda) \simeq L(\lambda^0) \otimes p^r\lambda^1$ with $L(\lambda^0)$ the simple G-module of highest weight λ^0, and an isomorphism $\hat{\nabla}(\lambda) \simeq \hat{\nabla}(\lambda^0) \otimes p^r\lambda^1$ with $\hat{\nabla}(\lambda^0)$ having a unique simple submodule $L(\lambda^0)$ [J, II.9.2, 6]. Let $i : \hat{L}(\lambda) \to \hat{\nabla}(\lambda)$ and $i_0 : L(\lambda^0) \to \hat{\nabla}(\lambda^0)$ denote the inclusions.

By the commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_{\mathcal{B}_r}(\hat{L}(\lambda)) & \xrightarrow{\mathcal{L}_{\mathcal{B}_r}(i)} & \mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}_r(\lambda)) \\
\sim\downarrow & & \downarrow\sim \\
\mathcal{L}_{\mathcal{B}_r}(L(\lambda^0)) \otimes_{\mathcal{B}_r} \mathcal{L}_{\mathcal{B}_r}(p^r\lambda^1) & \xrightarrow[\mathcal{L}_{\mathcal{B}_r}(i_0)\otimes_{\mathcal{B}_r}\mathcal{L}_{\mathcal{B}_r}(p^r\lambda^1)]{} & \mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}_r(\lambda^0)) \otimes_{\mathcal{B}_r} \mathcal{L}_{\mathcal{B}_r}(p^r\lambda^1),
\end{array}$$

in order to show that $\mathcal{L}_{\mathcal{B}_r}(i)$ splits, we may assume $\lambda \in \Lambda_r$.

(1.2) Keep the notations of (1.1), and assume $\lambda \in \Lambda_r$. A splitting of $\mathcal{L}_{\mathcal{B}_r}(i)$ exists iff

$$\mathbf{Mod}_{\mathcal{B}_r}(\mathcal{L}_{\mathcal{B}_r}(i), \mathcal{L}_{\mathcal{B}_r}(L(\lambda))) : \mathbf{Mod}_{\mathcal{B}_r}(\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)), \mathcal{L}_{\mathcal{B}_r}(L(\lambda))) \to \mathbf{Mod}_{\mathcal{B}_r}(\mathcal{L}_{\mathcal{B}_r}(L(\lambda)), \mathcal{L}_{\mathcal{B}_r}(L(\lambda)))$$

is surjective. Then, by the commutative diagram

$$\begin{array}{ccc}
\Gamma(\mathcal{B}_r, \mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)^* \otimes L(\lambda))) & \xrightarrow{\Gamma(\mathcal{B}_r,\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(i^*\otimes L(\lambda)))} & \Gamma(\mathcal{B}_r, \mathcal{L}_{\mathcal{B}_r}(L(\lambda)^* \otimes L(\lambda))) \\
\sim\big\downarrow & & \big\downarrow\sim \\
\mathrm{ind}_{G_rB}^G(\hat{\nabla}(\lambda)^* \otimes L(\lambda)) & \xrightarrow{\mathrm{ind}_{G_rB}^G(i^*\otimes L(\lambda))} & \mathrm{ind}_{G_rB}^G(L(\lambda)^* \otimes L(\lambda)) \\
\sim\big\downarrow & & \big\downarrow\sim \\
\mathrm{ind}_{G_rB}^G(\hat{\nabla}(\lambda)^*) \otimes L(\lambda) & \xrightarrow[\mathrm{ind}_{G_rB}^G(i^*)\otimes L(\lambda)]{} & \mathrm{ind}_{G_rB}^G(L(\lambda)^*) \otimes L(\lambda),
\end{array}$$

we have only to show that $\mathrm{ind}_{G_rB}^G(i^*) : \mathrm{ind}_{G_rB}^G(\hat{\nabla}(\lambda)^*) \to \mathrm{ind}_{G_rB}^G(L(\lambda)^*)$ is surjective. As $\hat{\nabla}(\lambda)^* \simeq \hat{\nabla}(2(p^r-1)\rho - \lambda)$ by [J, II.9.2] and as $L(\lambda)^* \simeq L(-w_0\lambda)$ is the head of $\hat{\nabla}(2(p^r-1)\rho-\lambda)$ with w_0 denoting the element of the Weyl group W of G such that $w_0R^+ = -R^+$, if we denote by j the quotient $\hat{\nabla}(2(p^r-1)\rho-\lambda) \to L(-w_0\lambda)$, we are to show that $\mathrm{ind}_{G_rB}^G(j) : \mathrm{ind}_{G_rB}^G(\hat{\nabla}(2(p^r-1)\rho-\lambda)) \to \mathrm{ind}_{G_rB}^G(L(-w_0\lambda))$ is surjective. If $\nabla = \mathrm{ind}_B^G$, $\mathrm{ind}_{G_rB}^G(\hat{\nabla}(2(p^r-1)\rho-\lambda)) \simeq \nabla(2(p^r-1)\rho-\lambda)$. Letting ev denote the evaluations [J, I.3], one has also an isomorphism $\mathrm{ev} : \mathrm{ind}_{G_rB}^G(L(-w_0\lambda)) \to L(-w_0\lambda)$ by the tensor identity [J, I.3.6]. We have thus a commutative diagram

$$\begin{array}{ccccc}
\nabla(2(p^r-1)\rho-\lambda) & \overset{\sim}{\text{———}} & \mathrm{ind}_{G_rB}^G(\hat{\nabla}(2(p^r-1)\rho-\lambda)) & \xrightarrow{\mathrm{ind}_{G_rB}^G(j)} & \mathrm{ind}_{G_rB}^G(L(-w_0\lambda)) \\
& \searrow^{\mathrm{ev}'} & \mathrm{ev}\big\downarrow & & \sim\big\downarrow\mathrm{ev} \\
& & \hat{\nabla}(2(p^r-1)\rho-\lambda) & \underset{j}{\twoheadrightarrow} & L(-w_0\lambda).
\end{array}$$

It follows that $\mathcal{L}_{\mathcal{B}_r}(i)$ splits iff the evaluation morphism $\mathrm{ev}' : \nabla(2(p^r-1)\rho-\lambda) \to \hat{\nabla}(2(p^r-1)\rho-\lambda)$ is surjective. As $\{2(p^r-1)\rho-\lambda\}-(p^r-1)\rho$ is dominant, however, the surjectivity has been shown in [AK, 8.2]. We have thus proved

(1.3) **Theorem:** *If $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^0 \in \Lambda_r$ and $\lambda^1 \in \Lambda$, the imbedding $\mathcal{L}_{\mathcal{B}_r}(\hat{L}(\lambda)) \to \mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}_r(\lambda))$ splits to yield $L(\lambda^0)\otimes\mathcal{L}_{\mathcal{B}_r}(p^r\lambda^1)$ as a direct summand of $\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}_r(\lambda))$.*

(1.4) Let $\lambda \in \Lambda$. Dualizing $i : \hat{L}(\lambda) \hookrightarrow \hat{\nabla}(\lambda)$, $i^* : \hat{\nabla}(\lambda)^* \twoheadrightarrow \hat{L}(\lambda)^*$ reads by (1.2) as the quotient $j : \hat{\nabla}(2(p^r-1)\rho-\lambda) \to L(-w_0\lambda^0)\otimes(-p^r\lambda^1)$ to the head of $\hat{\nabla}(2(p^r-1)\rho-\lambda)$. As $\mathcal{L}_{\mathcal{B}_r}(i)$ splits, so does $\mathcal{L}_{\mathcal{B}_r}(i)^\vee = \mathcal{L}_{\mathcal{B}_r}(i^*) = \mathcal{L}_{\mathcal{B}_r}(j)$. We have shown

Theorem: *For each $\lambda\in\Lambda$ the sheafification $\mathcal{L}_{\mathcal{B}_r}(j) : \mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)) \to \mathcal{L}_{\mathcal{B}_r}(\mathrm{hd}_{G_rB}\hat{\nabla}(\lambda))$ on $\mathcal{B}_r$ of the quotient $j : \hat{\nabla}(\lambda) \to \mathrm{hd}_{G_rB}\hat{\nabla}(\lambda)$ splits.*

(1.5) Recall from [J, II.11.8] that $\hat{\nabla}(\lambda)$ is simple iff $\lambda \in (p^r-1)\rho + p^r\Lambda$.

Corollary: *For any $\lambda \in \Lambda\setminus\{(p^r-1)\rho+p^r\Lambda\}$, $\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda))$ has $\mathcal{L}_{\mathcal{B}_r}(\mathrm{soc}\hat{\nabla}(\lambda))\oplus\mathcal{L}_{\mathcal{B}_r}(\mathrm{hd}\hat{\nabla}(\lambda))$ as a direct summand. In particular, $\mathcal{O}_{\mathcal{B}^{(r)}}\oplus\{\mathcal{L}_{\mathcal{B}}(-\rho)^{(r)}\otimes L((p^r-2)\rho)\}$ is a direct summand of $F_*^r\mathcal{O}_{\mathcal{B}}$.*

Proof: We may assume $\lambda \in \Lambda_r$. By (1.4) we have a decomposition $\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)) = \mathcal{L}_{\mathcal{B}_r}(\mathrm{rad}\hat{\nabla}(\lambda)) \oplus \mathcal{L}_{\mathcal{B}_r}(\mathrm{hd}\hat{\nabla}(\lambda))$. Let $\theta : \mathrm{soc}\hat{\nabla}(\lambda) \hookrightarrow \mathrm{rad}\hat{\nabla}(\lambda)$ and $\eta : \mathrm{rad}\hat{\nabla}(\lambda) \hookrightarrow \hat{\nabla}(\lambda)$. As in (1.2), $\mathcal{L}_{\mathcal{B}_r}(\theta)$ splits iff $\mathrm{ind}_{G_rB}^{G}(\theta^*) : \mathrm{ind}_{G_rB}^{G}((\mathrm{rad}\hat{\nabla}(\lambda))^*) \to \mathrm{ind}_{G_rB}^{G}((\mathrm{soc}\hat{\nabla}(\lambda))^*)$ is surjective. The latter follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{ind}_{G_rB}^{G}(\hat{\nabla}(\lambda)^*) & \xrightarrow{\mathrm{ind}_{G_rB}^{G}(\eta^*)} & \mathrm{ind}_{G_rB}^{G}((\mathrm{rad}\hat{\nabla}(\lambda))^*) & \xrightarrow{\mathrm{ind}_{G_rB}^{G}(\theta^*)} & \mathrm{ind}_{G_rB}^{G}((\mathrm{soc}\hat{\nabla}(\lambda))^*) \\
{\scriptstyle \mathrm{ev}}\big\downarrow & & {\scriptstyle \mathrm{ev}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}}\ \wr \\
\hat{\nabla}(\lambda)^* & \underset{\eta^*}{\twoheadrightarrow} & (\mathrm{rad}\hat{\nabla}(\lambda))^* & \underset{\theta^*}{\twoheadrightarrow} & (\mathrm{soc}\hat{\nabla}(\lambda))^*.
\end{array}$$

2° The general case

We now consider the case $P = P_I$, $I \subseteq R^{\mathrm{s}}$. For short put $\hat{\nabla}_P = \mathrm{ind}_P^{G_rP}$, $\mathcal{P} = G/P$, $\mathcal{P}_r = G/G_rP$, and let $q_P : \mathcal{P} \to \mathcal{P}_r$, $\bar{\pi} : \mathcal{B} \to \mathcal{P}$, $\bar{\pi}_r : \mathcal{B}_r \to \mathcal{P}_r$ denote the natural morphisms. We have thus $q_P \circ \bar{\pi} = \bar{\pi}_r \circ q$.

(2.1) Let $\Lambda_P = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in I\}$ and put $\rho_P = \frac{1}{2}\sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in \Lambda_P$ with $R_I^+ = R^+ \cap \mathbb{Z}I$. Let $\lambda = \lambda^0 + p^r\lambda^1 \in \Lambda_P$ with $\lambda^0 \in \Lambda_r$ and $\lambda^1 \in \Lambda$.

Recall from [AbK, 1.4] that $\hat{\nabla}_P(\lambda)$ has a unique simple submodule $L(\lambda^0) \otimes p^r\lambda^1$, which we will denote by $\hat{L}(\lambda)$ again. Let $i_P : \hat{L}(\lambda) \to \hat{\nabla}(\lambda)$ be the inclusion. Recall also from [J, I.5.18] the functorial isomorphism $\bar{\pi}_{r*} \circ \mathcal{L}_{\mathcal{B}_r} \simeq \mathcal{L}_{\mathcal{P}_r} \circ \mathrm{ind}_{G_rB}^{G_rP}$. By the tensor identity one has isomorphisms $\bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)) \simeq \mathcal{L}_{\mathcal{P}_r}(\hat{\nabla}_P(\lambda))$ and $\bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(L(\lambda)) \simeq \mathcal{L}_{\mathcal{P}_r}(L(\lambda))$, and hence, using the inclusion $i : L(\lambda) \to \hat{\nabla}(\lambda)$ from §1, obtains a commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_{\mathcal{P}_r}(L(\lambda)) & \xrightarrow{\mathcal{L}_{\mathcal{P}_r}(i_P)} & \mathcal{L}_{\mathcal{P}_r}(\hat{\nabla}_P(\lambda)) \\
\wr\big\uparrow & & \big\uparrow\wr \\
\bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(L(\lambda)) & \underset{\bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(i)}{\longrightarrow} & \bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(\hat{\nabla}(\lambda)).
\end{array}$$

As $\mathcal{L}_{\mathcal{B}_r}(i)$ splits by (1.3), so does $\bar{\pi}_{r*}\mathcal{L}_{\mathcal{B}_r}(i)$, and hence also $\mathcal{L}_{\mathcal{P}_r}(i_P)$. We have proved

Theorem: *For any $\lambda \in \Lambda_P$ the inclusion $i_P : \hat{L}(\lambda) \to \hat{\nabla}_P(\lambda)$ splits upon sheafification on $\mathcal{P}_r$ to yield a direct summand $L(\lambda^0) \otimes \mathcal{L}_{\mathcal{P}_r}(p^r\lambda^1)$ of $\mathcal{L}_{\mathcal{P}_r}(\hat{\nabla}_P(\lambda))$.*

(2.2) **Corollary:** *For any $\lambda \in \Lambda_r \cap \lambda_P$ one has $\mathcal{O}_{\mathcal{P}^{(r)}}$ as a direct summand of $(F^r)_*\mathcal{L}_{\mathcal{P}}(\lambda)$.*

(2.3) Let $\lambda \in \Lambda_P$. Recall from [AbK, 1.2] an isomorphism of G_rP-modules $\hat{\nabla}_P(\lambda)^* \simeq \hat{\nabla}_P(2(p^r-1)\rho_P - \lambda)$. Thus, putting $\mu = 2(p^r-1)\rho_P - \lambda$, the $\Bbbk$-linear dual of i_P reads as the quotient $j_P : \hat{\nabla}_P(\mu) \to \mathrm{hd}_{G_rP}\hat{\nabla}_P(\mu)$ to the head of $\hat{\nabla}_P(\mu)$. We have

Theorem: *For any $\lambda \in \Lambda_P$ the quotient $j_P : \hat{\nabla}_P(\lambda) \to \mathrm{hd}_{G_rP}\hat{\nabla}_P(\lambda)$ splits upon sheafification on $\mathcal{P}_r$.*

(2.4) Note that the existence of a splitting of $\mathcal{L}_{\mathcal{P}_r}(i_P)$ in (2.1) is equivalent, as in (1.2), to the surjectivity of the evaluation morphism $\nabla(2(p^r-1)\rho_P-\lambda)=\mathrm{ind}_P^G(2(p^r-1)\rho_P-\lambda)\to\hat{\nabla}_P(2(p^r-1)\rho_P-\lambda)$, $\lambda\in\Lambda_r\cap\Lambda_P$. Thus

Corollary: *Let $\lambda\in\Lambda_P$.*

(i) If $\lambda\in\Lambda_r$, any nonzero G_rP-linear morphism $\nabla(2(p^r-1)\rho_P-\lambda)\to\hat{\nabla}_P(2(p^r-1)\rho_P-\lambda)$ is surjective.

(ii) If $\hat{\nabla}_P(\lambda)$ is not simple, $\mathcal{L}_{\mathcal{P}_r}(\mathrm{soc}\hat{\nabla}_P(\lambda))\oplus\mathcal{L}_{\mathcal{P}_r}(\mathrm{hd}\hat{\nabla}_P(\lambda))$ is a direct summand of $\mathcal{L}_{\mathcal{P}_r}(\hat{\nabla}_P(\lambda))$.

(2.5) **Remark:** If $\lambda\in\Lambda_r\cap\Lambda_P$, any nonzero G_rP-linear morphism $\nabla(\lambda)\to\hat{\nabla}_P(\lambda)$ is injective, as $\nabla(\lambda)$ has a unique simple G_r-submodule $L(\lambda)$.

(2.6) Let $\lambda\in\Lambda_P$. Recall from [AbK, 1.4] that $\mathrm{soc}\hat{\nabla}_P(\lambda)=L(\lambda^0)\otimes p^r\lambda^1$ and that $\mathrm{hd}\hat{\nabla}_P(\lambda)=L((w^I\bullet\lambda)^0)\otimes p^r\{(w^I)^{-1}\bullet(w^I\bullet\lambda)^1\}$, where $w^I=w_0w_I$ with $w_I\in\langle s_\alpha|\alpha\in I\rangle$ such that $w_II=-I$. Put $\mathrm{soc}^1\hat{\nabla}_P(\lambda)=\lambda^1$ and $\mathrm{hd}^1\hat{\nabla}_P(\lambda)=(w^I)^{-1}\bullet(w^I\bullet\lambda)^1$. Recall from [AbK, 1.1] that $2\rho_P=\sum_{\alpha\in R^{\mathrm{s}}\setminus I}n_\alpha\varpi_\alpha$ for some $n_\alpha\in[2,h]$, h the Coxeter number of W. If $\lambda^0=\sum_{\alpha\in R^{\mathrm{s}}\setminus I}\lambda_\alpha\varpi_\alpha$, $\lambda_\alpha\in[0,p^r[$, define $r_\alpha\in\mathbb{N}$, $\alpha\in R^{\mathrm{s}}\setminus I$, such that $r_\alpha p^r-(n_\alpha+\lambda_\alpha)\in[0,p^r[$. Then

$$(w^I\bullet\lambda^0)^1=-\sum_{\alpha\in R^{\mathrm{s}}\setminus I}r_\alpha\varpi_{-w_0\alpha}\tag{1}$$

and

$$\mathrm{hd}^1\hat{\nabla}_P(\lambda)=(w^I)^{-1}\bullet(w^I\bullet\lambda^0)^1+\lambda^1=\lambda^1+\sum_{\alpha\in R^{\mathrm{s}}\setminus I}(r_\alpha-n_\alpha)\varpi_\alpha.\tag{2}$$

Note, in particular, that $\mathrm{hd}^1\hat{\nabla}_P(\lambda)$ depends not only on λ but also on p.

Proposition: *Assume $I\neq R^{\mathrm{s}}$ and that $p\geq h-1$. Write $\lambda^0=\sum_{\alpha\in R^{\mathrm{s}}\setminus I}\lambda_\alpha\varpi_\alpha$, $\lambda_\alpha\in[0,p^r[$ $\forall\alpha\in R^{\mathrm{s}}\setminus I$, and define $r_\alpha\in\mathbb{N}$ as above. If all $r_\alpha=1$, $\alpha\in R^{\mathrm{s}}\setminus I$, then $\forall i\in\mathbb{N}$, $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda)),\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda)))=0$ while $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda)),\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda)))\simeq\delta_{i0}\nabla(2\rho_P-\sum_{\alpha\in R^{\mathrm{s}}\setminus I}\varpi_\alpha)$.*

Proof: We may assume $\lambda\in\Lambda_r$. Put $J=R^{\mathrm{s}}\setminus I$. $\forall i\in\mathbb{N}$,

$$\mathrm{Ext}^i_{\mathcal{P}^{(r)}}(\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda))^{(r)},\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda))^{(r)})\simeq\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda)),\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda)))^{(r)}$$

with

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda)),\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda)))&=\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{O}_{\mathcal{P}},\mathcal{L}_{\mathcal{P}}((w^I)^{-1}\bullet(-\sum_{\alpha\in J}\varpi_{-w_0\alpha})))\\&\simeq\mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}((w^I)^{-1}\bullet(-\sum_{\alpha\in J}\varpi_{-w_0\alpha})))\\&\simeq\mathrm{H}^i(\mathcal{B},\mathcal{L}_{\mathcal{B}}((w^I)^{-1}\bullet(-\sum_{\alpha\in J}\varpi_{-w_0\alpha})))\quad\text{as }(w^I)^{-1}\bullet(-\sum_{\alpha\in J}\varpi_{-w_0\alpha})\in\Lambda_P\\&=0\quad\text{by [J, II.5.5]}\end{aligned}$$

as $-\sum_{\alpha\in J}\varpi_{-w_0\alpha}$ belongs to the closure of the bottom dominant alcove by the hypothesis that $p\geq h-1$. Likewise

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\mathrm{hd}^1\hat{\nabla}_P(\lambda)),\mathcal{L}_{\mathcal{P}}(\mathrm{soc}^1\hat{\nabla}_P(\lambda)))&=\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\sum_{\alpha\in J}(r_\alpha-n_\alpha)\varpi_\alpha),\mathcal{O}_{\mathcal{P}})\quad\text{by (2)}\\ &\simeq \mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\sum_{\alpha\in J}(n_\alpha-1)\varpi_\alpha))\\ &\simeq \delta_{i0}\nabla(\sum_{\alpha\in J}(n_\alpha-1)\varpi_\alpha)=\delta_{i0}\nabla(2\rho_P-\sum_{\alpha\in J}\varpi_\alpha)\quad\text{by Kempf's vanishing [J, II.4.5].}\end{aligned}$$

(2.7) **Remark:** If $\lambda=0$ and $p\geq h$, we have from [K, 1.4] that all $r_\alpha=1$ to satisfy the assumption of (ii).

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