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Imaginary Verma Modules and Kashiwara Algebras for $U_q(\widehat{\mathfrak{sl}(2)})$.

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Abstract. We consider imaginary Verma modules for quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$ and construct Kashiwara type operators and the Kashiwara algebra $\mathcal{K}_q$. We show that a certain quotient $\mathcal{N}_q^-$ of $U_q(\widehat{\mathfrak{sl}(2)})$ is a simple $\mathcal{K}_q$-module.

1. Introduction

Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and Δ denote the set of roots with respect to the Cartan subalgebra $\widehat{\mathfrak{h}}$. Then we have a natural (standard) partition of $\Delta = \Delta_+ \cup \Delta_-$ into set of positive and negative roots. With respect to this standard partition we have a standard Borel subalgebra from which we may induce the standard Verma module. A partition $\Delta = S \cup -S$ of the root system Δ is said to be a closed partition if whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ we have $\alpha + \beta \in S$. It is well known that for any finite dimensional complex simple Lie algebra, all closed partitions of the root system are Weyl group conjugate to the standard partition. However, this is not the case for affine Lie algebras. The classification of closed subsets of the root system for affine Lie algebras was obtained by Jakobsen and Kac [**JK85, JK89**], and independently by Futorny [**Fut90, Fut92**]. In fact for affine Lie algebras there exists a finite number (≥ 2) of inequivalent Weyl orbits of closed partitions. Corresponding to each such non-standard partitions we have non-standard Borel subalgebras from which we can induce other non-standard Verma-type modules and these typically contain both finite and infinite dimensional weight spaces. The imaginary Verma module [**Fut94**] is a non-standard Verma-type module associated with the simplest non-standard partition of the root system Δ which is the focus of our study in this paper.

For generic q, the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is the q-deformations of the universal enveloping algebras of $\widehat{\mathfrak{g}}$ ([**Dri85**], [**Jim85**]). It is known [**Lus88**] that integrable highest weight modules of $\widehat{\mathfrak{g}}$ can be deformed to those over $U_q(\widehat{\mathfrak{g}})$ in such a way that the dimensions of the weight spaces are invariant under the deformation. Following the framework of [**Lus88**] and [**Kan95**], *quantum imaginary Verma modules* for $U_q(\widehat{\mathfrak{g}})$ were constructed in ([**CFKM97**], [**FGM98**]) and it was shown that these modules are deformations of those over the universal enveloping algebra $U(\widehat{\mathfrak{g}})$

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in such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved.

Lusztig [**Lus90**] from a geometric view point and Kashiwara [**Kas91**] from an algebraic view point introduced the notion of canonical bases (equivalently, global crystal bases) for standard Verma modules $V_q(\lambda)$ and integrable highest weight modules $L_q(\lambda)$. The crystal base ([**Kas90, Kas91**]) can be thought of as the $q = 0$ limit of the global crystal base or canonical base. An important ingredient in the construction of crystal base by Kashiwara in [**Kas91**], is a subalgebra $\mathcal{B}_q$ of the quantum group which acts on the negative part of the quantum group by left multiplication. This subalgebra $\mathcal{B}_q$, which we call the Kashiwara algebra, played an important role in the definition of the Kashiwara operators which defines the crystal base.

In this paper we construct an analog of Kashiwara algebra $\mathcal{K}_q$ for the imaginary Verma module $M_q(\lambda)$ for the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ by introducing certain Kashiwara-type operators. Then we prove that certain quotient $\mathcal{N}_q^-$ of $U_q(\widehat{\mathfrak{g}})$ is a simple $\mathcal{K}_q$-module. This generalizes the corresponding result in [**?**] for the quantum affine algebra $U_q(\widehat{sl(2)})$. However, it is worth pointing out that some of the arguments involving explicit calculations in [**?**] do not extend to this general case.

The paper is organized as follows. In Sections 2 and 3 we recall necessary definitions and some results that we need. In Section 4 we recall some facts about the imaginary Verma modules for $U_q(\widehat{\mathfrak{g}})$. In particular, for any dominant weight λ with $\lambda(c) = 0$ we give a necessary and sufficient condition for the reduced imaginary Verma module $\tilde{M}_q(\lambda)$ to be simple. In Section 5 we define certain operators we call Ω-operators acting on certain subalgebra $\mathcal{N}_q^-$ of $\tilde{M}_q(\lambda)$ and prove generalized commutation relations among them. We define the Kashiwara algebra $\mathcal{K}_q$ in terms of certain Drinfeld generators and the Ω-operators in Section 6 and show that $\mathcal{N}_q^-$ is a left $\mathcal{K}_q$-module and define a symmetric invariant bilinear form on $\mathcal{N}_q^-$. Finally, in Section 7 we prove that $\mathcal{N}_q^-$ is simple as a $\mathcal{K}_q$-module and that the form defined in Section 6 is nondegenerate.

2. Notation

2.1. The *quantum group* $U_q(A_1^{(1)})$ is the $\mathbb{F}(q^{1/2})$-algebra with 1 generated by

$$e_0, e_1, f_0, f_1, K_0^{\pm 1}, K_1^{\pm 1}, D^{\pm 1}$$

with defining relations:

$$\begin{aligned}
&DD^{-1} = D^{-1}D = K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
&e_if_j - f_je_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&K_ie_iK_i^{-1} = q^2e_i, \quad K_if_iK_i^{-1} = q^{-2}f_i,\\
&K_ie_jK_i^{-1} = q^{-2}e_j, \quad K_if_jK_i^{-1} = q^2f_j, \quad i \neq j,\\
&K_iK_j - K_jK_i = 0, \quad K_iD - DK_i = 0,\\
&De_iD^{-1} = q^{\delta_{i,0}}e_i, \quad Df_iD^{-1} = q^{-\delta_{i,0}}f_i,\\
&e_i^3e_j - [3]e_i^2e_je_i + [3]e_ie_je_i^2 - e_je_i^3 = 0, \quad i \neq j,\\
&f_i^3f_j - [3]f_i^2f_jf_i + [3]f_if_jf_i^2 - f_jf_i^3 = 0, \quad i \neq j,
\end{aligned}$$

where, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

The quantum group $U_q(A_1^{(1)})$ can be given a Hopf algebra structure with a comultiplication given by

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i,\\
\Delta(D) &= D \otimes D,\\
\Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i,\\
\Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i,
\end{aligned}$$

and an antipode given by

$$\begin{aligned}
s(e_i) &= -e_iK_i^{-1},\\
s(f_i) &= -K_if_i,\\
s(K_i) &= K_i^{-1},\\
s(D) &= D^{-1}.
\end{aligned}$$

There is an alternative realization for $U_q(A_1^{(1)})$, due to Drinfeld [**Dri85**], which we shall also need. Let U_q be the associative algebra with 1 over $\mathbb{F}(q^{1/2})$ generated by the elements $x_k^{\pm}$ $(k \in \mathbb{Z})$, h_l $(l \in \mathbb{Z} \setminus \{0\})$, $K^{\pm 1}$, $D^{\pm 1}$, and $\gamma^{\pm \frac{1}{2}}$ with the following defining relations:

$$DD^{-1} = D^{-1}D = KK^{-1} = K^{-1}K = 1, \tag{2.1}$$

$$[\gamma^{\pm\frac{1}{2}}, u] = 0 \quad \forall u \in U, \tag{2.2}$$

$$[h_k, h_l] = \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{2.3}$$

$$[h_k, K] = 0, \quad [D, K] = 0, \tag{2.4}$$

$$Dh_kD^{-1} = q^k h_k, \tag{2.5}$$

$$Dx_k^{\pm}D^{-1} = q^k x_k^{\pm}, \tag{2.6}$$

$$Kx_k^{\pm}K^{-1} = q^{\pm 2}x_k^{\pm}, \tag{2.7}$$

$$[h_k, x_l^{\pm}] = \pm\frac{[2k]}{k}\gamma^{\mp\frac{|k|}{2}}x_{k+l}^{\pm}, \tag{2.8}$$

$$\begin{aligned} x_{k+1}^{\pm}x_l^{\pm} &- q^{\pm 2}x_l^{\pm}x_{k+1}^{\pm} \\ &= q^{\pm 2}x_k^{\pm}x_{l+1}^{\pm} - x_{l+1}^{\pm}x_k^{\pm}, \end{aligned} \tag{2.9}$$

$$[x_k^+, x_l^-] = \frac{1}{q - q^{-1}}\left(\gamma^{\frac{k-l}{2}}\psi(k+l) - \gamma^{\frac{l-k}{2}}\phi(k+l)\right), \tag{2.10}$$

$$\text{where } \sum_{k=0}^{\infty}\psi(k)z^{-k} = K\exp\left((q - q^{-1})\sum_{k=1}^{\infty}h_k z^{-k}\right), \tag{2.11}$$

$$\sum_{k=0}^{\infty}\phi(-k)z^{k} = K^{-1}\exp\left(-(q - q^{-1})\sum_{k=1}^{\infty}h_{-k} z^{k}\right). \tag{2.12}$$

The algebras $U_q(A_1^{(1)})$ and U_q are isomorphic [**Dri85**]. The action of the isomorphism, which we shall call the *Drinfeld Isomorphism,* on the generators of $U_q(A_1^{(1)})$ is:

$$\begin{aligned} e_0 &\mapsto x_1^- K^{-1}, \quad f_0 \mapsto Kx_{-1}^+, \\ e_1 &\mapsto x_0^+, \quad f_1 \mapsto x_0^-, \\ K_0 &\mapsto \gamma K^{-1}, \quad K_1 \mapsto K, \quad D \mapsto D. \end{aligned}$$

If one uses the formal sums

$$\phi(u) = \sum_{p\in\mathbb{Z}}\phi(p)u^{-p}, \;\; \psi(u) = \sum_{p\in\mathbb{Z}}\psi(p)u^{-p}, \;\; x^{\pm}(u) = \sum_{p\in\mathbb{Z}}x^{\pm}(p)u^{-p} \tag{2.13}$$

Drinfeld's relations (3), (8)-(10) can be written as

$$[\phi(u), \phi(v)] = 0 = [\psi(u), \psi(v)] \tag{2.14}$$

$$\phi(u)x^{\pm}(v)\phi(u)^{-1} = g(uv^{-1}\gamma^{\mp 1/2})^{\pm 1}x^{\pm}(v) \tag{2.15}$$

$$\psi(u)x^{\pm}(v)\psi(u)^{-1} = g(vu^{-1}\gamma^{\mp 1/2})^{\mp 1}x^{\pm}(v) \tag{2.16}$$

$$(u - q^{\pm 2}v)x^{\pm}(u)x^{\pm}(v) = (q^{\pm 2}u - v)x^{\pm}(v)x^{\pm}(u) \tag{2.17}$$

$$[x^+(u), x^-(v)] = (q - q^{-1})^{-1}(\delta(u/v\gamma)\psi(v\gamma^{1/2}) - \delta(u\gamma/v)\phi(u\gamma^{1/2})) \tag{2.18}$$

where $g(t) = g_q(t) = \sum_{k\geq 0} g(r)t^k$ is the Taylor series at $t = 0$ of the function $(q^2t - 1)/(t - q^2)$ and $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ is the formal Dirac delta function.

REMARK 2.1.1. *Writing* $g(t) = g_q(t) = \sum_{r\geq 0} g(r)t^r$ *we have*

$$g(r) = g_q(r) = g_{q^{-1}}(r) = \begin{cases} q^2 & \text{if} \quad r=0 \\ (1-q^{-4})q^{2(r+1)} = (q^4-1)q^{2(r-1)}, & \text{if} \quad r>0. \end{cases} \tag{2.19}$$

Considering Serre's relation with $k = l$, we get

$$x_k^- x_{k+1}^- = q^2 x_{k+1}^- x_k^- \tag{2.20}$$

The product on the right side is in the correct order for a basis element. If $k+1 > l$ and $k \neq l$ in (2.9), then $k+1 > l+1$ so that $k \geq l+1$, and thus we can write

$$x_l^- x_{k+1}^- = q^2 x_{k+1}^- x_l^- - x_k^- x_{l+1}^- + q^2 x_{l+1}^- x_k^- \tag{2.21}$$

and then after repeating the above identity, we will eventually arrive at sums of terms that are in the correct order. This is the opposite ordering of monomials as we had previously.

3. Ω-operators and their relations

Let $\mathbb{N}^{\mathbb{N}^*}$ denote the set of all functions from $\{k\delta \,|\, k \in \mathbb{N}^*\}$ to $\mathbb{N}$ with finite support. Then we can write

$$h^+ = h_+^{(s_k)} := h_{r_1}^{s_1} \cdots h_{r_l}^{s_l}, \quad h^- := h_-^{(s_k)} = h_{-r_1}^{s_1} \cdots h_{-r_l}^{s_l}$$

for $f = (s_k) \in \mathbb{N}^{\mathbb{N}^*}$ whereby $f(r_k) = s_k$ and $f(t) = 0$ for $t \neq r_i, 1 \leq i \leq l$.

Consider now the subalgebra $\mathcal{N}_q^-$, generated by $\gamma^{\pm 1/2}$, and x_l^-, $l \in \mathbb{Z}$. Note that the corresponding relations (9) hold in $\mathcal{N}_q^-$. Consider $x^-(v) = \sum_m x_m^- v^{-m}$ as a formal power series of left multiplication operators $x_m^- : \mathcal{N}_q^- \to \mathcal{N}_q^-$.

As in our previous paper we set

$$\bar{P} = x^-(v_1) \cdots x^-(v_k)$$
$$\bar{P}_l = x^-(v_1) \cdots x^-(v_{l-1}) x^-(v_{l+1}) \cdots x^-(v_k),$$

and

$$G_l = G_l^{1/q} := \prod_{j=1}^{l-1} g_{q^{-1}}(v_j/v_l), \quad G_l^q = \prod_{j=1}^{l-1} g(v_l/v_j)$$

where $G_1 := 1$. As in our previous work we define a collection of operators $\Omega_\psi(k), \Omega_\phi(k) : \mathcal{N}_q^- \to \mathcal{N}_q^-$, $k \in \mathbb{Z}$, in terms of the generating functions

$$\Omega_\psi(u) = \sum_{l\in\mathbb{Z}} \Omega_\psi(l) u^{-l}, \quad \Omega_\phi(u) = \sum_{l\in\mathbb{Z}} \Omega_\phi(l) u^{-l}$$

by setting

$$\Omega_\psi(u)(\bar{P}) := \gamma^m \sum_{l=1}^{k} G_l \bar{P}_l \delta(u/v_l\gamma) \tag{3.1}$$

$$\Omega_\phi(u)(\bar{P}) := \gamma^m \sum_{l=1}^{k} G_l^q \bar{P}_l \delta(u\gamma/v_l). \tag{3.2}$$

Note that $\Omega_\psi(u)(1) = \Omega_\phi(u)(1) = 0$. More generally let us write

$$\bar{P} = x^-(v_1) \cdots x^-(v_k) = \sum_{n\in\mathbb{Z}} \sum_{\substack{n_1,n_2,\dots,n_k\in\mathbb{Z} \\ n_1+\cdots+n_k=n}} x_{n_1}^- \cdots x_{n_k}^- v_1^{-n_1} \cdots v_k^{-n_k}$$

Then

$$\begin{aligned}&\psi(u\gamma^{-1/2})\Omega_\psi(u)(\bar{P})\\&=\sum_{k\geq 0}\sum_{p\in\mathbb{Z}}\sum_{n_i\in\mathbb{Z}}\gamma^{k/2}\psi(k)\Omega_\psi(p)(x^-_{n_1}\cdots x^-_{n_k})v_1^{-n_1}\cdots v_k^{-n_k}u^{-k-p}\\&=\sum_{n_i\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\geq 0}\gamma^{k/2}\psi(k)\Omega_\psi(m-k)(x^-_{n_1}\cdots x^-_{n_k})v_1^{-n_1}\cdots v_k^{-n_k}u^{-m}\end{aligned}$$

while

$$[x^+(u),\bar{P}]=\sum_{m\in\mathbb{Z}}\sum_{n_1,n_2,\dots,n_k\in\mathbb{Z}}[x^+_m,x^-_{n_1}\cdots x^-_{n_k}]v_1^{-n_1}\cdots v_k^{-n_k}u^{-m}.$$

Thus for a fixed m and k-tuple $(n_1,\dots,n_k)$ the sum

$$\sum_{k\geq 0}\gamma^{k/2}\psi(k)\Omega_\psi(m-k)(x^-_{n_1}\cdots x^-_{n_k})$$

must be finite. Hence

$$\Omega_\psi(m-k)(x^-_{n_1}\cdots x^-_{n_k})=0, \tag{3.3}$$

for k sufficiently large.

Proposition 3.0.2. *Then*

$$\Omega_\psi(u)x^-(v)=\delta(v\gamma/u)+g_{q^{-1}}(v\gamma/u)x^-(v)\Omega_\psi(u), \tag{3.4}$$

$$\Omega_\phi(u)x^-(v)=\delta(u\gamma/v)+g(u\gamma/v)x^-(v)\Omega_\phi(u) \tag{3.5}$$

$$(q^2u_1-u_2)\Omega_\psi(u_1)\Omega_\psi(u_2)=(u_1-q^2u_2)\Omega_\psi(u_2)\Omega_\psi(u_1) \tag{3.6}$$

$$(q^2u_1-u_2)\Omega_\phi(u_1)\Omega_\phi(u_2)=(u_1-q^2u_2)\Omega_\phi(u_2)\Omega_\phi(u_1) \tag{3.7}$$

$$(q^2\gamma^2u_1-u_2)\Omega_\phi(u_1)\Omega_\psi(u_2)=(\gamma^2u_1-q^2u_2)\Omega_\psi(u_2)\Omega_\phi(u_1) \tag{3.8}$$

The identities in Proposition 3.0.2 can be rewritten as

$$(q^2v\gamma-u)\Omega_\psi(u)x^-(v)=(q^2v\gamma-u)\delta(v\gamma/u)+(q^2v\gamma-u)x^-(v)\Omega_\psi(u), \tag{3.9}$$

$$(q^2v-u\gamma)\Omega_\phi(u)x^-(v)=(q^2v-u\gamma)\delta(v/u\gamma)+(v-q^2u\gamma)x^-(v)\Omega_\phi(u) \tag{3.10}$$

which may be written out in terms of components as

$$\begin{aligned}&q^2\gamma\Omega_\psi(m)x^-(n+1)-\Omega_\psi(m+1)x^-_n\\&\quad=(q^2\gamma-1)\delta_{m,-n-1}+\gamma x^-_{n+1}\Omega_\psi(m)-q^2x^-_n\Omega_\psi(m+1),\end{aligned} \tag{3.11}$$

$$\begin{aligned}&q^2\Omega_\phi(m)x^-(n+1)-\gamma\Omega_\phi(m+1)x^-(n)\\&\quad=(q^2-\gamma)\delta_{m,-n-1}+x^-(n+1)\Omega_\psi(m)-q^2\gamma x^-(n)\Omega_\psi(m+1).\end{aligned} \tag{3.12}$$

We also have by (3.8)

$$\Omega_\psi(k)\Omega_\phi(m)=\sum_{r\geq 0}g(r)\gamma^{2r}\Omega_\phi(r+m)\Omega_\psi(k-r), \tag{3.13}$$

as operators on $\mathcal{N}_q^-$.

We can also write (3.4) in terms of components and as operators on $\mathcal{N}_q^-$

$$\Omega_\psi(k)x^-(m)=\delta_{k,-m}\gamma^k+\sum_{r\geq 0}g_{q^{-1}}(r)x^-(m+r)\Omega_\psi(k-r)\gamma^r. \tag{3.14}$$

The sum on the right hand side turns into a finite sum when applied to an element in $\mathcal{N}_q^-$, due to (3.3).

4. The Kashiwara algebra $\mathcal{K}_q$

The Kashiwara algebra $\mathcal{K}_q$ is defined to be the $\mathbb{F}(q^{1/2})$-subalgebra of End $(\mathcal{N}_q)$ generated by $\Omega_\psi(m), x_n^-, \gamma^{\pm 1/2}$, $m, n \in \mathbb{Z}$, $\gamma^{\pm 1/2}$. Then the $\gamma^{\pm 1/2}$ are central and the following relations (which are implied by (3.14)) are satisfied

$$(4.1)\quad \begin{aligned} &q^2\gamma\Omega_\psi(m)x_{n+1}^- - \Omega_\psi(m+1)x_n^- \\ &\qquad = (q^2\gamma - 1)\delta_{m,-n-1} + \gamma x_{n+1}^-\Omega_\psi(m) - q^2 x_n^-\Omega_\psi(m+1)\end{aligned}$$

$$(4.2)\quad q^2\Omega_\psi(k+1)\Omega_\psi(l) - \Omega_\psi(l)\Omega_\psi(k+1) = \Omega_\psi(k)\Omega_\psi(l+1) - q^2\Omega_\psi(l+1)\Omega_\psi(k)$$

$$(4.3)\quad x_l^- x_{k+1}^- - q^2 x_{k+1}^- x_l^- = q^2 x_{l+1}^- x_k^- - x_k^- x_{l+1}^-$$

together with

$$\gamma^{1/2}\gamma^{-1/2} = 1 = \gamma^{-1/2}\gamma^{1/2}.$$

Proposition 4.0.3 (CFM). *There is a unique symmetric bilinear form (,) defined on $\mathcal{N}_q^-$ satisfying*

$$(x_m^- a, b) = (a, \Omega_\psi(-m)b), \quad (1,1) = 1.$$

For $\mathbf{m} = (m_1, \dots, m_n)$ set

$$x_{\mathbf{m}} = x_{m_1}^- \cdots x_{m_n}^-$$

and define the length of such a Poincare-Birkhoff-Witt basis element to be $|\mathbf{m}| = n$.

Proposition 4.0.4. *For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, and $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}^l$, if $n > l$, then*

$$(4.4)\quad (x_{\mathbf{m}}, x_{\mathbf{k}}) = 0.$$

On the other hand if $n = l$ with

$$m_1 \geq m_2 \geq \cdots \geq m_n, \quad k_1 \geq k_2 \geq \cdots \geq k_n,$$
$$\sum_{i=1}^n m_i = \sum_{i=1}^n k_i$$

we have

$$(4.5)\quad (x_{\mathbf{m}}, x_{\mathbf{k}}) \equiv \delta_{\mathbf{m},\mathbf{k}} \mod q^2\mathbb{Z}[\![q]\!].$$

and the form is symmetric.

Proof. The fact that the form is symmetric comes from Proposition 4.0.3 above. Suppose $n > l$. Then

$$\begin{aligned}(x_{m_1}\cdots x_{m_n}, x_{k_1}\cdots x_{k_l}) &= (x_{m_2}\cdots x_{m_n}, \Omega_\psi(-m_1)x_{k_1}\cdots x_{k_l}) \\ &= \delta_{m_1,k_1}(x_{m_2}\cdots x_{m_n}, x_{k_2}\cdots x_{k_n}) \\ &\qquad + \sum_{r\geq 0} g_{q^{-1}}(r)(x_{m_2}\cdots x_{m_n}, x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2}\cdots x_{k_l}).\end{aligned}$$

By the Serre relations (2.20) and (2.21)

$$x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2}\cdots x_{k_l}$$

is a sum of monomials of length $l-1$ we can use induction to see that

$$(x_{m_2}\cdots x_{m_n}, x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2}\cdots x_{k_l}) = 0.$$

Hence $(x_{m_1}\cdots x_{m_n}, x_{k_1}\cdots x_{k_l})=0$.

Now suppose $n=l$. For $n=1$ we have

$$(x_m,x_k)=(1,\Omega_\psi(-m)x_k^-)=\delta_{m,k}$$

by (3.14).

For $n=2$ we have by (3.14) for $m_1\geq m_2$, $k_1\geq k_2$ and $m_1+m_2=k_1+k_2$

$$\begin{aligned}(x_{m_1}x_{m_2},x_{k_1}x_{k_2})&=(x_{m_2},\Omega_\psi(-m_1)x_{k_1}x_{k_2})\\ &=\delta_{m_1,k_1}(x_{m_2},x_{k_2})+\sum_{r\geq 0}g_{q^{-1}}(r)(x_{m_2},x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2})\\ &=\delta_{\mathbf{m},\mathbf{k}}+\sum_{r\geq 0}g_{q^{-1}}(r)(x_{m_2},x_{k_1+r})\delta_{m_1+r,k_2}\\ &=\delta_{\mathbf{m},\mathbf{k}}+\sum_{r\geq 0}g_{q^{-1}}(r)\delta_{m_2,k_1+r}\delta_{m_1+r,k_2}\\ &=\delta_{\mathbf{m},\mathbf{k}}+H(k_2-m_1)g_{q^{-1}}(k_2-m_1)\delta_{m_2-k_1,k_2-m_1}\end{aligned}$$

where H is the Heaviside function given by $H(n)=1$ if $n\geq 0$ and $H(n)=0$ otherwise. Interchanging $(m_1,m_2)\leftrightarrow(k_1,k_2)$ in the above calculation we see that the $(x_{m_1}x_{m_2},x_{k_1}x_{k_2})=(x_{l_1}x_{k_2},x_{m_1}x_{m_2})$. Now if $k_2-m_1\neq 1$, then it is clear from (2.19), that $(x_{m_1}x_{m_2},x_{k_1}x_{k_2})\in\delta_{\mathbf{m},\mathbf{k}}+q^2\mathbb{Z}[\![q]\!]$. If $k_2-m_1=1$, then the second summand above is nonzero if and only if $m_2-k_1=1$. But then

$$m_1\geq m_2=k_1+1>k_1\geq k_2=m_1+1$$

which is impossible. Hence for $n=2$, we have (4.5).

Assume that (4.5) holds up to Poincare-Birkhoff-Witt monomials of length $n-1$. Let us first prove by induction that for all $1\leq i\leq n-1$ and any $p\in\mathbb{N}$,

$$(x_{m_2}^-\cdots x_{m_n}^-,x_{s_1}^-x_{s_2}^-\cdots x_{s_i}^-\Omega_\psi(-m_1-p)x_{s_{i+1}}^-\cdots x_{s_n}^-)\in\mathbb{Z}[\![q]\!],\tag{4.6}$$

$$(x_{m_2}^-\cdots x_{m_n}^-,x_{s_2}^-\cdots x_{s_n}^-)\in\mathbb{Z}[\![q]\!],\tag{4.7}$$

for any $\mathbf{s}=(s_2,\dots,s_n)\in\mathbb{Z}$ (so that $x_{s_2}^-\cdots x_{s_n}^-$ is not necessarily a PBW monomial). We say that $(s_2,\dots,s_n)$ has k ascending inversions if the number of pairs of indices (i,l) with $i<j$ and $s_i<s_j$ is k. Recall the Serre relations (2.20) and (2.21). Suppose there is an ascending inversion at the pair of indices $(i,i+1)$ with $s_i=k$ and $s_{i+1}=k+1$, then

$$(x_{m_2}^-\cdots x_{m_n}^-,x_{s_2}^-\cdots x_{s_i}^-x_{s_{i+1}}^-\cdots x_{s_n}^-)=q^2(x_{m_2}^-\cdots x_{m_n}^-,x_{s_2}^-\cdots x_{s_{i+1}}^-x_{s_i}^-\cdots x_{s_n}^-).\tag{4.8}$$

Then we have decreased the number of ascending inversions and by induction on the number of inversion on products of length $n-1$ we conclude

$$(x_{m_2}^-\cdots x_{m_n}^-,x_{s_2}^-\cdots x_{s_i}^-x_{s_{i+1}}^-\cdots x_{s_n}^-)\in\mathbb{Z}[\![q]\!].$$

Suppose there is an ascending inversion at the pair of indices $(i, i+1)$ with $s_i = l$ and $s_{i+1} = k+1$ with $l < k$, then

$$\begin{aligned}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+1}}\cdots x^-_{s_n}) &= (x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{l}x^-_{k+1}\cdots x^-_{s_n})\\ &= q^2(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{k+1}x^-_{l}\cdots x^-_{s_n})\\ &- (x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{k}x^-_{l+1}\cdots x^-_{s_n})\\ &+ q^2(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{l+1}x^-_{k}\cdots x^-_{s_n}).\end{aligned}$$

Observe that the number of ascending inversions in the first two summands has decreased by one and the last summand can also be rewritten as a sum of terms that have a decrease in the number of ascending inversions. By induction on the number of inversion on products of length $n-1$ we again conclude

$$(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+1}}\cdots x^-_{s_n}) \in \mathbb{Z}[\![q]\!]. \tag{4.9}$$

For the first statement (4.6) we begin at $i = n-1$. By (3.14) and (4.9) this is

$$\begin{aligned}&(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_{n-1}}\Omega_\psi(-m_1-p)x^-_{s_n})\\ &\qquad= \delta_{m_1+p,s_n}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_{n-1}}) \in \mathbb{Z}[\![q]\!].\end{aligned} \tag{4.10}$$

Suppose (4.6) is true for $i+1 \le n-1$. Then

$$\begin{aligned}&(x^-_{m_2}\cdots x^-_{m_n}x^-_{s_1}x^-_{s_2}\cdots x^-_{s_i}\Omega_\psi(-m_1-t)x^-_{s_{i+1}}\cdots x^-_{s_n})\\ &\quad= \delta_{m_1+t,s_{i+1}}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+2}}\cdots s^-_{k_n})\\ &\qquad+ \sum_{r\ge 0} g_{q^{-1}}(r)(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+1}+r}\Omega_\psi(-m_1-t-r)x^-_{s_{i+2}}\cdots x^-_{s_n})\\ &\quad\equiv \delta_{m_1+t,s_{i+1}}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+2}}\cdots x^-_{s_n})\\ &\qquad+ g_{q^{-1}}(1)\delta_{m_1+t+1,s_{i+2}}\left(x^-_{m_2}\cdots x^-_{m_n}, x^-_{s_1}x^-_{s_2}\cdots x^-_{s_i}x^-_{s_{i+1}+1}x^-_{s_{i+3}}\cdots x^-_{s_n}\right) \mod \mathbb{Z}[\![q]\!]\\ &\quad\equiv 0 \mod \mathbb{Z}[\![q]\!].\end{aligned}$$

Hence (4.6) is proved.

Now we want to prove a refined special case of (4.6): For any $1 \le i \le n-1$ and $t \in \mathbb{Z}_{\ge 0}$ one has

$$\begin{aligned}&(x^-_{m_2}\cdots x^-_{m_n}x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}\Omega_\psi(-m_1-t)x^-_{k_i}\cdots x^-_{k_n})\\ &\qquad\equiv \delta_{m_1+t,k_i}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_{i+1}}\cdots k^-_{k_n}) \mod q^2\mathbb{Z}[\![q]\!].\end{aligned} \tag{4.11}$$

Here we assume $k_1 \ge k_2 \ge \cdots \ge k_n$. For $i = n-1$ this is just (4.10). Now for any $t \ge 0$ we assume that (4.11) is true for $i+1 \le n-1$. Then by (4.6) and induction we have

$$\begin{aligned}
&(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}\Omega_\psi(-m_1-t)x^-_{k_i}\cdots x^-_{k_n})\\
&\quad=\delta_{m_1+t,k_i}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_{i+1}}\cdots x^-_{k_n})\\
&\qquad+\sum_{r\geq 0} g_{q^{-1}}(r)(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_i+r}\Omega_\psi(-m_1-t-r)x^-_{k_{i+1}}\cdots x^-_{k_n})\\
&\quad=\delta_{m_1+t,k_i}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_{i+1}}\cdots x^-_{k_n})\\
&\qquad+ g_{q^{-1}}(1)(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_i+1}\Omega_\psi(-m_1-t-1)x^-_{k_{i+1}}\cdots x^-_{k_n})\mod q^2\mathbb{Z}[\![q]\!]\\
&\quad\equiv\delta_{m_1+t,k_i}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_{i+1}}\cdots x^-_{k_n})\\
&\qquad+ g_{q^{-1}}(1)\delta_{m_1+t+1,k_{i+1}}\left(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_i+1}x^-_{k_{i+2}}\cdots x^-_{k_n}\right)\mod q^2\mathbb{Z}[\![q]\!]\\
&\quad\equiv\delta_{m_1+t,k_i}(x^-_{m_2}\cdots x^-_{m_n}, x^-_{k_1+1}x^-_{k_2+1}\cdots x^-_{k_{i-1}+1}x^-_{k_{i+1}}\cdots x^-_{k_n})\\
&\qquad+ g_{q^{-1}}(1)\delta_{m_1+t+1,k_{i+1}}\delta_{m_2,k_1+1}\cdots\delta_{m_{i+1},k_i+1}\delta_{m_{i+2},k_{i+2}}\cdots\delta_{m_n,k_n}\mod q^2\mathbb{Z}[\![q]\!].
\end{aligned}$$

where we used the fact that all monomials appearing are of PBW type with weakly decreasing indices. But the second summand in the last congruence above is nonzero only if

$$m_1\geq m_{i+1}=k_i+1>k_i\geq k_{i+1}=m_1+t+1$$

which is impossible for $t\geq 0$. Hence the second summand is zero modulo mod $q^2\mathbb{Z}[\![q]\!]$. This completes the proof of (4.11).

Now we show the induction step to complete the proof of the proposition:

$$\begin{aligned}
&(x_{m_1}\cdots x_{m_n}, x_{k_1}\cdots x_{k_n})=(x_{m_2}\cdots x_{m_n},\Omega_\psi(-m_1)x_{k_1}\cdots x_{k_n})\\
&\quad=\delta_{m_1,k_1}(x_{m_2}\cdots x_{m_n}, x_{k_2}\cdots x_{k_n})+\sum_{r\geq 0}g_{q^{-1}}(r)(x_{m_2}\cdots x_{m_n}, x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2}\cdots x_{k_n})\\
&\quad\equiv\delta_{\mathbf{m},\mathbf{k}}+g_{q^{-1}}(1)(x_{m_2}\cdots x_{m_n}, x_{k_1+1}\Omega_\psi(-m_1-1)x_{k_2}\cdots x_{k_n})\mod q^2\mathbb{Z}[\![q]\!]\\
&\quad\equiv\delta_{\mathbf{m},\mathbf{k}}+g_{q^{-1}}(1)\delta_{m_1+1,k_2}(x_{m_2}\cdots x_{m_n}, x_{k_1+1}x_{k_3}\cdots x_{k_n})\mod q^2\mathbb{Z}[\![q]\!]\\
&\quad\equiv\delta_{\mathbf{m},\mathbf{k}}+g_{q^{-1}}(1)\delta_{m_1+1,k_2}\delta_{m_2,k_1+1}\delta_{m_3,k_3}\cdots\delta_{m_n,k_n}\mod q^2\mathbb{Z}[\![q]\!]
\end{aligned}$$

where we used (4.6) in the third line and (4.11) in the fourth line. The second summand in the last congruence is nonzero if and only if $m_1+1=k_2, m_2=k_1+1, m_3=k_3,\dots m_n=k_n$. But this means that

$$m_1\geq m_2=k_1+1>k_1\geq k_2=m_1+1$$

which is a contradiction. This completes the proof of the proposition.

□

Corollary 4.0.5. *The form* (,) *is non-degenerate.*

Proof. Suppose $u\in\mathcal{N}_q^-$, with $(u,v)=0$ for all $v\in\mathcal{N}_q^-$ and say $u=\sum_{\mathbf{m}}a_{\mathbf{m}}x_{m_1}\cdots x_{m_n}$, then in particular this holds for any $v=x_{k_1}\cdots x_{k_n}$. Hence

$$0=(u,x_{k_1}\cdots x_{k_n})=\sum_{\mathbf{m}}a_{\mathbf{m}}(x_{m_1}\cdots x_{m_n},x_{k_1}\cdots x_{k_n})=a_{\mathbf{k}}.$$

Thus $a_{\mathbf{k}}=0$ for all $\mathbf{k}$. □

5. Imaginary Verma Modules for $A_1^{(1)}$

We begin by recalling some basic facts and constructions for the affine Kac-Moody algebra $A_1^{(1)}$ and its imaginary Verma modules. See [**Kac90**] for Kac-Moody algebra terminology and standard notations.

5.1. Let $\mathbb{F}$ be a field of characteristic 0. The algebra $A_1^{(1)}$ is the affine Kac-Moody algebra over field $\mathbb{F}$ with generalized Cartan matrix $A = (a_{ij})_{0\le i,j\le 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The algebra $A_1^{(1)}$ has a Chevalley-Serre presentation with generators $e_0, e_1, f_0, f_1, h_0, h_1, d$ and relations

$$\begin{aligned}
&[h_i, h_j] = 0, \quad [h_i, d] = 0,\\
&[e_i, f_j] = \delta_{ij} h_i,\\
&[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,\\
&[d, e_j] = \delta_{0,j} e_j, \quad [d, f_j] = -\delta_{0,j} f_j,\\
&(\mathrm{ad}\, e_i)^3 e_j = (\mathrm{ad}\, f_i)^3 f_j = 0, \quad i \neq j.
\end{aligned}$$

Alternatively, we may realize $A_1^{(1)}$ through the loop algebra construction

$$A_1^{(1)} \cong \mathfrak{s}l_2 \otimes \mathbb{F}[t, t^{-1}] \oplus \mathbb{F}c \oplus \mathbb{F}d$$

with Lie bracket relations

$$\begin{aligned}
&[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)c,\\
&[x \otimes t^n, c] = 0 = [d, c], \qquad [d, x \otimes t^n] = nx \otimes t^n,
\end{aligned}$$

for $x, y \in \mathfrak{s}l_2$, $n, m \in \mathbb{Z}$, where $(\ ,\ )$ denotes the Killing form on $\mathfrak{s}l_2$. For $x \in \mathfrak{s}l_2$ and $n \in \mathbb{Z}$, we write $x(n)$ for $x \otimes t^n$.

Let Δ denote the root system of $A_1^{(1)}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for Δ. Let $\delta = \alpha_0 + \alpha_1$, the minimal imaginary root. Then

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

5.2. The universal enveloping algebra $U(A_1^{(1)})$ of $A_1^{(1)}$ is the associative algebra over $\mathbb{F}$ with 1 generated by the elements $h_0, h_1, d, e_0, e_1, f_0, f_1$ with defining relations

$$\begin{aligned}
&[h_0, h_1] = [h_0, d] = [h_1, d] = 0,\\
&h_i e_j - e_j h_i = a_{ij} e_j, \quad h_i f_j - f_j h_i = -a_{ij} f_j,\\
&d e_j - e_j d = \delta_{0,j} e_j, \quad d f_j - f_j d = -\delta_{0,j} f_j,\\
&e_i f_j - f_j e_i = \delta_{ij} h_i,\\
&e_j e_i^3 - 3e_i e_j e_i^2 + 3e_i^2 e_j e_i - e_i^3 e_j = 0 \text{ for } i \neq j,\\
&f_j f_i^3 - 3f_i f_j f_i^2 + 3f_i^2 f_j f_i - f_i^3 f_j = 0 \text{ for } i \neq j.
\end{aligned}$$

Corresponding to the loop algebra formulation of $A_1^{(1)}$ is an alternative description of $U(A_1^{(1)})$ as the associative algebra over $\mathbb{F}$ with 1 generated by the elements

$e(k), f(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l \in \mathbb{Z} \setminus \{0\})$, c, d, h, with relations

$$\begin{aligned}
&[c, u] = 0 \quad \text{for all } u \in U(A_1^{(1)}),\\
&[h(k), h(l)] = 2k\delta_{k+l,0}c,\\
&[h, d] = 0, \quad [h, h(k)] = 0,\\
&[d, h(l)] = lh(l), \quad [d, e(k)] = ke(k), \quad [d, f(k)] = kf(k),\\
&[h, e(k)] = 2e(k), \quad [h, f(k)] = -2f(k),\\
&[h(k), e(l)] = 2e(k+l), \quad [h(k), f(l)] = -2f(k+l),\\
&[e(k), f(l)] = h(k+l) + k\delta_{k+l,0}c.
\end{aligned}$$

5.3. A subset S of the root system Δ is called *closed* if $\alpha, \beta \in S$ and $\alpha+\beta \in \Delta$ implies $\alpha+\beta \in S$. The subset S is called a *closed partition* of the roots if S is closed, $S \cap (-S) = \emptyset$, and $S \cup -S = \Delta$ [**JK85**],[**JK89**],[**Fut90**],[**Fut92**]. The set

$$S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$$

is a closed partition of Δ and is $W \times \{\pm 1\}$-inequivalent to the standard partition of the root system into positive and negative roots [**Fut94**].

For $\mathfrak{g} = A_1^{(1)}$, let $\mathfrak{g}_{\pm}^{(S)} = \sum_{\alpha \in S} \mathfrak{g}_{\pm\alpha}$. In the loop algebra formulation of $\mathfrak{g}$, we have that $\mathfrak{g}_{+}^{(S)}$ is the subalgebra generated by $e(k)$ $(k \in \mathbb{Z})$ and $h(l)$ $(l \in \mathbb{Z}_{>0})$ and $\mathfrak{g}_{-}^{(S)}$ is the subalgebra generated by $f(k)$ $(k \in \mathbb{Z})$ and $h(-l)$ $(l \in \mathbb{Z}_{>0})$. Since S is a partition of the root system, the algebra has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{-}^{(S)} \oplus \mathfrak{h} \oplus \mathfrak{g}_{+}^{(S)}.$$

Let $U(\mathfrak{g}_{\pm}^{(S)})$ be the universal enveloping algebra of $\mathfrak{g}_{\pm}^{(S)}$. Then, by the PBW theorem, we have

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_{-}^{(S)}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_{+}^{(S)}),$$

where $U(\mathfrak{g}_{+}^{(S)})$ is generated by $e(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l \in \mathbb{Z}_{>0})$, $U(\mathfrak{g}_{-}^{(S)})$ is generated by $f(k)$ $(k \in \mathbb{Z})$, $h(-l)$ $(l \in \mathbb{Z}_{>0})$ and $U(\mathfrak{h})$, the universal enveloping algebra of $\mathfrak{h}$, is generated by h, c and d.

Let $\lambda \in P$, the weight lattice of $\mathfrak{g} = A_1^{(1)}$. A $U(\mathfrak{g})$-module V is called a *weight* module if $V = \oplus_{\mu \in P} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, c \cdot v = \mu(c)v, d \cdot v = \mu(d)v\}.$$

Any submodule of a weight module is a weight module. A $U(\mathfrak{g})$-module V is called an *S-highest weight module* with highest weight λ if there is a non-zero $v_\lambda \in V$ such that (i) $u^+ \cdot v_\lambda = 0$ for all $u^+ \in U(\mathfrak{g}_{+}^{(S)}) \setminus \mathbb{F}^*$, (ii) $h \cdot v_\lambda = \lambda(h)v_\lambda$, $c \cdot v_\lambda = \lambda(c)v_\lambda$, $d \cdot v_\lambda = \lambda(d)v_\lambda$, (iii) $V = U(\mathfrak{g}) \cdot v_\lambda = U(\mathfrak{g}_{-}^{(S)}) \cdot v_\lambda$. An S-highest weight module is a weight module.

For $\lambda \in P$, let $I_S(\lambda)$ denote the ideal of $U(A_1^{(1)})$ generated by $e(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l > 0)$, $h - \lambda(h)1$, $c - \lambda(c)1$, $d - \lambda(d)1$. Then we define $M(\lambda) = U(A_1^{(1)})/I_S(\lambda)$ to be the *imaginary Verma module* of $A_1^{(1)}$ with highest weight λ. Imaginary Verma modules have many structural features similar to those of standard Verma modules, with the exception of the infinite-dimensional weight spaces. Their properties were investigated in [**Fut94**], from which we recall the following proposition [**Fut94**, Proposition 1, Theorem 1].

PROPOSITION 5.3.1. *(i)* $M(\lambda)$ *is a* $U(\mathfrak{g}_-^{(S)})$*-free module of rank 1 generated by the* S*-highest weight vector* $1 \otimes 1$ *of weight* λ*.*
(ii) $\dim M(\lambda)_\lambda = 1$*;* $0 < \dim M(\lambda)_{\lambda - k\delta} < \infty$ *for any integer* $k > 0$*; if* $\mu \neq \lambda - k\delta$ *for any integer* $k \geq 0$ *and* $M(\lambda)_\mu \neq 0$*, then* $\dim M(\lambda)_\mu = \infty$*.*
(iii) Let V *be a* $U(A_1^{(1)})$*-module generated by some* S*-highest weight vector* v *of weight* λ*. Then there exists a unique surjective homomorphism* $\varphi : M(\lambda) \to V$ *such that* $\varphi(1 \otimes 1) = v$*.*
(iv) $M(\lambda)$ *has a unique maximal submodule.*
(v) Let $\lambda, \mu \in P$*. Any non-zero element of* $Hom_{U(A_1^{(1)})}(M(\lambda), M(\mu))$ *is injective.*
(vi) $M(\lambda)$ *is irreducible if and only if* $\lambda(c) \neq 0$*.* □

Suppose now that $\lambda(c) = 0$. Consider an ideal $J(\lambda)$ of $U(\mathfrak{sl}(2, \mathbb{C})$ generated by $I_S(\lambda)$ and $h(l)$ for all l. Set

$$\tilde{M}(\lambda) = U(\mathfrak{sl}(2, \mathbb{C}))/J(\lambda).$$

Then $\tilde{M}(\lambda)$ is a homomorphic image of $M(\lambda)$ which we call the *reduced imaginary Verma module*. The module $\tilde{M}(\lambda)$ has a Λ-gradation:

$$\tilde{M}(\lambda) = \sum_{\xi \in \Lambda} \tilde{M}(\lambda)_\xi.$$

THEOREM 5.3.2 ([**?**]). *Let* $\lambda \in \Lambda$ *be such that* $\lambda(c) = 0$*. Then module* $\tilde{M}_q(\lambda)$ *is simple if and only if* $\lambda(h) \neq 0$*.*

6. The category $\mathcal{O}_{\mathbf{red,im}}$

Let G be the Heisenberg subalgebra, $G = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{\mathfrak{g}}_{k\delta} \oplus \mathbb{C}c$. We say that a nonzero $\hat{\mathfrak{g}}$-module V is G-compatible if

i). V has a decomposition $V = TF(V) \oplus T(V)$ into a sum of nonzero G-submodules such that
ii). G is bijective on $TF(V)$ (that any nonzero element $g \in G$ is a bijection on $TF(V)$) and $TF(V)$ has no nonzero $\hat{\mathfrak{g}}$-submodule,
iii). $G \cdot T(V) = 0$.

Consider the set

$$\mathfrak{h}^*_{red} := \{\lambda \in \mathfrak{h}^* \mid \lambda(c) = 0, \lambda(h) \notin \mathbb{Z}_{\geq 0}\}.$$

As usual let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The category $\mathcal{O}_{\text{red,im}}$ has as objects $\hat{\mathfrak{g}}$-modules M such that

(1)
$$M = \bigoplus_{\nu \in \mathfrak{h}^*_{red}} M_\nu, \quad \text{where} \quad M_\mu = \{m \in M \mid hm = \nu(h)m\}.$$
Note $\dim M_\nu$ may be infinite dimensional.

(2) $e_n = e \otimes t^n$ acts locally nilpotently for any $n \in \mathbb{Z}$.

(3) M is G-compatible.

The morphisms in the category are $\hat{\mathfrak{g}}$-module homomorphisms. For example direct sums of reduced imaginary Verma modules $\bar{M}(\lambda)$ are in the category $\mathcal{O}_{\text{red,im}}$. In this case $TF(\bar{M}(\lambda)) = \oplus_{k \in \mathbb{Z}, n \in \mathbb{Z}_{>0}} \bar{M}(\lambda)_{\lambda - n\alpha + k\delta}$ and $T(\bar{M}(\lambda)) = \bar{M}(\lambda)_\lambda \simeq \mathbb{C}$.

A loop module for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ is any representation of the form $\hat{M} := M \otimes \mathbb{C}[t, t^{-1}]$ where M is a highest weight module for $\mathfrak{sl}(2,\mathbb{C})$ and

$$(x \otimes t^k)(m \otimes t^l) := x \cdot m \otimes t^{k+l}, \quad c(m \otimes t^l) = 0.$$

Here $x \cdot m$ is the action of $x \in \mathfrak{sl}(2,\mathbb{C})$ on $m \in M$.

Proposition 6.0.3. *(1) The loop modules $\hat{M}$ with M in the category $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$ are not in $\mathcal{O}_{red,im}$.*
(2) For $\lambda, \mu \in \mathfrak{h}^*_{red}$ one has $Ext^1_{\hat{\mathfrak{g}}}(\bar{M}(\lambda), \bar{M}(\mu)) = 0$.

Proof. Suppose $\hat{M}$ is a loop module with $M \in \mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$. Then $\hat{M}$ satisfies condition (1) and (2) from above. Assume $\hat{M} = TF(\hat{M}) \oplus T(\hat{M})$ satisfies (i)- (iii) above. Now take any $\sum_{i=-k}^{k} m_i \otimes t^i \in T(\hat{M})$ with $m_i \in M_\mu$ for some weight μ. Then by (iii) we have

$$0 = h \otimes t^r \cdot \left(\sum_{i=-k}^{k} m_i \otimes t^i \right) = \lambda(h) \left(\sum_{i=-k}^{k} m_i \otimes t^{i+k} \right)$$

so that $\lambda(h) = 0$ which contradicts $\lambda \in \mathfrak{h}^*_{red}$. Then $T(\hat{M}) = 0$ and $\hat{M} = TF(\hat{M})$ which is a $\hat{\mathfrak{g}}$-module contradicting (i) and (ii) and thus (3).

For (2) we need to show that there are no nontrivial extensions between reduced imaginary Verma modules $\bar{M}(\lambda)$ and $\bar{M}(\mu)$. If $\mu = \lambda + k\delta$ for some integer k then any extension of $\bar{M}(\lambda)$ by itself has a two dimensional highest weight space of weight λ. Any highest weight vector in this space generates an irreducible submodule and thus the extension splits as a direct sum of two submodules each isomorphic to $\bar{M}(\lambda)$.

Indeed suppose now $\mu = \lambda + k\delta - s\alpha$ for some integers k and $s > 0$. Consider a short exact sequence

$$0 \longrightarrow \bar{M}(\lambda) \xrightarrow{\iota} M \xrightarrow{\pi} \bar{M}(\mu) \longrightarrow 0, \tag{6.1}$$

where we view ι as just the inclusion map. For any preimage weight vector $\bar{v}_\mu$ of a highest weight (w.r.t. $\mathfrak{sl}(2,\mathbb{C})$) vector v_μ in $\bar{M}(\mu)$ one has $G\bar{v}_\mu \in \bar{M}(\lambda)$. On the other hand $G\bar{v}_\mu = 0$. Suppose $0 \neq v = h_m\bar{v}_\mu$. Then $h_m\bar{v}_\mu = h_m v'$ for some $v' \in \bar{M}(\lambda)$ (one cannot have $h_m\bar{v}_\mu = \alpha v_\lambda$, $\alpha \in \mathbb{C}$ as otherwise $\mu + m\delta = \lambda$ and $s = 0$). Then $h_n(v' - \bar{v}_\mu) = 0$ and so $v' - \bar{v}_\mu \in T(M) = \mathbb{C}v_\lambda$ which is a contradiction to the fact $\bar{v}_\mu \notin \bar{M}(\lambda)$.

Recall that e_0 acts locally nilpotently on $\bar{v}_\mu$. Moreover, $e_0^t\bar{v}_\mu \neq 0$ if $t < s$, otherwise $e_0^{t-1}\bar{v}_\mu$ would generate a submodule in $\bar{M}(\lambda)$ which is a contradiction. So, $e_0^s\bar{v}_\mu = 0$ if $k = 0$ and $e_0^{s-1}\bar{v}_\mu = 0$ if $k \neq 0$. Without loss of generality we assume the latter. Suppose $s > 1$. Consider an $\mathfrak{sl}(2)$-subalgebra $\mathfrak{a}$ generated by e_0 and f_0 and an $\mathfrak{a}$-module generated by $\bar{v}_\mu$. This module is a non trivial extension of two Verma modules over $\mathfrak{a}$ with highest weights $\lambda + k\delta - \alpha$ and $\lambda + k\delta - s\alpha$. But this is impossible (e.g. these modules have different central characters). Suppose now $s = 1$. Then apply the same argument to an $\mathfrak{sl}(2)$-subalgebra generated by e_k and f_k. Assuming $e_k\bar{v}_\mu \neq 0$ we obtain a contradiction as above. Therefore, $M = \bar{M}(\lambda) \oplus \bar{M}(\mu)$ completing the proof.

□

Proposition 6.0.4. *If $M \in \mathcal{O}_{red,im}$ is a simple object, then $M \simeq \bar{M}(\lambda)$ for some $\lambda \in \mathfrak{h}^*_{red}$.*

PROOF. Consider any simple $M \in \mathcal{O}_{\mathrm{red,im}}$. Let $v \in T(M)$ be a nonzero element of some weight $\lambda \in \mathfrak{h}^*_{red}$. Then $Gv = 0$ and $e_0^N v = 0$ for some positive integer N. Choose N to be the least possible with such property. If $N = 1$ then $e_n v = 0$ for all integers n and hence M is a quotient of the reduced imaginary Verma module $\bar{M}(\lambda)$ with highest weight λ. Since $\lambda \in \mathfrak{h}^*_{red}$ then $\bar{M}(\lambda)$ is simple and thus $M \simeq \bar{M}(\lambda)$. Assume now that $N > 1$ and set $w = e_0^{N-1}v$. Then $e_0 w = 0$. We have $0 = h_{k\delta}e_0^N v = 2Ne_k e_0^{N-1}v = 2Ne_k w$ for all integers k. Therefore M is a quotient of the loop module induced from $U(G)w$ (with $e_k U(G)w = 0$ for all integers k). If $w \in T(M)$ then we are done. Suppose $w \notin T(M)$ so $0 \neq w \in TF(M)$ may be assumed to be a weight vector of weight μ. Then $W = U(G)w$ is a simple G-module in $TF(M)$. By a result of Chari [?] $W \simeq \mathbb{C}[t, t^{-1}]$. Consider the induced module $I(W) = Ind^{\hat{\mathfrak{g}}}_{G+N_++H} W$ where $N_+ = \oplus_{n\in\mathbb{Z}}\mathbb{C}e_n$ acts by zero on W, $H = \mathbb{C}h + \mathbb{C}d$ acts by $hw = \mu(h)w$ and $dw = \mu(d)w$. Since $U(G)w \subset TF(M)$ then it is easy to see that $TF(I(W)) = I(W)$. Hence the same holds for any of its quotients by the Short Five Lemma, i.e. $TF(M) = M$ which is a contradiction. Therefore $w \in T(M)$ which completes the proof. □

THEOREM 6.0.5. *If $M \in \mathcal{O}_{red,im}$ is any object then $M = \oplus_{\lambda_i \in \mathfrak{h}^*_{red}} \bar{M}(\lambda_i)$, $i \in I$ for some weights λ_i's.*

PROOF. Consider the subspace $T(M)$. Since the weights of M are in $\mathfrak{h}^*_{red}$, $T(M)$ is not a $\hat{\mathfrak{g}}$-submodule. Let $w \in T(M)$ be a nonzero element, $W = U(G)w \subset T(M)$. Arguing as in the proof of Proposition 6.0.4 we find a nonzero element $w' \in M$ such that $e_k w' = 0$ for all integers k. If $U(G)w' \neq \mathbb{C}w'$ then $w' \in TF(M)$ which is a contradiction. Hence w' generates a submodule isomorphic to a reduced imaginary Verma module containing W. Thus each nonzero element of $T(M)$ generates $\bar{M}(\lambda)$ for some λ. □

COROLLARY 6.0.6. *The category $\mathcal{O}_{red,im}$ is closed under taking subquotients and direct sums so it is a Serre category.*

7. Quantized Imaginary Verma modules

Let Λ denotes the weight lattice of $A_1^{(1)}$, $\lambda \in \Lambda$. Denote by $I^q(\lambda)$ the ideal of $U_q = U_q(\hat{\mathfrak{s}l}(2))$ generated by $x^+(k)$, $k \in \mathbb{Z}$, $a(l), l > 0$, $K^{\pm 1} - q^{\lambda(h)}1$, $\gamma^{\pm\frac{1}{2}} - q^{\pm\frac{1}{2}\lambda(c)}1$ and $D^{\pm 1} - q^{\pm\lambda(d)}1$. The imaginary Verma module with highest weight λ is defined to be ([**CFKM97**])

$$M_q(\lambda) = U/I^q(\lambda).$$

THEOREM 7.0.7 ([**CFKM97**], Theorem 3.6). *The imaginary Verma module $M_q(\lambda)$ is simple if and only if $\lambda(c) \neq 0$.*

Suppose now that $\lambda(c) = 0$. Then $\gamma^{\pm\frac{1}{2}}$ acts on $M_q(\lambda)$ by 1. Consider the ideal $J^q(\lambda)$ of U_q generated by $I^q(\lambda)$ and $a(l)$ for all l. Denote

$$\tilde{M}_q(\lambda) = U_q/J^q(\lambda).$$

Then $\tilde{M}_q(\lambda)$ is a homomorphic image of $M_q(\lambda)$ which we call the *reduced quantized imaginary Verma module*. The module $\tilde{M}_q(\lambda)$ has a Λ-gradation:

$$\tilde{M}_q(\lambda) = \sum_{\xi\in\Lambda} \tilde{M}_q(\lambda)_\xi.$$

Theorem 7.0.8 ([**?**]). *Let $\lambda \in \Lambda$ be such that $\lambda(c) = 0$. Then module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h) \neq 0$.*

8. The category $\mathcal{O}^q_{\mathbf{red,im}}$

Consider the set

$$\mathfrak{h}^*_{red} := \{\lambda \in \mathfrak{h}^* \mid \lambda(c) = 0, \lambda(h) \neq 0\}.$$

The category $\mathcal{O}^q_{\text{red,im}}$ has as objects $U_q(\hat{\mathfrak{g}})$-modules M such that there exists $\lambda_i \in \mathfrak{h}^*_{red}$, $i \in I$, with

$$M \cong \bigoplus_{i \in I} \tilde{M}_q(\lambda_i).$$

The morphisms in the category are just $U_q(\hat{\mathfrak{g}})$-module homomorphisms. Since $\bar{M}_q(\lambda)$ is a quantization of $\bar{M}(\lambda)$ in the sense of Lusztig, modules in $\mathcal{O}^q_{\text{red,im}}$ are quantizations of modules in $\mathcal{O}_{\text{red,im}}$. So equivalently the category $\mathcal{O}^q_{\text{red,im}}$ can be defined as follows:

Let G_q be the quantized Heisenberg subalgebra, $G_q = \sum_{k \in \mathbb{Z}\setminus\{0\}} \mathbb{F}K_{k\delta} \oplus \mathbb{F}\gamma$. We say that a nonzero $U_q(\hat{\mathfrak{sl}}(2))$-module V is G_q-compatible if

i). V has a decomposition $V = TF(V) \oplus T(V)$ into a sum of nonzero G_q-submodules such that

ii). G_q is bijective on $TF(V)$ (that any nonzero element $g \in G_q$ is a bijection on $TF(V)$) and $TF(V)$ has no nonzero $U_q(\widehat{\mathfrak{sl}(2,\mathbb{C})})$-submodule,

iii). $G_q \cdot T(V) = 0$.

The category $\mathcal{O}^q_{\text{red,im}}$ has as objects $U_q(\widehat{\mathfrak{sl}(2,\mathbb{C})})$-modules M such that

(1)

$$M = \bigoplus_{\nu \in \mathfrak{h}^*_{red}} M_\nu, \quad \text{where} \quad M_\nu = \{m \in M \mid Km = K^{\nu(h)}m,\ Dm = q^{\nu(d)}m\},$$

(2) x^+_n, $n \in \mathbb{Z}$ act locally nilpotently,

(3) M is G_q-compatible.

If $M \in \mathcal{O}^q_{\text{red,im}}$, we can write $M = \oplus_i \tilde{M}_q(\lambda_i)$ with $\tilde{M}_q(\lambda_i) = \oplus \mathbb{C}x^-_{n_1} \cdots x^-_{n_k} v_{\lambda_i}$. We define $\tilde{\Omega}_\psi(m)$ and $\tilde{x}^-_m$ on each $\tilde{M}_q(\lambda_i)$ as in (3.1):

$$\tilde{\Omega}_\psi(m)(x^-_{n_1} \cdots x^-_{n_k} v_{\lambda_i}) := \Omega_\psi(m)(x^-_{n_1} \cdots x^-_{n_k}) v_{\lambda_i} \tag{8.1}$$

$$\tilde{x}^-_m(x^-_{n_1} \cdots x^-_{n_k} v_{\lambda_i}) := x^-_m x^-_{n_1} \cdots x^-_{n_k} v_{\lambda_i}. \tag{8.2}$$

Hence the following result follows.

Theorem 8.0.9. *The operators $\tilde{\Omega}_\psi(m)$ and $\tilde{x}^-_m$ are well defined on objects in the category $\mathcal{O}^q_{red,im}$. Moreover on each summand $\tilde{M}_q(\lambda_i) \cong \mathcal{N}^-_q$ they agree with the $\Omega_\psi(m)$ respectively left multiplication by x^-_m defined as in (3.1).*

9. Imaginary $\mathbb{A}$-lattices and imaginary crystal basis

Let $\mathbb{A}_0$ (resp. $\mathbb{A}_\infty$) to be the ring of rational functions in $q^{1/2}$ with coefficients in a field $\mathbb{F}$ of characteristic zero, regular at 0 (resp. at ∞). Let $\mathbb{A} = \mathbb{C}[q^{1/2}, q^{-1/2}, \frac{1}{[n]_q}, n > 1]$, and $P = \{-k\alpha + m\delta \mid k > 0, m \in \mathbb{Z}\} \cup \{0\}$. Let M be a $U_q(\hat{\mathfrak{g}})$-module in the category. We call a free $\mathbb{A}_0$-submodule $\mathcal{L}$ of M an *imaginary crystal* $\mathbb{A}_0$*-lattice* of M if the following hold

(i). $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{A}_0} \mathcal{L} \cong M$,
(ii). $\mathcal{L} = \oplus_{\lambda \in P} \mathcal{L}_\lambda$ and $\mathcal{L}_\lambda = \mathcal{L} \cap \mathcal{M}_\lambda$,
(iii). $\tilde{\Omega}_\psi(m)\mathcal{L} \subseteq \mathcal{L}$ and $\tilde{x}_m^- \mathcal{L} \subseteq \mathcal{L}$ for all $m \in \mathbb{Z}$.

We now show that the above definition is not vacuous. Let $\lambda \in \mathfrak{h}^*$ and define

$$\mathcal{L}(\lambda) := \sum_{\substack{k \geq 0 \\ i_1 \geq \cdots \geq i_k, i_j \in \mathbb{Z}}} \mathbb{A} x_{i_1}^- \cdots x_{i_k}^- v_\lambda \subset \mathcal{N}_q^- v_\lambda = \tilde{M}_q(\lambda)$$

and also operators $\tilde{\Omega}_\psi(m) : \tilde{M}_q(\lambda) \to \tilde{M}_q(\lambda)$ and $\tilde{x}_m^- : \tilde{M}_q(\lambda) \to \tilde{M}_q(\lambda)$ where $\tilde{x}_m^-$ is the left multiplication operator by x_m^- and $\tilde{\Omega}_\psi(m)(x_{i_1}^- \cdots x_{i_k}^- v_\lambda) := \Omega_\psi(m)(x_{i_1}^- \cdots x_{i_k}^-)v_\lambda$ for $i_1 \geq \cdots \geq i_k$.

For $\mu = \lambda - k\alpha + m\delta$,

$$\tilde{M}_q(\lambda)_\mu = \begin{cases} \bigoplus_{\sum_{j=1}^k i_j = m, i_1 \geq \cdots \geq i_k} \mathbb{Q}(q^{1/2}) x_{i_1}^- \cdots x_{i_k}^- v_\lambda & \text{if } k > 0, \\ \mathbb{Q}(q^{1/2}) v_\lambda & \text{if } k = 0 \end{cases}$$

Now observe (i) is satisfied for $\mathcal{L} = \mathcal{L}(\lambda)$ as well as

(1) for $\mathcal{L}(\lambda)_\mu := \mathcal{L}(\lambda) \cap \tilde{M}_q(\lambda)_\mu$ one has $\mathcal{L}(\lambda) = \bigoplus_{\lambda \in P} \mathcal{L}(\lambda)_\mu$, and
(2)

$$\tilde{x}_m^- \mathcal{L}(\lambda) \subseteq \mathcal{L}(\lambda), \qquad \text{and} \qquad \tilde{\Omega}_\psi(m)\mathcal{L}(\lambda) \subseteq \mathcal{L}(\lambda)$$

where first statement follows from (2.20) and (2.21) and the last statement follows from (2.20), (2.21), (3.14) and the fact that $g_q(r) \in \mathbb{A}$ for $r \in \mathbb{Z}$ by (2.19). Thus $\mathcal{L}(\lambda)$ is an imaginary crystal lattice.

Proposition 9.0.10.

$$\mathcal{L}(\lambda) = \left\{ u \in \tilde{M}_q(\lambda) \,|\, (u, \tilde{M}_q(\lambda)) \subset \mathbb{A}_0 \right\}$$

If $\mathbb{F} = \mathbb{Q}$, *then*

$$\mathcal{L}(\lambda) = \left\{ u \in \tilde{M}_q(\lambda) \,|\, (u, u) \in \mathbb{A}_0 \right\}$$

Proof. Let R denote the right hand side of the above equality. We have the inclusion $\mathcal{L}(\lambda) \subseteq R$ by Proposition 4.0.4. For the other inclusion let $u \in R$ and by clearing denominators we can find a smallest $n \geq 0$ such that $q^{n/2}u \in \mathcal{L}(\lambda)$. If $n > 1$ then

$$(q^{n/2}u, \tilde{M}_q(\lambda)) \equiv 0 \mod q^{n/2}\mathbb{A}_0.$$

By Proposition 4.0.4 (,) is non-degenerate modulo $q^2\mathcal{L}(\lambda)$, we must have $q^{n/2}u \equiv 0 \mod q^2\mathcal{L}(\lambda)$. Hence $q^{(n/2)-2}u \in \mathcal{L}(\lambda)$ which contradicts the minimality of n. Thus $u \in \mathcal{L}(\lambda)$. □

For $\lambda \in \mathfrak{h}^*$ define

$$\mathcal{B}(\lambda) := \left\{ \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \,|\, i_1 \geq \cdots \geq i_k \right\}.$$

An *imaginary crystal basis* of a $U_q(\hat{\mathfrak{g}})$-module M in the category $\mathcal{O}^q_{\text{red,im}}$ is a pair $(\mathcal{L}, \mathcal{B})$ satisfying

(i). $\mathcal{L}$ is an imaginary crystal lattice of M,
(ii). $\mathcal{B}$ is an $\mathbb{F}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{F} \otimes_{\mathcal{A}_0} \mathcal{L}$,
(iii). $\mathcal{B} = \cup_{\mu \in P} \mathcal{B}_\mu$, where $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu / q\mathcal{L}_\mu)$,
(iv). $\tilde{x}_m^- \mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$ and $\tilde{\Omega}_\psi \mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$,

(v). For $m \in \mathbb{Z}$, then $\tilde{x}_m^- \tilde{\Omega}_\psi(-m) = \tilde{\Omega}_\psi(-m)\tilde{x}_m^-$. Moreover for fixed $b \in B$ there exists $m \gg 0$ such that $\tilde{x}_m^- \tilde{\Omega}_\psi(-m)b = \tilde{\Omega}_\psi(-m)\tilde{x}_m^- b$.

THEOREM 9.0.11. *For $\lambda \in \hat{\mathfrak{h}}^*_{red,im}$, the pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is an imaginary crystal basis of the reduced imaginary Verma module $\tilde{M}_q(\lambda)$.*

PROOF. Conditions (i)-(ii) are clear. For (iii) consider $b = \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda)$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. If $m \geq i_1$, then

$$\tilde{x}_m^- b = \tilde{x}_m^- \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{B}.$$

If $m = i_1 - 1$, then by (2.20) we have

$$\tilde{x}_m^- b = q^2 \tilde{x}_m^- \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) = 0 \mod q\mathcal{L}(\lambda).$$

If $m < i_1 - 1$, ($l = m$ and $k + 1 = i_1$ so $k = i_1 - 1$) then by (2.21) we have

$$\tilde{x}_m^- b \equiv -\tilde{x}_{i_1-1}^- \tilde{x}_{m+1}^- \tilde{x}_{i_2}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda)$$

and $i_1 - 1 \geq m + 1$. By induction this is either $0 \mod q\mathcal{L}(\lambda)$ or in $\pm\mathcal{B}$.

Next we have

$$\begin{aligned}\tilde{\Omega}_\psi(k)\tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda &= \delta_{k,-i_1}\tilde{x}_{i_2} \cdots \tilde{x}_{i_k}^- v_\lambda + \sum_{r\geq 0} g_{q^{-1}}(r)\tilde{x}_{i_1+r}^- \tilde{\Omega}_\psi(k-r)\tilde{x}_{i_2}^- \cdots \tilde{x}_{i_k}^- v_\lambda \\ &\equiv \delta_{k,-i_1}\tilde{x}_{i_2} \cdots \tilde{x}_{i_k}^- v_\lambda - \tilde{x}_{i_1+1}^- \tilde{\Omega}_\psi(k-1)\tilde{x}_{i_2}^- \cdots \tilde{x}_{i_k}^- v_\lambda\end{aligned}$$

Condition (v) is satisfied by (3.14). Indeed we begin by induction. If $b = v_\lambda$, then

$$\tilde{\Omega}_\psi(k)\tilde{x}_m^- b = \delta_{k,-m} b + \sum_{r\geq 0} g_{q^{-1}}(r)\tilde{x}_{m+r}^- \tilde{\Omega}_\psi(k-r) b = \delta_{k,-m} b$$

as $\tilde{\Omega}_\psi(k)v_\lambda = 0$ for all k. Now if $b = \tilde{x}_{i_1}^- v_\lambda$, then we have

$$\begin{aligned}\tilde{\Omega}_\psi(k)\tilde{x}_m^- b &= \delta_{k,-m} b + \sum_{r\geq 0} g_{q^{-1}}(r)\tilde{x}_{m+r}^- \Omega_\psi(k-r)\tilde{x}_{i_1}^- v_\lambda \\ &= \delta_{k,-m} b - \tilde{x}_{m+1}^- \Omega_\psi(k-1)\tilde{x}_{i_1}^- v_\lambda \\ &= \delta_{k,-m} b - \delta_{k-1,-i_1}\tilde{x}_{m+1}^- v_\lambda.\end{aligned}$$

Thus if $k = -m$, then $\tilde{\Omega}_\psi(k)\tilde{x}_m^- b = b - \delta_{-m-1,-i_1}\tilde{x}_{m+1}^- v_\lambda$. Next take $b = \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda$ with $i_1 \geq i_2 \geq \cdots \geq i_k$ and then

□

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References

[CFKM97] Ben Cox, Viatcheslav Futorny, Seok-Jin Kang, and Duncan Melville, *Quantum deformations of imaginary Verma modules*, Proc. London Math. Soc. (3) **74** (1997), no. 1, 52–80. MR 97k:17014

[CFM96] B. Cox, V. Futorny, and D. Melville, *Categories of nonstandard highest weight modules for affine Lie algebras*, Math. Z. **221** (1996), no. 2, 193–209. MR 97c:17036

[Cox94] Ben Cox, *Structure of the nonstandard category of highest weight modules*, Modern trends in Lie algebra representation theory (Kingston, ON, 1993), Queen's Papers in Pure and Appl. Math., vol. 94, Queen's Univ., Kingston, ON, 1994, pp. 35–47. MR 95d:17026

[Dri85] V. G. Drinfel′d, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064. MR MR802128 (87h:58080)

[FGM98] Viatcheslav M. Futorny, Alexander N. Grishkov, and Duncan J. Melville, *Quantum imaginary Verma modules for affine Lie algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **20** (1998), no. 4, 119–123. MR MR1662112 (99k:17029)

[Fut90] V. M. Futorny, *Parabolic partitions of root systems and corresponding representations of the affine Lie algebras*, Akad. Nauk Ukrain. SSR Inst. Mat. Preprint (1990), no. 8, 30–39.

[Fut92] ———, *The parabolic subsets of root system and corresponding representations of affine Lie algebras*, Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989) (Providence, RI), Contemp. Math., vol. 131, Amer. Math. Soc., 1992, pp. 45–52.

[Fut94] ———, *Imaginary Verma modules for affine Lie algebras*, Canad. Math. Bull. **37** (1994), no. 2, 213–218. MR 95a:17030

[Jim85] Michio Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69. MR MR797001 (86k:17008)

[JK85] H. P. Jakobsen and V. G. Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras*, Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984), Springer, Berlin, 1985, pp. 1–20. MR 87g:17020

[JK89] Hans Plesner Jakobsen and Victor Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II*, J. Funct. Anal. **82** (1989), no. 1, 69–90. MR 89m:17032

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990. MR 92k:17038

[Kan95] Seok-Jin Kang, *Quantum deformations of generalized Kac-Moody algebras and their modules*, J. Algebra **175** (1995), no. 3, 1041–1066. MR MR1341758 (96k:17023)

[Kas90] Masaki Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260. MR MR1090425 (92b:17018)

[Kas91] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516. MR MR1115118 (93b:17045)

[Lus88] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. in Math. **70** (1988), no. 2, 237–249. MR MR954661 (89k:17029)

[Lus90] ———, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498. MR MR1035415 (90m:17023)

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